

MANIFOLDS WITH CANONICAL COORDINATE CHARTS:
SOME EXAMPLES ¹⁾

by Dennis SULLIVAN and William THURSTON

We will consider examples of low dimensional manifolds with inversive, projective, and affine structures (see below). The geometry of the associated developing maps is a problem like the qualitative study of dynamical systems involving as it does the “infinite composition” of finitely many operations. Our goal will be to answer in the negative certain questions about affine manifolds—by producing examples where the developing map is either not a covering of its image or has a rather complicated image.

These structures, classically called locally homogeneous spaces [1, 3, 8] or spaces with a flat Cartan connection [4], are determined by “canonical coordinate charts” on manifolds and may be defined in general as follows:

One starts with a model manifold A (see table below) and a transitive group \mathcal{A} (usually a Lie group) of real analytic homeomorphisms of A . Then one constructs all possible manifolds by choosing open sets of A and pasting these together using restrictions of homeomorphisms from the given group \mathcal{A} of analytic homeomorphisms. Such a manifold M is called an \mathcal{A} -manifold. More precisely, an \mathcal{A} -manifold is a manifold M together with an atlas $\kappa_i : U_i \rightarrow A$ such that the changes of charts are restrictions of elements of \mathcal{A} ; a “canonical chart” is any chart on M which is \mathcal{A} -compatible with these.

We will consider the following cases:

¹⁾ Our first draft of this paper was done in January 1977. The present version contains notes and clarifications by N. Kuiper bringing the paper to a more precise form. The authors acknowledge their thanks.

structure	dimension	model manifold A	group \mathcal{A}
affine	2, 3, 4	affine space	affine motions
inversive	2	two sphere	group generated by inversions in circles
projective	2	real projective plane	projective transformations

We are basically interested in the compact case.

The dynamical nature of such a structure on M arises from the ability to “roll” or “develop” the manifold M along paths of M (by pasting the open sets of A) into the model manifold A . We can do this in particular for closed paths representing generators of the fundamental group $\pi_1 M$ of M . The words in $\pi_1 M$ then determine a dynamical system on A .

Starting from one of the open sets $U \subset A$, the development produces a covering space $M' \rightarrow M$, a representation $\pi_1 M \rightarrow G$ called the holonomy and a structure preserving immersion, the developing map $M' \rightarrow A$, which is equivariant via ρ with respect to the action of $\pi_1 M$ on M' and \mathcal{A} on A .

The ambiguity in the development $M' \rightarrow A$ is the choice of one canonical chart $\kappa: U \rightarrow A$ about a base point $x \in U$ on M , uniquely determined up to multiplication on the right by an element of \mathcal{A} . In other words, the development d may be found by choosing an arbitrary structure-preserving map in one patch, then extending this choice by analytic continuation. This process works globally because \mathcal{A} is a group of globally analytic diffeomorphisms.

In Note 1 at the end of this paper Ehresmann’s neat definition of the development is given.

In Note 2 one sees how an \mathcal{A} -structure can be viewed as a fibrebundle $A \rightarrow E \rightarrow M$ with fibre A and structure group \mathcal{A} , fixed cross section $s(M)$, and a foliation \mathcal{F} transverse to the fibres and to $s(M)$; the foliation \mathcal{F} defines a “parallel transport” of the “tangent” fibres such that, the holonomy is in \mathcal{A} .

In Note 3 one finds the development of curves in a manifold with Cartan connection as described by Ehresmann in [4], and how this specialises for flat connections to the above developing map into one fibre.

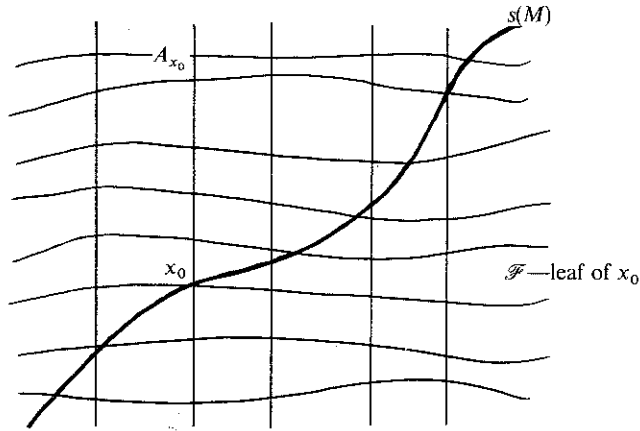


FIGURE 1

Before going to our examples we remark that considerations of the developing map (an immersion of manifolds of equal dimension),

$$\begin{array}{ccc} M' & \xrightarrow{\quad} & A \\ p \downarrow & & \\ M & & \end{array}$$

immediately shows such things as

- i) there are no compact manifolds with finite fundamental group which have affine structures.
- ii) the only compact n -manifolds with finite fundamental group with projective or inversive structures are actually covered by the n -sphere S^n .

Actually i) is true whenever the model manifold A is non-compact and ii) is true whenever the model manifold A has a compact universal covering like S^n .

