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HOMOLOGY THEORY AND DYNAMICAL SYSTEMS

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INTRODUCTION

THE MOST simple discrete time structurally stable dynamical systems are the Morse–Smale diffeomorphisms having a finite recurrence set. In this paper we study which components of the space of all diffeomorphisms contain Morse–Smale systems. If the dimension is greater than five we reduce this question to the algebraic topology of the manifold and the component in question, namely the cells, the fundamental group, and the induced transformations. In the simply connected case we find the simple characterization—a power of some component contains a Morse–Smale system iff the homology eigenvalues are roots of unity.

In this simply connected case we give a precise homological characterization which calls for some algebraic work relating to ideal classes in cyclotomic fields.

In the non-simply connected case, more geometric–algebraic work is called for in the 2-cells to yield a better algebraic characterization. This would yield by our construction results in the missing dimensions at least for specific manifolds, like torii.

In general, the constructions reveal a relationship between homological properties and recurrence properties of diffeomorphisms for example entropy and homology eigenvalues. Finally part of the construction works in all dimensions to yield proofs of the

THEOREM (Smale, Shub, Williams). *Every diffeomorphism on a compact manifold can be smoothly isotoped to a C^0 close structurally stable diffeomorphism.*

BACKGROUND AND DESCRIPTION OF RESULTS AND PROBLEMS

The study of structurally stable diffeomorphisms has dominated much of the recent work in geometric dynamical systems. At first it was hoped that the structurally stable diffeomorphisms might be open and dense in $\mathrm{Diff}^r(M)$ as was the case for M the circle as proven by Peixoto [14].

The structurally stable diffeomorphisms of S^1 have finitely many periodic points and clear geometrical structure. Examples of such diffeomorphisms can be constructed on any manifold by perturbing the identity map along the gradient lines of a non-degenerate Morse function. Palis and Smale [13] proved these were structurally stable and characterized geometrically structurally stable diffeomorphisms with finitely many periodic points.

At an early stage Smale [20] proved that these Morse–Smale diffeomorphisms were not dense even on the 2-dimensional sphere via his horseshoes, and Thom pointed out that on the 2-dimensional torus the linear map $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ is not even homotopic to a map with only finitely many generic periodic points because the Lefshetz numbers of the iterates are not bounded.† Anosov [1] finally showed this mapping and its geometric generalizations were structurally stable. One striking feature of the example is that the periodic points are dense in the manifold.

In 1965 Smale combined the Anosov diffeomorphism on the torus with another construction to show that the structurally stable diffeomorphisms were not dense in the C topology of Diff $M(r \geq 1)$. After a long history of weakening the notion of structural stability (Ω-stability, etc.) to try to find a dense (generic) set of Diff M attention was again focused on the structurally stable diffeomorphisms regarding more gross topological properties than C^r genericity. For example, does every component of Diff M contain a structurally stable diffeomorphism? Which components contain Morse–Smale diffeomorphisms? Which manifolds admit Anosov diffeomorphisms?

In 1971, Smale generalized the basic horseshoe example to show any isotopy class contains a diffeomorphism which is Ω-stable. At about the same time R. F. Williams and Shub extended Smale's argument to replace Ω-stable by structurally stable and Shub [18] showed that structurally stable diffeomorphisms are C^0-dense in Diff M. In these examples the closure of the periodic points are zero-dimensional Cantor sets or finite.

The work of Bowen [2] allows one to obtain a very clear picture of the diffeomorphism on these invariant sets. A beautiful fact emerges—for these examples the complete recurrence structure of the diffeomorphism can be described by geometric intersection matrices associated to the classical chain mappings induced by these diffeomorphisms. This construction is described in §1. The discussion is simplified by the idea of a fitted handle decomposition which perhaps has some independent geometric interest.

This relationship between the algebraic topology of a diffeomorphism and the picture of a diffeomorphism for this C^0 dense set of Diff M opens a new point of view in geometric dynamical systems and offers many interesting questions to geometric topologists. For example, the problem of constructing the simplest diffeomorphism in an isotopy class from the point of view of entropy is discussed in general in §4.

One very interesting question is the relationship for a diffeomorphism between the logarithm of the modulus of the largest eigenvalue of f on homology and the entropy.

The case of zero entropy or equivalently (for these examples) finitely many periodic points is discussed in §3. The problem becomes the one mentioned above—which components of Diff M contain Morse–Smale diffeomorphisms. We characterize such components in the simply connected case by a homology property of the component. If we could ignore torsion

† One can drop the word "generic" here using [25].

and the ideal classes of the cyclotomic fields the condition would be—f is isotopic to a Morse–Smale diffeomorphism iff the eigenvalues of f_* in homology lie on the unit circle.†

Motivated by Thom's example Shub [17] had already observed using a careful Lefshetz number argument that this eigenvalue condition was necessary. However, the Smale picture of these diffeomorphisms implies more, namely the diffeomorphism can be represented on the chain level by "virtual permutation matrices"

$$\begin{pmatrix} P_1 & * & \cdots & * \\ \vdots & P_2 & \ddots & \vdots \\ 0 & . & \cdots & P_n \end{pmatrix}$$

where each P_i is a permutation matrix (with ± 1's).

The geometrical discussion of §2 shows how to realize abstract chain mappings by handle preserving diffeomorphisms with good intersection matrices. Combining this with the Smale, Shub, Williams construction of §1 yields Morse–Smale diffeomorphisms and the converse.

We note that the virtual permutation condition can be expressed in terms of the homology class of the graph of f in $H_m(M \times M, Z)$. We also outline a program for the non-simply connected case.

This discussion of Morse–Smale diffeomorphisms is at the center of several interesting connections between geometrical and algebraic phenomena which should be explored further. For example, one could ask for

(i) the precise relationship between ideal classes in cyclotomic fields and virtual permutation chain mappings (see §3 and [26]).

(ii) the relationship between non-negative matrices and ideal classes (see §3 and [7]).

(iii) the meaning of the fact that the diffeomorphisms constructed by monodromy in algebraic geometry are isotopic to Morse–Smale diffeomorphisms (see §3).

We conclude by noting that the construction of Morse–Smale diffeomorphisms is a special case of the following:

THEOREM. *Suppose f is a diffeomorphism of a simply connected manifold M^m of dimension at least six, and $\{C_i \xrightarrow{F_i} C_i\}$ is an abstract chain mapping representing the homology class of graph f where $C_m = C_o = Z$, and $C_1 = C_{m-1} = 0$. Then we can deform f to a structurally stable diffeomorphism with a zero dimensional non-wandering set Ω so that Ω and f restricted to Ω can be explicitly constructed from symbolic dynamical systems determined by the chain matrices F_i after dropping signs.*

The proof of this theorem is given in §2 and §3 and non-simply connected generalizations are indicated.

† Our homology condition does imply some power of f is isotopic to a Morse-Smale diffeomorphism iff the eigenvalues of f_* are on the unit circle.

§0. BACKGROUND DEFINITIONS AND FACTS

If X and Y are topological spaces and $f: X \to X$, $g: Y \to Y$ are continuous, then f and g are topologically conjugate if and only if there is a surjective homeomorphism $h: X \to Y$ such that $gh = hf$. $f \in \mathrm{Diff}^r(M)$ is structurally stable if and only if there is a neighborhood $U_f \subseteq \mathrm{Diff}^r(M)$ of f with the property that any $g \in U_f$ is topologically conjugate to f. So structural stability means that the orbit structure is locally constant in a neighborhood of f up to continuous changes of coordinates.

Given $f \in \mathrm{Diff}^r(M)$, $\Omega(f)$ the nonwandering set of f (or simply Ω if no confusion is possible) is the set of points $x \in M$ such that given U_x a neighborhood of x in M there is an $n > 0$ so that $f^n(U_x) \cap U_x \neq \emptyset$. A point x in M is periodic if there is an $n > 0$ with $f^n(x) = x$. All periodic points are, of course, nonwandering.

