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QUASICONFORMAL MAPPINGS, OPERATORS ON HILBERT SPACE, AND LOCAL FORMULAE FOR CHARACTERISTIC CLASSES†

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1. INTRODUCTION

By rolling, or better pressing, a sphere $S^{2\ell}$ all around the manifold $M^{2\ell}$, we will construct bounded operators on the space of L^2 middle dimensional forms of $M^{2\ell}$ analogous to the classical Ahlfors–Beurling operator on the Riemann sphere $\hat{\mathbb{C}}$,

$$\varphi(z, \bar{z})\, d\bar{z} = \eta \mapsto S\eta = \left(\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\varphi(\zeta, \bar{\zeta})\, d\bar{\zeta}\, d\zeta}{(z-\zeta)^2}\right) dz. \tag{1}$$

The kernel of this classical operator is the biform $dz\, dz'/(z-z')^2$ on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Using the generalized operators and their kernels we obtain explicit local cycle representatives of the Hirzebruch–Thom characteristic classes for any quasiconformal manifold. At the end of the introduction we explain how the construction applies to topological manifolds. This answers a question raised by Bill Browder in lectures at Princeton in 1964.

Generalizations of (1) to all even spheres $S^{2\ell}$ with any bounded measurable pointwise $*$ operator on middle forms were constructed for the quasiconformal Yang Mills theory [7] by explicit formulae. For example, here we write

$$S_\mu = (1+\mu)^{-1}(\mu + S)S(\mu + S)^{-1}(1+\mu), \quad \|\mu\| < 1 \tag{2}$$

where S is the conformally invariant operator for the standard sphere (S is the identity on exact forms and $-$ identity on coexact forms), S_μ is the corresponding operator related to the new bounded measurable $*$ and μ relates the new $*$ and the standard $*$ (Section 2). The operator S for the standard sphere $S^{2\ell}$ can be written out on $R^{2\ell}$ as a quadratic expression in Riesz transforms, see [13]. When ℓ is even, these bounded operators could be called *conformal signature operators*, being the phases of the usual signature operators when the $*$ is smooth (see the proof of Theorem 3.2(4). When ℓ is odd they are generalizations of the Ahlfors–Beurling operator to general "curved" measurable conformal structures. The formulae (2) take us beyond the usual pseudodifferential calculus to operators where symbols (if defined) would only be measurable. However, the important property that S_μ commutes with multiplication by continuous functions modulo compact operators is still true and evident from (2).

† Dedicated to Bill Browder on his 60th birthday.

For a general even dimensional quasiconformal manifold provided with a bounded measurable conformal structure or, more generally, a bounded measurable $*$ on middle forms, we do the following:

(i) locally copy via charts the $*$ structure on $M^{2\ell}$ by $*$ structures on $S^{2\ell}$;
(ii) get locally defined operators on $M^{2\ell}$ from the operators on $S^{2\ell}$;
(iii) collect these together on $M^{2\ell}$ using a partition of unity to construct global operators S on the Hilbert space $\mathscr{H}$ of middle dimensional forms (Section 2).

Let $I(\ell)$ denote the ideal of compact operators A on a Hilbert space satisfying $\mu_n = \mathrm{O}(n^{-1/2\ell})$, where μ_n is the distance in norm between A and rank-n operators. The reader may recall that any degree minus one smoothing operator in dimension 2ℓ, e.g. the Poincaré lemma operator, belongs to the ideal $I(\ell)$. The standard notation for $I(\ell)$ is $\mathscr{L}^{2\ell,\infty}$ (Section 2).

THEOREM 1.1. *Given quasiconformal $M^{2\ell}$ with a bounded measurable conformal structure or, more generally, a bounded measurable $*$-structure on the ℓ-forms, the local construction yields an operator S which is determined by $*$ up to the ideal $I(\ell)$. Moreover, any such S satisfies:* (i) *S agrees* mod $I(\ell)$ *with the identity on exact ℓ-forms;* (ii) *S anticommutes modulo $I(\ell)$ with the involution γ associated to $*$ ($\gamma = *$ if ℓ is even, $\gamma = i*$ if ℓ is odd).*

Let us say an operator S on the Hilbert space $\mathscr{H}$ of middle dimensional forms satisfying (i) and (ii) of Theorem 1.1 belongs to the *Hodge class*. It is clear from the Hodge decomposition (Section 2) that given $*$ any two Hodge class operators differ by a compact operator in $I(\ell)$. Note also that (i) and (ii) imply: (iii) $S^2 \equiv I \bmod I(\ell)$.

There is a canonical nonlocal Hodge class operator S_* for the pair $(M, *)$. It is the involution defined, up to finite rank, by S_* is the identity on exact forms and S_* anticommutes with γ. The projectors associated with S_* are compatible with the usual Hodge decomposition of middle dimensional forms. Examples are the S_μ defined above for $S^{2\ell}$.

An interesting analytical consequence corresponding to the part relating to formula (2) of the proof of Theorem 1.1 is the following corollary.

COROLLARY 1.2. *Any* Hodge class operator *S on $\mathscr{H}$ defines a Fredholm module in the precise sense that for the sup norm dense subalgebra of continuous functions on $M^{2\ell}$ satisfying $\int_M |df|^{2\ell} < \infty$, the commutators $[S, f]$, where f denotes the multiplication operator associated to f, belong to the ideal $I(\ell)$ of compact operators.*

The interest in this corollary comes from Atiyah's paper [1]. The Dirichlet norm condition is sharp (see the appendix).

The theorem and corollary answer anew the question of Singer [21] about "constructing the operator", cf. [27, 11]. This time the context is bounded operators.

To our locally constructed Hodge class operator which satisfies $S\gamma + \gamma S = 0$ exactly, we apply the algebraic procedure of [6] to construct a *refined* Hodge class operator H satisfying the conditions (a) $H^2 = I$ on $\mathscr{H}$ exactly, (b) $H\gamma + \gamma H$ is trace class. This algebraic process (Theorem 3.2) preserves locality.

The formula for H in terms of S is

$$H = \gamma p(\theta) + \left(1 - \frac{1+\gamma}{2} p(\theta)\right) q(\theta) S$$

where $\theta = S^2 - I$, $q(t)$ is the polynomial of degree 2ℓ defined by $(1+t)^{-1/2} = q(t) + \mathrm{O}(t^{2\ell+1})$ and $p(t)$ is the polynomial given by $p(t) = (1+t)q(t)^2 - 1$.

The trace class operator $L = (H\gamma + \gamma H)H = H\gamma H + \gamma$ can be used in a simple way to construct representations of the Hirzebruch–Thom characteristic classes or rather their Poincaré dual homology classes. Assume $M^{2\ell}$ is oriented; then the kernel of L, $L(x, x')$, is a biform on $M \times M$ of bidegree (ℓ, ℓ). The support of $L(x, x')$ is near the diagonal since we started with a locally constructed Hodge class operator. The trace $L(x, x')$ on the diagonal is a 2ℓ form, or since M is oriented it is a measure in the Lebesgue measure class.

More generally, consider the cyclic expression $\{\text{trace } L(x_0, x_1)L(x_1, x_2) \cdots L(x_{2q}, x_0)\}$ which can be considered as either a top dimensional form or as a measure on $M \times M \times \cdots \times M$ ($2q + 1$ factors), supported near the diagonal.

THEOREM 1.3. *The cyclic expressions* $\{\text{trace } L(x_0, x_1)L(x_1, x_2)\cdots L(x_{2q}, x_0)\}$, *when considered as measures on* $(M)^{2q+1}$ *near the diagonal, define Alexander Spanier cycles. If* ℓ *and* q *are both even or both odd, these cycles represent the dual Hirzebruch–Thom characteristic homology classes times* $2^{2q+1}(2\pi i)^{-q}q!/2q!$. *In particular, if* ℓ *is even,* $L(x) = \text{trace } L(x, x')$ *is a locally constructed measure whose total mass is twice the signature of* $M^{2\ell}$. *All these measures are absolutely continuous with respect to the Lebesgue measure class.*

The algebraic construction used above to refine a Hodge class operator, $S \to H$, and the check that the odd cyclic expressions in the kernel $L(x, y)$ of $L = H\gamma H + \gamma$ define cycles are quite brief. However, the basic idea of this calculation is that any Hodge class operator defines a K-homology element because of the corollary and [1], and we know from [23, 27] what this element and its Chern character should be. We also know from [3] an explicit construction of the Chern character starting from K-theory of an algebra and arriving at the cyclic cohomology of the algebra which, for a manifold, is related to Alexander Spanier [6]. The connection with Hirzebruch–Thom classes is made by using the index theorem as in [27, 7, 6].

