The transversality characteristic class and linking cycles in surgery theory

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This paper contains two interlocking results. The origin of the first is the idea of Thom that the rational Pontrjagin classes of a manifold can be defined by transversality. This procedure is a beautiful juxtaposition of algebra and geometry. On the algebraic side is the fact that a rational cohomology class is determined uniquely by its "periods" or values on the rational homology. On the geometric side is the fact that transversality produces enough submanifolds of $M \times R^N$ with trivial normal bundles to generate the rational homology of M.

The geometric properties of these submanifolds are invariants of the manifold M. In fact when the submanifold has dimension $4k$, the signature of its intersection pairing on $2k$-cycles only depends on the homology class that it represents in M. Thom uses these submanifold signatures as the periods and produces a sequence of rational cohomology classes

$$L_1, L_2, \cdots, \quad \text{in } H^{4*}(M; Q) .$$

These Thom characteristic classes are expressed by the famous Hirzebruch polynomials in the Pontrjagin classes

$$L_1 = p_1/3, \ L_2 = (7p_2 - p_1^2)/45, \cdots,$$

when M is smooth. For a more general class of manifolds these classes of Thom can be thought of as a natural consequence of transversality and the existence of the signature cobordism invariant of closed manifolds.

In this paper we make use of the stronger facts that the signature is defined in Z for manifolds with or without boundary, and that it is additive under the operation of pasting manifolds together along components of their boundary.

This last property of the signature imposes significant arithmetic conditions on the Thom characteristic classes $L_1, L_2, \cdots$ of general manifolds. These are naturally expressed for odd primes by the statement that the Poincaré dual of $1 + L_1 + L_2 + \cdots$ is the character of a natural K-theory duality (at odd primes) in the manifold. This result is discussed in [S2] and [S4].

In this paper we will discuss the arithmetic properties of the L-characteristic classes at the prime 2. The discussion is surprisingly strenuous. For motivation and justification, we offer the remark that a complete determination of these arithmetic conditions leads one to a complete list of homeomorphism invariants for simply connected n-manifolds ($n > 4$) beyond those of homotopy type [S3]. The problem of homeomorphism invariants is the origin of the second result of the paper.

In order to fix ideas, let us see where one is lead by trying to define a canonical 2-integral L-characteristic class for topological manifolds. By a 2-integral L-class we mean a sequence of cohomology classes

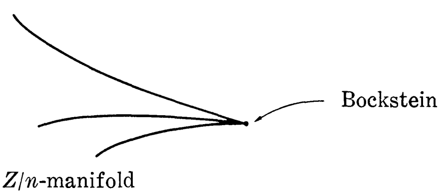
$$l_i \in H^{4i}(M; Z_{(2)}) , \qquad i = 1, 2, 3, \cdots ,$$

where M is the manifold and $Z_{(2)}$ is the ring of rational numbers with odd denominator.

We will discuss our procedure in relation to the construction of Thom. First of all, in Section 2 we describe an algebraic result to the effect that an integral cohomology class is precisely determined by a set of compatible "Q-periods" and "Q/Z-periods" (Theorem 2.1). The "Q-periods" are defined as before for rational homology and the "Q/Z-periods" are defined for Q/Z-homology. The appropriate local form of this result uses ($Q/Z_{(2)} = Z_{2^\infty}$)-periods instead of Q/Z-periods for determining $Z_{(2)}$-cohomology classes.

The Q/Z-homology classes of M are determined by Z/n-homology classes of M. We can try to represent these geometrically by a nice cycle $M \times R^N$. The appropriate geometric object is a "Z/n-manifold" — a space obtained by identifying isomorphic boundary components of a compact manifold in "bunches of n". These identifications are to respect orientations. Z/n-manifolds can be treated in many ways like oriented manifolds.

This discussion occurs in Section 1. For example, Z/n-manifolds have oriented tangent bundles and a signature in Z/n which is a cobordism invariant. A Gauss map can be defined by embedding the Z/n-manifold so that the n-sheets are all tangent at the Bockstein.



The signature is defined by taking the residue class of the integral signature of the corresponding manifold with boundary (modulo n). The additivity property of Novikov implies that this residue class is a cobordism invariant in Z/n for Z/n-manifolds.

Now we can use transversality to represent any Z/n-homology class by an embedded Z/n-manifold in $M \times R^N$ *when n is a power of* 2. This follows from the cobordism work of Thom and its extensions by Wall and Conner-Floyd.

We can say nothing about the normal bundle, however, except that it is smooth. So this geometric representation is highly non-unique. Thus we cannot directly use the signatures of these submanifolds in $Z/2^k$ as the "Z_{2^∞}-periods" of our 2-integral L-class.

