Structure on k-cycles of a k-dimensional Space

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Abstract

Definition 1. A *coordinate space* is a finite dimensional real vector space V with a choice of $dimV$ co-dimension one subspaces in general position. By general position we mean that the dimension of the intersection of any n of these hyperplanes is $dimV - n$.

In this note we attach, in a topologically invariant manner, a finite length chain complex of finite dimensional coordinate spaces to any compact triangulable space. The top homology of the chain complex agrees with the top homology of the space and thus inherits an oriented matroid structure from the coordinate space structure on the top chain groups. In dimension two, this chain complex invariant can be refined to give a complete topological invariant for a class of two dimensional spaces called taut. Taut two complexes exist in every possible homotopy type of connected two complexes, and are characterized by being built out of surfaces glued to graphs by maximally efficient attaching maps. This paper was motivated by our attempt to understand the possible content of a lost manuscript by Ralph Reid at MIT circa 1970.

Section 1

Let X be a compact Hausdorff space underlying a finite simplicial complex of dimension d (equivalently underlying a regular cell complex structure of dimension d). A point in such a space is called a k-manifold point if it has a neighborhood homeomorphic to an open set in Euclidean space of dimension k.

Now we define a filtration of X by subcomplexes. For every $0 \leq k \leq d$ we define $X_k := X$ if $k = d$ and $X_k := X_{k+1} \setminus \{k+1 \text{ manifold points}\}$ for k less than d. One can show the space X_k is a subcomplex of the cell complex structure on X of dimension at most k.

For every pair (X_k, X_{k-1}) we have the following long exact sequence (all coefficients are real numbers in this note).

$$\ldots \to H_{n+1}(X_k, X_{k-1}) \to H_n(X_{k-1}) \to H_n(X_k) \to \ldots$$

Using the exact sequences for (X_k, X_{k-1}) and (X_{k-1}, X_{k-2}), we get a map $H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1}, X_{k-2})$ and the following sequence of groups and morphisms,

$$\ldots \to H_{k+1}(X_{k+1}, X_k) \xrightarrow{\partial_k} H_k(X_k, X_{k-1}) \xrightarrow{\partial_{k-1}} H_{k-1}(X_{k-1}, X_{k-2}) \to \ldots \tag{1}$$

Definition 2. We call the connected components of the set of k-manifold points of X_k the *k-strata* of X. In other words, a *k-stratum* is a connected component of $X_k \setminus X_{k-1}$.

In the proof of the main theorem below, one sees that the orientable k-strata correspond to natural generators, up to scalar factors, of the chain groups $H_k(X_k, X_{k-1})$.

Theorem 1. *The filtration $X = X_d \supseteq X_{d-1} \supseteq \ldots$ has the following properties:*

(i) Each chain group $C_k = H_k(X_k, X_{k-1})$ has the structure of a coordinate space whose codimension one hyperplanes are defined by elements of C_k not supported on a particular k-stratum.

(ii) The maps induced by the exact sequences of pairs make (1) into a chain complex whose top homology is isomorphic to the top homology of X.

(iii) The intersection of the coordinate hyperplanes of C_k with the space of k-cycles in C_k defines an oriented matroid, up to re-orientation.
(See page 10 of [1])

(iv) Any homeomorphism between two such spaces induces an isomorphism between the above structures.

Proof. i) X_k has an underlying cell complex structure which gives rise to chain groups A_* ($\mathbb{R}$ coefficients). $H_k(X_k, X_{k-1})$ is the top homology group for the relative chain complex,

$$0 \to A_k(X_k)/A_k(X_{k-1}) \xrightarrow{\delta} A_{k-1}(X_k)/A_{k-1}(X_{k-1}) \to \ldots \tag{2}$$

So $H_k(X_k, X_{k-1}) = ker\delta$ and is a real vector space. Now we construct a basis up to multiplication by scalars which will give rise to a coordinate space structure on these relative homology groups.

For every orientable k-stratum, we pick an orientation and for each such stratum we construct a formal sum of oriented cells in A_k that are contained in this stratum. The coefficient of a cell is $+1$ if the orientation of the cell matches with that of the component and -1 if it does not. Now the boundary of such a sum only has $(k-1)$-cells which are in X_{k-1}. Hence these sums are in $ker\delta$. Also these sums for the various strata form a linearly independent set as the cells occuring in each sum for two different strata are distinct. It remains to show that every chain whose boundary is in X_{k-1} is a linear combination of the above mentioned sums.

