ON THE DYNAMICAL STRUCTURE NEAR AN ISOLATED COMPLETELY UNSTABLE ELLIPTIC FIXED POINT (*)

by

Dennis Sullivan

(CUNY-IHES)

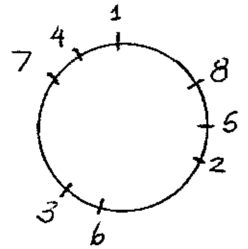
We study here the following problem:

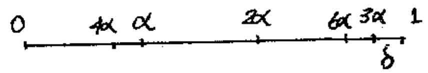
Consider the map $F_w\colon \mathbb{C} \to \mathbb{C}$ $F_w(z) = wz+z^2$, $|w| = 1$. Suppose F_w and $F_{w'}$ are (orientably) topologically conjugate on some neighborhood of zero. We ask if $w = w'$. This is equivalent to say that the derivative at the isolated fixed point is a topological invariant.

We will prove the following more general result: Let f be the germ of a local diffeomorphism of $(R^2,0)$. Furthermore let f be completely unstable at 0, an isolated fixed point. Then the irrational argument of the derivative at 0 is determined by the topological dynamics in any neighborhood of zero.

1. Consider first an irrational rotation of the circle by an angle $2\pi\alpha$, $\alpha \in [0,1]\setminus\mathbb{Q}$.

(*) Notes of Lecture at Poços de Caldas by Dennis Sullivan taken by Sergio Roberto Fenley.





We iterate the initial point. On the fourth iteration it crosses to the other side of 1. Then $\alpha = \frac{1}{3+x}$. The second approximation is determined by how many δs are there in α, or equivalently, when the crossing of 1 will change sides. In this way a sequence of integers $n_1, n_2, \ldots$ is constructed, completely determined by the successive points of the orbit and their orbit structure. We see clearly that

$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\cdots}}}$$

which is the continued fraction expansion of α.

2. ABSTRACT SITUATION

Let f be a homeomorphism of the circle without periodic points. Then

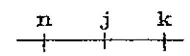
i) (Poincaré). The order type of each orbit is the same as any other.

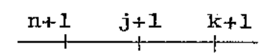
ii) The order type is the same as that of an unique irrational

rotation, (and this defines the rotation number).

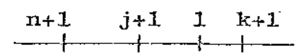
<u>Proof</u>: i) Suppose this is false. There are two points with different order structures. Let n be the first time their orbit order structure differs. By continuity there is a point between these two where there is a transition between the two structures. This generates a periodic point, contradiction.

ii) The order type is determined by the nearest neighborhoods at each step.

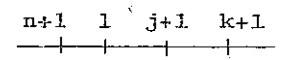




This changes when there is another point between $n+1$, $k+1$. The first point which makes this happens has to be 1 because one can always iterate back. Therefore there are 2 choices



R-right



L-left

A string of left-right choices is then created:

$$\underbrace{LLL}_{n_1} \quad \underbrace{RRRR}_{n_2} \quad \underbrace{LL}_{n_3} \quad \underbrace{RR}_{n_4} \quad LL \quad RRRRR$$

these are exactly the $n_1, n_2, \ldots$ from before and the order type is that of an irrational rotation of α where

$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots}}$$

3. Remarks.

a) These arguments break down for maps of higher degree. Consider $z \to z^2$ on the unit circle. There is a point whose orbit has the order type of any given irrational rotation, or $\{\beta^n\} \sim \{n\alpha\}$ mod 1; which means the orbit has arbitrarily long blocks having the same order structure of an irrational rotation.

b) Iaci Malta (PUC-RJ) showed Poincaré's discussion works for any continuous map of S^1 without periodic points using the numerical rather than the order approach to the rotation number.

4. We will define now the derivative at 0 of a germ of a diffeomorphism using the order geometry of the dynamics near zero.

Let f be a local diffeomorphism near 0 satisfying:

i) 0 is an isolated fixed point

ii) the derivative at 0 is an irrational rotation

iii) f is <u>completely unstable</u> (c.u.)

