arXiv:math/9201281v1 [math.DS] 28 Mar 1991

Expanding Direction of the Period Doubling Operator

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October 15, 1990

Abstract

We prove that the period doubling operator has an expanding direction at the fixed point. We use the induced operator, a "Perron-Frobenius type operator", to study the linearization of the period doubling operator at its fixed point. We then use a sequence of linear operators with finite ranks to study this induced operator. The proof

is constructive. One can calculate the expanding direction and the rate of expansion of the period doubling operator at the fixed point.

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§1 Introduction.

Perron-Frobenius operator. Suppose M^n is a compact, connected, oriented and smooth Riemannian manifold, $\Omega \subset M^n$ is an open set and $\sigma : \Omega \to M^n$ is an expanding mapping. Let $\mathcal{B} = \{v | v$ is a complex Lipschitz vector field on $\Omega\}$ and let ϕ be a real Lipschitz function on Ω. The Perron-Frobenius operator $\mathcal{L}_\phi : \mathcal{B} \to \mathcal{B}$ is defined by

$$(\mathcal{L}_\phi v)(x) = \sum_{y \in \sigma^{-1}(x)} (e^{\phi(y)}) v(y)$$

for $v \in \mathcal{B}$. An eigenvalue of $\mathcal{L}_\phi$ is a complex number λ such that $\mathcal{L}_\phi v = \lambda v$ for a nonzero vector field $v \in \mathcal{B}$. D. Ruelle [R1] proved the following theorem:

Theorem A. *The operator $\mathcal{L}_\phi$ has a positive and single maximal eigenvalue λ with an eigenvector h, and the remainder of the spectra of $\mathcal{L}_\phi$ is contained in a disk of radius strictly less than λ. Moreover, if σ, ϕ are C^k for $k = 1, 2, \cdots, \omega$, then h is a C^k vector field.*

More recently, in [P], [T] and [R2], the function ϕ was allowed to be a complex function and the spectral radius and the essential spectral radius of $\mathcal{L}_\phi$ on the $C^{k+\alpha}-$setting for $k = 1, 2, \cdots, 0 \leq \alpha \leq 1$ were estimated.

Feigenbaum's universality. Consider a family of unimodal mappings defined on $[-1,1]$, which is like the family $f_t(x) = t-(1+t)x^2$ for $t \in [0,1]$. Suppose t_n is the bifurcation value of parameters t such that f_t, $t < t_n$, does not have any periodic orbit of period 2^n and f_t, $t > t_n$, has an periodic orbit of period 2^n. M. Feigenbaum [F], and independently, P. Coullet and C. Tresser [CT], observed that the ratio $\delta_n = \frac{t_n - t_{n-1}}{t_{n+1}-t_n}$ converges to a universal number $\delta = 4.669\cdots$ as n goes to infinity. To explain this universality, Feigenbaum [F] posed the following conjecture.

Suppose $f : [-1,1] \to [-1,1]$ is a symmetric analytic folding mapping with a unique non-degenerate critical point 0 and satisfies $f(0)=1$ and $f^{\circ 2}(0) < 0 < f^{\circ 4}(0) < -f^{\circ 2}(0) < f^{\circ 3}(0)$. Let $q = f^{\circ 2}(0)$ and $I_q = [q, -q]$. The mapping $f\circ f | I_q : I_q \to I_q$ is again a folding mapping with a unique non-degenerate critical point. Suppose α_f is the linear rescaling of I_q to $[-1,1]$ with $\alpha_f(q) = -1$. Then $F = \alpha_f \circ f \circ f \circ \alpha_f^{-1}$ is a symmetric analytic folding mapping defined on $[-1,1]$. Denote F by $R(f)$. Then R is called the period doubling operator.

Conjecture A. *The operator R has a hyperbolic fixed point g with (i) codimension one contracting manifold and (ii) dimension one expanding manifold.*

O. Lanford proved this conjecture with computer assistance [L1]. After him some mathematicians proved the existence of the fixed point of R without computer assistance, for example, [CE] and [E]. Recently, one of us proved the existence of the fixed point g and part (i) using quasi-conformal theory [S]. The proof in [S] not only works for the period doubling operator but also works for its generalization, the renormalization operator (see Remarks in this introduction). However, part (*ii*) still lacks of a conceptual proof (which hopefully, is valid also for the periodic points of the renormalization operator). O. Lanford [L2] asked for a completely conceptual proof.

What we would like to say in this paper. We give a proof of part (*ii*). We use an induced operator $\mathcal{L}_\varphi$ to study the linearization T_gR of the period doubling operator R at the fixed point g. The operator $\mathcal{L}_\varphi$ is a "Perron-Frobenius type operator", but it is not a positive operator. The eigenvalues of $\mathcal{L}_\varphi$ agree with the eigenvalues of the linearization T_gR except for the value 1. We use the linear operator $\mathcal{L}_n$ with the finite rank 2^{n-1} to approximate $\mathcal{L}_\varphi$ in the C^b-setting (C^b is the space of bounded vector fields on $g(I)$). Under *the assumption that* g *is a concave function [L1]*, we prove the following statements:

(1) Each $\mathcal{L}_n$ has an eigenvalue $\lambda_n > 1$ with a positive eigenvector v_n, this means that each component of v_n is positive.