Let B be a chain whose boundary is contained in X_{k-1}. Let σ be a cell appearing with multiplicity m in B and let $\mathcal{S}$ be the stratum in which σ

is contained. If τ is another cell in $\mathcal{S}$, it is connected to σ by a sequence of k-cells in $\mathcal{S}$ with consecutive cells sharing a $(k-1)$-face not in X_{k-1}. This gives a path of alternating k and $(k-1)$ cells from σ to τ. There may be several such sequences. For any such sequence, each consecutive cell must appear in B with multiplicity m and appropriate sign so that the boundary of B does not contain $(k-1)$-cells from the corresponding path. This is true whether or not $\mathcal{S}$ is orientable.

Suppose $\mathcal{S}$ is not orientable. Two connecting sequences exist such that the orientations on a cell τ induced from a given orientation of some cell σ by these two sequences do not agree. Therefore, the multiplicity of τ in B must be both $+m$ and $-m$. So m must be zero.

If $\mathcal{S}$ is orientable, the above discussion shows that the entire sum associated with $\mathcal{S}$ appears in B with multiplicity m.

This proves that every element in C_k is a linear combination of oriented strata. A particular co-dimension one hyperplane of the coordinate space structure on C_k consists of those elements which have zero coefficient on a particular stratum.

ii) Now we show (by a familiar argument) that (1) is a chain complex. The map ∂_k is the composition of $H_{k+1}(X_{k+1}, X_k) \xrightarrow{f_{k+1}} H_k(X_k)$ and $H_k(X_k) \xrightarrow{f_k} H_k(X_k, X_{k-1})$. So $\partial_k \partial_{k-1} = f_{k+1}(f_k)^2 f_{k-1}$. Since $(f_k)^2 = 0$, the sequence (1) of groups and morphisms forms a chain complex. To find the top homology, consider the following part of the chain complex,

$$0 \to H_d(X_d, X_{d-1}) \xrightarrow{\partial} H_{d-1}(X_{d-1}, X_{d-2}).$$

It is easily seen that the top homology group is $ker\partial$. Now, ∂ is the composition of the following maps which come from two different exact sequences.

$$H_d(X_d, X_{d-1}) \xrightarrow{f_d} H_{d-1}(X_{d-1}) \xrightarrow{f_{d-1}} H_{d-1}(X_{d-1}, X_{d-2})$$

f_{d-1} is injective, as its kernel is the same as the image under a map from $H_{d-1}(X_{d-2}) = 0$. Thus $ker\partial = ker f_d$. This is the image under the map from $H_d(X_d) = H_d(X)$ in the following exact sequence,

$$0 \to H_d(X) \to H_d(X_d, X_{d-1}) \to \dots$$

This image, in turn, is isomorphic to $H_d(X)$ and hence the top homology of chain complex (1) is the same as the top homology of the topological space X.

iii) Intersecting a subspace of a coordinate space with the n hyperplanes gives rise to an oriented matroid up to re-orientation. (See page 10 of [1]).

iv) A homeomorphism from Y to X restricts to a homeomorphism from Y_k to X_k for every k, where Y_k is obtained from Y the same way in which X_k is obtained from X. An isomorphism is induced between the two chain complexes associated with the pairs (Y_k, Y_{k-1}) and (X_k, X_{k-1}). This, in turn, induces an isomorphism between $H_k(Y_k, Y_{k-1})$ and $H_k(X_k, X_{k-1})$ for every k, which respects the boundary maps. Thus there is an isomorphism between the two chain complexes preserving the coordinate space structure.

□

Corollary 1 (Inspired by Ralph Reid)**.** *For a d-dimensional regular cell complex, the real cellular d-cycles in C_d of minimal support define the circuits of an oriented matroid which is well-defined up to reorientation by the underlying topological space.*

Proof. This follows immediately from Theorem 1 (*iii*). (See page 10 of [1]).

□

The top homology group $H_d(X)$ is equal to the space of cycles in the top dimension. By the proof of Theorem 1 (*ii*), these cycles inject into $H_d(X_d, X_{d-1})$. Theorem 1 (*i*) shows that every element of $H_d(X_d, X_{d-1})$ corresponds to the linear combination of d-strata. Together, these facts give another corollary.

