Summary of 5.4:

(o) **Zero Columns (Rows):** If A has a column or a row with only zero entries then $\det A = 0$. This follows in many ways, for example from (1), since any term there contains exactly one factor from each column (row). One can also use (2) as easily, directly for columns, recursively for rows. Another argument is to add a different row to the zero row, thus producing a matrix with two equal rows. For columns one could transpose and use (ii).

(i) **Upper (Lower) Triangular Matrices A:** This means $A_{ij} = 0$ for $i > j$ ($i < j$). In that case the determinant is simply the product of the diagonal elements,

$$\det A = A_{11} \cdots A_{nn} \; .$$

This follows immediately from Laplace Expansion (2) with respect to the first (last) column, by induction. Alternatively, in (1) all terms but one (for the identity permutation) vanish. Why?

(ii) **Transposed Matrix:** If A is any $m \times n$ matrix, then the *transposed* of A is defined to be the $n \times m$ matrix A^t with $(A^t)_{ij} = A_{ji}$. So, A and A^t have their rows and columns interchanged (in order), and one is obtained from the other by reflection of all entries in the main diagonal. For square $n \times n$ matrices we have

$$\det A^t = \det A \; .$$

One can argue directly with (1), as in the text. Even simpler is the following recursive argument based on Laplace Expansion: Symmetrize (2) by summing also over j to obtain $n \det A = \sum_{i,j=1}^n (-1)^{i+j} A_{ij} \det \hat{A}_{ij} = \sum_{i,j=1}^n (-1)^{j+i} A_{ji} \det \hat{A}_{ji} = \sum_{i,j=1}^n (-1)^{i+j} (A^t)_{ij} \det \hat{A^t}_{ij} = n \det A^t$. Observe that $\hat{A^t}_{ij} = (\hat{A}_{ji})^t$ and $\hat{A}_{ji}$ are transposed matrices of size $(n-1) \times (n-1)$, and thus have the same determinant, by the induction assumption.

(iii) **Row and Column Operations:** Except possibly for small size matrices or special situations, the best way to compute determinants numerically is by elementary row operations and reduction to upper triangular form and using (i). Note that adding any multiple of one row to another does not change the determinant, as it is n-linear and alternating. However, one has to keep track of multiplying a row by a scalar, or switching rows. In light of (ii), column operation can be used as well, or any combination with row operations.

(iv) **Invertibility:** Recall that the row (column) rank of A is the dimension of the span of the rows (columns) of A, and these numbers are equal and called the rank $\mathrm{rk}A$. If $\mathrm{rk}A$ is not maximal, i.e. $\mathrm{rk}A < n$, then the rows (columns) of A are linearly dependent, and A is singular or not invertible. In this case $\det A = 0$. Why? Some row must be a linear combination of the others. Expanding the determinant by linearity in this row yields a linear combination of determinants of matrices with two equal rows, which are all zero.

On the other hand, assume A has maximal rank $\mathrm{rk}A = n$. Then A has an inverse A^{-1} so that $AA^{-1} = I$. It follows that $\det A \cdot \det A^{-1} = \det I = 1$, so

$$\det A^{-1} = \frac{1}{\det A} \; .$$

In particular, $\det A \neq 0$. Therefore, A is invertible precisely when $\det A \neq 0$, and singular when $\det A = 0$.

(v) **The Classical Adjoint:** For any $n \times n$ matrix A define the *classical adjoint* $\operatorname{adj} A$ as a matrix of same size by $(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det \hat{A}_{ji}$. Note the transposition, which is essential for the following identity,

$$\operatorname{adj} A \cdot A = A \cdot \operatorname{adj} A = \det A \cdot I_n \ .$$

This is an immediate consequence of Laplace Expansion. We have $(\operatorname{adj} A \cdot A)_{ij} = \sum_{k=1}^{n} (\operatorname{adj} A)_{ik} A_{kj} = \sum_{k=1}^{n} (-1)^{i+k} A_{kj} \det \hat{A}_{ki}$. Now for $j = i$, the last sum is simply $\det A$ by (2). It is zero when $j \neq i$. Why? Simply replace the ith column of A by its jth column. The last sum then becomes the Laplace Expansion of the determinant of this new matrix with two equal columns, which is zero. The computation for $A \cdot \operatorname{adj} A$ is carried out analogously, but with Laplace Expansion taken with respect to rows.

As A is invertible iff $\det A \neq 0$, the above identity gives an explicit formula for the inverse,

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A \ .$$

This formula is not very important from a numerical point of view (where Gauss-Jordan reduction is quite efficient), but it has considerable theoretical interest.

(vi) **Cramer's Rule:** Another situation where explicit solutions are possible, but again mostly of theoretical value, is a non-homogeneous linear system $AX = Y$ of n equations in n unknowns. Multiplying both sides by $\operatorname{adj} A$ and using (v) yields $\det A \cdot X = \operatorname{adj} A \cdot Y$. If x_j is the j-th component of the solution column vector X we now have $\det A \cdot x_j = \sum_{i=1}^{n} (\operatorname{adj} A)_{ji} \cdot y_i = \sum_{i=1}^{n} (-1)^{i+j} y_i \det \hat{A}_{ij} = \det B_j$, where B_j is obtained from A by replacing its jth column with Y. When A is invertible one therefore has *Cramer's Rule*,

$$x_j = \frac{\det B_j}{\det A} \ .$$

(vii) **Determinants of Linear Operators:** For a linear transformation $T : V \to V$ the determinant $\det T$ can be defined as follows: If $A = [T]_{\mathcal{B}}$ is the matrix of T with respect to any ordered basis $\mathcal{B}$ set $\det T = \det A$. This is well defined since the matrix of T with respect to another basis will be similar to A, i.e. is of the form $P^{-1}AP$ for some invertible matrix P, and thus has the same determinant by (v). Determinants are *invariants* of linear operators. The geometric interpretation of determinants in a real vecor space is that $\det T$ is the ratio of the volume distortion under the mapping T (as soon as a natural volume measurement is given).