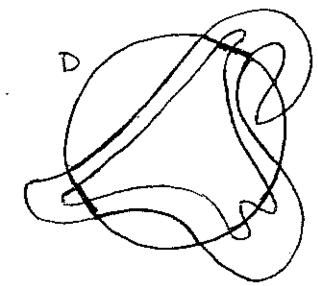
- g is stable at 0 iff $\exists$ U, V neighborhoods of 0 such that $g^n(U) \subset V \ \forall n \geq 0$. f is completely unstable if both f and f^{-1} are not stable at 0.

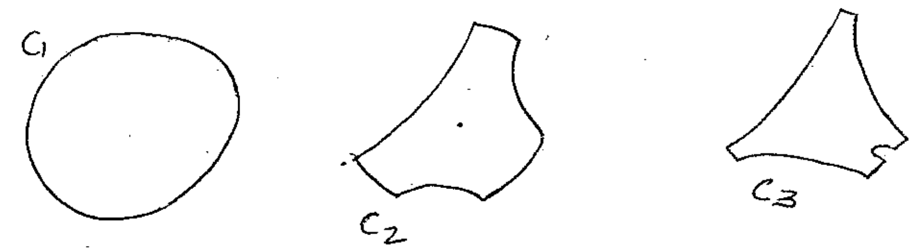
5. Let $g = f^{-1}$ and D a disk containing 0. Set

$$C_1(g,D) = D$$
$$C_2(g,D) = \text{component containing } 0 \text{ of } (D \cap fC_1(g,D))$$
$$\vdots$$
$$C_{n+1}(g,d) = \text{component containing } 0 \text{ of } (D \cap f(C_n(g,d)))$$

and

$$C(g,D) = \bigcap_{n>0} C_n(g,D), \qquad C^*(g,D) = C(g,D)\setminus\{0\}$$





Notice that $f(C_n(g,D)) \not\subset D \ \forall n$ because f is c.u. Also $C_n(g,D)$ is homeomorphic to a disk $\forall n$.

6. <u>Lemma</u>. If g and $C(g,D)$ are like described above then $C(g,D)$ satisfies

i) connectivity

ii) $C(g,D) \cap \partial D \neq \emptyset$

iii) invariant by g

iv) $x \in C(g,D)$ iff $\exists K$ connected; $0,x \in K$ and $g^n(K) \subset D$ $\forall n > 0$

v) $C(g,D)$ does not separate the plane

vi) $C(g,D) \subseteq C(g^n,D)$ $\forall n \in \mathbb{N}$

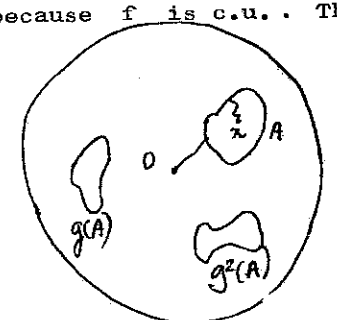
$C(g,D_1) \subseteq C(g,D_2)$ if $D_1 \subset D_2$.

<u>Proof</u>: Notation $C_n = C_n(g,D)$ $C = C(g,D)$ $C^* = C^*(g,d)$, $r \in \mathbb{N}$.

C_1 consists of the points $x \in D$ for which $\exists K$ connected $0,x \in K$ and $g^r(K) \subset D$ $0 \le r \le 1$. Similarly C_n satisfies the same properties, but now $g^r(K) \subset D$ for $0 \le r \le n$. Then C is clearly the set described in iv). It is obvious now that iii) is true. i) also follows because if K connected $0,x \in K$ and $g^n(K) \subset D \Rightarrow K \subset D$.

To prove ii) we note that since $g(C_n)$ always leaves the disk, $B_n = g(C_n) \cap \partial D \neq \phi$ is a disjoint union of closed intervals. Since B_n decreases it follows $\bigcap_{n>0} B_n \neq 0$ which proves ii).

Suppose $\exists$ closed arc $A \subset C$; $0 \notin \text{int } A$ (the limited component of $\mathbb{R}^2$-A) because f is c.u.. Therefore the interior of



A goes by g in the interior of $g(A) \subset D$ and similarly for higher powers of g. For $x \in \text{int } A$ $\exists \gamma$ connecting A to x and by iv) $x \in C$. Therefore int $A \subset C$ which proves v). C is simply

connected the statements in vi) follow easily from iv).