Corollary 2. *The top dimensional cycles can be written as linear combinations of top dimensional strata.*

This chain complex of coordinate spaces is a topological invariant of the underlying space which, with some enrichment, becomes a complete topological invariant in dimension one and in dimension two.

Remark 1. In dimension one, the connected components of 1-manifold points are either circles or open intervals. Thus for connected non-trivial spaces, the one dimensional strata are either a circle or the edges of a graph. The circle case is specified by having no degree zero term in the chain complex. The graph case for connected spaces of dimension one is essentially determined by the above coordinate space chain complex. For such a graph, X_1 is a finite collection of open intervals and X_0 is a finite collection of points. Every open interval and every vertex corresponds to a generator of the respective chain groups. The boundary map takes the generator corresponding to an interval to the difference of the generators corresponding to its ending vertices. Thus for every such open interval, which does not form a loop after being attached, the above chain complex completely determines its position in the graph. Thus in either case, circle or graph, the above chain complex, along with the function giving the number of loops attached to every vertex, is a complete topological invariant for connected compact one dimensional triangular spaces.

In the two dimensional case, we restrict our discussion to a special class of spaces.

Definition 3. We call a two dimensional cell complex *taut* if the attaching maps of the boundary components of the 2-strata to the intrinsic one skeleton are either locally injective or constant.

Note that for a general finite two complex the attaching maps can be homotoped to locally injective or to constant maps. Thus every two complex is homotopy equivalent to a taut two complex.

For a taut cell complex, the attaching map in the non-constant case is given by a cyclic word on the set of 1-strata. The boundary map in chain complex (1) is given by abelianizing the above word. The following theorem about taut complexes with only orientable 2-strata follows quite directly.

Theorem 2. *The following information forms a complete homeomorphism invariant of* 2*-dimensional taut cell complexes with all* 2*-strata orientable.*

(i) The chain complex (1), with the loop counting enrichment described in Remark (1)

(ii) The Euler characteristic and number of boundary components of each stratum

(iii) Boundary components of 2*-strata which are attached by constant maps labeled by the one or zero strata where they are attached*

(iv) The non-abelian version of $\partial_1(s)$ *associated with each boundary component of each* 2*-stratum,* s*, whose attaching maps are non-constant*

Proposition. *The free homotopy class of a non-constant attaching map of a boundary component of a* 2*-strata in a taut* 2*-dimensional cell complex determines the attaching map up to reparametrization.*

Proof. Consider a lift of the attaching map as a map of the interval into the universal cover of X_1 (which is a tree). Locally injective maps into a tree are just the shortest paths. These are determined by their endpoints up to reparametrization. Thus the conjugacy classes of the attaching maps give the attaching maps up to reparametrization. ☐

Theorem 3. *The following information forms a complete homeomorphism invariant of* 2*-dimensional taut cell complexes, with or without non-orientable* 2*-strata.*

(i) The chain complex (1), along with the loop counting enrichment described in Remark (1)

(ii) The Euler characteristic, the orientability and the number of boundary components of each 2*-stratum*

(iii) The labeling of the boundary components of 2*-strata which are attached by constant maps by the zero or one strata containing the constant images*

(iv) *The labeling of directed boundary components of* 2*-strata with non-constant attaching maps by conjugacy classes in the fundamental group of the intrinsic one skeleton* X_1

Proof. By the proposition, each locally injective attaching map of a boundary component is determined up to reparametrization by the corresponding free homotopy class of maps into the one skeleton. This free homotopy class is determined by the conjugacy class associated by the map to the directed boundary component. The rest of the proof is by inspection, extending, using the proposition, the homeomorphism from Remark 1 between the corresponding graphs over the 2-strata. □

References

[1] Björner, Anders. "Oriented Matroids". Cambridge: Cambridge UP, 1999. (see page 10).

[2] Gelfand, I., R. Goresky, R. Macpherson, and V. Serganova. "Combinatorial Geometries, Convex Polyhedra, and Schubert Cells". Advances in Mathematics 63.3 (1987): 301-16. (see for the use of matroids to label cells in the grassmanian).

[3] Whitehead, J. H. C., "Combinatorial homotopy". II, Bull. Amer. Math. . Soc. 55 (1949): 453-496. (see for a discussion of crossed modules).

[4] Whitehead, J. H. C., "On simply connected, 4-dimensional polyhedra". Comment. Math. Helv. , 22 (1949): 48-92. (see for other invariants of two-complexes up to homotopy).