The local construction of the operators and the analytical content of the corollary are easy consequences of formula (2) on spheres relating the (canonical nonlocal) Hodge operator S_μ for any measurable $*$ to the (canonical nonlocal) Hodge operator S for the standard structure on $S^{2\ell}$. Formula (2) also relates to the other discussions, which we mention now.

On the Riemann sphere one knows the remarkable measurable Riemann mapping theorem that any bounded measurable conformal structure is related to the standard one by a quasiconformal homeomorphism $w = \varphi(z)$. *Since* S_μ is given by the kernel $dw\,dw'/(w - w')^2$ on (0, 1)-forms while S is given by the kernel $dz\,dz'/(z - z')^2$ on (0, 1)-forms, the basic formula (2) has a direct relationship with the measurable Riemann mapping theorem. Namely, we can calculate $w = \varphi(z)$ from the kernel of S_μ by expanding out the formula (2).

In dimension 4 the operators S_μ were used in the analytical underpinnings of the Yang–Mills discussion [7]. Formulae (2) show, since $\|\mu\| < 1$, that S_μ determines isomorphisms on L^p-forms for p a little greater than two. Applied to the curvature 2-forms this gives the extra regularity to get past the critical Sobolev exponent for the Yang–Mills connections and gauge transformations. A corollary of this theory and [9] was that some closed M^4 have infinitely many distinct quasiconformal structures, and that some topological M^4 have no quasiconformal structure.

Thus our local constructions for characteristic classes based on (2) are higher dimensional relatives of the measurable Riemann mapping in dimension two and the Yang–Mills theory in dimension four.

Outside dimension four there is a proof [24] independent of the theory of [17] that *stable* topological manifolds have a quasiconformal structure unique up to isotopy. Here stable refers to the pseudogroup of homeomorphisms of R^n in the connected components of the identity or of a reflection [15]. Thus this paper defines local characteristic classes for *stable* topological manifolds independent of Novikov's theory [17].

We can also apply our constructions and those of [24] to general topological manifolds outside dimension 4, but this uses Kirby's result on the stable homeomorphism conjecture [15]. The proof in [15] properly contains Novikov's theory needed for his original proof of the topological invariance of rational Pontryagin classes. Thus our constructions or those of [26] cannot be construed as a new proof of Novikov's theorem except for the stable category. A more appropriate title for [26] would have been "A new analytical proof of the invariance of rational Pontryagin classes for stable homeomorphisms". We are indebted to Sergei Novikov for this clarification.

Historical remark. If $g_{\alpha\beta}$ denotes the Jacobian matrices of the overlap homeomorphisms of charts covering a manifold M^n, the curving or nonflatness of M^n is measured by $\theta_{\alpha\beta} = \{g_{\alpha\beta}^{-1}\, \mathrm{d}g_{\alpha\beta}\}$, which is a Cech 1-cocycle with twisted values in matrices of 1-forms. Thus if the overlap homeomorphisms have Lipschitz derivatives (or even second derivatives in L^n) there is a Chern Weil type construction of characteristic forms by forming products and traces.

By considering normal bundles to smooth foliations or discrete group actions and the Bott vanishing theorem [2] one finds serious obstructions [28] to the possibility of reducing this smoothness requirement and staying in the context of differential forms.

In our context of Lipschitz or quasiconformal manifolds we have exactly one less derivative than required above. It seems natural then to try to interpret $g^{-1}\,\mathrm{d}g$ as a distribution or as an Alexander Spanier cochain. This was attempted in 1976 by the second author when the possibility of having Lipschitz or quasiconformal coordinates appeared. However, the distribution idea fails because of the impossibility of forming products. This difficulty is removed in Alexander Spanier at the expense of noncommuting products. But then the trace step in the classical Chern–Weil procedure becomes problematic. In other words, there is either an analytical or an algebraical barrier to copying the pointwise "curvature" route to characteristic classes for quasiconformal or Lipschitz charts.

In this paper these difficulties are surmounted by using trace ideals of operators on Hilbert space [3] and an algebraic addition to the Chern–Weil algorithm coming from cyclic cohomology [3]. The quasiconformal charts provide enough analysis to "quantize the manifold" in the sense of constructing a Hilbert space and a relevant operator replacing curvature.

This "quantized curvature" is then treated algebraically in a manner guided by the formulae of cyclic cohomology. The essence of this algebraic point is that the cocycles in the cyclic context are just those multilinear functionals which when applied to (projector, projector, . . .) remain constant when *the* projector is varied by a homotopy. The reader may recall that this kind of consideration appears classically when showing that the Chern–Weil forms are cohomology invariants.

In summary we have treated a problem with one missing derivative in a classical context using the ideas and tools of "noncommutative geometry" [5].

2. PRELIMINARIES ON QUASICONFORMAL GEOMETRY

A quasiconformal (qc) homeomorphism h between two open domains Ω_1, Ω_2 in $\mathbb{R}^n$ is a homeomorphism with the property that relative distances are boundedly distorted, i.e., for each x in Ω_1,

$$\overline{\lim_{r\to 0}} \frac{\max\{|h(x)-h(y)|;\ |x-y|=r\}}{\min\{|h(x)-h(y)|;\ |x-y|=r\}} = K(x) \leq K < \infty\ . \tag{3}$$

We also assume the analogous statement for h^{-1}.

Gehring [10] proved that when $n > 1$ a qc-homeomorphism is a.e. differentiable; moreover, the first-order partial derivatives of the component functions of h belong to the Banach space $L^{n+\varepsilon}_{\mathrm{loc}}$, where $\varepsilon = \varepsilon(K) > 0$. It follows that h is a Hölder continuous function with exponent ε, h is nonsingular with respect to the Lebesgue measure class, and the best K that works in (3) almost everywhere for h also works for h^{-1} almost everywhere.

Let g be an arbitrary Euclidean metric on the tangent space to $\mathbb{R}^n$ at some of its points x_0. Recall that the metric g and all other similar metrics rg, where r is an arbitrary positive real number, define the conformal class $[g]$ of the metric g.

If $[g_0]$ and $[g_1]$ are two conformal structures, the conformal distance between them is by definition

$$d([g_0],[g_1]) = \log \frac{\max\{|v|_{g_1};|v|_{g_0}=1\}}{\min\{|v|_{g_1};|v|_{g_0}=1\}}\,. \tag{4}$$

From now on we suppose that the dimension $n = 2\ell$ = even, and we choose the standard orientation on $\mathbb{R}^n$.

Let Λ denote the vector space of all differential forms of degree ℓ at x_0. For any Euclidean metric g as above, the Hodge star operator $*_g$ associated to g defines an endomorphism

$$*_g : \Lambda \to \Lambda$$

with

$$*_g^2 = (-1)^\ell\,.$$

The main property of the operator $*_g$ acting on Λ is that it remains unchanged under dilations of the metric g, i.e. it depends only on the conformal class of g.

We let γ_g be the involution of the complexification $\Lambda_{\mathbb{C}}$ of Λ given by

$$\gamma_g = i^\ell *_g\,. \tag{5}$$

We let $\Lambda^\pm(g)$ be the ± 1 eigenspaces of γ_g. These subspaces are maximal definite subspaces for the quadratic form

$$\omega \to \omega \wedge \omega\,.$$

The conformal distance between two conformal classes $[g_0]$ and $[g]$ may be estimated in terms of the relative position of the eigenspaces $\Lambda^\pm$. Indeed, there exists a unique linear mapping

$$\mu : \Lambda^-(g_0) \to \Lambda^+(g_0) \tag{6}$$

with the property that the graph of μ is precisely $\Lambda^-(g)$. The operator norm of μ relative to the metric g_0 satisfies

$$|\mu|_{g_0} < 1 \tag{7}$$

and

$$\tfrac{1}{2}\log\frac{1+|\mu|_{g_0}}{1-|\mu|_{g_0}} \leq d([g_0],[g_1]) \leq \log\frac{1+|\mu|_{g_0}}{1-|\mu|_{g_0}}\,. \tag{8}$$

A field $c(x)$ of conformal structures over a domain U in $\mathbb{R}^n$ is called a bounded measurable (bm) conformal structure on U if there exists a Riemannian metric g over U, whose components are measurable functions, such that for any x in U, $c(x) = [g(x)]$, and

$$d([g(x)], [e]) \leq C < \infty$$

where e denotes the standard Euclidean metric.

Equivalently, c is a bounded measurable conformal structure iff the corresponding field of endomorphisms μ, relative to e, is a matrix field with measurable entries, and

$$\|\mu\|_\infty \doteq \sup |\mu(x)| < 1 .$$

If c is a bounded measurable conformal structure on U, then γ_c is a field of matrices with bounded measurable entries. If c is a bounded measurable conformal structure on V and $h: U \to V$ is a qc-homeomorphism, then h^*c is a bounded measurable conformal structure on U because a qc-homeomorphism induces uniformly quasihomotheties on the tangent spaces, a.e.