(2) There is a subsequence $\{n_i\}_{i=0}^{\infty}$ of the integers such that the limit $\lambda = \lim_{i\to\infty} \lambda_{n_i} > 1$ is an eigenvalue of $\mathcal{L}_\varphi$ with an eigenvector $v = \lim_{n\to+\infty} v_{n_i}$ in C^b.

(3) The number λ is an eigenvalue of $\mathcal{L}_\varphi$ in the $C^{0,1}$-setting ($C^{0,1}$ is the space of Lipschitz continuous vector fields on $g(I)$).

(4) The limit λ is an eigenvalue of $\mathcal{L}_\varphi$ in the C^ω-setting (C^ω is the space of analytic vector fields on $g(I)$).

These yield a proof of part (ii). The proofs are constructive. One can calculate the approximating expanding manifolds and the rate of expansion of R by using $\mathcal{L}_n$.

We also learned that recently, J.-P. Eckmann and H. Epstein [EE] gave a different proof of part (ii) and R. Artuso, E. Aurell and P. Cvitanoviè [AAC] gave a rigorous mathematical proof of part (ii).

Some remarks on the renormalization operator. Suppose $f : [-1, 1] \to [-1, 1]$ is a symmetric analytic unimodal mapping with a unique non-degenerate critical point 0. Suppose there is an integer $n > 1$ such that there exists an interval I containing 0 and the restriction of the n-fold $f^{\circ n}$ of f maps I into itself. Let n be the smallest such integer and $I_q = [q, -q]$ or $[-q, q]$ be the maximal such interval. The point q is a fixed point of $f^{\circ n}$. Let α_f be the linear mapping which rescales I_q to $[-1, 1]$ with $\alpha_f(q) = -1$. Then $F = \alpha_f \circ f^{\circ n} \circ \alpha_f^{-1}$ is a symmetric analytic unimodal mapping defined on $[-1, 1]$. We say f is once renormalizable and $R : f \mapsto F$ is the renormalization operator.

Conjecture B. (I) *For every periodic kneading sequence* $\rho = (w_1 * w_2 * \cdots * w_k)^{*\infty}$, *where* ρ *is decomposed into the star product of primary sequences, (see [MT] and [CEc] for a definition of a kneading sequence, a definition of star product and a definition of a primary sequence), there is a hyperbolic periodic point* g_ρ *(with this kneading sequence) of period* k *of* R *with (i) codimension one contracting manifold and* (ii) *dimension one expanding manifold.*

Moreover, (II) R *is hyperbolic on its maximal invariant set with* (i) *codimension one stable manifold and* (ii) *dimension one unstable* *manifold.*

Topologically, we knew that the maximal invariant set of R is like the Smale horse shoe. Under the assumption that g_ρ is a concave function and some a prior estimate of linear rescale mapping α_{g_ρ}, one may use the methods in this paper to prove part (ii) of (I). But as H. Epstein pointed out to us if we also consider a power law critical point and the exponent of g at its power law critical point is large, then g is not a concave function any more. In this paper, the concave condition is used only in the proof of statement (1). We are expecting a proof of statement (1) without the assumption that g_ρ is a concave function. This seems to be a promising problem. The other option is to prove that g_ρ is a concave function for every periodic kneading sequence ρ in the case that the critical point of g_ρ is non-degenerate. But it seems to be a difficult problem.

The other observation is that the generalized Feigenbaum's δ_ρ only depends on the data related to the critical orbit.

Acknowledgement: We would like to thank Folkert Tangerman for many helpful conversations and Viviane Baladi for many useful suggestions and comments to this paper. We would also like to thank John Milnor for reading this paper and for his many helpful remarks. This manuscript was first written on June, 1989 when one of us visited IHES. He would like to thank IHES for its hospitality.

§2 The Period Doubling Operator and the Induced Operator

Suppose I is the interval $[-1,1]$ and $U \subset \mathbf{C}^1$ is a connected open subset containing I. Let $\mathcal{B}(I,U)$ be the space of folding mappings f from I into I with a unique non-degenerate critical point 0 and an analytic extension F on U which can be extended to the boundary ∂U continuously. Suppose $\mathcal{B}_s(I,U)$ is the subspace of even functions in $\mathcal{B}(I,U)$ and $\mathcal{B}_{s,0}(I,U)$ is the subspace of mappings which are in $\mathcal{B}_s(I,U)$ and satisfy the conditions $f(0)=1$ and $f^{\circ 3}(0) > -f^{\circ 2}(0) > f^{\circ 4}(0) > 0 > f^{\circ 2}(0)$. The period doubling operator R from $\mathcal{B}_{s,0}(I,U)$ into $\mathcal{B}_s(I,U)$ is defined by

$$R(f)(x) = -\alpha_f f \circ f(-\alpha_f^{-1} x), \quad x \in I,$$

for $f \in \mathcal{B}_{s,0}(I,U)$, where $\alpha_f = -\frac{1}{f(1)}$.