An invariant set $\Lambda \subset M$ for f, i.e. $f(\Lambda) = \Lambda$, has a hyperbolic structure if $T_\Lambda M$, the tangent bundle of M restricted to Λ, has a Tf invariant direct sum decomposition $E^s \oplus E^u$ such that there are constants $c > 0$, $\lambda < 1$ with $\|Tf^n | E^s\| \leq C\lambda^n$ for $n > 0$ $\|Tf^n | E^u\| \leq C\lambda^n$ for $n < 0$.

$f \in \mathrm{Diff}^r(M)$ satisfies Smale's Axiom A if and only if

(a) $\Omega(f)$ has a hyperbolic structure,

(b) $\Omega(f)$ is the closure of the periodic points of f.

If $f \in \mathrm{Diff}^r(M)$ then

$$W^s(x) = \{y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}, \quad \text{and}$$
$$W^u(x) = \{y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}.$$

If f satisfies Axiom A then it follows from [8] and [23] that $W^s(x)$ and $W^u(x)$ are 1–1 immersed Euclidean spaces. If, moreover, $W^s(x)$ and $W^u(x)$ are transversal for all $x \in M$, f is said to satisfy the strong transversality condition.

Definition. $f \in \mathrm{Diff}^r(M)$ is Morse–Smale if and only if:

(1) f is Axiom A and satisfies the strong transversality condition, and

(2) $\Omega(f)$ is finite.

Palis and Smale [13] proved:

THEOREM. *$f \in \mathrm{Diff}^r(M)$ is Morse–Smale if and only if $\Omega(f)$ is finite and f is structurally stable.*

Thus from the dynamical systems point of view Morse–Smale diffeomorphisms are the simplest diffeomorphisms.

Joel Robbin [15] later proved:

THEOREM. *If $f \in \mathrm{Diff}^1(M)$ is C^2 and satisfies Axiom A and the strong transversality condition, then f is structurally stable.*

Clark Robinson has recently removed the hypothesis that f be C^2. Basic references for the unreferenced material above are [16] and [22]. Below we will define the topological entropy of f, $h(f)$, via a theorem of Bowen [4].

Let (X, d) be a compact metric space and $T\colon X \to X$ continuous. A set $E \subseteq X$ is (n, ε) separated if for any $x, y \in E$ with $x \neq y$ there is a j with $0 \leq j < n$ such that $d(T^j(x), T^j(y)) > \varepsilon$. Let $r_n(\varepsilon)$ denote the largest cardinality of an (n, ε) separated set. $r_\varepsilon(T) = \limsup 1/n \log r_n(\varepsilon)$, and the topological entropy of T, $h(T) = \lim_{\varepsilon \to 0} r_\varepsilon(T)$. So in some sense the entropy is a measure of the asymptotic exponential growth rate of the number of orbits of T up to any accuracy. For $f \in \mathrm{Diff}^r(M)$, $0 \leq h(f) < \infty$. The following theorem is a special case of a theorem of Bowen [2].

THEOREM. *Let $f \in \mathrm{Diff}^r(M)$. Then $h(f) = h(f/\Omega)$. And, if f satisfies Axiom A and the strong transversality condition, then $h(f) = 0$ if and only if f is Morse–Smale.*

§1. CONSTRUCTING STRUCTURALLY STABLE DIFFEOMORPHISMS

The goal of this section is to isotope any diffeomorphism to a structurally stable diffeomorphism and to identify the non-wandering sets of diffeomorphisms produced in this way. We begin with some simple facts from differential topology. Recall that a handle decomposition, H, of a manifold M is a sequence of submanifolds $\phi \subseteq M_o \subseteq \cdots \subseteq M_m = M$ where $\overline{M_j - M_{j-1}} = \bigcup_{i=1}^{n_j} D_i^j \times D_i^{m-j}$ and the $D_i^j \times D_i^{m-j}$ are attached to the boundary of M_{j-1} by disjoint embeddings of the $S_i^{j-1} \times D_i^{m-j}$. A diffeomorphism f is said to preserve the handle decomposition H if $f(M_i) \subseteq$ interior M_i for $0 \leq i \leq m$. Thus the handle decomposition is a filtration for f (see [19]) and we may define $K_i = \bigcap_{n \in Z} f^n(\overline{M_i - M_{i-1}})$ $0 \leq i \leq m$, which is the maximal f invariant set contained in $\overline{M_i - M_{i-1}}$. Ω is also decomposed by $\Omega_i = \Omega \cap \overline{M_i - M_{i-1}}$ and $K_i \supseteq \Omega_i$.

If f preserves the handle decomposition H we will say $f \in T_H$ if moreover, $f(D_i^j \times 0)$ is transverse to $0 \times D_k^{m-j}$ for $1 \leq i, k \leq n_j$ and for all j. If $f \in T_H$, $f(D_i^j \times 0)$ intersects $0 \times D_k^{m-j}$ in a finite number, $g_{ik}{}^j$, of points. We may form the geometric intersection matrix $G_j = (g_{ik}{}^j)$. We may also form the algebraic intersection matrix $A_j = (a_{ik}{}^j)$ where $a_{ik}{}^j$ is the number of points of intersection of $f(D_i^j \times 0)$ with $0 \times D_k^{m-j}$ counted with their signs. G_j is a matrix of non-negative integers whereas A_j may have negative entries, but in any case we have the obvious inequality $|a_{ik}{}^j| \leq g_{ik}{}^j$. The A_j's determine an endomorphism of the chain complex

$$\begin{array}{ccc}
\cdots > H_j(M_j, M_{j-1}) & \xrightarrow{\partial} & H_{j-1}(M_{j-1}, M_{j-2}) \to \cdots \\
\Big\downarrow{\scriptstyle A_j} & & \Big\downarrow{\scriptstyle A_{j-1}} \\
\cdots > H_j(M_j, M_{j-1}) & \xrightarrow{\partial} & H_{j-1}(M_{j-1}, M_{j-2}) \to \cdots
\end{array}$$

which induces $f_*\colon H_*(M) \to H_*(M)$ on homology. The G_j's will be used to determine the K_j's and the Ω_j's of the structurally stable diffeomorphisms we produce.

Definition 1.1. The subset $\mathscr{H} \subset \mathrm{Diff}^r(M)$ is the subset of those diffeomorphisms $f \in \mathrm{Diff}^r(M)$ such that

(1) f satisfies Axiom A and the strong transversality condition.

(2) f is an element of T_H for some handle decomposition H of M.

(3) $f|K_i$ is topologically conjugate to the subshift of finite type associated to the geometric intersection matrix G_i, and $f|\Omega_i$ is topologically conjugate to the subshift of finite type† restricted to its non-wandering set.

The diffeomorphisms in $\mathscr{H}$ are structurally stable and $f|K_i$ is identified.

The main theorem of this section is

THEOREM 1.1. *If f is an element of T_H then f may be isotoped to an element of $\mathscr{H}$ without changing the geometric intersection matrices G_i.*

In the next section we will discuss how to move any f into T_H for any handle decomposition and relate the possible G_i to the algebraic topology.

This will lead to the construction of Morse–Smale diffeomorphisms and more generally structurally stable diffeomorphisms with the most simple non-wandering behaviour compatible with the topology.

The C^0 length of the move is easily estimated in terms of the derivatives of f and the size of the handles so we will also have achieved the

COROLLARY (i) *Any diffeomorphism is smoothly isotopic to a structurally stable diffeomorphism.*

(ii) *Any diffeomorphism can be C^0 approximated by a structurally stable diffeomorphism.*

These results were described more briefly in [18] and were motivated by [24] where (i) was proved for Ω-stability rather than structural stability.

We will first discuss the construction of subshifts of finite type from non-negative matrices and then describe the isotopy of any diffeomorphism to an element of $\mathscr{H}$.