A qc-manifold is a topological manifold equipped with an atlas whose changes of coordinates are qc-homeomorphisms. It possesses a well-defined measure class, the Lebesgue measure class, since qc-homeomorphisms are absolutely continuous. The tangent bundle of a qc-manifold is a measurable real vector bundle.

A bounded measurable conformal structure on a qc-manifold is a field of conformal structures on its tangent spaces whose restriction to any qc-chart is bounded measurable. Any paracompact qc-manifold has such structures.

On a compact smooth manifold, a conformal structure is bounded iff the conformal distance (defined point by point) between it and the underlying conformal structure of a smooth Riemannian metric is a bounded function.

On a compact qc-manifold M, the space $L^{n/r}(M, \wedge^r)$ of r-forms with coefficients in $L^{n/r}$, $n = \dim M$, is well defined. Any bounded measurable conformal structure specifies a Banach space norm on $L^{n/r}$ by

$$(\|\omega\|)^{n/r} = \int_M |\omega|^{n/r} .$$

Given $\omega_1 \in L^{n/r}(M, \wedge^r)$, $\omega_2 \in L^{n/r+1}(M, \wedge^{r+1})$ we write $d\omega_1 = \omega_2$ iff this holds, in the sense of distributions, in any qc-local chart. This yields [25] a densely defined closed operator $d: L^{n/r} \to L^{n/r+1}$ which commutes with qc-homeomorphisms. We let Im d be the image of d; it is closed in $L^{n/r+1}$ provided $r \geqslant 1$ [7, 25], see the proof of Proposition 4.5(4).

The underlying topological vector spaces only depend on the quasiconformal structure.

3. STATEMENT OF THE MAIN RESULT

Let M be a compact oriented quasiconformal manifold of even dimension 2ℓ. Let $[g]$ be a bounded measurable conformal structure on M and γ, $\gamma^2 = 1$, the associated $*$ operator in the Hilbert space $\mathscr{H} = L^2(M, \wedge^\ell)$ of square integrable forms of degree ℓ on M.

Given an open neighborhood U of the diagonal in $M \times M$ and any bounded operator T in $\mathscr{H} = L^2(M, \wedge^\ell)$, we say that Support $(T) \subset U$ iff the following holds for any open $V \subset M$:

$$\omega \in \mathscr{H}, \text{ Support } \omega \subset V \;\Rightarrow\; \text{Support}(T\omega) \subset U \circ V \tag{9}$$

where $U \circ V = \{x \in M;\ \exists y \in V, (x, y) \in U\}$.

For any $p \in [1, \infty[$ the following conditions define two-sided ideals of compact operators in Hilbert space. We let, for any compact operator T in $\mathscr{H}$, $\mu_n(T)$ be the nth characteristic value of T, i.e. the nth eigenvalue of $|T| = (T^*T)^{1/2}$, or the distance in operator norm between T and rank-n operators.

$$\mathscr{L}^p(\mathscr{H}) = \left\{T \text{ compact}; \sum_1^\infty \mu_n(T)^p < \infty\right\} \tag{10}$$

$$\mathscr{L}^{(p,\infty)}(\mathscr{H}) = \{T \text{ compact}; \mu_n(T) = \mathrm{O}(n^{-1/p})\}. \tag{11}$$

Recall that $\mathscr{L}^1$ are the trace class operators, $\mathscr{L}^2$ are the Hilbert–Schmidt operators, and $\mathscr{L}^{(n,\infty)}$ contains degree-one smoothing operators on R^n [20, 29]. We can now define the following key notion refining that of Hodge class operators in the introduction.

Definition 3.1. Let U be an open neighborhood of the diagonal in $M \times M$. A U-local Hodge decomposition is a bounded operator H in $L^2(M, \wedge^\ell T^*_{\mathbb{C}})$ such that

(α) $H^2 = 1$,
(β) Support $H \subset U$,
(γ) $(H - 1)/\mathrm{Im}\, d \in \mathscr{L}^{(2\ell,\infty)}$ ($= I(\ell)$ of the introduction),
(δ) $H\gamma + \gamma H \in \mathscr{L}^1$ (= trace class).

Giving H is the same as giving the decomposition of $\mathscr{H} = L^2(M, \wedge^\ell T^*_{\mathbb{C}})$ as the linear sum of the two closed subspaces:

$$\{\xi \in \mathscr{H};\, H\xi = \pm\xi\}. \tag{12}$$

We shall now explain how to construct for each $q \in \{0, 1, \ldots, \ell\}$ an Alexander Spanier cycle on M from a U-local Hodge decomposition. To define Alexander Spanier homology on a compact space X we consider for each integer d the linear space A_d of totally antisymmetric measures σ on X^{d+1}. Such a measure σ is uniquely determined by the value of $\sigma(\varphi) = \int \varphi\, d\sigma$ on bounded Borel antisymmetric functions φ on X^{d+1}. We let $\delta: A_d \to A_{d-1}$ be the boundary operator given by the equality

$$(\delta\sigma)(\varphi) = \sum_0^d (-1)^j \int \varphi(x_0, \ldots, \check{x}_j, \ldots, x_d)\, d\sigma \quad \forall \varphi. \tag{13}$$

Let U be a neighborhood of the diagonal in $X \times X$. We shall say that $\sigma \in A_d$ is U-local iff

$$\text{Support}\, \sigma \subset \{(x_j) \in X^{d+1}; (x_i, x_j) \in U \quad \forall i, j \in \{0, 1, \ldots, d\}\}. \tag{14}$$

One checks that condition (14) is preserved by δ. This defines the complex (A_U, δ) of U-local elements of (A, δ). The Alexander Spanier homology $H_*(X, \mathbb{R})$ is obtained as the projective limit $\varprojlim H^*(A_U, \delta)$, when U runs through all open neighborhoods of the diagonal. The limit is actually achieved on appropriate sufficiently small neighborhoods.

Given a measure space (X, ν) and a measurable Hermitian vector bundle Λ on X, the Hilbert–Schmidt operators in $\mathscr{H} = L^2(X, \Lambda)$ are all given [20] by measurable kernels,

$$k(x, y) \in \mathrm{Hom}(\Lambda_y, \Lambda_x), \quad x, y \in X \tag{12'}$$

such that

$$\int_X \mathrm{trace}(k(x, y)^* k(x, y))\, d\nu(x)\, d\nu(y) < \infty. \tag{13'}$$

In particular, for any such kernel k the following expression defines a measure σ on X^{d+1} for any $d \geq 1$:

$$\sigma(\varphi) = \int_{X^{d+1}} \operatorname{trace}(k(x_0, x_1)k(x_1, x_2)\cdots k(x_d, x_0))\varphi(x_0, \ldots, x_d)\prod dv(x_i) \tag{14'}$$

as follows from the inequality

$$|\sigma(\varphi)| \leq \|k\|_2^{d+1}\|\varphi\|_\infty \quad \forall\varphi \in L^\infty(X^{d+1}, v^{d+1}). \tag{15}$$

We shall use the notation

$$\operatorname{trace}(\wedge^{d+1}k) = \text{total antisymmetrization of } \sigma \tag{16}$$

(where σ is associated to k by (14')). For $d = 0$ this formula continues to make sense provided the operator in $\mathscr{H} = L^2(X, \Lambda)$ associated to k is of *trace class* [20].

We can now state the main result of this paper. The proof occupies the next two sections.

THEOREM 3.2. *Let M be a compact oriented quasiconformal manifold of even dimension 2ℓ, γ the $\mathbb{Z}/2$ grading of $\mathscr{H} = L^2(M, \wedge^\ell T^*_{\mathbb{C}})$ associated to a measurable bounded conformal structure $[g]$ on M and U a neighborhood of the diagonal in $M \times M$.*

(1) *There exists a locally constructed U-local Hodge decomposition H.*
(2) *Let H be a U-local Hodge decomposition and $L = H\gamma H + \gamma$ with kernel $L(x, y)$. Then the measure $\sigma = \operatorname{trace}(\wedge^{2q+1}L)$ is a U^{2q}-local Alexander Spanier cycle of dimension $2q$.*
(3) *The homology class of σ among U^r-local cycles, $r = 2q(6\ell + 2)$, is independent of the choice of H.*
(4) *The homology class of σ is equal to $\lambda_{2q}(L_{2\ell-2q} \cap [M])$, where L is the Hirzebruch–Thom L-class and $\lambda_{2q} = 2^{2q+1}(2\pi i)^{-q}\, q!/2q!$.*

4. LOCAL CONSTRUCTION OF A U-LOCAL HODGE DECOMPOSITION

Let M be a quasiconformal manifold and $[g]$ a bounded measurable conformal structure on M. In this section we shall show how to construct local Hodge decompositions H using a covering of M by domains of qc local charts:

$$\rho_\alpha : V_\alpha \to S^{2\ell}.$$

The obtained formula for H will be algebraic in terms of the following ingredients:

(1) A partition of unity subordinate to the covering (V_α) of M.
(2) The pull-back by ρ_α of Hodge decompositions on $S^{2\ell}$ associated to a bounded measurable conformal structure which agrees with $\rho_\alpha[g]$ on $\rho_\alpha(V_\alpha)$.