Suppose g is the fixed point of R ([VSK]) and U is an open set contained in the domain of g. Let $\alpha = -\frac{1}{g(1)}$, $J = g(I)$ and $\Omega = g(U)$.

Suppose $\mathcal{V}^{\omega}(J,\Omega)$ is the space of real vector fields v on J with a complex analytic extension V on Ω which can be extended to the boundary $\partial\Omega$ continuously. This space equipped with the uniformly convergent norm is a Banach space.

§2.1 From the period doubling operator to the induced operator

Suppose J_0 and J_1 are the intervals $[g(1), g^{\circ 3}(1)]$ and $[g^{\circ 2}(1), 1]$. We define σ from $J_0 \cup J_1$ onto J by

$$\sigma(x) = \begin{cases} -\alpha x, & x \in J_0, \\ -\alpha g(x), & x \in J_1. \end{cases}$$

The mapping σ is expanding with expansion constant α for $|g^{'}(x)| > 1$, $x \in J_1$, and it has an analytic extension, which we still denote as σ, on $\Omega_0 \cup \Omega_1 \supset J_0 \cup J_1$ with also expansion constant α. Here Ω_0 and Ω_1 are disjoint subdomains of Ω and contain J_0 and J_1, respectively. Moreover, the restrictions $\sigma|\Omega_0$ and $\sigma|\Omega_1$ of σ to Ω_0 and Ω_1 are bijective from Ω_0 and Ω_1 to Ω and can be extended continuously to the boundaries $\partial\Omega_0$ and $\partial\Omega_1$, respectively (see Figure 1).

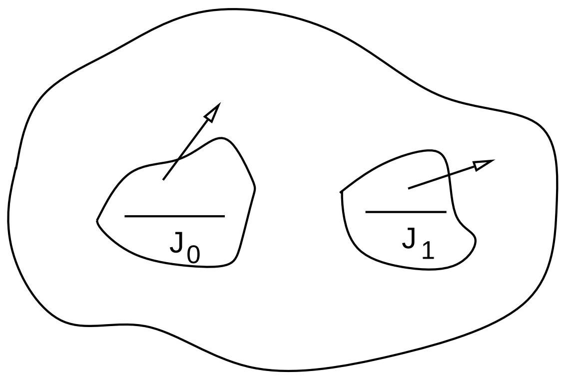

Figure 1

Suppose C is the attractor of g and Λ is the maximal invariant set of σ.

Lemma 1. *The set Λ and the set C are the same.*

Proof. The reader may check it by the equation $g(x) = -\alpha g \circ g(-\alpha^{-1}x)$.

Suppose $\varphi(z)$ is the derivative $\sigma'(z)$ of σ on $\Omega_0 \cup \Omega_1$. We define $\mathcal{L}_\varphi$ from $\mathcal{V}^\omega(J, \Omega)$ into $\mathcal{V}^\omega(J, \Omega)$ by

$$(\mathcal{L}_\varphi v)(z) = \sum_{w \in \sigma^{-1}(z)} \varphi(w)v(w)$$

and call it the induced operator. It is a "Perron-Frobenius type operator" but is not positive. It is clearly bounded and compact (by Montel's theorem).

Suppose $T_g\mathcal{B}_{s,0}(I,U)$ is the tangent space of $\mathcal{B}_{s,0}(I,U)$ at g and T_gR from $T_g\mathcal{B}_{s,0}(I,U)$ into $T_g\mathcal{B}_{s,0}(I,U)$ $(= T_g\mathcal{B}_s(I,U))$ is the tangent map of R at g.

Lemma 2. *The mapping g_* from $\mathcal{V}^\omega(J,\Omega)$ into $T_g\mathcal{B}_{s,0}(I,U)$ defined by $g_*(v)(x) = v(g(x))$ for $x \in \Omega$ and $v \in \mathcal{V}^\omega(J,\Omega)$ is an isomorphism.*

Proof. The proof is easy.

Lemma 3. *The operators $\mathcal{L}_\varphi$ and T_gR have the same eigenvalues (counted with multiplicity) except for the value* 1.

Proof. By some calculations, we can show that

$$\mathcal{L}_\varphi = g_*^{-1} \circ T_gR \circ g_* + e_1,$$

where e_1 is the projection from $\mathcal{V}^\omega(J,\Omega)$ to the eigenspace of eigenvalue one.

Remark. Suppose $V_{2m-1}(x) = g^{'}(x)x^{2m-1} - (g(x))^{2m-1} \in T_g\mathcal{B}_{s,0}(I,U)$ and $v_{2m-1} = g_*^{-1}(V_{2m-1}) \in \mathcal{V}^\omega(J,\Omega)$. The vector v_{2m-1} is an eigenvector of $\mathcal{L}_\varphi$ with eigenvalue $\lambda_{2m-1} = \alpha^{-(2m-2)}$ for $m = 1, 2, \cdots$.

