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Generalized Euler classes, differential forms and commutative DGAs

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In the context of commutative differential graded algebras over $\mathbb{Q}$, we show that an iteration of "odd spherical fibration" creates a "total space" commutative differential graded algebra with only odd degree cohomology. Then we show for such a commutative differential graded algebra that, for any of its "fibrations" with "fiber" of finite cohomological dimension, the induced map on cohomology is injective.

1. Introduction

In geometry, one would like to know which rational cohomology classes in a base space can be annihilated by pulling up to a fibration over the base with finite dimensional fiber. One knows that if $[x]$ is a $2n$-dimensional rational cohomology class on a finite dimensional CW complex X, there is a $(2n-1)$-sphere fibration over X so that $[x]$ pulls up to zero in the cohomology groups of the total space. In fact there is a complex vector bundle V over X of rank n whose Euler class is a multiple of $[x]$. Thus this multiple is the obstruction to a nonzero section of V, and vanishes when pulled up to the part of V away from the zero section, which deformation retracts to the unit sphere bundle.

Rational homotopy theory provides a natural framework to study this type of questions, where topological spaces are replaced by commutative differential

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graded algebras (commutative DGAs) and topological fibrations replaced by algebraic fibrations. This will be the context in which we work throughout the paper. The reader can read more in [2, 4, 6] about the topological meaning of the results of this paper from the perspective of rational homotopy theory of manifolds and general spaces.

The first theorem (Theorem 3.3) of the paper states that the above construction, when iterated, creates a "total space" commutative DGA with only odd degree cohomology.

Theorem A. *For each commutative DGA (A, d), there exists an iterated odd algebraic spherical fibration (TA, d) over (A, d) so that all even cohomology [except dimension zero] vanishes.*

Our next theorem (Theorem 5.7) then limits the odd degree classes that can be annihilated by fibrations whose fiber has finite cohomological dimension.

Theorem B. *Let (B, d) be a connected commutative DGA such that $H^{2k}(B) = 0$ for all $0 < 2k \leq 2N$. If $\iota : (B, d) \to (B \otimes \Lambda V, d)$ is an algebraic fibration whose algebraic fiber has finite cohomological dimension, then the induced map*

$$\iota_* : \bigoplus_{i \leq 2N} H^i(B) \to \bigoplus_{i \leq 2N} H^i(B \otimes \Lambda V)$$

is injective.

It follows from the two theorems above that the iterated odd spherical fibration construction is universal for cohomology classes that pull back to zero by any fibrations whose fiber has finite cohomological dimension.

The paper is organized as follows. In Sec. 2, we recall some definitions from rational homotopy theory. In Sec. 3, we use iterated algebraic spherical fibrations to prove Theorem A. In Sec. 4, we define bouquets of algebraic spheres and analyze their minimal models. In Sec. 5, we prove Theorem B.

2. Preliminaries

We recall some definitions related to commutative differential graded algebras. For more details, see [2, 4, 6].

Definition 2.1. A commutative differential graded algebra (commutative DGA) is a graded algebra $B = \oplus_{i \geq 0} B^i$ over $\mathbb{Q}$ together with a differential $d : B^i \to B^{i+1}$ such that $d^2 = 0$, $xy = (-1)^{ij} yx$, and $d(xy) = (dx)y + (-1)^i x(dy)$, for all $x \in B^i$ and $y \in B^j$.

Definition 2.2. (1) A commutative DGA (B, d) is called connected if $B^0 = \mathbb{Q}$.

(2) A commutative DGA (B, d) is called simply connected if (B, d) is connected and $H^1(B) = 0$.

(3) A commutative DGA (B, d) is of finite type if $H^k(B)$ is finite dimensional for all $k \geq 0$.
(4) A commutative DGA (B, d) has finite cohomological dimension d, if d is the smallest integer such that $H^k(B) = 0$ for all $k > d$.

Definition 2.3. A connected commutative DGA (B, d) is called a model algebra if as a commutative graded algebra it is free on a set of generators $\{x_\alpha\}_{\alpha \in \Lambda}$ in positive degrees, and these generators can be partially ordered so that dx_α is an element in the algebra generated by x_β with $\beta < \alpha$.