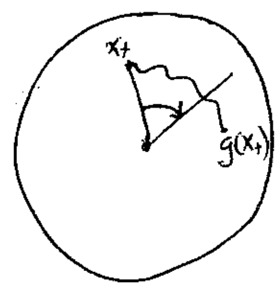
6a. The following result shows, among other facts, the pair f,g is not completely symmetric.

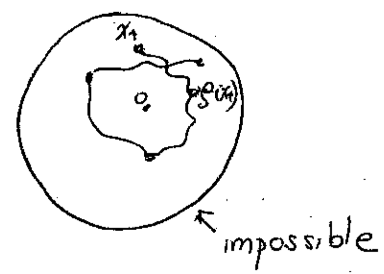
<u>Lemma</u>. Let f like above. Then $\nexists$ $C_+, C_- \subset \mathbb{R}^2 - \{0\}$ connected satisfying $x_+, g(x_+) \in C_+$; $x_-, f(x_-) \in C_-$ and $g^n(C_+) \subset D$, $f^n(C_-) \subset D$ $\forall n \ge 0$.

(Obs. only f and f^{-1} unstable at 0 is needed.)

<u>Proof</u>: By contradiction. Consider $\exists$ sets C_+, C_- and points x_+, x_- with the given properties.

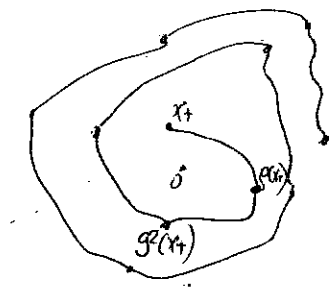
If C_+ winds around 0 g has an invariant open set and is therefore stable. So there is an extension of the argument function to all of C_+. Let $B_+ = C_+ \cup g(C_+) \cup \dots$. There is also an argument function defined on B_+ because otherwise B_+ separates zero from infinity and zero becomes stable.



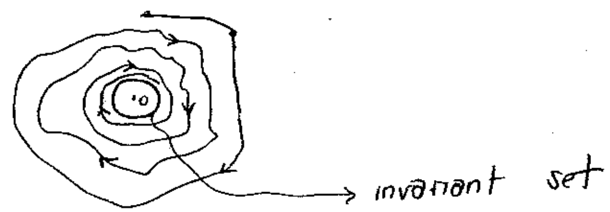


We can suppose D is so small that g is very close to a rigid rotation of angle α. Then the angle between x and $g(x)$ is close to α for any $x \in D$. Therefore as we iterate g, the argument function for the images of C_+ increases and goes to $+\infty$.

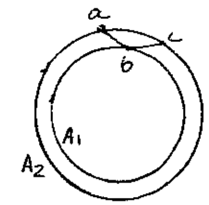
If $g^n(C_+)$ goes away from the origin, as the next picture shows, there would be an invariant open set (just consider $w(C_+)$ and then its interior).



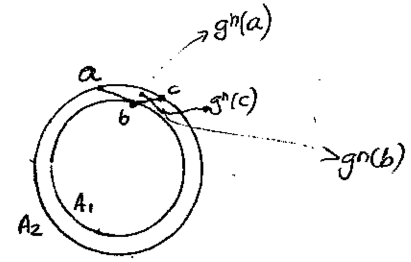
Then $g^n(C_+)$ goes nearer the origin. If $0 \notin \bar{B}_+$ there is also an invariant open set.



