This project concerns some simple applications of Linear Algebra to Fibonacci Numbers. Discuss the problem below in a concise and precise essay at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.

Let \( a_n \) denote the basic sequence of Fibonacci numbers defined by the recursive relation
\[
a_{n+2} = a_n + a_{n+1}, \quad a_0 = 1, a_1 = 1.
\]

Work out the problems below and find an explicit formula for \( a_n \).

1. Consider an operator \( T \) on the vector space \( \mathbb{R}^2 \) such that \( T \) maps the vector \((x, y)\) to the vector \((y, x + y)\). Show that \( T \) maps \((a_{n-2}, a_{n-1})\) to \((a_{n-1}, a_n)\), where \( a_n \) is the Fibonacci sequence. Write the matrix \( A \) of \( T \) in the standard basis and prove that
\[
A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.
\]

2. Diagonalize the operator \( T \) by finding its eigenvalues and eigenvectors. Show that the eigenvectors \( v_1, v_2 \) form a basis in \( \mathbb{R}^2 \). If \( B \) denotes the diagonal matrix of \( T \) with respect to this basis, verify that \( A = P^{-1}BP \), where the columns of the transition matrix \( P \) are \( v_1, v_2 \) in terms of the coordinates with respect to the standard basis.

3. Find \( B^n \) and conclude that \( A^n = P^{-1}B^nP \) from \( A = P^{-1}BP \). Now easily find \( A^n \) and write an explicit formula for the \( n \)-th Fibonacci number.