Definition 2.4. A model algebra (B, d) is called minimal if for each generator x_α, dx_α has no linear term, that is,

$$d(B) \subset B^+ \cdot B^+, \quad \text{where } B^+ = \oplus_{k>0} B^k.$$

Remark 2.5. For every connected commutative DGA (A, d_A), there exists a minimal model algebra $(\mathcal{M}(A), d)$ and a morphism $\varphi : (\mathcal{M}(A), d) \to (A, d_A)$ such that φ induces an isomorphism on cohomology. $(\mathcal{M}(A), d)$ is called a minimal model of (A, d), and is unique up to isomorphism. See p. 288 of [6] for more details, cf. [2, 4].

Definition 2.6. (i) An algebraic fibration (also called *relative model algebra*) is an inclusion of commutative DGAs $(B, d) \hookrightarrow (B \otimes \Lambda V, d)$ with $V = \oplus_{k \geq 1} V^k$ a graded vector space; moreover, $V = \bigcup_{n=0} V(n)$, where $V(0) \subseteq V(1) \subseteq V(2) \subseteq \cdots$ is an increasing sequence of graded subspaces of V such that

$$d : V(0) \to B \quad \text{and} \quad d : V(n) \to B \otimes \Lambda V(n-1), \quad n \geq 1,$$

where ΛV is the free commutative DGA generated by V.
(ii) An algebraic fibration is called minimal if

$$\mathrm{Im}(d) \subset B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V.$$

Let $\iota : (B, d) \hookrightarrow (B \otimes \Lambda V, d)$ be an algebraic fibration. Suppose B is connected. Consider the canonical augmentation morphism $\varepsilon : (B, d) \to (\mathbb{Q}, 0)$ defined by $\varepsilon(B^+) = 0$. It naturally induces a commutative DGA:

$$(\Lambda V, \bar{d}) := \mathbb{Q} \otimes_B (B \otimes \Lambda V, d).$$

We call $(\Lambda V, \bar{d})$ the algebraic fiber of the given algebraic fibration.

3. Iterated Odd Spherical Algebraic Fibrations

In this section, we show that for each commutative DGA, there exists an iterated odd algebraic spherical fibration over it such that the total commutative DGA has only odd degree cohomology.

Let (B, d) be a connected commutative DGA. An *odd algebraic spherical fibration* over (B, d) is an inclusion of commutative DGAs of the form

$$\varphi : (B, d) \to (B \otimes \Lambda(x), d),$$

such that $dx \in B$, where x has degree $2k-1$ and $\Lambda(x)$ is the free commutative graded algebra generated by x. The element $e = dx \in B$ is called the Euler class of this algebraic spherical fibration.

Proposition 3.1. *Let (B, d) be a commutative DGA. For every even dimensional class $\beta \in H^{2k}(B)$ with $k > 0$, there exists an odd algebraic spherical fibration $\varphi : (B, d) \to (B \otimes \Lambda(x), d)$ such that its Euler class is equal to β and the kernel of the map $\varphi_* : H^{i+2k}(B) \to H^{i+2k}(B \otimes \Lambda(x))$ is $H^i(B) \cdot \beta = \{a \cdot \beta \,|\, a \in H^i(B)\}$.*

Proof. Let $(B\otimes\Lambda(x), d)$ be the commutative DGA obtained from (B, d) by adding a generator x of degree $2k-1$ and defining its differential to be $dx = \beta$. We have the following short exact sequence

$$0 \to (B, d) \to (B \otimes \Lambda(x), d) \to (B \otimes (\mathbb{Q} \cdot x), d \otimes \mathrm{Id}) \to 0,$$

which induces a long exact sequence

$$\cdots \to H^{i-1}(B \otimes (\mathbb{Q} \cdot x)) \to H^i(B) \to H^i(B \otimes \Lambda(x)) \to H^i(B \otimes (\mathbb{Q} \cdot x)) \to \cdots.$$

Applying the identification $H^{i+(2k-1)}(B\otimes(\mathbb{Q}\cdot x)) \cong H^i(B)$, we obtain the following Gysin sequence

$$\cdots \to H^i(B) \xrightarrow{\cup e} H^{i+2k}(B) \xrightarrow{\varphi_*} H^{i+2k}(B \otimes \Lambda(x)) \xrightarrow{\partial_{i+1}} H^{i+1}(B) \to \cdots.$$

This finishes the proof. □

Definition 3.2. An *iterated odd algebraic spherical fibration* over (B, d) is algebraic fibration $(B, d) \hookrightarrow (B \otimes \Lambda V, d)$ such that $V^k = 0$ for k even. This fibration is called *finitely iterated odd algebraic spherical fibration* if $\dim V < \infty$.