To overcome this difficulty we begin a boot-strap operation which first solves the problem in case M is also smooth using the explicit homology properties of the universal grassmannians. This also gives 2-integral L-classes for vector bundles. Then we continue with a construction of the 2-integral L-class in the general topological context by an inductive procedure which produces first l_1, then l_2, and so on. At each stage we construct the Z_{2^∞}-periods for a topological $Z/2^k$-submanifold with smooth normal bundle from its signature and its inductively defined l-classes in conjunction with the smooth L-class of the normal bundle.

The construction of the smooth L-class is given in Section 3. The formulae of Hirzebruch only involve odd denominators and so define 2-integral classes (since the Pontrjagin classes are integral). These classes are not exactly right for our purposes, however. We construct a universal class

$$L_{SO} = 1 + l_1 + l_2 + \cdots$$

which agrees with the inverse Hirzebruch class rationally and reduces mod 2 to the squares of the even Wu classes

$$l_1 \equiv V_2^2,\ l_2 \equiv V_4^2,\ \cdots \pmod 2$$

by a mixed algebraic-geometric argument (Theorem 3.2). This class satisfies a good product formula because v_{2i+1} vanishes in the oriented context.

In Section 4 we develop the algebraic apparatus for the inductive argument indicated above. The result is, roughly speaking, that a cohomology class is determined by a set of periods on cobordism classes of submanifolds which satisfy certain product relations (Theorem 4.1).

The brunt of the mathematics in the paper is concerned in one way or another with these product relations. One difficulty arises because the topological product of two Z/n-manifolds has a codimension 2-singularity, where the transversal is the cone over (n points) join (n points). This singularity can be resolved in a natural way so we have a product operation for Z/n-manifolds. Now, however, it is not geometrically clear that the signature of this product of Z/n-manifolds is the product of the signatures. This is true (Theorem 6.6), but the proof for n even is quite difficult.

Once we have this product formula we are able to prove one of the two main results of this paper. That is we extend the 2-integral L-class to a natural characteristic class for all topological R^n-bundles. That is if $\xi^n \to X$ is such a bundle we have $l_i(\xi) \in H^{4i}(X; Z_{(2)})$. Rationally this class is the inverse of the L-class constructed by Thom. Reduced modulo 2, l_i is $(v_{2i})^2$. The defining property of l_i is that it measures the signatures of transversal intersections of manifolds and $Z/2^k$-manifolds mapping into the Thom space of ζ with the 0-section. This result is proved in Section 7.

This brings us to the surgery theory of the paper. Recall that one of the most interesting arguments in the basic paper of Kervaire and Milnor on exotic spheres showed how to produce by surgery a $4k$-manifold whose boundary was a given partially parallelized[1] $(4k-1)$-manifold. It was implicit in their discussion that the signature of this $4k$-manifold was well-defined modulo 8 by the $(4k-1)$-manifold with partial parallelization.

One of our favorite results in this paper provides an explicit geometric calculation of this mod 8 invariant of $(4k-1)$-manifolds without doing surgery. Let (M^{4k-1}, F) denote the manifold with its partial parallelization F. Let

$$T \otimes T \xrightarrow{l} Q/Z$$

denote the linking form on the torsion cycles of M in dimension $2k-1$. On the odd torsion subgroup of T, l can be algebraically refined to quadratic function $T \xrightarrow{q} Q/Z$, satisfying

(i) $q(x) = q(-x)$

(ii) $q(x+y) = q(x) + q(y) + l(x, y)$.

On the 2-torsion subgroup of T we can represent homology classes by embedded submanifolds with a normal field which is compatible with F (see Section 5). Using these normal fields we define q on the 2-torsion of T.

Then we form the Gaussian sum

$$\sum_{x \in T} e^{2\pi i q(x)} .$$

The argument of this complex number is an eighth root of unity which

[1] On the $2k-1$ skeleton.

determines the signature mod 8 mentioned above (Theorem 5.8).

We also prove an analogous result for a surgery problem

$$L^{4k-1} \xrightarrow[g]{} M^{4k-1}$$

between two $(4k-1)$-manifolds (Theorem 5.8). In this case, the quadratic function is defined on Tg, the torsion subgroup of the homology kernel of g in dimension $2k-1$. The argument of the corresponding Gauss sum now determines the signature mod 8 of a surgery cobordism of g to a homotopy equivalence.

We study this invariant $\Theta(g)$ in $Z/8$ determined by embedded cycles with normal fields in Section 5. The discussion is mostly geometric.

Then we use this *a priori* description to give a formula for the surgery obstruction in Z/n for a normal map between Z/n-manifolds of dimension $4k$,

$$V^{4k} \xrightarrow{f} W^{4k} .$$

This formula involves the $Z/8$ invariant of f along the Bockstein and the signatures of V and W (opened up) in $Z/8n$.[2] Namely,

$$\sigma_f = (1/8)(\text{signature } V + n\Theta(\delta f) - \text{signature } W)$$

in Z/n (Theorem 5.3). Note that the terms of this formula can all be computed by intersecting cycles and homologies in V, W, and δV. No surgery is required.