Lemma 3 tells us that we can use $\mathcal{L}_\varphi$ which has an explicit form to study the eigenvectors and eigenvalues of T_gR except the value 1. We will use it to find the expanding direction and the rate of R at the fixed point g.

§2.2 The induced operator $\mathcal{L}_\varphi$

Suppose v is a real vector field on Λ. We say it is a Lipschitz continuous if there is a constant $M > 0$ such that $|v(x) - v(y)| \leq M|x - y|$ for any x and y in Λ. We say it is bounded if there is a constant $M > 0$ such that $|v(x)| \leq M$ for any x in Λ. Let $\mathcal{V}^{0,1}(\Lambda)$ be the space of real Lipschitz continuous vector fields on Λ and $\mathcal{V}^b(\Lambda)$ be the space of bounded vector fields on Λ. Suppose $\varphi(x)$ is the derivative $\sigma'(x)$ on Λ. We define two linear operator by the same formula. One is $\mathcal{L}_{\varphi,L}$ from $\mathcal{V}^{0,1}(\Lambda)$ into $\mathcal{V}^{0,1}(\Lambda)$ defined by

$$(\mathcal{L}_{\varphi,L}v)(x) = \sum_{y\in\sigma^{-1}(x)} \varphi(y)v(y)$$

and the other is $\mathcal{L}_{\varphi,B}$ from $\mathcal{V}^b(\Lambda)$ into $\mathcal{V}^b(\Lambda)$ defined by

$$(\mathcal{L}_{\varphi,B}v)(x) = \sum_{y\in\sigma^{-1}(x)} \varphi(y)v(y).$$

They are bounded but not compact.

Lemma 4. *Suppose λ is an eigenvalue of $\mathcal{L}_{\varphi,B}$ and $\lambda > \alpha + 1$. Then it is an eigenvalue of $\mathcal{L}_{\varphi,L}$.*

Proof. There is a nonzero vector field v in $\mathcal{V}^b(\Lambda)$ such that

$$\mathcal{L}_{\varphi,B}v = \lambda v.$$

This is

$$-\alpha v(-\alpha^{-1}x) - \alpha g'(g^{-1}(-\alpha^{-1}x))v(g^{-1}(-\alpha^{-1}x)) = \lambda v(x) \qquad (*)$$

for any x in Λ. From this we can have an inequality

$$\max_{x\neq y\in\Lambda}\left(\frac{|v(x)-v(y)|}{|x-y|}\right) \leq \frac{M}{\lambda-\alpha-1}$$

where M is a positive constant. In the other words, v is Lipschitz continuous on Λ and is an eigenvector of $\mathcal{L}_{\varphi,L}$ with the eigenvalue λ.

Lemma 5. *Suppose λ is an eigenvalue of $\mathcal{L}_{\varphi,L}$ and $\lambda > \alpha + 1$. Then λ is an eigenvalue of $\mathcal{L}_{\varphi}$.*

Proof. The basic idea to proof this lemma is to use the fact that the grand preimage $\cup_{n=0}^{\infty} g^{-n}(\Lambda)$ is a dense subset on I and to use the equality (*) countably many times. We will not write down our proof in detail because recently, there is a more general theorem proved by D. Ruelle [R2]. One of us learned this theorem when he visited IHES. We outline some Ruelle's result here.

§2.3 A general theorem for operators like the induced operator

In this subsection, the notations J_0, J_1, Ω, Ω_0 and Ω_1 are the same as that in §2.1.

Suppose e from $J_0 \cup J_1$ into and onto J is an expanding mapping such that the restrictions $e|J_0$ and $e|J_1$ of e to J_0 and J_1 have bijective, expanding, analytic extensions F_0 and F_1 from Ω_0 and Ω_1 to Ω, respectively. Moreover, F_0 and F_1 can be extended to the boundaries $\partial\Omega_0$ and $\partial\Omega_1$ continuously, We use E to denote the expanding map

$$E(z) = \begin{cases} F_0(z), & z \in \Omega_0, \\ F_1(z), & z \in \Omega_1 \end{cases}$$

from $\Omega_0 \cup \Omega_1$ to Ω.

Suppose ϕ from $J_0 \cup J_1$ into the real line is a real analytic function with a complex analytic extension Φ from $\Omega_0 \cup \Omega_1$ into the complex plane which can be also extended continuously to the boundaries $\partial\Omega_0 \cup \partial\Omega_1$. Let θ be the expanding constant of E and Λ_E be its maximal invariant set. A linear operator $\mathcal{L}_\phi$ from $\mathcal{V}^\omega(J, \Omega)$ into $\mathcal{V}^\omega(J, \Omega)$ is defined by

$$(\mathcal{L}_\phi)(v)(z) = \sum_{w \in E^{-1}(z)} \phi(z) v(z).$$

Suppose $\mathcal{V}^{0,1}(\Lambda_E)$ is the space of real Lipschitz continuous vector fields on Λ_E. Let $|\phi|$ be the function which takes values $|\phi(z)|$ at all $z \in$