We shall begin by describing the canonical Hodge decomposition on $S^{2\ell}$ associated to a bounded measurable conformal structure.

(α) *Canonical Hodge decomposition on $S^{2\ell}$*

Let $[g_0]$ be the standard conformal structure on the sphere $S^{2\ell}$, and $[g]$ an arbitrary bounded measurable conformal structure on $S^{2\ell}$. Let γ_0, γ be the corresponding $*$ operations in the vector bundle Λ of middle dimensional forms. If we let $\Lambda_\pm$ be the two eigenspaces of γ_0 we get two subbundles of Λ and a unique measurable bundle homomorphism:

$$\mu_+ : \Lambda_- \to \Lambda_+$$

whose graph at each point $p \in S^{2\ell}$ gives the subspace

$$\{\omega \in \Lambda_p;\ \gamma\omega = -\omega\}.$$

We endow the vector bundle Λ with the metric associated to the standard conformal structure $[g_0]$. The boundedness of the measurable conformal structure $[g]$ then means that

$$\|\mu_+\|_\infty = \operatorname*{Sup}_{S^{2\ell}} |\mu_+(p)| < 1. \tag{17}$$

Let

$$\mu = \begin{bmatrix} 0 & \mu_+ \\ \mu_+^* & 0 \end{bmatrix}$$

viewed as an endomorphism of the vector bundle Λ. One has

$$\mu\gamma_0 = -\gamma_0\mu, \quad \mu = \mu^* \tag{18}$$

and so μ is self-adjoint with respect to the wedge pairing:

$$\gamma = (1+\mu)\gamma_0(1+\mu)^{-1}. \tag{19}$$

Indeed $\gamma^2 = 1$ and by definition $(1+\mu)$ transports the eigenspaces of γ_0 into those of γ using (18).

Finally, on the vector bundle Λ the metric associated to the conformal structure $[g]$ is given by

$$\langle\omega_1, \omega_2\rangle_g = \langle\omega_1, \gamma_0\gamma\omega_2\rangle_{g_0} = \left\langle\omega_1, \left(\frac{1-\mu}{1+\mu}\right)\omega_2\right\rangle_{g_0}. \tag{20}$$

Note that since $\|\mu\| < 1$ the operator $(1-\mu)(1+\mu)^{-1}$ is positive. We now consider the Hilbert space $\mathscr{H}_0 = L^2(S^{2\ell}, \Lambda)$ with the inner product given by $[g_0]$. We view all the above endomorphisms of the vector bundle Λ as operators in $\mathscr{H}_0$. The equalities (18) and (19) continue to hold.

The standard Hodge decomposition on $S^{2\ell}$ decomposes $\mathscr{H}_0$ as the direct sum of two orthogonal subspaces, the exact forms and the coexact forms. Let H_0 (denoted by S in the introduction and in formula (2)) be the linear operator such that $H_0\,\omega = \omega$ for any exact form and $H_0\,\omega = -\omega$ for any coexact form. One has

$$H_0 = H_0^*, \qquad H_0^2 = 1 \tag{21}$$

$$H_0\gamma_0 = -\gamma_0 H_0. \tag{22}$$

Moreover, since H_0 is a standard singular integral operator of order 0, the following subalgebra $A(S^{2\ell})$ of the algebra of continuous functions $C(S^{2\ell})$ contains all smooth functions [13] and is therefore norm dense:

$$A(S^{2\ell}) = \{f \in C(S^{2\ell});\ [H_0, f] \in \mathscr{L}^{(2\ell,\infty)}\} \tag{23}$$

where $f \in C(S^{2\ell})$ is considered as a multiplication operator in $L^2(S^{2\ell}, \Lambda)$ and recall that $\mathscr{L}^{(2\ell,\infty)}$ is the two-sided ideal of compact operators in $\mathscr{H}_0$ given by the condition

$$T \in \mathscr{L}^{(2\ell,\infty)} \Leftrightarrow \mu_n(T) = O(n^{-1/2\ell}) \tag{24}$$

where $\mu_n(T)$ is the nth characteristic value of T.

Let us now consider on the locally convex vector space $\mathscr{H}_0$ the new inner product given by the metric g; using (20) this can be expressed by

$$\langle \omega_1, \omega_2 \rangle = \left\langle \omega_1, \frac{1-\mu}{1+\mu}\omega_2 \right\rangle_0 . \tag{25}$$

The Hodge decomposition on $S^{2\ell}$ relative to the bounded measurable conformal structure $[g]$ is given by the following proposition.

PROPOSITION 4.1. (a) *The orthogonal complement for the inner product* (25) *of* $\operatorname{Im} d = \{\omega;\ H_0\omega = \omega\}$ *is equal to* $\gamma(\operatorname{Im} d)$.
(b) *Let H be the linear operator equal to* 1 *on* $\operatorname{Im} d$ *and to* -1 *on* $\gamma(\operatorname{Im} d)$; *then*

$$H = (1-\mu)^{-1}(H_0 - \mu)H_0(H_0-\mu)^{-1}(1-\mu).$$

Proof. Since $H_0^2 = 1$ and $\|\mu\| < 1$ the operator $H_0 - \mu$ is invertible. Let us consider the operator $T = (1-\mu)^{-1}(H_0-\mu)H_0(H_0-\mu)^{-1}(1-\mu)$. It is conjugate to H_0 so that $T^2 = 1$. Its eigenspaces are obtained from those of H_0 by applying $(1-\mu)^{-1}(H_0-\mu)$. Hence $\{\xi;\ T\xi = \xi\} = \operatorname{Im} d$. One has $\gamma = (1-\mu)^{-1}\gamma_0(1-\mu)$ and hence $\gamma T = -T\gamma$ which shows that $\{\xi;\ T\xi = -\xi\} = \gamma(\operatorname{Im} d)$. The orthogonality of $\operatorname{Im} d$ with $\gamma_0(\operatorname{Im} d)$ for the inner product $\langle\ \rangle_0$ implies the orthogonality of $\operatorname{Im} d$ with $\gamma(\operatorname{Im} d)$ for the inner product (25), using $\gamma_0\gamma = (1-\mu)/(1+\mu)$. Thus we have shown (a) and (b). □

COROLLARY 4.2. *For any $f \in L^\infty(S^{2\ell})$ and any two-sided ideal J of operators in $\mathscr{H}_0$ one has*

$$[H_0, f] \in J \iff [H, f] \in J .$$

Proof. Since μ commutes with f and $[H, \cdot]$ satisfies the Leibniz rule, direct calculation yields

$$[H, f] = -(1+\mu)(H_0-\mu)^{-1}[H_0, f](H_0-\mu)^{-1}(1-\mu). \tag{26}$$

This completes the proof. □

COROLLARY 4.3. *For any two-sided ideal J of operators in Hilbert space the class of functions $f \in L^\infty(S^{2\ell})$ such that $[H_0, f] \in J$ is invariant under qc-homeomorphisms.*

Proof. Let φ be a qc-homeomorphism of $S^{2\ell}$; then φ is a.e. differentiable and it defines a bounded operator $U(\varphi)$ in $\mathscr{H}_0 = L^2(S^{2\ell}, \Lambda)$ by the formula

$$U(\varphi)\omega = (\varphi^{-1})^*\omega \quad \forall \omega \in \mathscr{H}_0 . \tag{27}$$

This yields a bounded operator in $\mathscr{H}_0$ such that

$$U(\varphi) f U(\varphi)^{-1} = f \circ \varphi^{-1} \quad \forall f \in L^\infty(S^{2\ell}) \tag{28}$$

$$U(\varphi) H_0 U(\varphi)^{-1} = H_g \tag{29}$$

where $[g]$ is the mb conformal structure $(\varphi^{-1})^*[g_0]$. To prove (29) note first that $U(\varphi)$ $\operatorname{Im} d = \operatorname{Im} d$ (cf. [7]), while $\langle U(\varphi)\omega_1, U(\varphi)\omega_2\rangle_0 = \langle \omega_1, \omega_2\rangle_g$ $\forall \omega_1, \omega_2 \in \mathscr{H}_0$. Thus $U(\varphi)$ $H_0 U(\varphi)^{-1}$ is equal to 1 on $\operatorname{Im} d$ and to -1 on its orthogonal complement for the inner product $\langle\ \rangle_g$. Hence $U(\varphi)H_0U(\varphi)^{-1} = H_g$. Then let $f \in L^\infty(S^{2\ell})$; if $[H_0, f] \in J$ then $[U(\varphi)H_0U(\varphi)^{-1}, U(\varphi) f U(\varphi)^{-1}] \in J$ and $[H_g, f\circ\varphi^{-1}] \in J$ so that $[H_0, f\circ\varphi^{-1}] \in J$ by the above corollary. □

We shall now see that, modulo the ideal $J = \mathscr{L}^{(2\ell, \infty)}$, the class of the operator H_g is locally determined by the bounded measurable conformal structure $[g]$.