Now let us prove the main result of this section.

Theorem 3.3. *For each commutative DGA (A, d), there exists an iterated odd algebraic spherical fibration (TA, d) over (A, d) such that all even cohomology* [*except dimension zero*] *vanishes.*

Proof. We will construct TA by induction. In the following, for notational simplicity, we shall omit the differential d from our notation.

Let $\mathcal{A}_0 = A$. Suppose we have defined the iterated odd algebraic spherical fibration $\mathcal{A}_{m-1}$ over A. Fix a basis of $H^{2k}(\mathcal{A}_{m-1})$ for each $k > 0$. Denote the union of all these bases by $\{a_i\}_{i\in I}$. Define W_{m-1} to be a $\mathbb{Q}$ vector space with basis $\{x_i\}_{i\in I}$, where $|x_i| = |a_i| - 1$. We define $\mathcal{A}_m$ to be the iterated odd algebraic spherical fibration $\mathcal{A}_{m-1} \otimes \Lambda(W_{m-1})$ over $\mathcal{A}_{m-1}$ with $dx_i = a_i$ for all $i \in I$. The inclusion map $\iota : \mathcal{A}_{m-1} \hookrightarrow \mathcal{A}_m$ induces the zero map $\iota_* = 0 : H^{2k}(\mathcal{A}_{m-1}) \to H^{2k}(\mathcal{A}_m)$ for all $k > 0$. By construction, $\mathcal{A}_m$ is also an iterated odd algebraic spherical fibration.

Finally, we define TA to be the direct limit of $\mathcal{A}_m$ under the inclusions $\mathcal{A}_m \hookrightarrow \mathcal{A}_{m+1}$. Clearly, TA is an iterated odd algebraic spherical fibration over A. More precisely, let $V = \bigcup_{i=0}^{\infty} W_i$. We have $TA = A \otimes \Lambda V$ with the filtration of V given by $V(n) = \bigcup_{i=0}^{n} W_i$. Moreover, we have $H^{2k}(TA) = 0$ for all $2k > 0$. This completes the proof. □

Remark 3.4. If an element $\alpha \in H^\bullet(A)$ maps to zero in $H^\bullet(TA)$, then there exists a subalgebra S_α of TA such that S_α is a *finitely* iterated odd algebraic spherical fibration over A and α maps to zero in $H^\bullet(S_\alpha)$.

4. Bouquets of Algebraic Spheres

In this section, we introduce a notion of bouquets of algebraic spheres. It is an algebraic analogue of usual bouquets of spheres in topology.

Definition 4.1. For a given set of generators $X = \{x_i\}$ with x_i having odd degree $|x_i|$, we define the bouquet of odd algebraic spheres labeled by X to be the following commutative DGA

$$\mathcal{S}(X) = \left(\bigwedge_{x_i \in X} \mathbb{Q}[x_i] \right) \Big/ \langle x_i x_j = 0 \mid \text{all } i, j \rangle$$

with the differential $d = 0$.

Proposition 4.2. *Let $\mathcal{S}(X)$ be a bouquet of odd algebraic spheres, and $\mathcal{M}(X) = (\Lambda V, d)$ be its minimal model. Then $\mathcal{M}(X)$ satisfies the following properties*:

(i) *$\mathcal{M}$ has no even degree generators, that is, V does not contain even degree elements*;
(ii) *each element in $H^{\geq 1}(\mathcal{M}(X))$ is represented by a generator, that is, an element in V.*

Proof. This is a special case of Koszul duality theory, cf. [5, Chaps. 3, 7 and 13]. Since $\mathcal{S} = \mathcal{S}(X)$ has zero differential, we may forget its differential and view it as a graded commutative algebra. An explicit construction of a minimal model of $\mathcal{S}$ is given as follows: first take the Koszul dual coalgebra $\mathcal{S}^{¡}$ of $\mathcal{S}$; then apply the cobar construction to $\mathcal{S}^{¡}$, and denote the resulting commutative DGA by $\Omega\mathcal{S}^{¡}$. By Koszul duality, $\mathcal{M}(X) \coloneqq \Omega\mathcal{S}^{¡}$ is a minimal model of $\mathcal{S}$.