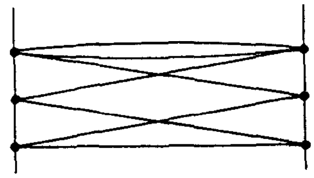
Let $B = (b_{ij})$ be an $n \times n$ matrix. We say that B is a 0–1 matrix if $b_{ij} = 0$ or 1 for all i and j. Given a 0–1 matrix B it determines a subshift of finite type as follows. Let $N = (1, \ldots, n)$ with the discrete topology and $\Sigma = \prod_{-\infty}^{\infty} N$ have the product topology. A typical element of this product is a bi-infinite sequence $(\ldots a_1, a_0, a_{-1} \ldots)$ where $a_i \in N$. Σ_B is the closed subset of Σ defined by $\{a_i\}_{\in iZ} \in \Sigma_B$ iff $b_{a_i a_{i+1}} = 1$ for all $i \in Z$. The shift map $\sigma\colon \Sigma_B \to \Sigma_B$ is defined by $\sigma(\{a_i\}_{i\in Z}) = \{a_i'\}_{i\in Z}$ where $a_i' = a_{i+1}$. Note that we have called $\sigma\colon \Sigma_B \to \Sigma_B$ a subshift of finite type even if it is not topologically transitive.

There is an appealing geometric picture‡ for this construction which shows we can start from a matrix with arbitrary non-negative integral entries to construct subshifts.

Identify any $k \times n$ matrix (g_{ij}) of non-negative integers with the set of lines between two parallel planes, one containing k ordered points the other n ordered points and with g_{ij} lines connecting the ith point on the first plane with the jth point on the second plane.

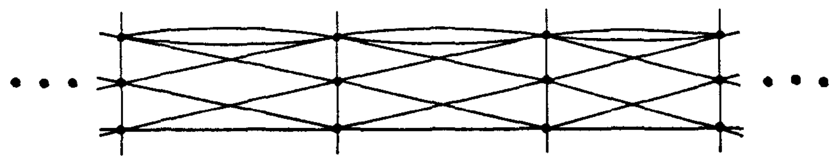
† See definition below.

‡ Communicated by Bob Williams and René Thom.



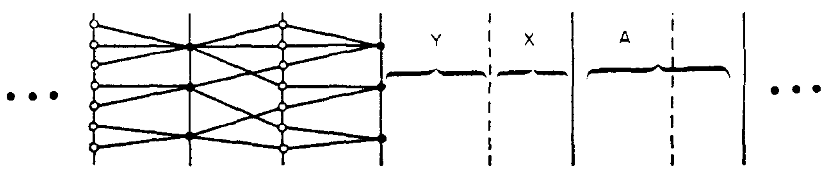
If $k = n$ we can repeat this figure infinitely often to the right and to the left and we obtain an infinite graph with a shift symmetry.

The space of all infinite lines in graph with the shift operation is just the



symbolic dynamics described above for 0–1 matrices. The space of lines is naturally topologized by saying two lines are close if they agree between $-n$ and n.

Slicing the figure differently into new periodic pieces gives a new presentation of the dynamical system. For example, if we add one slice midway between each of the given slices (the dotted lines in the figure)



then we generate two 0, 1 matrices X and Y satisfying $X \cdot Y = A$ and $Y \cdot X$ is a 0–1 matrix which we denote B.

We will see this operation below when we pass from big handles and the geometric matrix G to little handles and a 0–1 matrix B.

First we note the

PROPOSITION 1.1. *Let G be a non-negative $n \times n$ integral matrix and let B be the 0–1 matrix associated to G. Then there is a surjective linear map $Y: Z^m \to Z^n$ such that $YB = GY$ and $B/\ker Y = 0$. In particular G and B have the same non-zero eigenvalues.*

Proof. Let Y be the matrix above. Then $Y(XY) = (YX)Y$ shows $YB = GY$. Clearly $B = XY$ is zero on kernel Y. Also Y is clearly onto.

To isotope any f to an element of H we will carry through an argument which is an extension of Smale's procedure for isotoping a diffeomorphism to an Ω-stable diffeomorphism [24]. Given a handle decomposition we will call the discs $p \times D_i^{m-j}$ transverse discs and the discs $D_i^j \times q$ core discs.

Definition 1.2. We will say that a handle decomposition H of M is fitted iff any core disc which intersects a core disc of lower dimension contains it. Given H, a diffeomorphism f of M will be called fitted if f(core disc) contains any core disc it intersects for all core discs.

Now we prove two key propositions needed for Theorem 1.1.

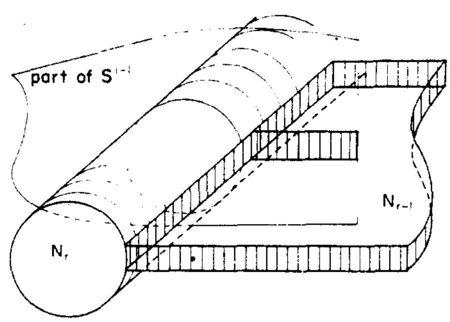
PROPOSITION 1.2. *Let H be any handle decomposition. Then the attaching maps for H can be isotoped so that H becomes fitted.*

PROPOSITION 1.3. *If H is a fitted handle decomposition and f preserves H then f can be isotoped (preserving H) so that f becomes fitted relative to H.*

Proof of 1.2. Assume $H = \bigcup M_i$ is fitted on M_{i-1}. To fit H on M_i we have to isotope the attaching maps of $S^{i-1} \times D$ (we don't write the unimportant superscripts) to fit well with the induced geometrical structure on ∂M_{i-1}. What is this structure? Well, inductively ∂M_{i-1} is divided into closed $(n-1)$ manifolds $\bigcup_r N_r$ which meet along their boundaries which are manifolds with corners. Here $0 \leq r \leq i-1$.

Each N_r is the product of D_r and a "core" manifold possibly with boundary, the boundary possibly with corners. The complexity of N_r increases as r decreases. Thus for r maximal N_r is a union of spheres product D^r. For r one less N_r is a manifold with smooth boundary product D^r, and so on.

Now consider the attaching map of $S^{i-1} \times D$. We think of the disc factor as a small normal disc to the central sphere. Now beginning with $r = i-1$ we isotope S^{i-1} so that it is transversal to the core of N_{i-1}. Then we further



isotope $S^i \times D$ so that $S^i \times D \cap N_{i-1}$ is made up entirely of the transverse $(i-1)$ discs of N_{i-1}.

Now S^i is then also transverse on the ∂N_{i-2} with the boundary core of the N_{i-2} because the transverse discs of N_{i-2} fit nicely into those of N_{i-1}. So we can move S^i in the interior of N_{i-2} to make it completely transverse to the core of N_{i-2}. Then we further move $S^i \times D$ to make $S^i \times D \cap N_{i-2}$ consist entirely of transverse discs.

We proceed inductively down the line to make the entire attaching map of $S^i \times D$ fit well with our geometrical structure on ∂M_{i-1}. In this way we make M_i fitted.

By induction over i we deform a given handle decomposition to a fitted one.

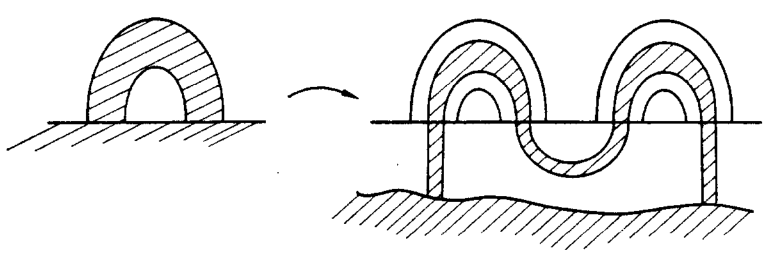
Proof of 1.3. Now consider a diffeomorphism $f\colon M \to M$ so that $fM_i \subset$ interior M_i for each i. We can move f by an isotopy preserving the condition $fM_i \subset$ interior M_i so that f becomes fitted, i.e. if f(core disc) intersects any core disc then f(core disc) contains the core disc. To do this we repeat the argument above and isotope f so that $f(D^i \times D)$ moves (relative $S^{i-1} \times D$) into good position relative to the geometrical structure provided by the handle decomposition. The argument here is easier because the pieces are now familiar handles and are easier to visualize.

Remark. If f belonged to T_H then it follows easily from the proof that H can be isotoped to a fitted decomposition and the f isotoped to a fitted map without changing the geometric intersection matrices.

Proof of Theorem 1.1. In our isotopy of f into a fitted map we may just as well make f uniformly expanding in the core discs and uniformly contracting in the transverse discs (see figure). Thus the set K_i has a hyperbolic structure. We have the advantage here that the local unstable manifolds of the K_i are the core discs, while the local stable manifolds are the transverse discs. Since the global unstable manifolds are the unions of the forward iterates of the local unstable manifolds we have done more than verify the strong transversality hypothesis to apply Robbin's theorem to see that f is now structurally stable. By making f fitted we have essentially constructed the tubular families of Palis and Smale [13].