PROPOSITION 4.4. *Let $U \subset S^{2\ell}$ be an open subset and f_1, f_2 be continuous functions with support in U. Let g_1, g_2 be two bm conformal structures on $S^{2\ell}$ which agree on U. Then $f_1(H_{g_1} - H_{g_2})f_2 \in \mathscr{L}^{(2\ell, \infty)}$.*

Proof. We can replace f_i by smooth functions equal to 1 on the support of the previous ones. Thus we can assume that $f_i \in A(S^{2\ell})$. By Proposition 4.1 both operators H_{g_i} are the sum of a geometrically norm convergent series:

$$H_{g_i} = (1 - \mu_i)^{-1}(H_0 - \mu_i)\sum_0^\infty (\mu_i H_0)^n (1 - \mu_i) \tag{30}$$

whose terms are monomials of the form

$$a_{1,i} H_0 a_{2,i} H_0 \cdots a_{n,i} H_0 a_{n+1,i} = T_{n,i} \tag{31}$$

where $a_{k,i}$ belongs to the commutant of $L^\infty(S^{2\ell})$ in $\mathscr{H}_0$ and $f_1 a_{k,1} = f_1 a_{k,2}$ for all k. It follows, using $[H_0, f_1] \in \mathscr{L}^{(2\ell, \infty)}$, that

$$f_1(T_{n,1} - T_{n,2}) \in \mathscr{L}^{(2\ell, \infty)}. \tag{32}$$

One thus expresses $f_1(H_{g_1} - H_{g_2})$ as the sum of a series convergent in the Banach space $\mathscr{L}^{(2\ell, \infty)}$, and the conclusion follows. □

(β) *The class of H modulo $\mathscr{L}^{(2\ell, \infty)}$*

Now let M be a quasiconformal manifold of dimension 2ℓ.

Let $A(M)$ be the subalgebra of $C(M)$ of functions f such that for any qc-local chart: $V \xrightarrow{\rho} S^{2\ell}$ and for any $h \in C_c(\rho(V)) \cap A(S^{2\ell})$ one has $h\, f \circ \rho^{-1} \in A(S^{2\ell})$. By Corollary 4.3 one has $\rho^*(A(S^{2\ell}) \cap C_c(\rho(V))) \subset A(M)$ for any qc-local chart $V \xrightarrow{\rho} S^{2\ell}$. It follows that $A(M)$ is a norm dense subalgebra of $C(M)$ and that it has partitions of unity subordinate to any finite open covering of M.

Let $[g]$ be a bounded measurable conformal structure on M.

Let (V_α) be a finite open cover of M by domains of qc-local charts $\rho_\alpha: V_\alpha \to S^{2\ell}$ and g_α a bounded measurable conformal structure on $S^{2\ell}$ which agrees with $\rho_\alpha[g]$ on $\rho_\alpha(V_\alpha)$. For each α let $H_\alpha = H_{g_\alpha}$ be the corresponding Hodge decomposition on $S^{2\ell}$. Let (φ_α) be a partition of unity, $\varphi_\alpha \in A(M)$, Support $\varphi_\alpha \subset V_\alpha$ and for each α let $\psi_\alpha \in A(M)$ be equal to 1 in a neighborhood of Support φ_α, with Support $\psi_\alpha \subset V_\alpha$. We let, as above, Λ be the measurable vector bundle $\wedge^\ell T^*_{\mathbb{C}}$ on M and we consider the following locally constructed operator in $L^2(M, \Lambda)$:

$$S = \sum \psi_\alpha (H_\alpha)^{\rho_\alpha} \varphi_\alpha \tag{33}$$

where we used ρ_α to let H_α act in $L^2(M, \Lambda)$.

PROPOSITION 4.5. *The class of S modulo $\mathscr{L}^{(2\ell, \infty)}$ only depends upon the bm conformal structure $[g]$ on M and one has*

(1) $S\gamma + \gamma S \in \mathscr{L}^{(2\ell, \infty)}$ (*the construction yields* $S\gamma + \gamma S = 0$),
(2) $[S, f] \in \mathscr{L}^{(2\ell, \infty)}\ \forall f \in A(M)$,
(3) $S^2 - 1 \in \mathscr{L}^{(2\ell, \infty)}$,
(4) $(S - 1)/\mathrm{Im}\, d \in \mathscr{L}^{(2\ell, \infty)}$.

Proof. We first need to show that $S - S' \in \mathscr{L}^{(2\ell,\infty)}$ for any two operators S, S' constructed by formula (33). Using the compactness of M it is enough to show that for any $x \in M$ there exists $f \in A(M)$, $f(x) \neq 0$ such that $(S - S')f \in \mathscr{L}^{(2\ell,\infty)}$. Using a local qc-chart the proof follows from Proposition 4.4. To check (1) note that any of the operators S given by (33) satisfies $S\gamma + \gamma S = 0$.

The condition (2) follows from Corollary 4.3.

To check (3) we take a representative $S = \sum \psi_\alpha H_\alpha \varphi_\alpha$ constructed from a covering (V_α) by domains of qc-charts, such that for any α the open set $\bigcup V_\beta$, $V_\beta \bigcap V_\alpha \neq \emptyset$ is the domain of a qc-chart. The result then follows from Corollary 4.3 and Proposition 4.4. Actually (3) also follows formally from (1) and (4).

Let us check (4). On the sphere $S^{2\ell}$ the operators H_g, g a bounded measurable conformal structure, are equal to 1 on Im d (Proposition 4.1). Thus, using Corollary 4.3, the operator $S = \sum \psi_\alpha H_\alpha \varphi_\alpha$ satisfies $(S - 1)/E_\alpha \in \mathscr{L}^{(2\ell,\infty)}$, where E_α is the closure in $L^2(M, \Lambda)$ of $\{d\omega;\, \omega \in L^{2\ell/\ell-1}(V_\alpha, \Lambda),\, d\omega \in L^2\}$. Thus (4) follows if we show that the map $(\omega_\alpha) \in \oplus E_\alpha \to \sum \omega_\alpha \in L^2(M, \Lambda)$ is surjective on Im d. Let P_α (resp. $Q_\alpha = \gamma P_\alpha \gamma$) be the orthogonal projection on E_α (resp. γE_α). For any α, β the closed subspaces $E_\alpha \subset \operatorname{Im} d$ and $\gamma E_\beta \subset \gamma(\operatorname{Im} d)$ are orthogonal. Thus it is enough to show that the following operator is equal to $1 +$ compact:

$$T = \sum (P_\alpha + Q_\alpha)\varphi_\alpha .$$

(Its range is then closed by the Fredholm theory.)

Let us show that for each α the operator $(P_\alpha + Q_\alpha)\varphi_\alpha - \varphi_\alpha$ is compact. We can assume that $M = S^{2\ell}$ is the sphere. By Proposition 4.1(a) one has orthogonal decomposition $L^2(S^{2\ell}, \Lambda) = \operatorname{Im} d \oplus \gamma \operatorname{Im} d$, where $d: L^{2\ell/\ell-1}(S^{2\ell}, \Lambda^{\ell-1}) \to L^2(S^{2\ell}, \Lambda^\ell)$ has closed range. (We assume $\ell \neq 1$.)

Using a compact operator $R: L^2(S^{2\ell}, \Lambda) \to L^{2\ell/\ell-1}(S^{2\ell}, \Lambda^{\ell-1})$ such that $dR = 1$ on Im d we thus get a pair of compact operators $R_j: L^2(S^{2\ell}, \Lambda) \to L^{2\ell/\ell-1}(S^{2\ell}, \Lambda^{\ell-1})$ such that $dR_1 + \gamma d R_2 = 1$. The conclusion then follows using the following equality, with ψ a smooth function with support in V_α and equal to 1 in a neighborhood of Support φ_α:

$$\psi\omega = d(\psi\omega_1) + \gamma d(\psi\omega_2) - d\psi \wedge \omega_1 - \gamma(d\psi \wedge \omega_2), \quad \omega_i = R_i\omega . \qquad \square$$

Remark The discussion of (4) is tantamount to a proof that d from $(\ell - 1)$-forms of class L^p, where $p = 2\ell/(\ell - 1)$), to ℓ-forms of class L^2 has closed image, $\ell > 1$.