More precisely, set $W = \bigoplus_{i\geq 0} W_i$ to be the graded vector space spanned by X. Let sW (resp. $s^{-1}W$) be the suspension (resp. desuspension) of W, that is, $(sW)_{i-1} = W_i$ (resp. $(s^{-1}W)_i = W_{i-1}$). Let $\mathcal{L}^c = \mathcal{L}^c(sW)$ be the cofree Lie coalgebra generated by sW. More explicitly, let $T^c(sW) = \bigoplus_{n\geq 0}(sW)^{\otimes n}$ be the tensor coalgebra, and $T^c(sW)^+ = \bigoplus_{n\geq 1}(sW)^{\otimes n}$. The coproduct on $T^c(sW)$ naturally induces a Lie cobracket on $T^c(sW)$. Then we have $\mathcal{L}^c(sW) = T^c(sW)^+/T^c(sW)^+ * T^c(sW)^+$, where $*$ denotes the shuffle multiplication. With the above notation, we have $\mathcal{S}^{¡} \cong \mathcal{L}^c$. The cobar construction of $\mathcal{L}^c$ is given explicitly by

$$\mathbb{Q} \to s^{-1}\mathcal{L}^c \xrightarrow{d} \Lambda^2(s^{-1}\mathcal{L}^c) \to \cdots \to \Lambda^n(s^{-1}\mathcal{L}^c) \to \cdots$$

with the differential d determined by the Lie cobraket of $\mathcal{L}^c$. Now the desired properties of $\mathcal{M}(X)$ follow from this explicit construction. □

Remark 4.3. In the special case of a bouquet of odd algebraic spheres where the cohomology of a commutative DGA model is that of a circle or the first Betti number is zero, this was discussed by Baues [1, Corollary 1.2] and by Halperin and Stasheff [3, Theorem 1.5].

5. Main Theorem

In this section, we show that if a commutative DGA has cohomology, up to a certain degree, isomorphic to that of a bouquet of odd algebraic spheres, then its minimal model is isomorphic to that of the bouquet of odd algebraic spheres, up to that given degree. Then we apply it to prove that if a commutative DGA has only odd degree cohomology up to a certain degree, then all nonzero cohomology classes up to that degree will never pull back to zero by any algebraic fibration whose fiber has finite cohomological dimension.

Suppose B is a connected commutative DGA of finite type such that $H^{2k}(B) = 0$ for all $0 < 2k \leq 2N$. Let X_i be a basis of $H^i(B)$ and $X = \bigcup_{i=1}^{2N+1} X_i$. Let $M = \mathcal{M}(X)$ be the bouquet of odd algebraic spheres labeled by X from Definition 4.1. Then we have $H^i(M) \cong H^i(B)$ for all $0 \leq i \leq 2N$. Let $M_k \subset M$ be the subalgebra generated by the generators of degree $\leq k$.

Lemma 5.1. *Let k be an odd integer. Then $H^{k+2}(M_k) = H^{k+1}(M_k) = 0$.*

Proof. $H^{k+1}(M_k) = 0$ as $H^{k+1}(M_k) \to H^{k+1}(M) = 0$ is injective.

By Proposition 4.2 above, M has no even-degree generators. In particular, we have $M_k = M_{k+1}$. Moreover, $H^{\geq 1}(M)$ is spanned by odd-degree generators. From the first observation it follows that the map $H^{k+2}(M_k) \to H^{k+2}(M)$ is injective, and from the second that its range is 0. □

It follows that for an odd k, we have $M_{k+2} = M_k \otimes \Lambda(V[k+2])$ as an algebra, where the vector space $V = V_1 \oplus V_2$ is placed at degree $(k+2)$, with $V_1 \cong H^{k+2}(M)$ and $V_2 = H^{k+3}(M_k)$. The differential can be described as follows. It suffices to define $d : V \to M_k$. We define $d = 0$ on V_1. To define d on V_2, let us choose a basis $\{a_i\}_{i\in I}$ of $H^{k+3}(M_k)$. Let $\{\tilde{a}_i\}_{i\in I}$ be the corresponding basis of V_2. Then we define $d\tilde{a}_i = a_i$.