All that's left for the proof of the theorem is to identify the K_i and Ω_i. We haven't verified that the periodic points are dense in Ω_i. This follows from Newhouse [11], but also from the direct analysis of $f|K_i$ and below.

On the $\overline{M_i - M_{i-1}}$ level our diffeomorphism looks something like



that is the image of a handle h_i inside another slice all the way through expanding on the core discs and contracting on the transverse discs. Denote the i-handles by h_1, h_2, etc. and the geometric intersection $(g_{jk}{}^i)$. The number of components of $f(h_j) \cap h_k$ is exactly $g_{jk}{}^i$. Call the connected components of the images of $f|h_j$ in h_k, $e_{kj_1}, \ldots$. The image of each e_{kj_p} goes through e_{lm_q} once or not at all. We can record this behaviour with a 0–1 matrix B.

B is exactly the 0–1 matrix associated to G_i. Let $S = \{e_{kl}\}$. Given $x \in K_i$ we map x into $\prod_{n \in Z} S$ by $\phi(x) = \{a_i\}_{i \in Z}$ where a_n is that e_{kl} such that $f^n(x) \in e_{kl}$. It follows from the fact that f expands along the core discs and f^{-1} expands along the transverse discs that ϕ mapping K_i into the full shift Σ is injective and continuous. By definition it follows that ϕ maps K_i into Σ_B. And from the fact that the nested intersection of non-empty compact sets is non-empty it follows that ϕ maps K_i onto Σ_B. By construction ϕ is a conjugacy.† Finally one can check by the construction and the definition of non-wandering points that ϕ maps Ω_i onto the non-wandering set of σ: $\Sigma_B \to \Sigma_B$. Alternatively, one could observe that this condition is always satisfied because the periodic points are dense in each non-wandering set and ϕ is a conjugacy. *Q.E.D.*

Given $f \in H$ we may calculate the number of periodic points of f of period m, $N_m(f)$ via the geometric intersection matrices G_i.

PROPOSITION 1.5. *Let $f \in H$. $N_m(f|K_i) = \text{trace}(G_i^m)$.*

Proof. By Bowen and Lanford [6], $N_m(\sigma\colon \Sigma_B \to \Sigma_B) = \text{trace } B^m$. But by Proposition 1.1 the eigenvalues of B are the same as the eigenvalues of G_i taken with multiplicities. As ϕ is a conjugacy $N_m(f|K_i) = \text{trace } B^m = \text{trace } G_i^m$.

Let A be an $n \times n$ matrix. We say that A is pseudo-unipotent if all the eigenvalues of A are roots of unity or 0.

PROPOSITION 1.6. *Let $f \in H$. Then $N_m(f) = \sum_i \text{trace } G_i^m$.*

PROPOSITION 1.7. *Let $f \in H$. Then f is Morse–Smale iff G_i is pseudo-unipotent for all i.*

Proof. Σ trace G_i^m is bounded iff f is Morse-Smale and consequently by a theorem of algebra every eigenvalue is 0 or a root of unity.

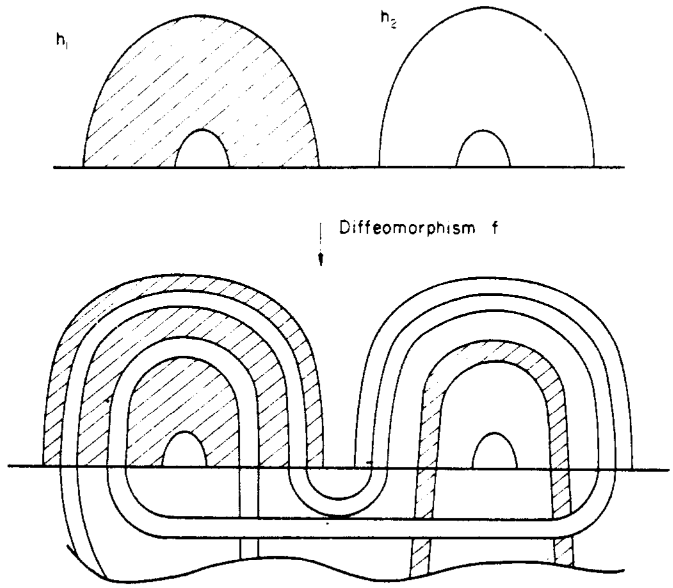
We can in fact compute a sort of asymptotic Lefshetz inequality for diffeomorphisms in H.

PROPOSITION 1.8. *Let $f \in H$. Then* $\limsup \frac{1}{n} \log N_n(f) \geq \max|\log|\lambda||$ *where the max is taken over all eigenvalues of f_*: $H_*(M, Q) \to H_*(M, Q)$.*

Proof. By Proposition 1.6 $\limsup \frac{1}{m} \log N_m(f) = \log \gamma$ where $\gamma = \max|\lambda|$ and the max is taken over all eigenvalues of the G_i. Now recall that any eigenvalue of f_* is an eigenvalue of A_i and $|a_{kj}{}^i| \leq g_{kj}{}^i$. By theorems about non-negative matrices see Gantmacher [7], the maximum absolute value of an eigenvalue of A_i is less than or equal to the maximum absolute value of those for G_i. This proves the proposition without the outside absolute value signs. The fact that we could put the outside absolute value signs in was pointed out to us by Bowen. The reason is that if $f \in H$, f^{-1} is also an element of H. The A_i, G_i for f^{-1} are the inverses of the A_i, G_i for f but $N_m(f^{-1}) = N_m(f)$.

† See more detailed discussion in the example below.

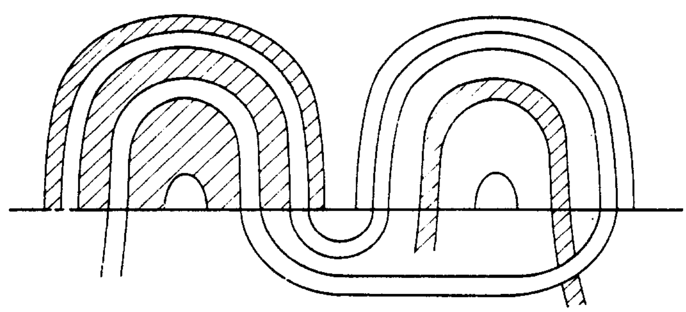
Examples. Suppose we deform our diffeomorphism so that it looks like



The geometric intersection matrix for f in terms of the h_1, h_2 basis is

$$\begin{array}{cc} h_1 & \multirow{2}{*}{$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$} \\ h_2 & \end{array}.$$

The algebraic intersection matrix is also $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$. If we had drawn the picture differently



The geometric matrix would be unchanged but the algebraic matrix is now

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.^{\dagger}$$

† If we use non-abelian algebra in the case of one handles then we don't lose this geometry (see next section).

In either case we have four little handles e_{11}, e_{12}, e_{21}, e_{22} constructed from the components of $f(h_i) \cap h_j$. The geometric intersection matrix for f applied to these handles is

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

This is the 0–1 matrix associated to $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ by the algebraic discussion above.

Now the non-wandering set for f in $h_1 \cup h_2$ is certainly contained in the infinite intersection

$$K = \bigcap_{n=-\infty}^{\infty} f^n(h_1 \cup h_2)$$

which is equal to

$$\bigcap_{n=-\infty}^{\infty} f^n(e_{11} \cup e_{12} \cup e_{21} \cup e_{22}).$$

We can label a point x in K by a bi-infinite sequence of symbols from the four element set e_{11}, e_{12}, e_{21}, e_{22}.

$$x \rightarrow \ldots a_{-2}\, a_{-1}\, a_0\, a_1\, a_2 \ldots$$

where a_0 tells us in which little handle x lies, a_1 tells us where $f(x)$ lies, a_{-1} determines where $f^{-1}(x)$ lies, and so on.

This correspondence determines an equivariant homeomorphism between the invariant set K and the subset of bi-infinite sequences (with the shift operation) determine by the 0–1 intersection matrix for f above.

