AN ANALYTIC PROOF OF NOVIKOV'S THEOREM
ON RATIONAL PONTRJAGIN CLASSES

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We give here an analytic proof for the following:

Theorem 1 (S. P. Novikov [3]). — The rational Pontrjagin classes of any simply-connected compact oriented smooth manifold are topological invariants.

This problem was previously posed by I. M. Singer [4] and D. Sullivan [5]. Theorem 1 is a direct consequence of the following Theorems 2 and 3.

Theorem 2 (D. Sullivan [5]). — Any topological manifold of dimension $\neq 4$ has a Lipschitz atlas of coordinates, and for any two such Lipschitz structures $L_i$, $i = 1, 2$, there exists a Lipschitz homeomorphism $h : L_1 \to L_2$ close to the identity.

Remark 1. — The proof of theorem 2 in general uses Kirby's annulus theorem to know that topological manifolds are stable. The proof of Theorem 2 for stable manifolds is more elementary. Simply connected manifolds are stable and these are sufficient for proving Novikov's theorem.

Theorem 3 (N. Teleman [6]). — For any compact oriented boundary-free Riemannian $\mathbb{R}$-manifold $M$ and for any Lipschitz complex vector bundle $\xi$ over $M$, there exists a signature operator $D^*_\xi$, which is Fredholm, and its index is a Lipschitz invariant.

Theorem 2 allows a strengthening of the statement of Theorem 3.

Theorem 4. — For any simply-connected compact, oriented, boundary-free topological manifold $M$ of dimension $2 \mu + 4$, and for any complex continuous vector bundle $\xi$ over $M$, there exists a class $\mathcal{C}(M, \xi)$ of signature operators $D^*_\xi$ which are Fredholm operators. The index of any of these operators is the same and is a topological invariant of the pair $(M, \xi)$. When $M$ and $\xi$ are smooth, the smooth signature operators $D^*_\xi$ (cf. [1]) belong to this class $\mathcal{C}(M, \xi)$.

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(2) See also P. Tukia and J. Väisälä [7] and [8].
Proof. — Pick a Lipschitz structure $\mathcal{L}_i$ on $M$ by Theorem 2, and regularize the bundle $\xi$ up to a Lipschitz vector bundle $\xi_\mathcal{L}$. Theorem 3 says that the class $\mathcal{C}(M, \xi)$ is not void, and because the Lipschitz signature operators generalize the smooth signature operators, the last part of the theorem follows.

Suppose now that $\mathcal{L}_i$, $i = 1, 2$, are two Lipschitz structures on $M$ and that $\xi_\mathcal{L}$ are corresponding Lipschitz regularizations of $\xi$.

The Theorem 2 implies that there exists a Lipschitz homeomorphism $h : \mathcal{L}_1 \to \mathcal{L}_2$ close to the identity (isotopic to the identity). As $h$ is isotopic to the identity, the bundle $h^*\xi_\mathcal{L}$ is Lipschitz isomorphic to $\xi_\mathcal{L}$; let $\overline{h} : \xi_\mathcal{L} \to \xi_\mathcal{L}$ be such an isomorphism. Take any Lipschitz Riemannian metric $[\xi_\mathcal{L}^\Gamma]_i$ on $M$, $i = 1, 2$, and any connection $\Delta_i$ in $\xi_\mathcal{L}$; the signature operators $D_\Delta^\xi_\mathcal{L}$ are defined. From Theorem 3 we know that the index of $D_\Delta^\xi_\mathcal{L}$, $i$ fixed, is independent of the Riemannian metric $\Gamma_i$ and the connection $\Delta_i$ chosen. In order to compare Index $D_\Delta^\xi_\mathcal{L}$ and Index $D_\Delta^\xi_\mathcal{L}$ themselves, we chose $\Gamma_2$ and $\Delta_2$ arbitrarily, but we take

$$\Gamma_1 = h^*\Gamma_2, \quad \text{and} \quad \Delta_1 = \overline{h}^*\Delta_2.$$ 

From the very definition of the signature operators, we get that the homeomorphisms $h$, $\overline{h}$ allow us to identify the corresponding domains and codomains of the operators $D_\Delta^\xi_\mathcal{L}$, $D_\Delta^\xi_\mathcal{L}$; with these natural identifications, $D_\Delta^\xi_\mathcal{L}$ and $D_\Delta^\xi_\mathcal{L}$ coincide, and therefore, they have the same index.

Proof of theorem 1. — Suppose that $M^{2n}$ is a smooth manifold, and $\xi$ is a smooth complex vector bundle over $M$. The signature theorem due to F. Hirzebruch, and subsequently generalized by M. F. Atiyah and I. M. Singer [1], asserts that

$$\text{Index } D^\xi = \text{ch} \xi.L(p_1, p_2, \ldots, p_w)[M]$$

where $L$ is the Hirzebruch polynomial and $p_1, p_2, \ldots, p_w$ are the Pontrjagin classes of $M$. Theorem 4 implies that the right hand side of this identity is a topological invariant of the pair $(M, \xi)$. By letting $\xi$ to vary, $\text{ch} \xi$ generates over the rationals the whole even-cohomology subring of $H^*(M, \mathbb{Q})$. From the Poincaré duality we deduce further that the cohomology class $L(p_1, \ldots, p_w)$ is a topological invariant. It is known that the homogeneous cohomology part $L_i$ of degree $4i$ of $L(p_1, \ldots, p_w)$ is of the form (see e.g. [2])

$$L_i = a_i p_i + \text{polynomial in } p_1, p_2, \ldots, p_{i-1}, \quad a_i \in \mathbb{Q}, \quad a_i \neq 0.$$ 

Therefore $p_1, p_2, \ldots, p_w$ are polynomial combinations with rational coefficients of $L_1, L_2, \ldots, L_w$, which, as seen, are topological invariants.