Proposition 5.2. *For each odd integer $k \leq 2N$, there exists a morphism $\varphi_k : M_k \to B$ such that the induced map on cohomology $H^i(M_k) \cong H^i(M) \to H^i(B)$ is an isomorphism for $i \leq k$.*

Proof. We construct the maps φ_k by induction. By the previous lemma and the fact that M has no even degree generators, it suffices to define φ_k for odd integers k. The case where $k = 1$ is clear.

Now assume that we have constructed φ_n, with n an odd integer $\leq 2N-3$. We shall extend φ_n to a morphism φ_{n+2} on $M_{n+2} = M_n \otimes \Lambda(V[n+2])$, where the vector space $V = V_1 \oplus V_2$ is placed at degree $(n+2)$, with $V_1 \cong H^{n+2}(M)$ and $V_2 = H^{n+3}(M_n)$. It suffices to define φ_{n+2} on V. Let $\{b_j\}_{j\in J}$ be a basis of $H^{n+2}(B)$. Since $H^{n+2}(M) \cong H^{n+2}(B)$, let $\{\tilde{b}_j\}_{j\in J}$ be the corresponding basis of V_1. We define φ_{n+2} on V_1 by setting $\varphi_{n+2}(\tilde{b}_j) = b_j$. Similarly, choose a basis $\{c_\lambda\}_{\lambda\in K}$ of $H^{n+3}(M_n)$, and let $\{\tilde{c}_\lambda\}_{\lambda\in K}$ be the corresponding basis of V_2. Since $H^{n+3}(B) = 0$, for each $c_\lambda \in M_n$, there exists $\theta_\lambda \in B$ such that $\varphi_n(c_\lambda) = d\theta_\lambda$. We define φ_{n+2} on V_2 by setting $\varphi_{n+2}(\tilde{c}_\lambda) = \theta_\lambda$. By construction, the induced map $(\varphi_{n+2})_*$ on H^i agrees with $(\varphi_n)_*$ for $i \leq n+1$ and $(\varphi_{n+2})_*$ is an isomorphism on H^{2n+2}. This finishes the proof. □

Now let $\mathcal{M}_B$ be a minimal model of B and $(\mathcal{M}_B)_k$ be the subalgebra generated by the generators of degree $\leq k$. Combining the above results, we have proved the following proposition.

Proposition 5.3. *The commutative DGAs $(\mathcal{M}_B)_{2N-1}$ and M_{2N-1} are isomorphic.*

Moreover, we have the following result, which is an immediate consequence of the construction in Proposition 5.2.

Corollary 5.4. *Let B be a connected commutative DGA such that $H^{2i}(B) = 0$ for all $0 < 2i \leq 2N$. Let α be a nonzero class in $H^{2k+1}(\mathcal{M}_B)$ with $2k+1 < 2N$. Then there exists a morphism $\psi : \mathcal{M}_B \to (\Lambda(\eta), 0)$ such that $\psi_*(\alpha) = [\eta]$, where η has degree $2k+1$ and $\Lambda(\eta)$ is the free commutative graded algebra generated by η.*

Proof. From the description of the minimal model $\mathcal{M}_B$ of B, it follows that $\mathcal{M}_B$ has a set of generators such that all the cohomology groups up to degree $(2N-1)$ is generated by the cohomology classes of these generators; moreover we can choose these generators so that the given class α is represented by a generator, say, a. Then we define ψ by mapping a to η and the other generators to 0. □

An inductive application of the same argument above proves the following.

Proposition 5.5. *Suppose (C, d) is a connected commutative DGA with $H^{2k}(C) = 0$ for all $2k > 0$. Let X_i be a basis of $H^i(C)$ and $X_C = \bigcup_{i=1}^{\infty} X_i$. Then the bouquet of odd algebraic spheres $\mathcal{M}(X_C)$ is a minimal model of (C, d).*

Applying the above proposition to the commutative DGA (TA, d) from Theorem 3.3 immediately gives us the following corollary.