Recall these are the sequences where for each consecutive pair $a_n a_{n+1}$ the $a_n a_{n+1}$ entry of the matrix is 1 (thinking of the four symbols as labeled 1, 2, 3, 4).

In this specific case any symbol can follow any other eventually. (The ith state can eventually achieve the jth state for any i and j.) This irreducibility of the 0–1 matrix implies the periodic points for this subshift are dense (this is easy to prove). It follows that K is equal to the non-wandering set of f intersect the handles $h_1 \cup h_2$, so this part of Ω is a Cantor set and f/Ω is isomorphic to the shift.

If the 0–1 matrix had not been irreducible we could have permuted the basis elements to achieve a block upper triangular form with irreducible blocks on the diagonal. Then Ω breaks into pieces corresponding to the diagonal blocks and K is larger than if there are upper triangular entries. For example, if the matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ these are three types of admissible symbol sequences

$$\ldots a\, a\, a\, a \ldots$$
$$\ldots a\, a\, a\, b\, b\, b \ldots$$
$$\ldots b\, b\, b\, b\, b \ldots.$$

Ω consists of two points and K contains countably many other points "connecting" then via f. This situation occurs in the torus example in the Morse–Smale section.

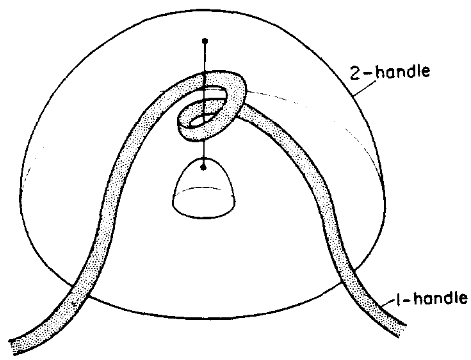
The general case is a mixture of these two examples.

§2. GEOMETRICAL CONSTRUCTIONS AND CHAIN MAPPINGS

Now we consider a diffeomorphism f and a handle decomposition $M = UM_i$. As stated above we can isotope f so that f preserves the decomposition,

$$f(M_i) \subset \text{interior } M_i.$$

Let us consider this general position construction more carefully. The idea is illustrated by the figure



By general position the 1-handle can be shifted slightly† to miss the transverse disc D^{m-j} of a j-handle if $j > 1$. Then the 1-handle can be pulled down below the j-handle by a radial isotopy in the j-handle. This process is essentially unique up to isotopy until the 2-handles are reached. Then we can create an arbitrary linking number of the 1-handle with the transverse disc of the 2-handle before performing the radial isotopy.

In this way we can obtain various handle-preserving isotopes of f. A little thought shows the possible induced chain maps.

$$C_i = H_i(M_i, M_{i-1}) \xrightarrow{f_i} H_i(M_i, M_{i-1})$$

fill out the chain homotopy class. The ambiguity of these linking numbers corresponds precisely to chain homotopies

$$C_i \xrightarrow{D_i} C_{i+1}.$$

So we have the

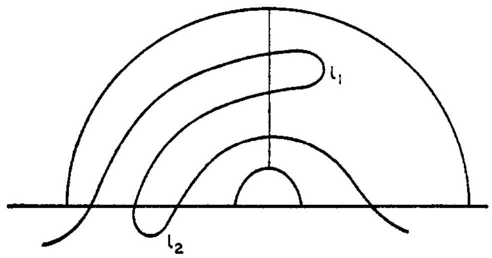
PROPOSITION 2.1. Given a diffeomorphism f and a handle decomposition $M = U_i M_i$ we can isotope f to preserve the decomposition and realize any chain map in the chain homotopy class. ‡

† After shrinking it transversally to a neighborhood of its core.

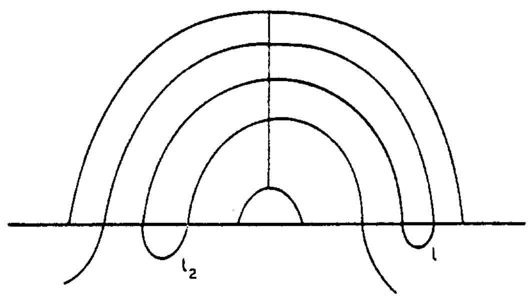
‡ With the natural geometric restriction in degree 0 and n.

Note that the C^0 length of the isotopy is less than a constant times the mesh size of the handle decomposition. The C^1 length of the isotopy is more subtle. It involves the reciprocals of the distances of the handles to the transverses discs after general position, the created linking numbers etc.

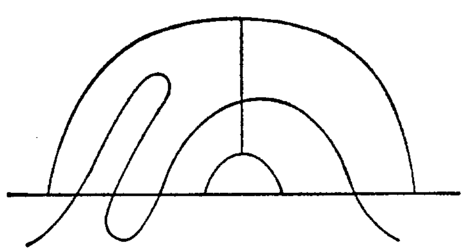
Now we can assume that after a further small perturbation f(core disc of i-handle) is transversal to the transverse discs of the various i-handles.



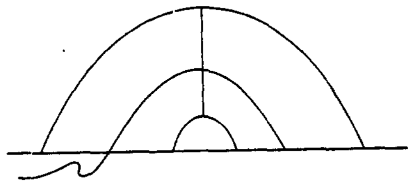
Now we can apply the process described in §1 to isotope f further to achieve structural stability. This involves



stretching the small loop l_1. We might however have first removed the small loop l_1 by a preliminary isotopy:



and then the structurally stable picture is much simpler:



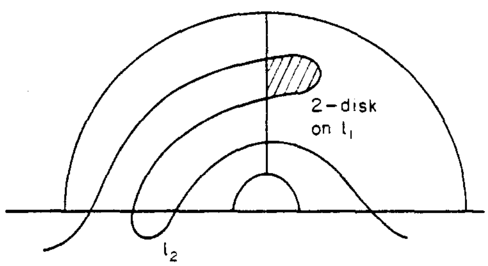
In general we can try to make the geometric intersections

$$f(\text{core } D_j{}^i) \cap \text{transverse } D_k^{m-i}$$

as close as possible to the algebraic intersections which determine the integral chain map

$$H_i(M_i, M_{i-1}) \to H_i(M_i, M_{i-1}).$$

The idea is to cancel opposite pairs of intersection points such as those associated to the loop l_1 by spanning l_1 by an embedded 2-disc which is disjoint from $fD_j{}^i$ and D_k^{m-i}.



Then we slide $fD_j{}^i$ over this 2-disc to remove these two geometric intersections. This argument can be employed quite successfully in simply connected manifolds to prove

PROPOSITION 2.2. *If* $\dim M \geq 5$, $\pi_1 M = e$ *and the handle decomposition has no* 1-*handles or* $(n-1)$ *handles we can further isotope* f *to make the geometric intersections agree up to sian with the algebraic intersections.*

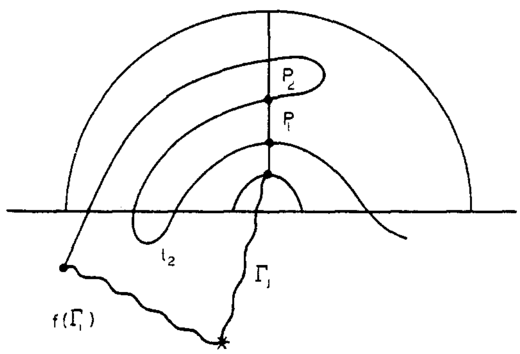
The Whitney device [see 10] with the 2-disc is used in a straightforward way as indicated above. For the 2-handles and $(n-2)$ handles one has only to note that the appropriate subspaces of M are simply connected so first a singular 2-disc can be found, then a non-singular one by general position.

With a view to further applications of this discussion we will briefly discuss the analog of Proposition 2.2 for non-simply connected manifolds.

First of all it is appropriate to work with "path based cells" rather than the ordinary cells generating $C_i = H_i(M_i, M_{i-1})$ in the non-simply connected case. Let $*$ be a base point in the interior of some zero handle, choose for each i-handle a path Γ from the i-handle to $*$. These path based cells give a geometric basis for the chains on the universal cover over the group ring of π_1.

