A TOPOLOGICAL INVARIANT OF FLOWS ON 1-DIMENSIONAL SPACES

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To a $k \times k$ matrix $A$ of non-negative integers whose sum is $s$ one can associate a shift invariant subset $X_A$ (see below) of the countable product of $\{1, \ldots, s\}$ with itself. The shift on this subset is a transformation $T_A$ which can be naturally studied in set theory [3], probability theory [3] and [4], and topology [1]. These points of view merge in the dynamical study of diffeomorphisms.

In this note we consider the flow with cross section $X_A$ and Poincaré map $T_A$ and show that $\det(1 - A)$ is a topological invariant of the underlying space of the flow. We compare this invariant to the classification of the transformation $T_A$ itself in the three contexts above. The argument here is purely topological but a related result was first discovered in [5] using variational principles for the entropy of flows. This invariant $\det(1 - A)$ allows us to answer anew Bowen's question [2] on the non-triviality of the equivalence relation between non-negative matrices induced by the topological classification of the corresponding flows (see also [5]).

Finally, it is an interesting problem to give a direct topological description of this invariant of flows.

Now we go into more detail. If $A = (a_{ij})_{X_A}$ can be described as a certain set of infinite paths in the complex $K_A$ made up of vertices $\{1, \ldots, k\} \times \{\text{integers}\}$ with $a_{ij}$ edges between $(i, n)$ and $(j, n + 1)$ for each $1 \leq i, j \leq k$ and $-\infty < n < \infty$. $X_A$ consists of those infinite paths in $K_A$ which map isomorphically to the real line under the natural projection $K_A \to \text{real line}$. There are $s$ intervals over $[n, n + 1]$ so $X_A$ is naturally a subset of the countable product of $\{1, \ldots, s\}$ with itself. We will assume that $A$ is irreducible in the sense that for any $1 \leq i, j \leq k$ there is a path in $K_A$ connecting $(i, 0)$ with $(j, m)$ for some $m$. Then $X_A$ is either finite or uncountable. We will dispense with the first case where $A$ is just a permutation matrix and study the transformation $T_A$, induced on $X_A$ by the shift $(\cdot, n) \to (\cdot, n - 1)$ of $K_A$ in the case where $X_A$ is uncountable.

In set theory $T_A$ is determined up to isomorphism by the orbit structure. It is easy to see that $T_A$ has a finite number $N_n$ of periodic points of order $n$ (in fact, $N_n = \text{trace } A^n$). So $T_A$ has uncountably many non-periodic points and is determined as a set transformation by the sequence of integers $N_n$. These can be neatly catalogued in the Weil zeta function of $T_A$,

$$\xi(t) = \exp \left( \sum_{n=1}^{\infty} \frac{N_n}{n} t^n \right)$$

which is just the rational function $1/\det(1 - tA)$, [3].

Now $X_A$ has a natural topology homeomorphic to the Cantor set. Open sets of paths in $X_A$ are obtained by restricting the position of the path at a finite number of places. In this topology the periodic points of the homeomorphism $T_A$ are dense (again easy to see) and there is a Baire set of points whose orbits are dense [1]. As a topological transformation $T_A$ is determined by the class of the matrix $A$ under the equivalence relation (shift equivalence) generated by $A \sim A'$ if $A = XY$ and $A' = YX$ where $X$ and $Y$ are (not necessarily square) matrices of non-negative integers (see [7] and the errata [8]).

$T_A$ can be treated measure theoretically as an "intrinsic Markov chain" (see [4]). There is a unique invariant measure which maximizes entropy. As a measure preserving transformation $T_A$ is determined by this entropy (which is just the log of the maximum eigenvalue of $A$; see [1] and [6]) and the periodicity of $A$.

Now we turn to the flow with cross section $X_A$ and Poincaré map $T_A$. The underlying space of the flow $F_A$ is $X_A \times \{\text{real line}\}$ modulo the action of $T_A \times \text{unit translation}$. We will say that the

*If $A$ has periodicity 1, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} has periodicity two.
homeomorphisms \( T_A \) and \( T_B \) (or the matrices \( A \) and \( B \)) are \textit{flow equivalent} if there is a homeomorphism between \( F_A \) and \( F_B \) carrying flow lines to flow lines with the proper direction. Since \( X_A \) is totally disconnected, the arc components of \( F_A \) are just the flow lines and flow equivalence is just "oriented homeomorphism". (A remark of Charles Pugh).