5. PROOF OF THEOREM 3.2

Proof of (1). First choose a neighborhood V of the diagonal such that $V^{2q} \subset U$, $q = 6\ell + 2$. Next (Proposition 4.5) let S be an operator such that

(a) Support$(S) \subset V$,
(b) $S^2 - 1 \in \mathscr{L}^{(2\ell,\infty)}$,
(c) $S\gamma = -\gamma S$,
(d) $(S - 1)/\operatorname{Im} d \in \mathscr{L}^{(2\ell,\infty)}$.

Then let $\theta = S^2 - 1$. By construction, θ commutes with S; it also commutes with γ by (c). Let $q(t)$ be the unique polynomial of degree 2ℓ such that

$$(1 + t)^{-1/2} = q(t) + O(t^{2\ell+1}) \quad (\text{for } t \text{ small}). \tag{34}$$

Let $p(t)$ be the polynomial given by

$$p(t) = (1 + t)q(t)^2 - 1 . \tag{35}$$

We shall define an operator H by the formula

$$H = \gamma\, p(\theta) + \left(1 - \left(\frac{1+\gamma}{2}\right) p(\theta)\right) q(\theta)\, S. \tag{36}$$

First note that θ is V^2-local; thus since q (resp. p) has degree 2ℓ (resp. $4\ell + 1$) we see that T is U-local as required. Next, as $\theta \in \mathscr{L}^{(2\ell, \infty)}$ by (b) and as $p(t) = O(t^{2\ell+1})$ we see that $p(\theta) \in \mathscr{L}^1$. We thus have

$$H\gamma + \gamma H = 2p(\theta) \in \mathscr{L}^1 \tag{37}$$

where we used (c) to get the equality. Since $p(\theta) \in \mathscr{L}^1$ and $\theta \in \mathscr{L}^{(2\ell, \infty)}$ we have $S - H \in \mathscr{L}^{(2\ell, \infty)}$ and hence, using (d),

$$(H - \mathrm{id})/\operatorname{Im} d \in \mathscr{L}^{(2\ell, \infty)}. \tag{38}$$

It remains to check that $H^2 = 1$. Note first that the two terms $\gamma\, p(\theta)$, $H - \gamma\, p(\theta)$ anticommute so that H^2 is the sum of their squares:

$$\begin{aligned} H^2 &= (\gamma\, p(\theta))^2 + \mathrm{q}(\theta)^2 \left(1 - \frac{1+\gamma}{2} p(\theta)\right)\left(1 - \frac{1-\gamma}{2} p(\theta)\right) S^2 \\ &= p(\theta)^2 + q(\theta)^2 (1 - p(\theta))(1 + \theta) = 1. \end{aligned}$$

Before we begin the proof of (2) we recall that the cyclic complex (C_λ^*, b) of an algebra $\mathscr{A}$ is given by

$$\begin{aligned} C_\lambda^n(\mathscr{A}) = \{&\text{multilinear forms } \tau \text{ on } \mathscr{A} \times \cdots \times \mathscr{A}\, (n+1 \text{ times}) \\ &\text{such that } \tau(a^1, \ldots, a^n, a^0) = (-1)^n \tau(a^0, \ldots, a^n) \quad \forall a^j \in \mathscr{A}\} \end{aligned} \tag{39}$$

$$\begin{aligned} (b\tau)(a^0, \ldots, a^{n+1}) = &\sum_0^n (-1)^j \tau(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) \\ &+ (-1)^{n+1} \tau(a^{n+1} a^0, \ldots, a^n) \quad \forall a^j \in \mathscr{A}. \end{aligned} \tag{40}$$

We also note that if J is an ideal in a larger algebra $\mathscr{A}$ and if $B \subset \mathscr{A}$ is a subalgebra such that $J \cap B = \{0\}$, then the natural extension by 0 on B of the cochains $\tau \in C_\lambda^*(J)$ satisfying

$$\begin{aligned} &\tau(a^0, a^1, \ldots, a^j \delta, a^{j+1}, \ldots, a^n) = \tau(a^0, a^1, \ldots, a^j, \delta a^{j+1}, \ldots, a^n) \\ &\forall a^i \in \mathscr{A}, \quad \forall j, \quad \forall \delta \in B \end{aligned} \tag{41}$$

commutes with the coboundary b of $C_\lambda^*(J + B)$.

Proof of (2). It follows from [6, Lemma 2], but we shall give the details here. By construction, $\sigma = \operatorname{trace}(\wedge^{d+1} k)$ is $U \circ U \circ \cdots \circ U$-local ($d$-factors), where $U \circ V$ is as defined in Section 3. We need to show that $b\sigma = 0$. We shall first show that, with J the algebra of trace class operators in $\mathscr{H}$, the following formula defines a morphism of complexes: $(A^*, \delta) \xrightarrow{\tau} (C_\lambda^*(J), b)$, where (A^*, δ) is the complex of bounded measurable totally anti-symmetric functions (straight cochains) on M.

$$\tau_\varphi(k^0, \ldots, k^n)$$

$$(-1)^n \int_{M^{n+1}} \operatorname{trace}(k^0(x_0, x_1) k^1(x_1, x_2) \ldots k^n(x_n, x_0))\, \varphi(x_0, \ldots, x_n) \prod dv(x_i) \quad \forall \varphi \in A^n. \tag{42}$$

The coboundary δ in (A^*, δ) is given by

$$\delta\varphi = \sum_0^{n+1} (-1)^j \varphi_j, \qquad \varphi_j(x_0, \ldots, x_{n+1}) = \varphi(x_0, \ldots, \check{x}_j, \ldots, x_{n+1}). \tag{43}$$

Next, when one computes τ_{φ_j}, $j \geq 1$, the variable x_j does not occur in φ_j and thus one gets

$$\tau_{\varphi_j}(k^0, \ldots, k^{n+1}) = -\tau_\varphi(k^0, \ldots, k^{j-1}k^j, \ldots, k^{n+1}). \tag{44}$$

For $j = 0$ it is the variable x_0 which does not occur in φ_0 and using the cyclicity of the trace one gets, using the antisymmetry of φ,

$$\tau_{\varphi_0}(k^0, \ldots, k^{n+1}) = (-1)^{n+1}\tau_\varphi(k^{n+1}k^0, k^1, \ldots, k^n). \tag{45}$$

Thus, using (43) we get

$$\tau_{\delta\varphi} = b\tau_\varphi \quad \forall \varphi \in A^n. \tag{46}$$

Next note that the compatibility condition (41) is fulfilled by any element of the algebra $B = \{\lambda_0 + \lambda_1\gamma;\ \lambda_j \in \mathbb{C}\}$ generated by the operator γ in $\mathscr{H}$. We shall still denote by τ the extension of the above morphism of complexes to $(C_\lambda^*(\mathscr{A}), b)$, $\mathscr{A} = J + B$.

Let us now check that $b\sigma = 0$, i.e. that $\sigma(\delta\varphi) = 0$ for any $\varphi \in A^d$. Using (46) we know that the cyclic cocycle $\tau_{\delta\varphi}$ is a coboundary and, thus, that it vanishes when evaluated on an idempotent:

$$\tau_{\delta\varphi}(P, P, \ldots, P) = 0 \quad \forall P, \quad P^2 = P, \quad P \in \mathscr{A}. \tag{47}$$

Applying this to $P = H((1-\gamma)/2)\,H = \frac{1}{2}((1+\gamma)/2) + \frac{1}{2}L$ gives the desired result.