Let us now try to put f into the simplest geometrical form compatible with the algebra. Suppose first f has been deformed to preserve the handle decomposition. Then the 0 and 1 handles together form a generalized solid torus T. We can suppose f fixes our base point $*$ in T and we think of T as obtained by attaching 1-handles $x_1, \ldots, x_n$ to a ball containing the fixed point $*$. f on T is determined up to homotopy by giving a word in $x_1, \ldots, x_n$ for each 1-handle x_j. If the dimension of M is at least four we can isotope f in T so that f(jth handle) intersects the transverse discs of $x_1, \ldots, x_n$ according to the jth word. So geometric intersections correspond to algebraic intersections described by a homomorphism of the free group on n-generators.

Now for the higher handles. We suppose each $f(D^k)$ is transversal to each of the transverse discs D^t of the k-handles. Each point of intersection $f(D_i^k) \cap D_j^t$ determines an element in $\pi_1(M)$ using $f(\Gamma_i)$ and Γ_j.



Note that in the figure the conjugacy class of the loop l_2 in $\pi_1 M$ is determined by the elements associated to P_1 and P_2. In this way we get a refined chain mapping corresponding to f

$$f_\#: \tilde{C}_k \to \tilde{C}_k$$

where $\tilde{C}_k$ is the free module on the k-handles over the group ring of $\pi_1 M$.

Then we can employ the Whitney device with the 2-disc to make the geometric intersections agree with these algebraic intersections. This works smoothly for $3 \leq k \leq n-3$. For $k = 2$ the Whitney argument works if we calculate in $\pi_1 T$ rather than $\pi_1 M$.† The cases $k = n-2, n-1, n$ are treated as for $k = 2, 1, 0$ using f^{-1} in place of f.

All this requires dimension M to be at least five.‡

Now once f is deformed to preserve the handle decomposition these algebraic data are all determined so the geometric form achieved above for f is best possible.

One can change this algebraic data by a "chain homotopy" by deforming f to a handle preserving map in a different way. Again all algebraic possibilities can be achieved by isotopy.

† More work is required here in the realm of model making for self-maps of 2-complexes.

‡ One needs dimension six to realize arbitrary handle decompositions compatible with the algebra.

§3. MORSE-SMALE DIFFEOMORPHISMS

Now we will apply the discussions of §1 and §2 to characterize which components of Diff M contain Morse–Smale diffeomorphisms.

We will describe Smales' picture of these structurally stable diffeomorphisms with only finitely many periodic points. This picture readily shows that such a diffeomorphism is in a certain sense a "virtual permutation" in homotopy theory.† In particular we find that f satisfies the following homology property: there is a finite length chain complex C of finitely generated free Abelian groups $\ldots \to C_{i+1} \xrightarrow{\partial} C_i \to \ldots$ with a chain automorphism

$$F = \{C_i \xrightarrow{F_i} C_i\}$$

so that

(i) The chain matrices F_i are virtual permutation matrices, that is of the form

$$\begin{pmatrix} P_1 & * & * & * \\ 0 & P_2 & * & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & P_k \end{pmatrix}.$$

where the P_i are signed permutation matrices.

(ii) The pair (C, F) is equivalent to a geometric chain map induced by the Morse–Smale diffeomorphism f.

This condition may be formulated in a more invariant way using the homology class of the graph of f in $H_m(M \times M)$.

Because $\{C_i\}$ is chain equivalent to the chains on M we have a duality chain equivalence $\{C_i\} \overset{\psi}{\rightleftarrows} \{C^{m-i}\}$ well defined up to homotopy. ($C^{m-i} = \mathrm{Hom}(C_{m-i}, Z)$.)

Each F_i determines an element in $\mathrm{Hom}(C_i, C_i)$ which is naturally isomorphic to $C^i \otimes C_i$, which is mapped by ψ into $C_{m-i} \otimes C_i$.

So $F = \{F_i\}$ determines a m-chain in the complex $\{C_i\} \otimes \{C_i\}$. This m-chain is a cycle in the homology class of (graph f).

The homology condition described above for Morse–Smale diffeomorphisms can be reformulated as follows: *the homology class of graph f can be constructed by a virtual permutation of a chain complex for the manifold.*

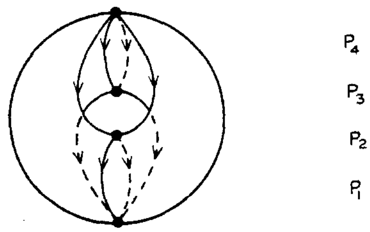
Then we will show that for simply connected manifolds of dimension at least 6 this purely algebraic homology property is a sufficient condition for a component of Diff M to contain a Morse–Smale diffeomorphism.

Finally we will briefly discuss a connection between this algebraic condition and the ideal classes of cyclotomic fields and also outline a characterization of Morse–Smale components of Diff M for non-simply connected manifolds.

Now for the Smale picture of f.

A good example to begin with is the ε-time map of the gradient flow on the torus associated to the usual height function. However to obtain genericity and structural stability we lean the torus back slightly and then use the height function.

† We are indebted to P. Deligne for the terminology "virtual permutation".



So this diffeomorphism has four fixed points. At each fixed point the derivative has eigenvalues not on the unit circle. There are complementary submanifolds emanating from each fixed point

$$W_p^u \text{ and } W_p^s$$

corresponding to the eigenvalues inside and outside of the unit circle. These are invariant by f and extend globally to embedded Euclidean spaces in the manifold. On the unstable manifold W_p^u f is expanding and on the stable manifold W_p^s f is contracting.

In our torus example, the unstable and stable manifold pairs are

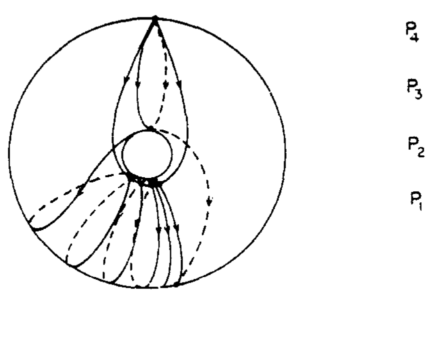
$$\begin{aligned} P_4 &: \ (R^2, 0) \\ P_3 &: \ (R^1, R^1) \\ P_2 &: \ (R^1, R^1) \\ P_1 &: \ (0, R^2). \end{aligned}$$

Now we can make a more interesting example which is not isotopic to the identity. In a small armband on the left side of the torus construct a diffeomorphism g by rotating each level circle by an angle which varies from 0 to 2π as we sweep the circle down through the armband. Extend g to be the identity outside the armband.

Note that g preserves our height function. Now push g slightly down the gradient lines of the height function. We obtain a Morse–Smale diffeomorphism with four periodic points (all fixed) whose map on first homology is represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The stable and unstable manifolds are as before but now since g is twisting the left side of the torus the unstable manifold of P_3 is forced to intersect the stable manifold of P_2. Since this point is moving out on $WP_3{}^u$ and in on $WP_2{}^s$, the forward orbit of this point gives us infinitely many points of intersection of $WP_2{}^s$ and $WP_3{}^u$.

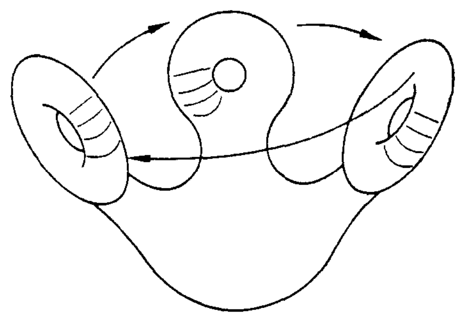


All these intersections are transversal and this diffeomorphism is then Morse–Smale by definition: there are only finitely many periodic points which are hyperbolic (no eigenvalues of the derivative on the unit circle) and whose nonstable and stable manifolds intersect transversally if at all.