$\Omega_0 \cup \Omega_1$. We assume $|\phi|$ is a positive function. Then the operator $\mathcal{L}_{|\phi|}$ from $\mathcal{V}^{0,1}(\Lambda_E)$ into $\mathcal{V}^{0,1}(\Lambda_E)$ defined by

$$(\mathcal{L}_{|\phi|})(v)(x) = \sum_{y \in E^{-1}(x)} |\phi|(y)v(y)$$

is an Perron-Frobenius operator. For the positive function $|\phi|$, we can define its pressure as

$$P(\log|\phi|) = \sup_{\mu}(h_\mu(E) + \int_{\Lambda_E} (\log|\phi|)d\mu)$$

where μ is an invariant measure of E and h_μ is the measure-theoretic entropy of E with respect to μ. By the variation principle (see, for example, [B]), we have that

$$P(\log|\phi|) = \lim_{n\to\infty} \frac{1}{n} \log(\sum_{x \in fix(E^{\circ n})} \prod_{i=0}^{n-1} |\phi(E^{\circ i}(x))|).$$

Let $A = exp(P(\log|\phi|))$. It is a simple eigenvalue of $\mathcal{L}_{|\phi|}$ and all other eigenvalues of $\mathcal{L}_{|\phi|}$ are in the open disk $D_A = \{z| \in \mathbf{C}^1, |z| < A\}$ (see the theorem in the introduction). Suppose $\mathcal{L}_{\phi,L}$ from $\mathcal{V}^{0,1}(\Lambda_E)$ into $\mathcal{V}^{0,1}(\Lambda_E)$ is defined by

$$(\mathcal{L}_{\phi,L})(v)(x) = \sum_{y \in E^{-1}(x)} \phi(y)v(y)$$

and $A_1 = \theta^{-1}A$.

Lemma 6. (see [R2]). *All the eigenvalues of $\mathcal{L}_{\phi,L}$ are in the open disk D_A and if λ is an eigenvalue of $\mathcal{L}_{\phi,L}$ and $|\lambda| > A_1$, then λ is an eigenvalue of $\mathcal{L}_{\phi}$.*

Suppose σ is the expanding mapping induced from the period doubling operator and φ is the derivative σ'. The expanding constant of σ is α. By some combinatorial arguments, we have that

$$\sum_{x\in fix(\sigma^{\circ n})} \prod_{i=0}^{n-1} \log|\phi(\sigma^{\circ i}(x))| \le (\alpha^2+\alpha)^n$$

Moreover, by using the variation principle,

$$A = exp(P(\log|\varphi|)) \le \alpha(\alpha+1)$$

and thus $A_1 = \alpha^{-1}A \le \alpha + 1$. From this, Lemma 6 gives a proof of Lemma 5. Moreover, if λ is an eigenvalue of $\mathcal{L}_{\phi,B}$ and $\lambda > \alpha+1$, then it is an eigenvalue of T_gR.

§3 The Construction of the Expanding Direction

We prove that the induced operator $\mathcal{L}_{\varphi}$ has an expanding direction and construct this direction in this section. The transformation of this direction under g_* is the expanding direction of the period doubling operator.

§3.1 An easy observation

Suppose $I = [-1, 1]$ is a closed interval of the real line $\mathbf{R}^1$ and $D \supset I$ is an open disk in the complex plan $\mathbf{C}^1$. Suppose I_0 and I_1 are disjoint closed subintervals of I and e is a piecewise linear expanding map from $I_0 \cup I_1$ onto and into I with the derivative

$$\phi(x) = e'(x) = \begin{cases} -a, & x \in I_0, \\ b, & x \in I_1 \end{cases}$$

where $b > a > 2$ are two constants. Let E be the extension of e from D_0 and D_1 onto and into D with also the derivative, we still denote it by ϕ,

$$\phi(z) = E'(z) = \begin{cases} -a, & z \in D_0, \\ b, & z \in D_1. \end{cases}$$

Let $\mathcal{V}^\omega(I, D)$ be the space of real vector fields v on I with a complex analytic extension V on D which can be extended to the boundary ∂D continuously. For $t \in [0, 1]$, we define

$$\phi_t(x) = \begin{cases} ae^{2\pi i t}, & x \in J_0, \\ b, & x \in J_1 \end{cases}$$

and $\mathcal{L}_{\phi_t}$ from $\mathcal{V}^\omega(I, D)$ into $\mathcal{V}^\omega(I, D)$ by

$$(\mathcal{L}_{\phi_t})(v)(z) = ae^{2\pi i t} v(E_0(z)) + bv(E_1(z))$$

where E_0 and E_1 are the inverse branches of E.

Proposition A. *The set* $\{\lambda_{n,t} = \frac{1}{b^{n-1}} + \frac{1}{e^{2\pi it(n-1)}a^{n-1}}\}_{n=0}^{\infty}$ *is the spectrum of* $\mathcal{L}_{\phi_t}$.