Proof of (3). Let H_0, H_1 be two U-local Hodge decompositions and $Y = H_1 - H_0$. We have

$$Y/\mathrm{Im}\, d \in \mathscr{L}^{(2\ell, \infty)} \tag{48}$$

$$Y\gamma + \gamma Y \in \mathscr{L}^1. \tag{49}$$

As $\mathrm{Im}\, d \oplus \gamma(\mathrm{Im}\, d)$ is of finite codimension in $\mathscr{H}$ it follows that

$$Y \in \mathscr{L}^{(2\ell, \infty)}. \tag{50}$$

Let then $H_t = H_0 + tY$. We have

$$\text{Support } H_t \subset U \tag{51}$$

$$H_t^2 - 1 \in \mathscr{L}^{(2\ell, \infty)} \tag{52}$$

$$(H_t - 1)/\mathrm{Im}\, d \in \mathscr{L}^{(2\ell, \infty)} \tag{53}$$

$$H_t\gamma + \gamma H_t \in \mathscr{L}^1. \tag{54}$$

It follows that $S_t = \frac{1}{2}(H_t - \gamma H_t \gamma)$ anticommutes with γ and satisfies (51)–(53). Thus with q and p as in (34) and (35) we get a family H_t' of U^n-local Hodge decompositions:

$$H_t' = \gamma p(\theta_t) + \left(1 - \left(\frac{1+\gamma}{2}\right)p(\theta_t)\right)q(\theta_t)S_t, \quad \theta_t = S_t^2 - 1. \tag{55}$$

For $t = 0$ (or for $t = 1$), the operator $\theta_0 = S_0^2 - 1$ belongs to $\mathscr{L}^1$; we can thus, keeping the relation (35), replace the polynomial q by $1 + \lambda(q-1)$, $\lambda \in [0, 1]$ and still get a family of U^n-local Hodge decompositions joining H_0' (for $\lambda = 1$) with

$$H_0'' = \gamma\theta_0 + \left(1 - \left(\frac{1+\gamma}{2}\right)\theta_0\right)S_0, \quad \theta_0 = S_0^2 - 1, \quad S_0 = \tfrac{1}{2}(H_0 - \gamma H_0 \gamma). \tag{56}$$

Since the obtained path of idempotents $(H_t'((1-\gamma)/2)H_t')$ in $\mathscr{A}$ are piecewise polynomial it follows that the corresponding straight cycles are U^n-homologous. This is, using (46), a restatement of the homotopy invariance of the pairing of K-theory with cyclic cohomology [3].

It remains to compare, in the same way, H_0'' with H_0. One has by hypothesis $S_0 - H_0 \in \mathscr{L}^1$, $\theta_0 \in \mathscr{L}^1$; thus

$$H_0'' - H_0 \in \mathscr{L}^1. \tag{57}$$

It follows that the idempotents

$$e = H_0\left(\frac{1-\gamma}{2}\right)H_0, \qquad e'' = H_0''\left(\frac{1-\gamma}{2}\right)H_0''$$

satisfy the equality $e'' = WeW^{-1}$, where $W = H_0'' H_0 \in 1 + J$ as well as W^{-1}, while W is U^2-local as well as W^{-1}. One then considers the following smooth path of idempotents $e(t) \in M_2(\mathscr{A})$ connecting $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ with $\begin{bmatrix} 0 & 0 \\ 0 & e'' \end{bmatrix}$ and with support in U^3:

$$e(t) = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} R_t \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} R_t^{-1} \begin{bmatrix} 1 & 0 \\ 0 & W^{-1} \end{bmatrix}$$

where R_t is a rotation matrix. Using again the homotopy invariance of the pairing of K-theory with cyclic cohomology, one gets the desired result.

Proof of (4). We shall first check directly that $\int_M L(x, x) = 2\,\mathrm{Sign}(M)$. Since the Alexander Spanier cocycle given by the constant function $\varphi(x^0) = 1$ is defined everywhere, the proof of (3) shows that $\int_M L(x, x)$ is independent of the choice of the Hodge decomposition H, without any U-locality hypothesis. We can thus choose $H = 2P - 1$, where P is the orthogonal projection on the closed subspace Im d. One has $P + \gamma P \gamma + K = 1$, where K is the harmonic projection [7]. Thus one gets

$$H\gamma H + \gamma = 2K\gamma$$

$$\mathrm{Trace}(H\gamma H + \gamma) = 2\,\mathrm{Trace}(K\gamma) = 2\,\mathrm{Sign}(M).$$

To compute the other homology classes let us first assume that M is a smooth manifold. By [6] these classes ω_{2q} represent 2^{2q+1} times the Chern character of the K-homology class of the operator $H = 2P - 1$, with P as above.

Thus it is enough to show that the K-theory class of the symbol of H, $[\sigma(H)] \in K^0(T^*M)$, is the same as the K-theory class of the symbol of the signature operator. The latter is given by the odd endomorphism $s(x, \xi) = e_\xi + i_\xi$, $\xi \in T_x^*(M)$, of the pull-back of $\wedge^* T_{\mathbb{C}}^*$ (oriented by $\gamma = i^r *(-1)^{p(p-1)/2}$) to T^*M. (Here e_ξ, i_ξ are, respectively, exterior and interior multiplication by ξ.) The symbol $\sigma(H)$ of H is the same as the symbol of $2P - 1$, where P is the orthogonal projection on the image of d. Its restriction to the unit sphere $\{\xi \in T^*V,\ \|\xi\| = 1\}$ is thus given by

$$\sigma(x, \xi) = e_\xi i_\xi - i_\xi e_\xi \text{ acting on } \wedge^r T_{\mathbb{C}}^*.$$

Let $u(x, \xi) = (1/\sqrt{2})(1 - e_\xi + i_\xi)$; then for $\|\xi\| = 1$, it is an invertible operator in $\wedge^* T_{\mathbb{C}}^*$ with inverse $u^{-1}(x, \xi) = (1/\sqrt{2})(1 + e_\xi - i_\xi)$. One has

$$(usu^{-1})(x, \xi) = e_\xi i_\xi - i_\xi e_\xi \text{ acting on } \wedge^* T_{\mathbb{C}}^*.$$

By construction, u commutes with γ. This shows that the K-theory class $[s] - [\sigma]$ is given by the symbol

$$\rho(x, \xi) = e_\xi i_\xi - i_\xi e_\xi \text{ acting on } 2 \wedge^* T_{\mathbb{C}}^* \ominus \wedge^r T_{\mathbb{C}}^*$$

with the $\mathbb{Z}/2$ grading γ. However, using the canonical isomorphism, for $p \neq \ell$,

$$\wedge^p T_{\mathbb{C}}^* \simeq (\wedge^p T_{\mathbb{C}}^* \oplus \wedge^{2\ell - p} T_{\mathbb{C}}^*)^{\pm}; \quad \omega \to \tfrac{1}{2}(\omega \pm \gamma\omega)$$

one checks that the class of ρ is equal to 0.

This completes the proof that the classes we construct agree with the Hirzebruch–Thom classes in the smooth case. The extension from the smooth case to the qc-case can be done using cobordism as in [27]. One can also use K-theory as in [12].

Remarks. (1) Let M be a compact quasiconformal manifold; one can characterize the subalgebra $A(M)$ of $C(M)$ (Section 4) by the following simple criterion:

$$f \in A(M) \Leftrightarrow df \in L^{2\ell}(M, T_{\mathbb{C}}^*).$$

A proof of $\Rightarrow$ follows from [4] and the converse from [19] plus an identification of the Besov space there with this Sobolev space. It is sufficient by Section 4 to work with the standard singular integral operator on the sphere. For a more complete discussion see the appendix.

(2) Let M be a polyhedron of dimension 2ℓ with homological properties to be specified. Let $\mathscr{H} = \oplus \mathscr{H}_\alpha$ be the direct sum of the Hilbert spaces $\mathscr{H}_\alpha$ of square integrable ℓ-forms on the 2ℓ dimensional simplices. For each vertex v let E_v be the closure of the subspace of $\mathscr{H}$ of boundaries of Whitney flat forms [29] with support in the star of v. Let γ be the $*$ operation on $\mathscr{H}$ given by the canonical flat metric on each 2ℓ-simplex (with equilateral length). Let b_v be the barycentric coordinate assigned to the vertex v. Then let us consider the following operator:

$$\sum_v (P_v - \gamma P_v \gamma) b_v = S$$

where P_v is the orthogonal projection on E_v. It is clear that S is localized and anticommutes with γ. Now we assume that E_v and $\gamma(E_v)$ are a local decomposition, up to compacts, i.e. $(1 - P_v - \gamma P_v \gamma)$ is compact on forms on the star. We also assume that $S = 1$ on the closure of boundaries of Whitney flat forms modulo $I(\ell)$. These are the prescribed homological conditions. They will be satisfied for pL manifolds by the discussion above for qc-manifolds. Then $S^2 - 1$ is in $\mathscr{L}^{(2\ell, \infty)}$ and one can use the above formulae to define characteristic homology classes. The involved Hilbert space theoretical data are of the same nature as those appearing in transfer matrix theory of statistical mechanics and suggest a purely combinatorial approach to the K-orientation of [23] in the extended context of spaces with singularities.

APPENDIX: AN INTERESTING ANALYTICAL POINT†

There is an interesting issue concerning operator theory and classical analysis which is related to the topics of this paper and which does not seem to have been treated in the literature. For the sake of clarity we discuss this issue in a restricted setting. Let T be a zeroth-order pseudodifferential operator on $\mathbb{R}^d$, $d > 1$, which we also assume to be translation and dilation invariant and nonzero. Thus T could be represented by a Fourier multiplier which is homogeneous of degree 0, and T is a bounded linear operator on $\mathscr{H} = L^2(\mathbb{R}^d)$.