For these diffeomorphisms Smale constructed a partial ordering of the periodic points

$$p < q \text{ iff } W_q^u \cap W_p^s \neq \phi \quad \text{or} \quad \dim W_p^u < \dim W_q^u.$$

Thus we can picture a general Morse–Smale diffeomorphism as a combination of these pictures and permutations, for example



More generally, the underlying manifold of a Morse–Smale diffeomorphism is the union of the unstable manifolds of the various periodic points. These come in layers determined by the partial ordering on the periodic points. The unstable manifolds in each layer are permuted by the diffeomorphism. If we allow ourselves to think of these unstable manifolds as cells in some computation of the homology of the manifold, we will find chain matrices of the permutation form indicated above.

THEOREM. *A Morse–Smale diffeomorphism determines a "virtual permutation" of the homology in the following sense: The diffeomorphism can be represented on the integral chain level by virtual permutation matrices.*

COROLLARY. *The eigenvalues on homology for a Morse–Smale diffeomorphism are roots of unity.*

This theorem can be proved in two ways from the picture. One way is to apply Cech cohomology to the filtration and build such a chain complex for the Cech cohomology of M (which is of course the usual cohomology).

A second way which gives more information for non-simply connected spaces is to look at the complementary filtration of M by open manifolds—the union of the unstable manifolds coming down from the top. These open manifolds are invariant by the diffeomorphism and have the homotopy type of finite cell complexes. It is a pleasant geometrical exercise to check that one cell is added in homotopy theory each time an unstable manifold is added to the dense growing open manifold coming down from the top.

This second argument shows that a Morse–Smale diffeomorphism is a "virtual permutation in homotopy theory". By this we mean

(i) There is a space made up of finitely many cells attached successively by dimension however each dimension may come in several layers.

(ii) There is a cellular map of this space which permutes the cells of each layer.

(iii) The cell complex and the map are homotopy equivalent† to the manifold and its Morse–Smale diffeomorphism.

Obviously, these virtual permutations of cell complexes have a special form on the fundamental group and on the chains of the universal cover. We will discuss this below.

Now we turn to the converse discussion of constructing Morse–Smale diffeomorphism in isotopy classes which induce "virtual permutations".

First consider the case when M is a simply connected manifold of dimension at least 6. Let (f) denote a component of Diff M which is a virtual permutation on the homology. That is, the homology class of the graph of f can be described on some chain complex for M by virtual permutation matrices.

First, we have an algebraic proposition that we can assume this complex has the form

$$0 \to Z \to 0 \to C_{m-2} \to \dots \to C_2 \to 0 \to Z \to 0$$

since $H_1 M = H_{m-1} M = 0$. This is proved in Appendix A.

Now choose a handle decomposition of M whose geometric chain complex is isomorphic to this one. By Proposition 2.1 we can isotope the diffeomorphism to a handle preserving diffeomorphism whose geometric chain map is the virtual permutation.

Since there are no 1-handles or $(m-1)$ handles we can further isotope f so that the geometric intersections are up to sign equal to the geometric intersections (Proposition 2.2).

Then we can apply the stretching and fitting of §1 to deform f to a structurally stable diffeomorphism. The structure of the non-wandering set Ω is determined by the geometric intersection matrices. But these are still virtual permutation matrices with eigenvalues on the unit circle. So Ω is finite and the new f is Morse–Smale.

THEOREM. *If M is simply connected and has dimension at least* 6, *a component of* Diff M *contains a Morse–Smale diffeomorphism iff it determines a virtual permutation of the homology.*

We close this section with some additional remarks about this homology condition and the non-simply connected case.

Let G denote the Grothendieck group generated by chain equivalence classes of finite rank chain complexes with an automorphism and direct sum.

We can consider the subgroup $P \subseteq G$ generated by those where the automorphism is described by virtual permutation matrices and the subgroup $F \subseteq G$ where the induced homology automorphisms are quasi-unipotent (the eigenvalues are on the unit circle).

Of course we have $P \subseteq F \subseteq G$. If we were working over Q instead of Z one can prove $P_Q = F_Q$, but we have it from Swan that P is strictly smaller than F.

† The equivalence is actually a simple homotopy equivalence.

To get some feeling for this consider an automorphism T of a free Abelian group A as an element of G. Now such a transformation can be put in upper triangular form

$$T = \begin{pmatrix} A_1 & * & * \\ 0 & A_2 & \vdots \\ \vdots & & * \\ 0 & \dots & A_r \end{pmatrix}$$

where each A_i has an irreducible characteristic polynomial.

If the eigenvalues of T are on the unit circle and each A_i is equivalent to the companion matrix of the corresponding cyclotomic polynomial it is fairly easy to show $[T] \in G$ lies in $P \subseteq G$. In general, however the Z-equivalence classes of the A_i are in one to one correspondence with the ideals classes of the cyclotomic field generated by the eigenvalues of A_i. Of course, A_i is equivalent over Q to its companion matrix and this explains why $P_Q = F_Q$. Now R. Swan claims that already the ideal classes in the field of 23rd roots of unity determine non-zero elements of F/P. Even the homology map multiplication by 2 on $Z/47$ can not be realized by an element of P [26] and so determines a nontrivial element in F/P.†

There are certain facts about elements in F which follow from our geometrical discussion. First of all if f is a periodic diffeomorphism of some manifold M then f determines an element X_f in F. However, one can also prove X_f lies in P because we can choose a non-degenerate Morse function on $M \xrightarrow{\psi} R$ invariant by $f, f \cdot \psi = \psi$. Then we can push f down the gradient lines of ψ to make it Morse–Smale. Then X_f lies in P. For example if $T \in \mathrm{Gl}(n, Z)$ has finite order, then $\{\Lambda^i T\}$ lies in P. For we can represent T as a periodic diffeomorphism on the n-torus T^n.‡

Arguments from algebraic geometry show X_f lies in P whenever f is a diffeomorphism constructed by monodromy of an algebraic variety. So in the simply connected case f is isotopic to a Morse–Smale diffeomorphism.

Another interesting example concerns matrices with non-negative elements. Suppose T in $\mathrm{Gl}(n, Z)$ is represented by a matrix (t_{ij}) with the following property: the matrix $|T| = (|t_{ij}|)$ has eigenvalues on the unit circle.§ It follows from matrix theory that T has eigenvalues on the unit circle and from our geometrical discussion that $[T]$ lies in P. We merely realize T by a Morse–Smale diffeomorphism inducing T on the 2nd homology of a manifold with boundary obtained by thickening a bouquet of 2-spheres.

For example, a quasi-unipotent transformation in $\mathrm{Gl}(n, Z)$ with non-negative entries determines the zero element of F/P.

Now let us discuss the situation for non-simply connected manifolds. It is clear from the cellular picture described above that a Morse–Smale diffeomorphism has a special form on π_1 and is represented on the (path based) chain level by matrices of the form

† The arguments of [26] have to be slightly expanded to prove this.

‡ This leads one to ask whether any quasi unipotent element in $\mathrm{Gl}(n, Z)$ determines a component of Diff T^n containing a Morse-Smale diffeomorphism.

§ Note the generators of P have this eigenvalue property.

$$\begin{pmatrix} P_1 & \cdots & * & \cdots & * \\ 0 & & P_2 & & \vdots \\ \vdots & & & & \\ 0 & & & \cdots & P_n \end{pmatrix}$$

where each P_i has permutation form but the non-zero elements are $\pm g$ where $g \in \pi_1 \subseteq$ group ring of π_1.

On the other hand, the discussion about algebraic and geometric intersection numbers and the Whitney device pointed out that algebraic operations on chain complexes over the group ring could be imitated on the geometrical level. So we can carry through our constructions leading to a Morse–Smale diffeomorphism in a component if we have enough algebraic information (on π_1 and the path based chain maps). We hope some interested worker will work out a neat formulation of the general case.

APPENDIX A

Let $\{C_i\}$ be a finite length chain complex for H_*M^m where $H_1M = H_{m-1}M = 0$. Then we can in a natural way construct a new chain complex of the form

$$0 \to Z \to 0 \to \tilde{C}_{m-2} \to \ldots \to \tilde{C}_2 \to 0 \to Z \to 0$$

for H_*M^m.

The idea is to keep folding $\{C_i\}$ (slightly modified) at the points $(m-2)$ and (2) until it is concentrated in the range $(2, m-2)$. We will describe the folding at position $(m-2)$.