Then we come to the most interesting geometric step of our discussion. This arises in the problem of giving a formula for the obstruction in a product of two surgery problems. Proposition 6.3 is a local geometric product formula which is quite fun to prove.

The result contains a surprise. It shows for example that the Θ-invariant of a map obtained by forming the cartesian product of a non-bounding orientable 5-manifold (e.g. SU_3/SO_3) with a surgery map in dimension $4k+2$ with non-zero Kervaire invariant is *non-zero*.

So the theory of the Kervaire invariant is subsumed by the theory of these signature invariants along Z/n-manifolds.

The general product formula for surgery problems over Z/n-manifolds $M \xrightarrow{f} N$ and $L \xrightarrow{g} Q$ (Theorem 6.5) reads

$$\sigma(f \otimes g) = i(Q)\sigma(f) + i(N)\sigma(g) + 8\sigma(f)\sigma(g)$$

where i is an absolute invariant of manifolds computed homologically. If L

[2] These surgery obstructions were introduced in 1966 by the second author to obtain the invariants for the Hauptvermutung.

has dimension l, then $i(L)$ is the signature of L in Z/n when $l \equiv 0 \pmod 4$, $i(L)$ is the de Rham invariant of L in $Z/2$ if $l \equiv 1 \pmod 4$, $i(L)$ is the de Rham invariant of the Bockstein of L if $l \equiv 2 \pmod 4$, and $i(L)$ is zero if $l \equiv 3 \pmod 4$.

The formula is the subject of Section 6 where the de Rham invariant is discussed and defined.

This paper concludes with two applications of the product formula. The first allows us to prove the product formula for the signature of Z/n-manifolds, Proposition 6.6. Then we can finish the construction of the 2-integral L-class for topological manifolds. This is done in Section 7 in the more general context of bundles. One corollary is a "Hirzebruch formula" for the signature of a $Z/2^k$-manifold in terms of the characteristic classes of its tangent bundle (Proposition 7.2).

In the final Section 8 we give the second application which concerns the universal spaces for surgery problems G/pl and G/Top. By the same general method used in constructing the transversality characteristic class we construct a 2-integral class $\mathfrak{L} \in H^{4*}(G/Top; Z_{(2)})$. This class figures in a cohomological surgery obstruction formula (Theorem 8.7):

$$\sigma(f) = \langle \mathfrak{L}_f \cdot \mathfrak{L}_M, M \rangle + \langle \delta^*(k_f \cdot W_M), M \rangle .$$

Here $M \xrightarrow{f} G/Top$ classifies the surgery problem f, $\mathfrak{L}_f$ is the restriction of the universal class $\mathfrak{L}$ in G/Top, $\mathfrak{L}_M$ is the transversality characteristic class for M, k_f is the restriction of the universal class constructed in [RS], and W_M is the characteristic class which tabulates the de Rham invariant

$$W_M = (1 + v_2 + v_4 + \cdots) \cdot Sq^1(1 + v_2 + v_4 + \cdots)(M) .$$

The classes $\mathfrak{L}$ and k in the cohomology of G/Top yield a canonical structure theorem for this H-space at the prime 2. We also obtain the analogous structure theorem for G/pl.

Historical note and acknowledgments

The existence of the two cohomology classes constructed in this paper was established by the second author in 1966–1967 by approximately these methods. A non-canonical construction of the G/pl-class was outlined in the Princeton notes [S2]. The existence of the transversality class was independently established in Brumfiel's thesis, MIT (1967), by algebraic calculation.

The new result of this paper is to give these classes canonically. The

geometrical analysis employed to overcome the main difficulty here was inspired by Rourke's immersed cycle idea in [RS].

The surprising twist about the Kervaire obstruction and the signature obstruction came as a by-product of the surgery product formula. This latter result for $Z/2$-manifolds was independently obtained by J. Milgram [M1] using homotopy theoretical methods for the purpose of [BMM]. This method generalizes the algebraic approach of W. Browder [B2] and later E. Brown [B1] to the Kervaire invariant, just as this work generalizes the geometric approach of [RS] to the Kervaire invariant.

In the algebraic discussion concerning the signature mod 8 and linking we owe a debt to J. Milnor who verified the key algebraic result by calculating the irreducible cases.

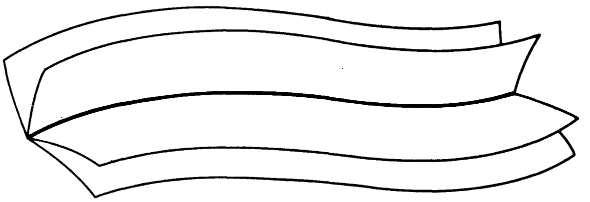
The Gauss sum formula is an elegant form of this result whose proof was supplied by Paul Monsky (see note in Section 5).

