Lecture Reference: Chapter 1, Sections 1.1 – 1.6

1. Basics

1.1. Differentiable manifolds and maps, tangent bundle, vector fields, Lie brackets (Review)
1.2. Integral curves and the flow of a vector field (Review)
1.3. Lie derivatives and geometric applications (Review)
1.4. Distributions and global Frobenius Theorem (Review)
1.5. Lie groups
1.6. Homogeneous spaces

Problems for Sections 1.5 and 1.6

28. Consider maps $P \rightarrow N \subset M$ between smooth manifolds, where $N$ is a weak differentiable submanifold of $M$, i.e. a subset and the inclusion is an immersion. Show that if $P \rightarrow N$ is continuous and $P \rightarrow M$ is differentiable, then so is $P \rightarrow N$. Give an example, where $P \rightarrow N$ is not continuous, but $P \rightarrow M$ is smooth. Why can this not happen when $N$ is a proper submanifold of $M$, i.e. carries the induced topology?

29. Let $G \supset H$, where $G$ is a Lie group and $H$ an abstract subgroup which is a weak differentiable submanifold. Show that $H$ is a Lie subgroup of $G$, i.e. the induced group multiplication $\mu: H \times H \rightarrow H$ is smooth as a mapping into $H$. [Hint: Argue that $H$ is integral manifold of a left-invariant involutive distribution $\Delta$ on $G$; then use $\Delta$-flat charts and Problem 28.]

30. Verify that the adjoint representation $ad: g \rightarrow gl(g)$ of a Lie algebra $g$ on itself (given by Lie multiplication) is a Lie algebra homomorphism.

31. $G$ Lie group. Why is $X$ defined by $X_h = L_{h\ast}v$ for $v \in T_eG$ a smooth left-invariant vector field on $G$?

32. Let $\varphi: G \rightarrow G$ be an abstract homomorphism of Lie groups. Argue that $\varphi$ is smooth if $\varphi$ is differentiable at $e$.

33. Suppose $H \subset G$ is an open Lie subgroup of the Lie group $G$. Show that $H$ is closed in $G$. Conclude that if $G$ is connected, then $G$ can be generated algebraically by the elements of an arbitrarily small neighborhood of $e$ in $G$.

34. Determine the right-invariant fields of the general linear group $GL(n, \mathbb{R})$ and their brackets.

35. (a) Show that the connected component $G_0$ of $e$ in $GL(n, \mathbb{R})$ is the (open and closed) normal Lie subgroup $G \supset G_0 = GL_n^+ = \{g | \det g > 0\}$ of orientation preserving transformations.
   (b) Show that the special linear group $SL_n = SL(n, \mathbb{R})$ is connected and not compact.

36. Prove that the special orthogonal group $SO(n)$ is connected and compact.

37. There is a canonical isomorphic imbedding $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ given as follows: If $GL(n, \mathbb{C}) \ni c = a + ib$ with $a, b \in M_n(\mathbb{R})$, then

\[ c \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \]
38. Give an example of a 2-dimensional non-abelian Lie group.

39. Let $G$ be a topological group, $H$ a subgroup. Prove that $G/H$ is hausdorff iff $H$ is closed in $G$. In particular, $G$ is hausdorff iff $\{e\}$ is closed.

40. Let $H \subset G$ be an invariant subgroup of a topological group $G$. Then $G/H$ is a topological group, i.e. the induced multiplication is continuous.

41. Prove the following consequence of Cantor’s Intersection Theorem: If a compact set $C \subset \mathbb{R}^n$ consists only of accumulation points, then $C$ is not countable.

42. Let $H \subset G$ be a Lie subgroup of the Lie group $G$. Discuss the converse of a theorem proved in class and fully establish that $H$ is closed in $G$ if and only if $H$ is a subspace, i.e. a proper submanifold.

43. Show that $G/H$ is a Lie group if $H$ is a closed invariant Lie subgroup.

44. Consider a differentiable atlas $\{\pi \circ T_g | g \in G\}$ as constructed in class for a homogeneous space $G/H$. Fill in the details and show that the projection $\pi: G \to G/H$ is a submersion and in fact a smooth (principal) fibration (with structure group $H$). Furthermore, $G$ acts smoothly (and transitively) on $G/H$ by left translations.

45. Write out the complete proof of the following basic theorem: Let $G^n$ act smoothly and transitively on the differentiable manifold $M$. Fix $p \in M$. Let $H$ denote the isotropy subgroup (= stabilizer) of $p$, i.e. $H = \{h \in G | h \cdot p = p\}$. Clearly, $H$ is a closed subgroup. Suppose that $H$ is a Lie subgroup. (This is always the case.) Then there is a canonical (unique and $G$-equivariant) diffeomorphism $\bar{f}: G/H \to M$ such that the orbit map $f: G \to M$ with $f(g) = g \cdot p$ factors as $f = \pi \circ \bar{f}$. Simply set $\bar{f}(gH) = f(g)$.

46. Discuss the different homogeneous structures for the spheres

$$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1) \cong SO(2n)/SO(2n-1) \cong O(2n)/O(2n-1).$$

47. Argue that $SO(3)$ is isomorphic to the group $S^3/\{−1,1\}$, where $S^3 \cong Sp(1) \cong Spin(3)$ is the group of unit quaternions, $\{−1,1\}$ is its center. Identifying $\mathbb{R}^3$ with the quaternions having real part 0 yields a map from $S^3$ to $SO(3)$ as follows: Associate with $q \in S^3$ the orthogonal map $\mathbb{H} \to \mathbb{H}$ via $x \to qxq^*$, leaving the span of $\{i,j,k\}$ among quaternions $\mathbb{H}$ invariant. This is a classical description of rotations in $\mathbb{R}^3$.

48. Prove inductively for the fundamental groups that $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$, $\pi_1(SO(2)) \cong \mathbb{Z}$. Note that $SO(3) \cong \mathbb{R}P^3$ and use the exact homotopy sequence for the fibration $SO(n-1) \to SO(n) \to S^{n-1}$.

49. Compute the dimension of the Grassmann manifold $G_{nk}$ of all $n$-dimensional linear subspaces of $\mathbb{R}^{n+k}$, as well as of the Stiefel manifold $V_{nk}$ of all oriented $n$-frames. $V_{nk}$ projects onto $G_{nk}$ in a natural way producing a fibration. What are the fibers?

50. The affine group $Aff(n)$ of transformations $\mathbb{R}^n \to \mathbb{R}^n$ of the form $x \mapsto Ax + b$ with $A \in GL_n$ and a translation $b \in \mathbb{R}^n$ has a linear representation in $GL_{n+1}$,

$$(x \mapsto Ax + b) \mapsto \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}.$$  

Explain why the homogeneous space $Aff(n)/GL(n)$ is canonically diffeomorphic to the affine subapece $\{(x,1) | x \in \mathbb{R}^n\}$ of $\mathbb{R}^{n+1}$. 