Bowen asked "what is the equivalence relation on non-negative integral matrices induced by flow equivalence?" To answer this consider the operation on matrices \( A \rightarrow \tilde{A} \) defined by

\[
A = \begin{pmatrix}
\sigma_{11} & \ldots & \sigma_{1n} \\
\vdots & \ddots & \vdots \\
\sigma_{n1} & \ldots & \sigma_{nn}
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & a_{11} & \ldots & a_{1n} \\
1 & 0 & \ldots & 0 \\
0 & a_{21} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n1} & \ldots & a_{nn}
\end{pmatrix} = \tilde{A}.
\]

Our result is the

\textbf{THEOREM.} The 'flow equivalence relation' on square non-negative integral matrices is generated by the relations

(i) \( XY \sim YX \) (\( X, Y \) not necessarily square)
(ii) \( A \sim \tilde{A} \).

\textbf{COROLLARY.} If \( A \) and \( B \) are flow equivalent then \( \det (1 - A) = \det (1 - B) \).

\textit{Example.} For every \( n \neq m \), the full \( n \)-shift is not flow equivalent to the full \( m \)-shift (this also follows from the result in [5]).

We summarize all these statements about the transformation \( T_A \) and induced flow in the classification table

<table>
<thead>
<tr>
<th>Context</th>
<th>Classifying invariant</th>
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</thead>
<tbody>
<tr>
<td>( T_A ) as a transformation of sets</td>
<td>the zeta function, ( 1/\det (1 - tA) )</td>
</tr>
<tr>
<td>( T_A ) as a measure preserving transformation</td>
<td>the entropy, the log of the maximum eigenvalue of ( A ) and the periodicity of ( A )</td>
</tr>
<tr>
<td>( T_A ) as a topological transformation</td>
<td>the shift equivalence class of ( A ), the relation being generated by ( XY \sim YX )</td>
</tr>
</tbody>
</table>
| The topological study of the flow associated to \( T_A \) | the equivalence class of \( A \) under the relation generated by:
  (i) \( XY \sim YX \)
  (ii) \( A \sim \tilde{A} \) |

We remark that the problem of interpreting the invariant \( \det (1 - A) \) topologically is just the problem of interpreting the value of the zeta function at 1. As noted above the zeta function near \( t = 0 \) is determined by the periodic point structure of \( T_A \) which is somewhat lost in the flow picture. The problem is complicated by the fact that 1 is beyond the radius of convergence of this power series for \( \zeta(t) \).

Now we discuss the proof of the theorem.

Consider self-homeomorphisms of Cantor sets under the relation of flow equivalence, i.e. oriented homeomorphisms of the corresponding flow spaces. Given one such \( X \rightarrow X \) we define its expansion \( X \rightarrow \tilde{X} \) given a decomposition \( X = \tilde{A} \cup \tilde{B} \) into disjoint open and closed sets. Let \( \hat{B} \rightarrow B \) be an identification between \( B \) and another copy \( \hat{B} \), let \( \tilde{X} = \tilde{A} \cup \hat{X} \cup \tilde{B} \), and define \( \tilde{f} : \tilde{X} \rightarrow X \) by \( f = i^{-1} \cdot \hat{f} \) on \( f^{-1}B, f = i \) on \( \hat{B}, f = f \) on \( f^{-1}A \). (See picture below). Then we have the

\textbf{PROPOSITION.} The flow equivalence relation on homeomorphisms of Cantor sets is generated by topological conjugacy and expansion.

\textit{Proof of Proposition.} First, that \( f \) and \( \tilde{f} \) are flow equivalent is clear from the picture

\footnote{Bob Williams pointed out that one could remove our original absolute value signs (which arose from the fact that \( \det (A - I) = -\det (\tilde{A} - I) \)).}
Conversely, if in one flow space we have two Cantor set cross sections $X$ and $Y$, we can easily find a third Cantor set cross section $Z$ so that the three are arrayed as in the figure.

In fact, $Z$ is a forward translate of that subset of $X$ which meets $Y$ before $X$ under the flow. Clearly, the homeomorphisms of $X$ and $Y$ induced by the flow are multiple expansions of the Poincaré map of $Z$. This proves the proposition.

This proposition is given in an equivalent form in [5] in terms of discrete rational reparametrizations of flows.

The proof of the theorem is completed by the following remarks:

(i) By changing $A$ in its shift equivalence class we can assume $A$ is a $k \times k$ matrix of 0's and 1's and the open and closed set for an expansion is just that subset of $\{1, \ldots, k\}$ in which the first coordinate is restricted to a subset of $\{1, \ldots, k\}$ (see [7]).

(ii) If we expand $T_A$ along the subset $\{ \cdots, 1, \cdots, \}$ we obtain $T_A$.

(iii) Expanding along a disjoint union is equivalent to a consecutive expansion on each piece.

Q.E.D.

REFERENCES

University of Warwick
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