In this context we remark that it is not totally unreasonable to hope that *all 3-manifolds have canonical charts relative to some subgroup of analytic homeomorphisms of S^3* . This statement by the above remark implies a strong form of the Poincaré conjecture; yet the statement itself only involves dimension 3 and not the fundamental group explicitly.

Now we turn to our 2-dimensional examples.

INVERSIVE 2-MANIFOLDS

Inversive structures on orientable two manifolds of genus > 1 form a rich theory properly containing for example the classical subject of Fuchsian and Kleinian surface groups.

If $Sl(2, \mathbf{C}) / \pm 1 = Gl(2, \mathbf{C}) / Gl(1, \mathbf{C})$ is the group of fractional linear transformations of \mathbf{CP}^1 , that is the group of orientable inversive (conformal) transformations of S^2 , and Γ is a discrete subgroup acting freely and discontinuously on a connected open set $\Omega \subset S^2$, then Ω/Γ is a 2-manifold M with inversive structure. M' is just Ω and the developing map is an embedding.

Example 1. If Γ is a Fuchsian group, that is, Ω is an open (round) disk in $\mathbf{C} \subset S^2$, then the inversive structure is actually a hyperbolic structure—corresponding to a metric of constant negative curvature. The structure is inversive and projective at the same time.

Example 2. If Γ as in Example 1 is deformed slightly (a so-called quasi-Fuchsian group; see [9]) then Ω remains an open disk whose boundary can be a rather remarkable non rectifiable Jordan curve. This curve has no tangent at a dense set.

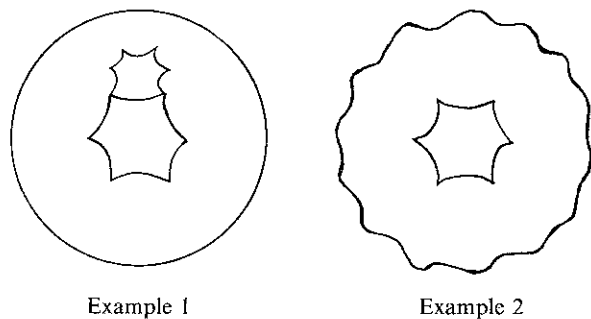


FIGURE 2

Example 3. Let Γ be generated by two general hyperbolic elements of sufficient strength so that the union of the fundamental domains of each covers the entire sphere. Then Ω is S^2 minus a Cantor set and Ω/F is a compact conformal 2 manifold whose developing image is Ω . (Shottky group)

In Figure 3, r_1, r_2 and r_3 are inversions (reflections) in three circles and Γ consists of all products of an even number of these inversions. Γ is generated by $r_1 r_2$ and $r_1 r_3$. A fundamental domain is $D \cup r_1 D$, $D = D_1 \cup D_2$. The Cantor set appears clearly on the line of symmetry m .

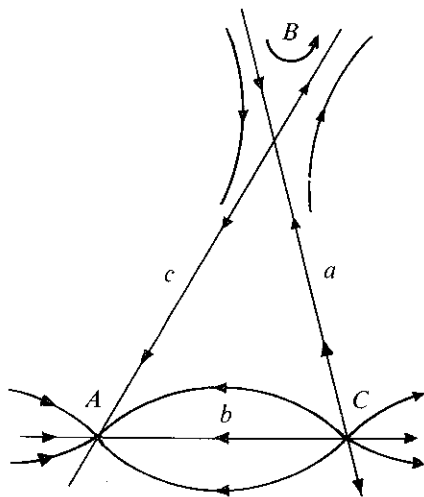


FIGURE 7a

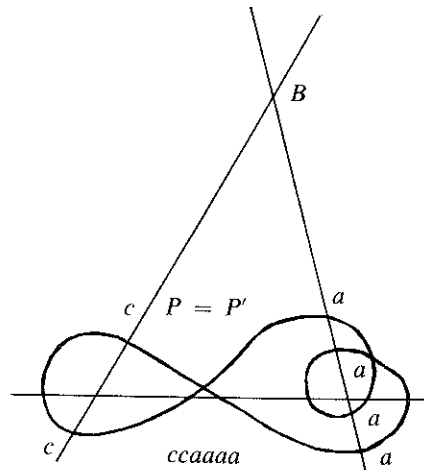


FIGURE 7b

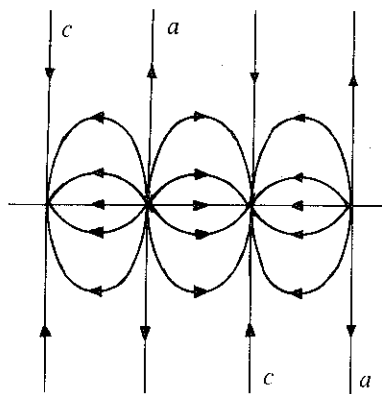


FIGURE 8a

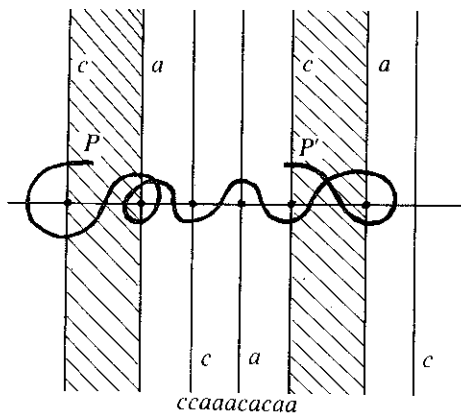


FIGURE 8b

Flowing the curve along for time t sweeps out a thickening of the immersed curve, an immersed annulus. We may identify the two boundary components of the annulus by the time t map, a locally projective isomorphism.