Consider the portion of $\{C_i\}$

$$0 \to C_n \to \ldots \to C_{m-1} \xrightarrow{\partial} C_{m-2}.$$

We can alter $C_{m+1} \xrightarrow{\partial} C_m$ to kill $H_m M = Z$ by adding a Z to C_{m+1} and redefining ∂. Now we have an exact sequence

$$0 \to C_n \to \ldots \to C_{m+1} \to C_m \to C_{m-1} \to B_{m-2} \to 0$$

where the boundaries B_{m-2} form a direct summand of C_{m-2} since $H_{m-2}(M^m)$ is torsion free by duality. We can apply Hom(, Z) to obtain a new exact sequence

$$0 \to B'_{m-2} \xrightarrow{i} C'_{m-1} \xrightarrow{\partial} \ldots \to C_n' \to 0.$$

Then

$$0 \to Z \to 0 \to C_{m-2} \xrightarrow{\partial''} \begin{matrix} C_{m-3} \\ \oplus \\ C'_{m-1} \end{matrix} \xrightarrow{\partial \oplus \partial} \begin{matrix} C_{m-4} \\ \oplus \\ C_m' \end{matrix} \longrightarrow$$

is a new chain complex for H_*M where ∂'' is defined by writing $C_m - 2 = B_{m-2} \oplus C$, choosing an isomorphism $B_{m-2} \simeq B_{m-2}$, and using i on the B_{m-2} summand and the old ∂ on the C summand. We continue in this way by folding back at 2, then $m-2$ again, and so on until the complex is supported in the range $(2, m-2)$.

This is the algebraic construction required for the Morse–Smale discussion.

§4. THE SIMPLEST DIFFEOMORPHISMS IN AN ISOTOPY CLASS

Dynamical systems is mainly concerned with the orbit structure of a diffeomorphism and of its perturbations. From this point of view the identity diffeomorphism is not a simple diffeomorphism because the orbit structure of its perturbations changes drastically. We have already noted that from this point of view the Morse–Smale diffeomorphisms are the simplest ones known which motivates us to make the following:

Definition. $f \in \mathrm{Diff}^r(M)$ is a simplest diffeomorphism in its isotopy class if:

(1) f is structurally stable, and

(2) f has the smallest topological entropy of any structurally stable diffeomorphism in the isotopy class.

There is no guarantee that an isotopy class of diffeomorphisms has a simplest diffeomorphism. Also if there is one there are infinitely many distinct ones even up to topological or Ω conjugacy.

For an Axiom A diffeomorphism Bowen [2] proved:

THEOREM. $h(f) = \limsup \frac{1}{n} \log N_n(f)$.

So it follows from Proposition 1.8 *that; if $f \in H$ then*

$$(*)\ h(f) \geq \max |\log|\lambda||$$

where the max is taken over all eigenvalues of $f_*: H_*(M, Q) \circlearrowleft$. And the main problem that we shall consider now is the extent to which (*) holds. From [18] and the construction of H we know:

PROPOSITION. *Any $f \in \mathrm{Diff}^r(M)$ is isotopic to an element of H by an arbitrarily small C^0 isotopy. So H is dense in* $\mathrm{Diff}^r(M)$ *with the C^0 topology.*

Combining this with Nitecki [12] yields:

PROPOSITION. (*) *holds for an open and dense subset of* $\mathrm{Diff}^r(M)$ *in the C^0-topology.*

This suggests the obvious:

Problem. Does (*) hold on all of $\mathrm{Diff}^r(M)$?

One of the beginning points of our investigations was the observation that (*) holds for Morse–Smale diffeomorphisms and hence the eigenvalues of f_* were roots of unity. We conjecture that (*) holds for all Axiom A and no cycle diffeomorphisms (see Smale [23] for a definition of these) and that in fact given f which is Axiom A and no cycle there is a $g \in H$ which is isotopic to f with $h(g) \leq h(f)$. The theorems of Bowen [3] and Manning [9] are very suggestive of this, and Bowen [5] has proven that (*) holds for Axiom A no-cycle diffeomorphisms with zero dimensional Ω.

Finally we ask, when can equality be achieved in (*) with an element of H, or with a sequence $f_n \in H$ such that $h(f_n) \to \max|\log|\lambda||$, etc. The theorem on the existence of Morse–Smale diffeomorphisms is the first step.

REFERENCES

1. D. V. ANOSOV: Geodesic flows on closed Riemann manifolds, *Proc. Steklov Inst. Math.* **90** (1967). Am. Math. Soc. translation, Providence, Rhode Island (1969).
2. R. BOWEN: Topological entropy and Axiom A, *Proc. Symp. Pure Math.*, Vol. XIV, *Global Analysis*, p. 23. Am. Math. Soc., Providence, Rhode Island (1970).
3. R. BOWEN: Markov partitions for Axiom A diffeomorphisms, *Am. J. Math.* **92** (1970), 725.
4. R. BOWEN: Entropy for group endomorphisms and homogeneous spaces, *Trans. Am. math. Soc.* **153** (1971), 401.
5. R. BOWEN: Entropy versus homology for certain diffeomorphisms (to appear).
6. R. BOWEN and O. LANFORD III: Zeta functions of restrictions of the shift transformation, *Proc. Symp. Pure Math.* Vol. XIV, *Global Analysis*, p. 43. Am. Math. Soc., Providence, Rhode Island (1970).
7. F. GANTMACHER: *The Theory of Matrices.* GITTL, Moscow (1953). English translation, Vol. 2, Chelsea, N.Y. (1959).
8. M. W. HIRSCH, J. PALIS, C. C. PUGH and M. SHUB: Neighborhoods of hyperbolic sets, *Invent. Math.* **9** (1969), 121.
9. A. MANNING: Axiom A diffeomorphisms have rational zeta functions, *Bull. Lond. math. Soc.* **3** (1971), 215.
10. J. MILNOR: *Lectures on the h Cobordism Theorem.* Princeton University Press (1965).
11. S. NEWHOUSE: On hyperbolic limit sets, *Trans. Am. math. Soc.* (to appear).
12. Z. NITECKI: On semi-stability for diffeomorphisms, *Invent. Math.* **14** (1971), 83.
13. J. PALIS and S. SMALE: Structural Stability Theorems, *Proc. Symp. Pure Math.*, Vol. XIV, *Global Analysis*, p. 223. Am. Math. Soc., Providence, R.I. (1970).
14. M. PEIXOTO: Structural stability on two-dimensional manifolds, *Topology* **1** (1962), 101.
15. J. ROBBIN: On structural stability, *Ann. Math.* **94** (1971), 447.
16. M. SHUB: Stability and genericity for diffeomorphisms, *Proc. Symp. Dynamical Systems, Salvador, 1971* (to appear).
17. M. SHUB: Morse–Smale diffeomorphisms are unipotent on homology, *Proc. Symp. Dynamical Systems, Salvador*, 1971 (to appear).
18. M. SHUB: Structurally stable diffeomorphisms are dense, *Bull. Am. math. Soc.* **78** (1972), 817.
19. M. SHUB and S. SMALE: Beyond hyperbolicity, *Ann. Math.* **96** (1972), 587.
20. S. SMALE: Diffeomorphisms with many periodic points, in *Differential and Combinatorial Topology.* Princeton University Press (1965).
21. S. SMALE: Structurally stable systems are not dense. *Am. J. Math.* **88** (1966), 491.
22. S. SMALE: Differentiable dynamical systems, *Bull. Am. math. Soc.* **73** (1967), 747.
23. S. SMALE: The Ω-stability Theorem, *Proc. Symp. Pure Math.*, Vol. XIV, *Global Analysis*, p. 289. Am. Math. Soc., Providence, Rhode Island (1970).
24. S. SMALE: Stability and isotopy in discrete dynamical systems, *Proc. Symp. Dynamical Systems, Salvador, 1971* (to appear).
25. M. SHUB and D. SULLIVAN: A remark on the Lefshetz fixed point formula for differentiable maps *Topology* **13** (1974), 189.
26. R. G. SWAN: Invariant rational functions and a problem of Steenrod, *Invent. Math.* **7** (1969), 148.
27. R. F. WILLIAMS: Classification of subshifts of finite type. *Annals* (1974).

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