Proof. Suppose the center of D is 0. Then every $v \in \mathcal{V}^{\omega}(I, D)$ has the Taylor expansion

$$v(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where a_k are all real numbers. To find $\lambda_{n,t}$ for $n = 0, \cdots, +\infty$, we may solve the equation

$$\mathcal{L}_{\phi_t} v_n = \lambda_{n,t} v_n$$

for $v_n(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{V}^{\omega}(I, D)$.

Under the condition $b > a > 2$, $||\lambda_{n,t}|| < 1$ for $n > 1$. The other two eigenvalues $\lambda_{1,t}(= 2$ for all $t \in [0,1])$ and $\lambda_{0,t} = b + e^{2\pi it}a$ are special. Here $2 = \exp(h(E))$ is a topological invariant where $h(E)$ is the topological entropy of E.

From this proposition, we can observe that $\lambda_{0,t}$ is the maximal eigenvalue of $\mathcal{L}_{\phi_t}$ for all $t \in [0,1]$ if and only if $b - a > 2$. In the other words, It is the maximal eigenvalue of $\mathcal{L}_{\phi_t}$ for all $t \in [0,1]$ if and only if $b - a > 2$: an unbalanced condition, the orientation preserving part is much stronger than the orientation reversing part or vice versa.

§3.2 The construction

We use the same notations as that in the previous section. We note that the derivative of σ on J_1 is strictly greater than one and the derivative of σ at the right end point of J is α^2.

Suppose φ_1 is the function defined by

$$\varphi_1 = \begin{cases} -\alpha, & x \in J_0 \cap \Lambda, \\ \alpha^2, & x \in J_1 \cap \Lambda \end{cases}$$

and $\mathcal{L}_1 : \mathcal{V}^b(\Lambda) \to \mathcal{V}^b(\Lambda)$ is the corresponding operator defined by

$$(\mathcal{L}_1 v)(x) = \sum_{y \in \sigma^{-1}(x)} \varphi_1(y) v(y).$$

The number $\lambda_1 = \alpha(\alpha - 1)$ is an eigenvalue of $\mathcal{L}_1$ with an eigenvector $v_1 = 1$ on Λ.

Suppose $\sigma^{-2}(J) = J_{21} \cup J_{22} \cup J_{23} \cup J_{24}$ and $J_{23} = [a_{21}, b_{21}]$, $J_{24} = [a_{22}, b_{22}]$ (see Figure 1). Let $\beta_{21} = |g'(b_{21})|$ and $\beta_{22} = |g'(b_{22})| = |g'(1)| = \alpha$. Because g is a concave function [L1], we have that $\beta_{21} \leq$

β_{22}. Suppose φ_2 is the function defined by

$$\varphi_2(x) = \begin{cases} -\alpha, & x \in J_0 \cap \Lambda, \\ \alpha\beta_{21}, & x \in J_{23} \cap \Lambda, \\ \alpha\beta_{22}, & x \in J_{24} \cap \Lambda \end{cases}$$

and $\mathcal{L}_2 : \mathcal{V}^b(\Lambda) \to \mathcal{V}^b(\Lambda)$ is the corresponding operator defined by

$$(\mathcal{L}_2 v)(x) = \sum_{y \in \sigma^{-1}(x)} \varphi_2(y) v(y).$$

Let k_{21} be the vector field on Λ defined by

$$k_{21}(x) = \begin{cases} 1, & x \in J_0 \cap \Lambda, \\ 0, & x \in J_1 \cap \Lambda \end{cases}$$

and $k_{22} = 1 - k_{21}$. The space $\mathbf{R}^2 = span\{k_{21}, k_{22}\}$ is a subspace of $\mathcal{V}^b(\Lambda)$. For any $v = x_{21}k_{21} + x_{22}k_{22}$,

$$(\mathcal{L}_2 v)(x) = (k_{21}, k_{22}) \begin{pmatrix} -\alpha, & \alpha\beta_{21} \\ -\alpha, & \alpha\beta_{22} \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}.$$

Let A_2 be the matrix

$$\begin{pmatrix} -\alpha, & \alpha\beta_{21} \\ -\alpha, & \alpha\beta_{22} \end{pmatrix}.$$

Proposition C. *The maximal eigenvalue of A_2 is*

$$\lambda_2 = \alpha \frac{(\beta_{22} - 1) + \sqrt{(\beta_{22} - 1)^2 + 4(\beta_{22} - \beta_{21})}}{2},$$

with an eigenvector $v_2 = (t_{21}, 1)$, $t_{21} < 1$.

Proof. The proof uses linear algebra.

Furthermore, suppose $\sigma^{-n}(J) = J_{n1} \cup J_{n2} \cup \cdots \cup J_{n2^{n-1}} \cup J_{n(2^{n-1}+1)} \cup \cdots \cup J_{n2^n}$ and $J_{n(2^{n-1}+i)} = [a_{ni}, b_{ni}]$ (see Figure 2).

