Special Basic Project 5

(due by 12/21/05 – 10:30am)

This project concerns only very basic aspects of Linear Algebra. It is meant to give those a chance who are in need of extra credit toward passing at a C level. It can **not** *be counted as credit toward the course grades A and B. Whenever you use a reference, quote it and do not copy. Use your own words.*

1. Let U and W be linear subspaces of a vector space V. Prove that the following two conditions are equivalent:

(a) $U + W = V$ and $U \cap W = 0$.

(b) For each vector $v \in V$ there are *unique* vectors $u \in U$ and $w \in W$ such that $v = u + w$.

In case V is finite dimensional, each of the above conditions is equivalent to:

(c) There exists a basis in V such that each vector in this basis belongs either to U or to W.

2. Consider the vectors in $\mathbb{R}^4$ defined by

$$v_1 = (1,0,1,1), \quad v_2 = (1,0,2,1), \quad v_3 = (1,2,0,1) \quad v_4 = (3,2,3,3)$$

(a) What is the dimension of the subspace W of $\mathbb{R}^4$ spanned by the four given vectors? Find a basis for W and extend it to a basis of $\mathbb{R}^4$.

(b) Use a basis of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space W. What can you say about the rank of such a T? What is therefore the precise condition on the values of T on that basis?

(c) Give an explicit example of an operator $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that the range of T is W.

3. Prove that the vectors

$$v_1 = (1,1,1,1), \quad v_2 = (1,1,2,1), \quad v_3 = (0,1,0,1), \quad v_4 = (1,1,1,0)$$

form a basis for $\mathbb{R}^4$. What are the coordinates of the vector (a,b,c,d) in this basis?

4. Let V be the vector space over $\mathbb{R}$ of all real polynomial functions p of degree at most 2.

(a) What are the coordinates of the polynomial function $a+bx+cx^2$ with respect to the ordered basis $\{1-x^2,\ 1+x+x^2,\ 1\}$ in V?

(b) For any fixed $h \in \mathbb{R}$ consider the *shift operator* $T\colon V \to V$ with $(Tp)(x) = p(x+h)$. Consider also the differentiation operator $D\colon V \to V$ with $Dp = p'$. Find the range, null space, rank and nullity of the operators TD, DT, D^2 and T^2. Which of these operators are isomorphisms? Write down the matrices of the operators TD, D^2 and T^2 with respect to the ordered basis $1, x, x^2$.

5. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1,x_2) = (-\frac{\sqrt{2}}{2}(x_1+x_2), \frac{\sqrt{2}}{2}(x_1-x_2))$.

(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

(b) Interpret the operation of T geometrically.

(c) What is the matrix of T in the ordered basis v_1, v_2, where $v_1 = (1,1)$ and $v_2 = (2,0)$?

(d) Prove that for every real number λ the operator $(T - \lambda I)$ is invertible.

(e) Find all *complex* numbers λ such that the operator $(T - \lambda I)$ is *not* invertible.

6. Let $T\colon V \to V$ be a linear operator on the vector space V with null space W_1 and range W_2. Suppose that $S\colon V \to V$ is another linear operator on V commuting with T, i.e. $ST = TS$. Prove that W_1, W_2 are invariant subspaces of both T and S.