Corollary 5.6. *With the same notation as above, the minimal model of (TA, d) is isomorphic to a bouquet of odd algebraic spheres.*

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Before proving the main theorem of this section, we shall prove the following special case first.

Theorem 5.7. *Let $(\Lambda(x), d)$ be the commutative DGA generated by x of degree $2k+1 \geq 1$ such that $dx = 0$. For any algebraic fibration $\varphi : (\Lambda(x), d) \to (\Lambda(x) \otimes \Lambda V, d)$ whose algebraic fiber $(\Lambda V, \bar{d})$ has finite cohomological dimension, the map $\varphi_* : H^j(\Lambda(x)) \to H^j(\Lambda(x) \otimes \Lambda V)$ is injective for all j.*

Proof. The case where $2k+1 = 1$ is trivial. Let us assume $2k+1 > 1$ in the rest of the proof.

Let $\varphi : (\Lambda V, d) \hookrightarrow (\Lambda(x) \otimes \Lambda V, d)$ be any algebraic fibration whose algebraic fiber has finite cohomological dimension. It suffices to show that $\varphi_* : H^{2k+1}(\Lambda(x)) \to H^{2k+1}(\Lambda(x) \otimes \Lambda V)$ is injective, since the induced map φ_* on H^i is automatically injective for $i \neq 2k+1$.

Now suppose to the contrary that

$$\varphi_*(x) = 0 \quad \text{in } H^{2k+1}(\Lambda(x) \otimes \Lambda V).$$

Then we have $x = d(w \cdot x + v)$ for some $w, v \in \Lambda V$. By inspecting the degrees on both sides, one sees that $w = 0$. Therefore, we have $x = dv$ for some $v \in \Lambda V$. It follows that $\bar{d}v = 0$.

Now let $n \in \mathbb{N}$ be the smallest integer such that $[v^n] = 0$ in $H^\bullet(\Lambda V, \bar{d})$. Such an integer exists since $(\Lambda V, \bar{d})$ has finite cohomological dimension. Then there exists $u \in \Lambda V$ such that $v^n = \bar{d}u$. Equivalently, we have

$$v^n = u_0 \cdot x + du,$$

for some $u_0 \in \Lambda V$. It follows that

$$0 = d^2 u = d(v^n - u_0 \cdot x) = nv^{n-1} \cdot x - (du_0) \cdot x.$$

Therefore, $v^{n-1} = \frac{1}{n} du_0$, which implies that $[v^{n-1}] = 0$ in $H^\bullet(\Lambda V, \bar{d})$. We arrive at a contradiction. This completes the proof. □

Now let us prove the main result of this section.

Theorem 5.8. *Let (B, d) be a connected commutative DGA such that $H^{2k}(B) = 0$ for all $0 < 2k \leq 2N$. If $\iota : (B, d) \to (B \otimes \Lambda V, d)$ is an algebraic fibration whose algebraic fiber has finite cohomological dimension, then the induced map*

$$\iota_* : \bigoplus_{i<2N} H^i(B) \to \bigoplus_{i<2N} H^i(B \otimes \Lambda V)$$

is injective.

Proof. Let $f : (\mathcal{M}_B, d) \to (B, d)$ be a minimal model algebra of B.

Claim. For any algebraic fibration $\iota : (B, d) \to (B \otimes \Lambda V, d)$, there exist an algebraic fibration $\varphi : (\mathcal{M}_B, d) \to (\mathcal{M}_B \otimes \Lambda V, d)$ and a quasi-isomorphism $g : (\mathcal{M}_B \otimes \Lambda V, d) \to (B \otimes \Lambda V, d)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_B & \xrightarrow{\ f\ } & B \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ \mathcal{M}_B \otimes \Lambda V & \xrightarrow{\ g\ } & B \otimes \Lambda V. \end{array}$$