† This appendix is a result of collaboration with Stephen Semmes.

Under what conditions on a function $f(x)$ on $\mathbb{R}^d$ is it true that $[f, T] \in \mathcal{L}^{(d,\infty)}(\mathcal{H})$? (Recall that $\mathcal{L}^{(d,\infty)}(\mathcal{H})$ denotes the space of compact operators A on $\mathcal{H}$ such that $\mu_n(A) = O(n^{-1/d})$, where $\mu_n(A)$ is the nth eigenvalue of $(A^*A)^{1/2}$.) This type of question has been studied extensively (see [14], for instance), but this particular case involves a critical index and has some special features. It follows from [19] that

$$[f, T] \in \mathcal{L}^{(d,\infty)} \Leftrightarrow f \in Osc^{d,\infty}(\mathbb{R}^d) \quad \text{when } d > 1 \tag{A1}$$

where $Osc^{d,\infty}(\mathbb{R}^d)$ is a variant of a Besov space whose definition will be reviewed soon. For many purposes it would be preferable to work with a Sobolev space instead of $Osc^{d,\infty}$. It was observed in [18, Theorem 2.2, p. 228] that

$$W^{1,d}(\mathbb{R}^d) \subseteq Osc^{d,\infty}(\mathbb{R}^d) \quad \text{when } d > 1 \tag{A2}$$

where $W^{1,d}(\mathbb{R}^d)$ is the Sobolev space of locally integrable functions on $\mathbb{R}^d$ whose distributional first derivatives all lie in $L^d(\mathbb{R}^d)$. In fact, we have the following theorem.

THEOREM. *When $d > 1$, $W^{1,d}(\mathbb{R}^d) = Osc^{d,\infty}(\mathbb{R}^d)$, and so $[f, T] \in \mathcal{L}^{(d,\infty)}(\mathcal{H})$ if and only if $f \in W^{1,d}(\mathbb{R}^d)$.*

It is very important here that the dimension d is the same as the exponent in the function spaces; otherwise this theorem would not work. This theorem is rather surprising from the perspective of classical analysis, because Sobolev spaces normally coincide with Besov-type spaces only when the exponent is 2. Indeed, the second half of the theorem had been conjectured by Jaak Peetre, and one of us (S.S.) was of the opposite mind.

We need only show that the inclusion opposite to (A2) holds. Our original proof of this was obtained by understanding the Dixmier trace of $|[f, T]|^d$, which in fact reduces to $\int_{\mathbb{R}^d} |\nabla f(x)|^d \, dx$ for certain T [4]. We shall sketch a more direct proof below.

Let us recall the definition of $Osc^{d,\infty}(\mathbb{R}^d)$. This space is somewhat nonstandard; at this critical index, the standard Besov space is distinct from this one.

Given $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+ = \mathbb{R}_+^{d+1}$, let $B(x, t)$ denote the ball with center x and radius t. Let $m_{x,t}(g)$ denote the average on $B(x, t)$ of the locally integrable function $g(y)$ on $\mathbb{R}^d$. For such a function we let $\Theta(x, t)$ denote its average oscillation on $B(x, t)$, which is given by

$$\Theta(x, t) = m_{x,t}(|g - m_{x,t}(g)|).$$

We shall define $Osc^{d,\infty}(\mathbb{R}^d)$ in terms of a global measurement of these localized oscillation quantities.

Let $\{(x_j, t_j)\}_j$ denote a reasonably thick hyperbolic lattice in $\mathbb{R}_+^{d+1}$: we require that every point in $\mathbb{R}_+^{d+1}$ be no further than 10^{-2} from some (x_j, t_j) in the hyperbolic metric and that no pair of the (x_j, t_j)'s are closer to each other than 10^{-3}. Thus the numbers $\Theta(x_j, t_j)$ measure the average oscillations of g at all possible locations and scales.

Let θ_n, $n = 1, 2, 3, \ldots$, denote the nth largest value of $\Theta(x_j, t_j)$. In other words, we reorder the $\Theta(x_j, t_j)$'s in decreasing size. Then

$$g \in Osc^{d,\infty}(\mathbb{R}^d) \Leftrightarrow \theta_n = O(n^{-1/d}).$$

This definition does not depend on the particular choice of the lattice (x_j, t_j). (The point of (A1), incidentally, is that the θ_n's for f can be related to the $\mu_n([f, T])$'s.)

The main step in the proof of the theorem is to show that if g is smooth then

$$\left(\int_{\mathbb{R}^d} |\nabla g(x)|^d \, dx\right)^{1/d} \le C \limsup_{n \to \infty} n^{1/d} \theta_n \tag{A3}$$

for some constant C which does not depend on g. Once we know this, then we know that the $Osc^{d,\infty}(\mathbb{R}^d)$ norm controls the $W^{1,d}(\mathbb{R}^d)$ norm for smooth functions, and the proof of the theorem can be finished. with a standard approximation argument (which we omit). (The main point is that if you convolve $g \in Osc^{d,\infty}(\mathbb{R}^d)$ with a function in $L^1(\mathbb{R}^d)$ with bounded norm, then you get a function in $Osc^{d,\infty}(\mathbb{R}^d)$ with bounded norm.) The proof of (A3) that follows is pretty sketchy, but it would not be too difficult to give a detailed argument.

Let us first try to understand the right-hand side of (A3) better. For each $\lambda > 0$ let $N(\lambda)$ be the number of j's such that $\Theta(x_j, t_j) > \lambda$. Thus $\theta_{N(\lambda)} > \lambda$, and so

$$\limsup_{n\to\infty} n^{1/d}\theta_n \geq \limsup_{\lambda\to 0} N(\lambda)^{1/d}\lambda. \tag{A4}$$

Now let us try to understand the left-hand side of (A3). For the time being we shall work on some fixed large cube K. Let $\{Q_\ell\}$ be a partition of K into tiny cubes of side-length s. If s is small enough, then ∇g will be almost constant on each Q_ℓ, because g is smooth. Let G_ℓ denote the approximate value of $|\nabla g|$ on Q_ℓ.

Set $\hat{Q}_\ell = Q_\ell \times (0, s) \subseteq \mathbb{R}^{d+1}_t$. If $(x, t) \in \hat{Q}_\ell$,then $\Theta(x, t)$ isapproximately tG_ℓ. For each ℓ let $N_\ell(\lambda)$ denote the number of j's such that $(x_j, t_j) \in \hat{Q}_\ell$ and $\Theta(x_j, t_j) > \lambda$. If $sG_\ell < \lambda$ then we should have that $N_\ell(\lambda)$ is usually 0, while if $sG_\ell > \lambda$ then $N_\ell(\lambda)$ should be approximately the same as the number of j's such that $(x_j, t_j) \in \hat{Q}_\ell$ and $t_j > \lambda G_\ell^{-1}$. Simple considerations of hyperbolic geometry imply that $N_\ell(\lambda)$ is roughly proportional to $(sG_\ell/\lambda)^d$ in this case. This is also about the same as what we got in the first case.

Since $N(\lambda) \geq \sum_\ell (N_\ell(\lambda)$, we conclude that $N(\lambda)$ should dominate $\lambda^{-d}\sum_\ell (sG_\ell)^d$, modulo controllable errors. From (A4) we get that

$$\limsup_{n\to\infty} n^{1/d}\theta_n \text{ is roughly larger than } \left(\sum_\ell (sG_\ell)^d\right)^{1/d}.$$

On the other hand, $\sum_\ell (sG_\ell)^d$ is just a Riemann sum for $\int_K |\nabla g(x)|^d\,dx$. In the limit we get that

$$\left(\int_K |\nabla g(x)|^d\,dx\right)^{1/d} \leq C \limsup_{n\to\infty} n^{1/d}\theta_n$$

where C does not depend on g or K. This implies (A3), and proves the theorem.

Notice, incidentally, that the proof of (A3) works also when $d = 1$. However, the approximation argument that gives $Osc^{d,\infty}(\mathbb{R}^d) \subseteq W^{1,d}(\mathbb{R}^d)$ when $d > 1$ gives only $Osc^{1,\infty}(\mathbb{R}) \subseteq BV(\mathbb{R})$, where $BV(\mathbb{R})$ denotes the space of functions on $\mathbb{R}$ of bounded variation (i.e. whose distributional derivatives are finite measures). The reciprocal inclusion is false, because jump discontinuities are bad for $Osc^{1,\infty}(\mathbb{R})$.

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