So $\exists\ c_n \in B_+$, $c_n \to 0$. Now arbitrarily near 0 the following situation cannot occur:

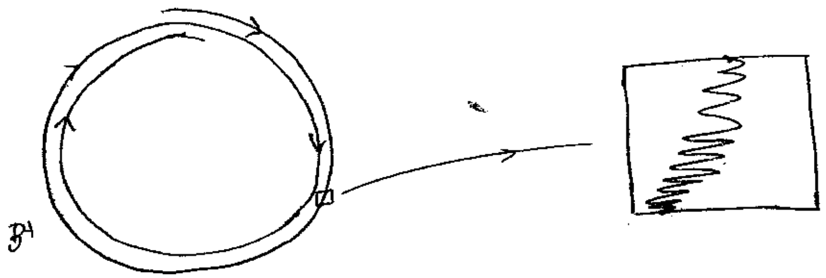


where A_2 and A_1 are circles having centers at 0 and $abc \subset B_+$. Since g is close to the rigid rotation of α there would be $n \in \mathbb{N}$ such that $g^n(abc) \cap abc \neq 0$ as show below, creating an invariant set.

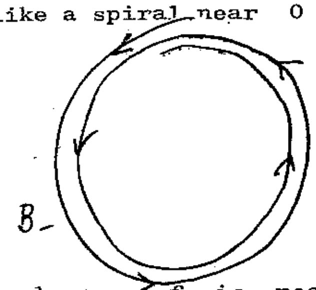


Similarly the case below cannot happen either

This means that arbitrarily near 0 parts of B_+ are close to circular arcs belonging to circles centered at the origin. Then, near the origin, B_+ "looks" like a very neat spiral (although microscopically B_+ can be highly non circular). Again because we are near 0 all points in B_+ converge to 0 when iterated by g



The same discussion applies to f creating an invariant set B_- which looks like a spiral near 0 notice the reversed



direction, which occurs because f is, near zero, close to a rotation of $-\alpha$.

The sets B_+, B_- have non empty intersection. Choose a in this intersection, a very near 0. Iteration by g sends a nearer and nearer the origin and iteration by f^{-1} (=g) sends a away from the origin. □

Therefore at least one of f and g satisfy: $\nexists\, x \in D$ such that $\exists$ k connected $x, f(x) \in k$ (or $x, g(x) \in k$) and $f^n(k) \subset D$ (or $g^n(k) \subset D$) $\forall\, n \geq 0$. This seems to imply there is a prefered direction in the pair f,g.

Next we check if our original g from the first lemma satisfy this property. If not we only exchange f and g bet-

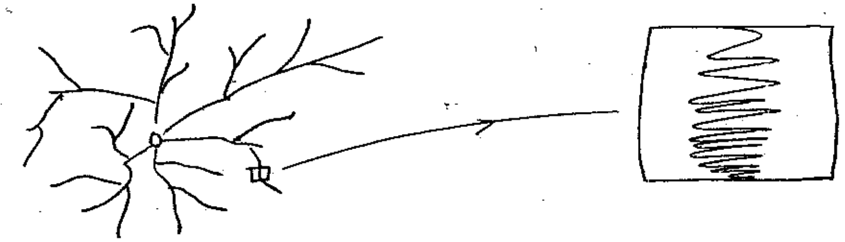
ween themselves. Now g has the property that for any $x \in C(g,D)$ $x, g(x)$ belong to different components of $C^*(g,D)$.

7. <u>Corollary</u>. $C^*(g,D)$ has infinitely many components which are permuted by the action generated by g.

<u>Proof</u>: If C^* had only finitely many components, at least one component would be periodic which contradicts the revious lemma for a power of g. Notice we are using vi) of the 1^{st} lemma.

8. Remarks

a) Possibly $C(g,D)$ is not locally connected.



b) The situation below cannot occur because iv) in the lemma. All components of C^* have 0 in its closure

$C(g,D)$

c) The order structure of a component can be defined using betweeness of components.

d) This order strucutre is completely determined by the determined by the derivative at 0 because arbitrarily near 0 g is arbitrarily close to a rigid irrational rotation. (needs an argument.)

9. The following result is therefore clear:

Theorem. The irrational argument of the derivative at an isolated elliptic fixed point of a completely unstable local diffeomorphism of $(\mathbb{R}^2,0)$ is determined by the topological dynamics.

10. The complete unstability is essential for the result above. That doesn't happen in the case of an analytic germ where classical results apply, as the following corollary shows.

Corollary. For an analytic germ of a diffeomorphism

$$z \xrightarrow{f} wz + a_2 z^2 + \dots$$

$$w = e^{i\theta} \qquad \theta/2\pi \in [0,1]\setminus\mathbb{Q},$$

w is a topological conjugacy invariant.