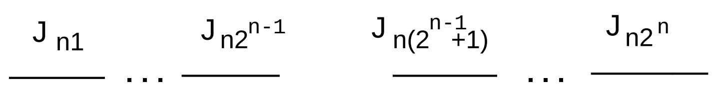

Figure 2

Let $\beta_{ni} = |g'(b_{ni})|$ for $i = 1, 2, \cdots 2^{n-1}$. Because g is a concave function, we have that

$$1 < \beta_{n1} < \cdots < \beta_{n2^{n-1}} = \alpha.$$

Suppose φ_n is the function defined by

$$\varphi_n(x) = \begin{cases} -\alpha, & x \in J_0 \cap \Lambda, \\ -\alpha\beta_{ni}, & x \in J_{n(2^{n-1}+i)} \cap \Lambda, \quad i = 1, 2, \cdots 2^{n-1}, \end{cases}$$

and $\mathcal{L}_n$ from $\mathcal{V}^b(\Lambda)$ into $\mathcal{V}^b(\Lambda)$ is the corresponding operator defined by

$$(\mathcal{L}_n v)(x) = \sum_{y \in \sigma^{-1}(x)} \varphi_n(y) v(y).$$

Let k_{ni} be the vector field on Λ defined by

$$k_{ni}(x) = \begin{cases} 1, & x \in (J_{n(2i-1)} \cup J_{n(2i)}) \cap \Lambda, \\ \\ 0, & x \in \Lambda \setminus ((J_{n(2i-1)} \cup J_{n(2i)}) \cap \Lambda) \end{cases}$$

for $i = 1,\, 2,\, \cdots,\, 2^{n-1}$. The space $\mathbf{R}^{2^{n-1}} = span\{k_{n1}, \cdots, k_{n2^{n-1}}\}$ is a subspace of $\mathcal{V}^b(\Lambda)$. For any $v = x_{n1}k_{n1} + \cdots x_{n2^{n-1}}k_{n2^{n-1}}$, we have that

$$\mathcal{L}_n v = K_n A_n X_n^t$$

where $K_n = (k_{n1}, \cdots, k_{n2^{n-1}})$ and $X_n = (x_{n1}, \cdots, x_{n2^n})$ and A_n stands for the $2^{n-1} \times 2^{n-1}$-matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha & \alpha\beta_{n1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\alpha & \alpha\beta_{n2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\alpha & 0 & 0 & \alpha\beta_{n3} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\alpha & 0 & 0 & \alpha\beta_{n4} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -\alpha & \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha\beta_{n(2^{n-1}-3)} & 0 \\ 0 & -\alpha & \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha\beta_{n(2^{n-1}-2)} & 0 \\ -\alpha & 0 & \cdots & 0 & 0 & 0 & & \cdots & 0 & \alpha\beta_{n(2^{n-1}-1)} \\ -\alpha & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha\beta_{n2^{n-1}} \end{pmatrix}.$$

Proposition D. *The matrix A_n has an eigenvalue λ_n which is greater than $\alpha(\alpha - 1)$.*

Proof. Suppose CN_n is the set $\{(x_{n1}, \cdots, x_{n2^{n-1}})| \in \mathbf{R}^{2^{n-1}}$, $x_{ni} \geq 0$ for $i = 1, \cdots, i = 2^{n-1}$ and $x_{n1} \leq x_{n2} \leq \cdots \leq x_{n2^{n-1}}\}$. It is easy to check that CN_n is a convex cone and A_n maps this cone into the interior of this cone and zero vector. By the Brouwer fixed point theorem, we conclude that there is a unique direction $\mathbf{R}^+ v_n$ in this cone which is preserved by A_n. Suppose $v_n = (t_{n1}, \cdots, t_{n2^{n-1}})$ with $t_{n2^{n-1}} = 1$ is an eigenvector with the eigenvalue λ_n. By the equation $A_n v_n = \lambda_n v_n$, we have that $-\alpha t_{n1} + \alpha^2 = \lambda_n$. Because $t_{n1} < 1$, we get $\lambda_n > \alpha(\alpha - 1)$.

Remark. Because the cone CN_n is a subset of the cone CN_{n+1} for any $n \geq 1$, we can prove more that $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence. But we will not use this fact because we would like the following arguments to be true even in the case that g is not a concave function.

Proposition E. *There is a subsequence $\{n_i\}$ of the integers such that the continuous extension of the limit $v = \lim_{i \mapsto \infty} v_{n_i}$ on the critical orbit $Or(g) = \{g^{\circ n}(0)\}_{n=1}^{\infty}$ is an eigenvector of $\mathcal{L}_{\varphi,B}$ with the eigenvalue $\lambda = \lim_{i \mapsto \infty} \lambda_{n_i}$.*

Proof. Because $Or(g)$ is a countable set, we can find a subsequence $\{n_i\}_{i=0}^{\infty}$ such that for every $a \in Or(g)$, the limit $v_{n_i}(a)$ exists as i goes to infinity. We denote this limit as $v(a)$. For the sequence $\{\lambda_{n_i}\}_{i=0}^{\infty}$, we can find convergent subsequence. Let λ be the limit of this subsequence. Then we have that $(\mathcal{L}_{\varphi,B}v)(a) = \lambda v(a)$ for any $a \in Or(g)$. Now by using the equation $(*)$ and the fact $\alpha(\alpha - 1) > \alpha + 1$ which can be implied by $\alpha > 1 + \sqrt{2}$, we can show that v has a continuous extension on Λ which is the closure of $Or(g)$.