We construct φ and g inductively. Consider the filtration $V = \cup_{n=0}^{\infty} V(k)$ from Definition 2.6. Choose a basis $\{x_i\}_{i \in I_0}$ of $V(0)$. Let $x = x_i$ be a basis element. If $dx = a \in B$, then $da = d^2x = 0$. It follows that there exists $\tilde{a} \in \mathcal{M}_B$ such that $f(\tilde{a}) = a + dc$ for some $c \in B$. We define an algebraic fibration $\varphi_0 : (\mathcal{M}_B, d) \hookrightarrow (\mathcal{M}_B \otimes \Lambda(x), d)$ by setting $dx = \tilde{a}$. Moreover, we extend $f : (\mathcal{M}_B, d) \to (B, d)$ to a morphism (of commutative DGAs) $g_0 : (\mathcal{M}_B \otimes \Lambda(x), d) \to (B \otimes \Lambda(x), d)$ by setting $g(x) = x + c$. By the Gysin sequence from Sec. 3, we see that g_0 is a quasi-isomorphism. Now apply the same construction to all basis elements $\{x_i\}_{i \in I_0}$. We still denote the resulting morphisms by $\varphi_0 : (\mathcal{M}_B, d) \to (\mathcal{M}_B \otimes \Lambda(V(0)), d)$ and $g_0 : (\mathcal{M}_B \otimes \Lambda(V(0)), d) \to (B \otimes \Lambda(V(0)), d)$.

Now suppose we have constructed an algebraic fibration

$$\varphi_k : (\mathcal{M}_B \otimes \Lambda(V(k-1)), d) \to (\mathcal{M}_B \otimes \Lambda(V(k)), d)$$

and a quasi-isomorphism $g_k : (\mathcal{M}_B \otimes \Lambda(V(k)), d) \to (B \otimes \Lambda(V(k)), d)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_B \otimes \Lambda(V(k-1)) & \xrightarrow{\ g_{k-1}\ } & B \otimes \Lambda(V(k-1)) \\ {\scriptstyle\varphi_k}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ \mathcal{M}_B \otimes \Lambda(V(k)) & \xrightarrow{\ g_k\ } & B \otimes \Lambda(V(k)) \end{array}$$

Let $\{y_i\}_{i \in I_{k+1}}$ be a basis of $V(k+1)$ that extends the basis $\{x_i\}_{i \in I_k}$ of $V(k) \subseteq V(k+1)$. Apply the same construction above to elements in $\{y_i\}_{i \in I_{k+1}} \backslash \{x_i\}_{i \in I_k}$, but with $B \otimes \Lambda(V(k))$ in place of B, and $\mathcal{M}_B \otimes \Lambda(V(k))$ in place of $\mathcal{M}_B$.

We define $(\mathcal{M}_B \otimes \Lambda V, d)$ to be the direct limit of $(\mathcal{M}_B \otimes \Lambda(V(k)), d)$ with respect to the morphisms $\varphi_k : (\mathcal{M}_B \otimes \Lambda(V(k-1)), d)$. We define φ to be the natural inclusion morphism $(\mathcal{M}_B, d) \hookrightarrow (\mathcal{M}_B \otimes \Lambda V, d)$. The morphisms g_k together also induce a quasi-isomorphism $g : (\mathcal{M}_B \otimes \Lambda V, d) \to (B \otimes \Lambda V, d)$, which makes the diagram in the claim commutative. This finishes the proof of the claim.

Now assume to the contrary that there exists $0 \neq \alpha \in H^{2k+1}(B)$ with $2k+1 < 2N$ such that $\iota_*(\alpha) = 0$. Let $\tilde{\alpha} \in H^{2k+1}(\mathcal{M}_B)$ be the class such that $f_*(\tilde{\alpha}) = \alpha$. In particular, we have $\varphi_*(\tilde{\alpha}) = 0$. By Corollary 5.4, there exists a morphism $\psi : (\mathcal{M}_B, d) \to (\Lambda(\eta), 0)$ such that $\psi_*(\tilde{\alpha}) = \eta$. Now let

$$\tau : (\Lambda(\eta), 0) \to (\Lambda(\eta) \otimes \Lambda V, d) = (\Lambda(\eta) \otimes_{\mathcal{M}_B} (\mathcal{M}_B \otimes \Lambda V), d)$$

be the push-forward algebraic fibration of $\varphi : (\mathcal{M}_B, d) \to (\mathcal{M}_B \otimes \Lambda V, d)$. It follows that

$$\tau_*(\eta) = \tau_* \psi_*(\tilde{\alpha}) = (\psi \otimes 1)_* \varphi_*(\tilde{\alpha}) = 0$$

which contradicts Theorem 5.7. This completes the proof. □

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