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A FOLIATION OF GEODESICS IS CHARACTERIZED BY HAVING NO "TANGENT HOMOLOGIES"

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Dedicated to the memory of George Cooke

1.

Say that a one dimensional foliation is *taut* if the leaves become geodesics for some Riemann metric. The flow whose parametrization is arc length is said to be geodesible (according to Herman Gluck*). In the oriented case one can characterize the situation by the

Theorem. (i) *A foliation is taut if and only if there is a one form* ω *so that* ω *(each foliation direction)* >0 *and* $d\omega$ *(any 2-plane tangent to foliation)* $=0$.

(ii) *A flow is geodesible if and only if there is a transverse field of codimension one planes invariant under the flow.*

(iii) *Either of these conditions can happen for a foliation (or flow) precisely when the following cannot occur — for some invariant measure the corresponding 1-dimensional foliation cycle can be arbitrarily well approximated by the boundary of a 2-chain tangent to the foliation.*

Proof. Let us begin with (ii). Consider a segment $[A, B]$ of an orbit in a flow of geodesics. Swing geodesics from A of length AB to obtain a surface T_B normal to the leaf at B. Similarly construct T_A. Elementary geometry shows T_A and T_B cut-off on leaves order ε near AB segments of length $AB \pm$ order ε^2. This implies the orthogonal plane field is invariant under the flow.

Conversely, suppose a flow has an invariant transversal codimension one-plane field. Take any metric on the codimension one-plane field orthogonal direct sum the parametrization to obtain a metric for which the flow lines are geodesic in the arc length parametrization. This follows because the geodesic tubular neighborhood of a segment on a first order neighborhood of the segment is up to second order metrically like a Riemannian submersion fibration (under our hypothesis).

* This work was directly motivated by a detailed and attractive letter from Herman Gluck about "filling manifolds by geodesics".

Thus a first order pertubation of a segment cannot make it shorter. This completes the proof of (ii).

Now condition (i) is a reformulation of the condition in (ii). Namely, the invariant transversal codimension one-plane field and the parametrization of (ii) determine a 1-form ω satisfying $i \cdot \omega = 1$ and $(di + id)\omega = 0$ (or $id\omega = 0$). Conversely, given a form as in (i) choose the parametrization so that $i \cdot \omega = 1$. The second condition becomes $id\omega = 0$ so $(di + id)\omega = 0$, and the kernel of ω is the desired invariant field. This proves (i) assuming (ii).

Now condition (iii) is clearly necessary using Stoke's theorem while its sufficiency follows from the Hahn-Banach theorem as in [1].

More precisely, if c_n is a sequence of 2-chains tangent to the foliation so that ∂c_n converges to a foliation cycle z (in the sense of integrating individual smooth forms), then

$$0 = \int_{c_n} d\omega = \int_{\partial c_n} \omega \to \int_z \omega > 0,$$

a contradiction. Conversely, if the closed linear sub-space of the dual space of forms generated by $\{\partial c\}$ where the c are 2-chains tangent to the foliation does not intersect the ("compact") cone of foliation *cycles* [1], we can find a closed hyperplane containing the subspace and supporting the cone of foliation *currents* [1] by hahn–Banach.

This subspace determines the form ω satisfying (i).

