A REMARK ON THE LEFSCHETZ FIXED POINT FORMULA
FOR DIFFERENTIABLE MAPS

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(Received 10 September 1973)

If 0 is an isolated fixed point for the continuous map \( f: U \to \mathbb{R}^m \), where \( U \) is an open subset of \( \mathbb{R}^m \), then the index of \( f \) at 0, \( \sigma_f(0) \), is the local degree of the mapping \( \text{Id}_f \) restricted to an appropriately small open set about 0. If 0 is an isolated fixed point of \( f^n \), then \( \sigma_{f^n}(0) \) is defined for all \( n > 0 \), where \( f^n \) means \( f \) composed with itself \( n \) times restricted to a small neighborhood of 0. We will use a little elementary calculus to show:

**Proposition.** Suppose that \( f: U \to \mathbb{R}^m \) is \( C^1 \) and that 0 is an isolated fixed point of \( f^n \) for all \( n \). Then \( \sigma_{f^n}(0) \) is bounded as a function of \( n \).

The proposition is not true for continuous functions as the mapping of the complex plane \( f(z) = 2z^2/\|z\| \) shows. In fact, for this \( f \), \( \sigma_{f^n}(0) = 2^n \). Our interest in the proposition arose from the Lefschetz fixed point formula as applied to a smooth endomorphism \( f \) of a compact differentiable manifold \( M \). The Lefschetz formula says that the Lefschetz numbers

\[
L(f^n) = \sum (-1)^i \text{tr} \; f_{*i}^n : H_i M \to H_i M
\]

can be computed locally by these fixed point indices,

\[
L(f^n) = \sum_{P \in \text{Fix}(f^n)} \sigma_{f^n}(P),
\]

provided that the fixed points of \( f^n \) are isolated.

**Corollary.** If \( f: M \to M \) is \( C^1 \), and the Lefschetz numbers \( L(f^n) \) are not bounded then the set of periodic points of \( f \) is infinite.

In particular, any \( C^1 \) degree two map of the two sphere, \( S^2 \), has an infinite number of periodic points and hence an infinite non-wandering set [see 1].† The corollary suggests the possibility of getting sharper estimates on the asymptotic growth rate of \( N_n(f) \), the number of fixed points of \( f^n \).

**Problem.** If \( f: M \to M \) is smooth, is

\[
\lim sup \frac{1}{n} \log |L(f^n)| \leq \lim sup \frac{1}{n} \log N_n(f)?
\]

† Note that the one-point compactification of \( f(z) = 2z^2/\|z\| \) is a **continuous** degree two map of \( S^2 \) with only two periodic points.
As remarked in [1] this inequality is rather obviously true for the set of $C^r$ endomorphisms $f$ of $M$ which have the property that all periodic points of $f$ are transversal. Then, of course, $|L(f^n)| \leq N_\omega(f)$.

We now proceed with the proof of the proposition. In all that follows below $f$ is $C^1$ and 0 is an isolated fixed point of $f^n$ for all $n$. The idea is to try to approximate $I - f^n$ by $(I + f + f^2 + \cdots + f^{n-1})(I - f)$ so that if $I + f + f^2 + \cdots + f^{n-1}$ is a local diffeomorphism then degree $(I - f^n) = \pm$ degree $(I - f)$. To make this precise and to do the estimates we work with the derivatives of $f^n$ at 0 which we denote by $Df^n$.

**Lemma 1.** If $\sum_{j=0}^{n-1} Df^j$ is non-singular then $\sigma_f(0) = \pm \sigma_f(0)$.

Before we prove Lemma 1 we will show how it proves the proposition. $\sum_{j=0}^{n-1} Df^j$ is singular precisely when $n = mk$, $k > 1$, and $Df$ has a primitive $k$th root of unity as an eigenvalue. For each integer $n$, let $\lambda$ be the least common multiple of these orders $k$. Then we may apply the proposition to see that $\sigma_f(0) = \pm \sigma_f(0)$. (If $(k_1, k_2, \ldots)$ are the orders of roots of unity in the spectrum of $Df$, then $(k_1, \gcd(k_1, \lambda), \ldots)$ are the orders for $Df^1$. But now $n/\lambda$ is not a multiple of any of these orders greater than 1.)

Since we only need finitely l.c.m.'s $\lambda$ to take care of all the integers $n$, this argument proves the proposition.

A standard fact that we shall use in proving Lemma 1 is:

**Lemma 2.** If $h, k : U \rightarrow \mathbb{R}^n$ are continuous, have 0 as an isolated 0 and $\|h(x) - k(x)\| < \|h(x)\|$ then degree $(h) = \text{degree} (k)$.

**Proof of Lemma 1.** Let $f = Df + \theta_1$ and $f^n = Df^n + \theta_n$.

Then $I - f^n = I - Df^n - \theta_n$,

$$= (I + Df + \cdots + Df^{n-1})(I - Df) - \theta_n$$

$$= (I + Df + \cdots + Df^{n-1})(I - f) + (I + Df + \cdots + Df^{n-1})\theta_1 - \theta_n.$$

We will show by induction that given $(n, \epsilon)$ there is a neighborhood $U_{n, \epsilon}$ of 0 such that

$$\left\| \left( \sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n \right)(x) \right\| < \epsilon \| (I - f)(x) \| \text{ for all } x \in U_{n, \epsilon}.$$

So that if $\sum_{j=0}^{n-1} Df^j$ is non-singular then by Lemma 2,

degree $(I - f^n) = \text{degree} \left( \sum_{j=0}^{n-1} Df^j \right)(I - f) = \pm$ degree $(I - f)$.

To estimate $\sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n$ first observe that $\theta_n = \sum_{j=0}^{n-1} Df^{n-j} \theta_1 f^j$ as can easily be seen by induction. So

$$\sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n = \sum_{j=0}^{n-1} Df^{n-j} \theta_1 - \sum_{j=0}^{n-1} Df^{n-j} \theta_1 f^j$$

$$= \sum_{j=1}^{n-1} Df^{n-j} (\theta_1 - \theta_1 f^j).$$

By the mean value theorem

$$\left\| \left( \sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n \right)(x) \right\| \leq \sum_{j=1}^{n-1} \| Df^{n-j} \| \| D\theta_1 \|_{U_{n, \epsilon}} \| (I - f^j)(x) \|.$$
where \(\|D\theta_1\|_{U_{n,\epsilon}} = \sup_{x \in U_{n,\epsilon}} \|D_x \theta_1\|\). Since \(D_0 \theta_1 = 0\) it clearly suffices to prove inductively that given \(j < n\) there is a neighborhood \(V_j\) of 0 and a \(0 \leq k_j < \infty\) such that
\[
\|(I - f^j)(x)\| \leq k_j \|(I - f)(x)\| \quad \text{for all} \quad x \in V_j.
\]
Since
\[
I - f^j = \left( \sum_{i=0}^{j-1} Df^i \right) (I - f) + \sum_{i=0}^{j-1} Df^i \theta_i - \theta_j,
\]
we can inductively choose \(U_{j,\epsilon}\) so that
\[
\|(I - f^j)(x)\| \leq \sum_{i=0}^{j-1} \|Df^i\| \|(I - f)(x)\| + \epsilon \|(I - f)(x)\|,
\]
and we are done.

REFERENCE


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