§3.3 A program

In §3.2, we use a subsequence of $\{v_n\}_{n=0}^{\infty}$ to prove that there is an expanding direction of $\mathcal{L}_{\varphi}$. Under the assumption that g is a concave function, we can say more on the sequence $\{v_n\}_{n=0}^{\infty}$ and the corresponding eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. For example, $\{\lambda_n\}$ is an increasing sequence and for every $a \in \Lambda$, $\{v_n(a)\}$ is a monotone sequence. In practice, we can use these good properties to give an effective program to find the expanding direction and the rate of the period doubling operator as follows:

Suppose v is a vector in $\mathbf{R}^k$. We use $(v)_i$ to denote its i^{th}-coordinate.

(1) Start from the constant function $v_1 = 1$. Consider it as a vector

in $\mathbf{R}^2$ and compute the limiting vector

$$v_2 = \lim_{l \mapsto \infty} \frac{A_2^l v_1}{(A_2^l v_1)_2}$$

and the corresponding eigenvalue $\lambda_2 = \alpha(\alpha - (A_2 v_2)_1)$.

(n) Let $v_{n-1} \in \mathbf{R}^{2^{n-2}}$ be the eigenvector of A_{n-1} with the eigenvalue λ_{n-1} . Consider v_{n-1} as a vector in $\mathbf{R}^{2^{n-1}}$ and compute the limiting vector

$$v_n = \lim_{l \mapsto \infty} \frac{A_n^l v_{n-1}}{(A_n^l v_{n-1})_{2^{n-1}}}$$

and the corresponding eigenvalue $\lambda_n = \alpha(\alpha - (A_n v_n)_1)$.

(∞) The limiting vector

$$V = \lim_{n \mapsto \infty} g_*(v_n)$$

is the expanding direction and the limiting value

$$\delta = \lim_{n \mapsto \infty} \lambda_n$$

is the rate of expansion of the period doubling operator at the fixed point g.

References

[AAC] R. Artuso, E. Aurell and P. Cvitanovič. Recycling of Strange Sets: I. Cycle Expansions and II. Applications. *Preprint.*

[B] R. Bowen. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. *Springer-Verlag, Berlin, Heidelberg, New York.*

[CEc] P. Collet and J. P. Eckmann. Iterated maps on the interval as dynamical systems. *Progress in Physics, Birkhäuser.*

[CE] M. Campanino and H. Epstein. On the existence of Feigenbaum's fixed-point. *Comm. Math. Phys.* **79***(1981), 261-302.*

[CT] P. Coullet and C. Tresser. Itérations d'endomorphismes et groupe de renormalisation. *C. R. Acad. Sci. Paris Ser. A-B 287(1978), A577-A580.*

[EE] J. P. Eckmann and H. Epstein. Bounds on the Unstable Eigenvalue for Period Doubling. *Universite De Geneve, UGUA-DP7, (1989)/06-6,6.*

[EW] J. P. Eckmann and P. Wittwer. A complete Proof of the Feigenbaum Conjectures. *J. Stat. Phys. 46 455-477 (1987).*

[E] H. Epstein. New Proofs of the Existence of the Feigenbaum Functions. *Comm. of Math. Phys. 106 (1986) 395-426.*

[F] M. Feigenbaum. The Universal Metric Properties of Non-linear Transformations. *J. Stat. Phys. 21 669(1979).*

[L1] O. E. Lanford III. A Computer-assisted Proof of Feigenbaum Conjectures. *Bull. Am. Math. Sco. 6 (1982), 427-434.*

[L2] O. E. Lanford III. Computer-assisted Proof in Analysis. *Preprint, IHES.*

[MT] J. Milnor and W. Thurston. On iterated maps of the interval I and II. *Preprint, Princeton University Press:Princeton.*

[P] M. Pollicot. A Complex Ruelle-Perron-Frobinus Theorem and Two Counterexamples. *Ergod. Th. and Dynam. Sys. 4 135-146(1984).*

[R1] D. Ruelle. Thermodynamic Formalism. The mathematical structures of classical equilibrium statistical mechanics. *Addison-Wesley: Reading.*

[R2] D. Ruelle. The Thermodynamical Formalism for Expanding Map. *Preprint, IHES, January, (1989).*

[T] F. M. Tangerman. Meromorphic Continuous of Ruelle Zeta Function. *Thesis, (1986), Boston University.*

[S] D. Sullivan. Class Notes, 1990.

[VSK] E. B. Vul, Ya.G. Sinia and K.M. Khanin. Feigenbaum universality and the thermodynamic formalism. *Russian Math. Surveys* **(39)***: 3(1984), 1-40.*