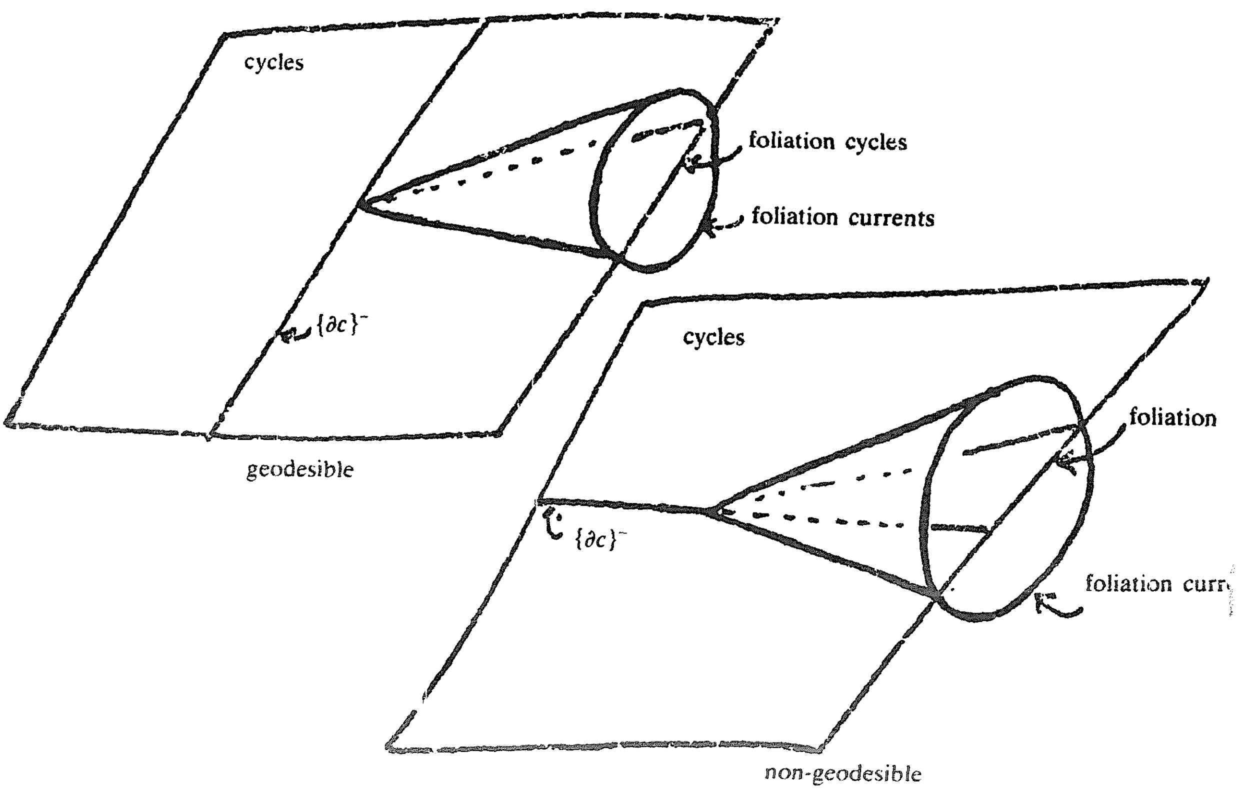

Fig. 1.

2. Examples and further remarks

Corollary. *In* dim 3 *we can record the strict inclusions,*

$$\left\{\begin{matrix}\textit{contact flows union}\\ \textit{“flows with section”}\end{matrix}\right\} \subset \left\{\begin{matrix}\textit{geodesible}\\ \textit{flows}\end{matrix}\right\} \subset \left\{\begin{matrix}\textit{“partially volume preserving”}\\ \textit{union “flows with section”}\end{matrix}\right\}$$

$$\cap$$

$$\{\textit{not generalized horocycle flows}\}$$

and we note the horocycle flows are completely volume preserving and not geodesible.

Explanation. Relative to condition (i) $d\omega$ identically zero implies the flow has a cross section and so is transversal to a fibration over S^1. Conversely, such a flow is geodesible (by Gluck's direct calculation, or use (i)). Furthermore if $d\omega$ is non-zero somewhere we have a smooth invariant measure which we have denoted "partially volume preserving" above. This shows the right hand inclusion and half of the left.

Now a contact flow is determined (without parametrization) by kernel $(d\eta)$ where η is a 1-form to that $\eta \wedge d\eta$ is a volume form. Thus (i) is fulfilled, and this foliation is taut.

A (generalized) horocycle flow is defined by kernel $(d\omega)$ where ω is a 1-form satisfying $\omega \wedge d\omega \equiv 0$ (the foliation defined by ω in the classical case is the foliation of asymptotic geodesics in the unit tangent bundle of a negatively curved surface). Any such $(\ker d\omega)$ foliation on a compact 3-manifold (where $d\omega$ is nowhere zero and $\omega \wedge d\omega$ is identically zero) is not geodesible or taut. In fact, the 2-current defined by ω can be approximated by pieces of leaves of ω which by the way contain the leaves of $(\ker d\omega)$. These pieces define 2-chains whose boundaries approach the foliation cycle $d\omega$ (thought of as current) and we find ourselves in the forbidden circumstance of (iii) of the Theorem. (See Fig. 2(b).)

Remark. Of course having a cross section is an open condition, while having a smooth invariant measure is a very unstable condition. The theorem suggests that being geodesible is a rather general mixture of these two properties with the marriage being supervised by the "no tangent homology" condition.

This homology condition first arose in the preparation of [1] when we tried to characterize contact flows. The homology condition was only necessary to be contact (and not sufficient — think of flows with cross sections) and was omitted from [1]. Finally one might recall that the stronger homology condition "no foliation cycle is homologous to zero" exactly characterizes the class of flows with cross section. This is due to Schwartzman [2], later Fried in a more geometric form, and was demonstrated in [1] along with other similar results in the language used above.

