Summary of 5.1:

All that follows here generally works for coefficients in any commutatative ring R with 1. But it's good enough to think of the real numbers $R = \mathbb{R}$ or perhaps the complex numbers $R = \mathbb{C}$.

Summary of 5.2 and 5.3:

Consider a function D that assigns to every $n \times n$ matrix $A \in R^{n\times n}$ a number $D(A) \in R$. Let $\alpha_i = [A_{i1} \cdots A_{in}]$ be the i^{th} row of A. We can view $D(A) = D(\alpha_1, ..., \alpha_n)$. D is called *n-linear* if $D(A)$ is a linear function of the row α_i when keeping all the other rows fixed, whatever the position $1 \leq i \leq n$.

The n-linear function D is *alternating* or *skew symmetric* if $D(A) = 0$ for any A with two equal rows. This implies that $D(A') = -D(A)$ whenever A' is obtained from A by switching two rows. Why? Let α, β be two different rows of A and call $d(\alpha, \beta) = D(A)$. Now by n-linearity, clearly

$$0 = d(\alpha+\beta), \alpha+\beta) = d(\alpha, \alpha+\beta)+d(\beta, \alpha+\beta) = d(\alpha, \alpha)+d(\alpha, \beta)+d(\beta, \alpha)+d(\beta, \beta) = d(\alpha, \beta)+d(\beta, \alpha)\,.$$

Thus $d(\beta, \alpha) = -d(\alpha, \beta)$. The converse holds as well, certainly for real or complex matrix entries: Since $d(\alpha, \alpha) = -d(\alpha, \alpha)$ it follows that $2d(\alpha, \alpha) = 0$, which implies $d(\alpha, \alpha) = 0$ (unless $1+1 = 2 = 0$, or $d(\alpha, \alpha) = u \neq 0$, but $2u = 0$, which can happen in certain rings R).

Uniqueness: An alternating n-linear function D is completely determined by its value $D(I) = D(\varepsilon_1, ..., \varepsilon_n)$, where $\varepsilon_j = (\delta_{1j}, ..., \delta_{nj})$ is the standard unit row [vector] so that $\alpha_i = \sum_{j=1}^{n} A_{ij}\varepsilon_j$. Then $D(A) = D(\alpha_1, ..., \alpha_n) = D(\sum_{j_1=1}^{n} A_{1j_1}\varepsilon_{j_1}, ..., \sum_{\jmath_n=1}^{n} A_{nj_n}\varepsilon_{\jmath_n}) = \sum_{j_1,...,j_n=1}^{n} A_{1j_1} \cdots A_{nj_n} D(\varepsilon_{j_1}, ..., \varepsilon_{j_n})$, using n-linearity. Here the n indices $j_1, ..., j_n$ run independently from 1 to n. We can consider j as a function from the set of integers $\{1, ..., n\}$ into itself, with values $j(k) = j_k$. Note that $D(\varepsilon_{j_1}, ..., \varepsilon_{j_n}) = 0$ whenever two indices are the same, as D is alternating. Therefore the original sum of n^n terms reduces to a sum taken over only the $n!$ permutations j of the numbers $1, ...n$, i.e. j is one-to-one and then also onto. Let us denote the set of all these permutations by S_n [the *symmetric group of n letters*]. We now have

(1) $$D(A) = D(\alpha_1, ..., \alpha_n) = \sum_{j\in S_n} A_{1j_1} \cdots A_{nj_n} D(\varepsilon_{j_1}, ..., \varepsilon_{j_n})\,.$$

A *switch* is a permutation that interchanges two different numbers. Any permutation can be obtained as a sequence (composition) of switches, in many ways. Why? It follows that $D(\varepsilon_{j_1}, ..., \varepsilon_{j_n}) = \pm D(\varepsilon_1, ..., \varepsilon_n)$, and $D(I)$ determines $D(A)$ uniquely.

Existence: For any $c \in R$ there exists an n-linear alternating function D_c with $D_c(I) = c$, and D_c is unique by the above. For $c = 1$, we call $D_1(A) = \det A$, the *determinant* of A.

Proof: It suffices to construct $D_1(A) = \det A$. Then set $D_c(A) = c \cdot \det(A)$.

Construct $\det A$ recursively via *Laplace Expansion* with respect to column j. For $n = 1$ we set $\det A = A_{11}$. Suppose an alternating $(n-1)$-linear function det has been onstructed for $(n-1) \times (n-1)$ matrices such that $\det I_{n-1} = 1$. For an $n \times n$ matrix A and $1 \leq i, j \leq n$ consider the $(n-1) \times (n-1)$ matrix $\hat{A}_{ij}$ = "$A(i|j)$" obtained from A by deleting row i and column j. We define for a fixed number $1 \leq j \leq n$,

(2) $$\det A = \det_j A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det \hat{A}_{ij}\,.$$

The above uniqueness result shows that $\det_j(A)$ is independent of the row j. In order to prove (2) we need to verify:
(a) $\det_j(A)$ is linear in each row of A.
(b) $\det_j$ is alternating,
(c) $\det_j I_n = 1$.

To verify (a), suppose some row α_{i_0} is the sum of rows α and β. Then each term of the sum in (2) is additive in α and β. For $i \neq i_0$, this follows since $\det \hat{A}_{ij}$ is additive by induction assumption, and for $i = i_0$, the entry $A_{i_0 j}$ is of course additive, while the determinant factor does not involve the row α_{i_0}. Similarly, a scalar factor of row α_{i_0} can be taken out of each summand in (2).

If we have two equal rows $\alpha_{i_1} = \alpha_{i_2} = \alpha$ for $i_1 < i_2$ then all terms in (2) with $i \neq i_1$ and $i \neq i_2$ vanish by induction, since $\hat{A}_{ij}$ has two equal rows. The remaining two summands have opposite sign and cancel each other out. Why? Note that the matrix $\hat{A}_{i_2 j}$ differs from $\hat{A}_{i_1 j}$ only in the position of the row $\hat{\alpha}$ which remains when deleting $A_{i_1 j} = A_{i_2 j}$ from α. The first position is i_1, the second $i_2 - 1$ (since we lost i_1). As it takes $(i_2 - 1) - i_1$ succesive neighbor switches to move α from one position into the other, we have $\det \hat{A}_{i_2 j} = (-1)^{(i_2-1)-i_1} \det \hat{A}_{i_1 j}$. Thus the sign for the i_1-term is $(-1)^{i_1+j}$ and for the i_2-term $(-1)^{i_2+j} \cdot (-1)^{(i_2-1)-i_1}$, which are clearly opposite. This proves (b).

Finally, (c) follows immediately by induction as the only non-zero term in (2) is $(-1)^{j+j} A_{jj} \det \hat{A}_{jj}$. But $A_{jj} = 1$ and $\det \hat{A}_{jj} = \det I_{n-1} = 1$ for $A = I_n$.

The Product Formula: If A, B are $n \times n$-matrices then

$$\det AB = \det A \cdot \det B \; . \tag{3}$$

Proof: Fix B and consider the function $D(A) = \det AB$. Clearly, $D(A) = \det(\alpha_1 B, ..., \alpha_n B)$ is n-linear and alternating. Therefore, by the above uniqueness and existence results, $D(A) = \det A \cdot D(I)$. Since $D(I) = \det B$, we are done.