The identification space is a projective structure on the torus M whose developing map is the map: $M' = S^1 \times \mathbf{R} \rightarrow \mathbf{RP}^2$, obtained by spreading the immersed curve around by the flow for all time $t \in \mathbf{R}$.

The developing map is not a covering and the image is the projective plane minus three points for any word different from aa or cc . Note that the covering

space M' is obtained by gluing, each time along one of the two segments of a or c , as many copies of open sectors bounded by the lines a and c , (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the $t = 1$ flow map. In suitable homogeneous coordinates the last is expressed as $f_1 : f_t : (x, y, z) \rightarrow (xe^{at}, ye^{bt}, ze^{ct})$ $\alpha < \beta < \gamma$, $t = 1$.

Remark. Following the curve from its initial point P to its endpoint P' , one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \rightarrow (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with f_1 :

$$g : (x, y, z) \rightarrow (xe^\lambda, ye^\mu, ze^\nu)$$

$\lambda, \mu, \nu \in \mathbf{R}$.

AFFINE STRUCTURES IN 2, 3, AND 4 DIMENSIONS

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$ (where $\mathbf{R}^* = \mathbf{R} - \{0\}$) lifts to an affine transformation of $V = \mathbf{R}^3 - \{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops

over the part outside the coordinate axes of $\{X > 0\} \cup \{Z > 0\} \subset \mathbf{R}^3 = \{(x, y, z)\}$, but not as a covering. In these examples the 3-manifold M is a 3-torus.

ii) Similarly, a projective transformation of the complex projective line $\mathbf{CP}^1 = \mathbf{C}^2 - \{0\}/\mathbf{C}^*$, that is to say an orientable conformal or inversive transformation of $S^2 = \mathbf{CP}^1$, lifts to a complex affine transformation of $V = \mathbf{C}^2 - \{0\}$, unique but for scalar multiplication and commuting with scalar multiplication.

We can build a four dimensional affine manifold from an inversive 2-manifold, which is actually a complex affine manifold of \mathbf{C} -dimension 2, and this construction is the analogue of the above over \mathbf{C} , thinking of S^2 as \mathbf{CP}^1 and the conformal transformations as the \mathbf{C} -projective transformation.

Again compactness is achieved if we divide by $\alpha = 2$. Thus using the inversive Example 2 we obtain affine 4-manifolds whose developing image has a complicated boundary related to the non-differentiable Jordan curve. Using Example 3, we obtain an affine four-manifold whose developing image in R^4 omits a Cantor set of two planes passing through the origin.

Using Example 4, we can build affine manifolds whose developing map is not a covering of its image (which is all of $\mathbf{C}^2 - 0$). And we repeat, all the above are actually complex affine structures on compact 4-manifolds.

NOTE 1 (see page 16). *Ehresmann* defined the *development map* as follows. Let $\mathcal{P} \rightarrow M$ be the principle \mathcal{A} -bundle over M , whose points are germs $[x, \kappa]$ of canonical charts $\{x \in U \subset M, \kappa: U \rightarrow A\}$. Define a new topology $\mathcal{F}(\mathcal{P})$ in the set \mathcal{P} by taking as open set the germs at all points $x \in U$ of any given chart $\kappa: U \rightarrow A$. The natural map $d: \mathcal{F}(\mathcal{P}) \rightarrow A$ is an immersion. Choose one component of $\mathcal{F}(\mathcal{P})$ and call it M' . The restriction $d: M' \rightarrow A$ is a development map. The restriction of the natural fibre bundle projection $p: \mathcal{F}(\mathcal{P}) \rightarrow M$ is a covering $M' \rightarrow M$.

NOTE 2 (see page 16). *The fibre bundle picture*. For the simple local discussion of *one canonical chart* $U \subset A$, we can describe a trivial fibre bundle $E_U = U \times A \rightarrow U$ by assigning to any $x \in U$ the "heavily osculating" model space $A_x = A$. The manifold U is embedded as the diagonal cross section. $s(U) = \{(y, y)\} = \text{diag}(U \times U) \subset U \times U \subset U \times A$. Its points are the points of tangency of fibre and base manifolds. Finally a foliation \mathcal{F} is defined as the one with horizontal leaves $U \times \{v\} \subset E_U = U \times A$, for $v \in A$.

For the *global* discussion of an \mathcal{A} -structure on a manifold M , we assume \mathcal{A} -compatible canonical charts that are topological embeddings $\kappa: U \hookrightarrow A$ for