The tangent homology condition can be illustrated by a finite example (Fig. 2(a)) and an infinite example (Fig. 2(b)).

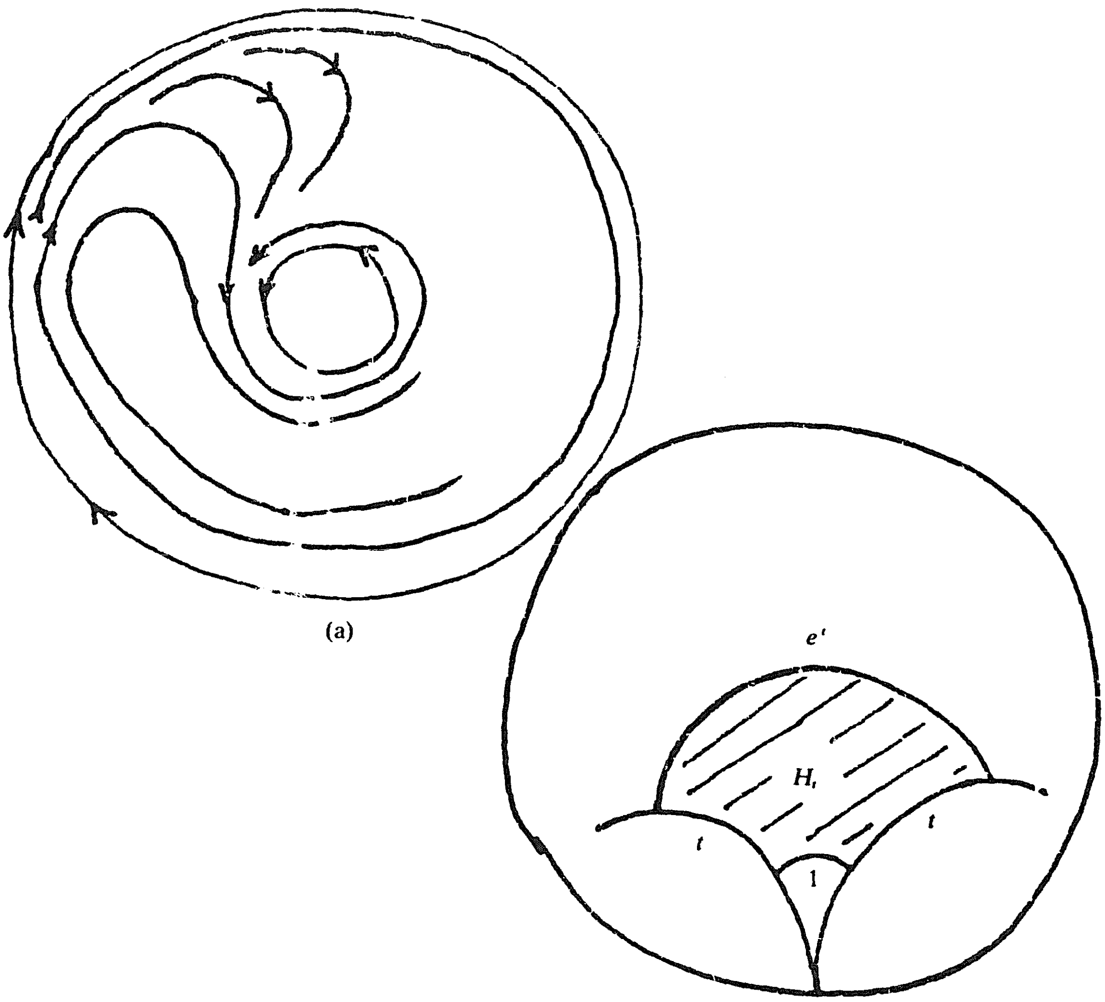

Fig. 2.

The flow on the annulus (Fig. 2(a)) was already observed by Gluck to be non-geodesible. By considering the Euler characteristic, one sees any finite tangent homology example has to occur on an annulus.

The chain H_t (Fig. 2(b)) bounded by horocycles and geodesics of indicated lengths in the Poncaré disc was already in Plante's thesis. The sequence $e^{-t} \cdot H_t$ has boundaries approaching (the unique) foliation cycle for the horocycle flow. It provides an infinite example of the tangent homology condition showing the horocycle flow is not geodesible.

References

[1] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds. Inventiones Math. 36 (1976) 225–255.

[2] S. Schwartzmann, Asymptotic cycles, Ann. Math. 66 (1957) 270–284.

[3] H. Gluck, Open letter on geodesible flows.