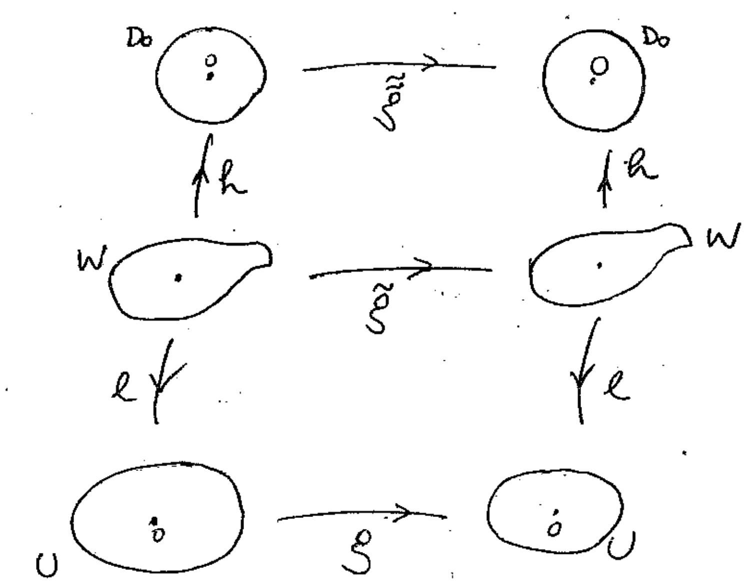
Proof: If f is completely unstable the result follows from the theorem.

Suppose f^n is stable at 0 and A, B are neighborhoods of 0 such that $g^m(A) \subset B$ $\forall\, n > 0$ where $g = f^n$.

Now g is analytic and $g(0) = 0$.

Set $U = A \cup f(A) \cup f^2(A) \dots$.

Clearly U is open and $f(U) \subset U$. If U is not simply connected consider the universal covering surface W of U which is a Riemann surface, and the projection map $\ell: W \to U$. Lift g to W in a way that 0 goes to a fixed point of the lift $\tilde{g}$. By the uniformization theorem W is conformally diffeomorphic to either the sphere, the plane or the unit disk D_o. The only possible choice here is the disk. We have an analytic map $\tilde{\tilde{g}}: D_o \to D_o$, $\tilde{\tilde{g}}(0) = 0$, which commutes the upper part of the diagram



Because ℓ is locally a conformal diffeomorphism and h is conformal $|\tilde{\tilde{g}}(0)| = |g(0)| = 1$. By Schwarz Lemma $\tilde{\tilde{g}}(z) = cz$ where $c = e^{i\theta}$, $\theta/2\pi \in [0,1]\setminus\mathbb{Q}$. Restricting to a smaller neighborhood of 0 we see that g is $\mathbb{C}$-analytically conjugate to a rigid irrational rotation and the result follows.

11. Remark. Siegel has shown that f is stable if $n_o, n_1, \ldots$ doesn't grow too fast. There are unstable examples when the $n_o, n_1, \ldots$ grow very fast.

Genericity and Full Measure

by

Floris Takens

Mathematisch Instituut
Rijksuniversiteit
Groningen

1. Introduction

There are two ways in which one usually formalizes the notion of "almost all elements": one, which can be used in case we have a measure μ on a set X, defines a subset $X' \subset X$ to contain almost all elements of X if $\mu(X - X') = 0$. According to the second definition, which can be used in case we have a topological space X satisfying the Baire property, a subset $X' \subset X$ contains almost all elements of X if X' contains a countable intersection of dense open subsets of X. In the first case, X' is called a subset of full measure, in the second case X' is called a residual subset. Also, the property of belonging to X' is called in the first case a property with full measure, in the second case a generic property.

It should be noted that these two interpretations of "almost all elements" can become quite contradictory in cases where we have both a topology and a measure. For example in the unit interval $[0,1]$, with the usual topology and the Lebesgue measure, there is a decomposition $[0,1] = A \cup B$, $A \cap B = \emptyset$, such that A