This formulation was also found independently by Milgram.

1. Z/k-Manifolds

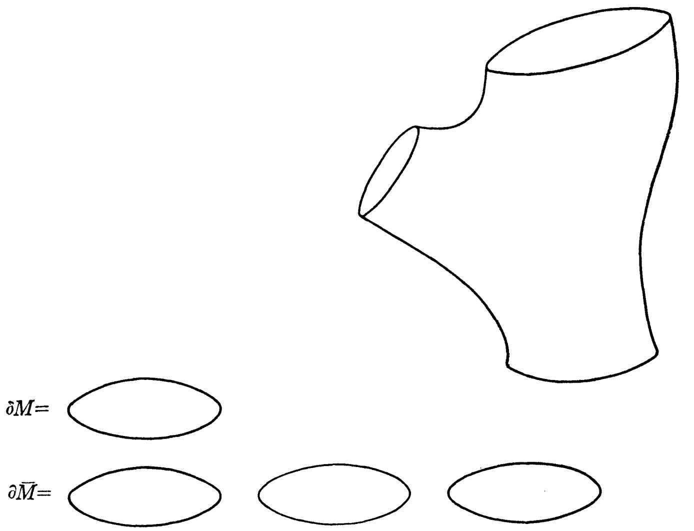
In this section we discuss the techniques of geometric topology which apply to Z/k-manifolds. Many standard theorems and constructions in manifold theory have "mod k" analogues. Z/k-manifolds carry fundamental Z/k-homology classes (though they do not satisfy Poincaré duality with Z/k-coefficients), have signatures in Z, and have orientable tangent bundles. Z/k-bordism groups exist and are related to ordinary bordism by the coefficient sequence. In fact these geometrically defined groups are the "homotopy theoretic bordism with Z/k-coefficients" groups. The bordism theory is also equipped with an anti-commutative associative multiplication defined geometrically.

A Z/k-manifold is a space which is non-singular outside a codimension one submanifold where k-sheets come together. Thus each point of the singularity has a neighborhood isomorphic to (cone on k points) $\times$ (Euclidean space).



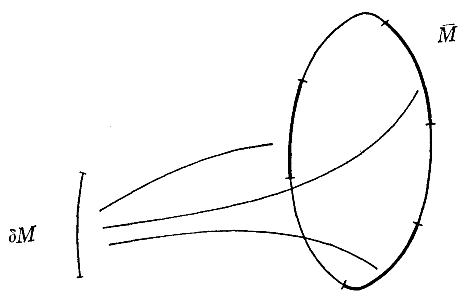
part of a $Z/4$-manifold

More precisely let $\bar{M}^n$ be an oriented[1] n-manifold with boundary together with an identification of $\partial\bar{M}^n$ to k copies of a closed oriented $(n-1)$-manifold, δM.



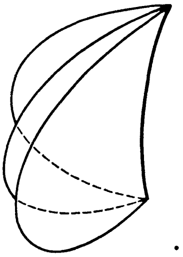
These data determine a (closed) "oriented Z/k-manifold" by collapsing the k copies of δM together. Call the quotient M_k. We will always assume without further mention that the sheets are labeled $1, 2, \cdots, k$ near the Bockstein δM. This is important for gluing Z/k-manifolds together. A closed oriented manifold is thus a $Z/0 = Z$-manifold.

More generally a Z/k-manifold with boundary is determined by (1) an n-manifold with boundary $\bar{M}$, and (2) k disjoint embeddings of a compact $(n-1)$ manifold δM in $\partial\bar{M}$ (everything oriented).



These data upon collapsing yield a Z/k-manifold M, pictured.

[1] We adopt the following convention. If a manifold with boundary is oriented, then ∂M receives an orientation by taking the inward normal as the last vector. With these conventions $\partial(M^m \times N^n) = M^m \times \partial N^n + (-1)^n \partial M^m \times N^n$.



The boundary of the Z/k-manifold is formed from $\partial\bar{M} - \operatorname{int} k(\delta M)$ by collapsing the k copies of $(\partial(\delta M))$ together.

From this description it is clear that closed Z/k-manifolds have fundamental homology classes with Z/k-coefficients, and Z/k-manifolds with boundary have a relative fundamental class. Thus they serve as geometric representatives of Z/k-homology in any space. This is one of the main reasons for studying them here.

A map between Z/k-manifolds is a map of the quotient spaces, which sends the singularity to the singularity, and preserves the (cone on k) structure (with the labeling) near the singularity. Thus embeddings have normal bundles, and we have a theory of transversality.

We turn to the bordism theory of Z/k-manifolds, denoted $\Omega_*(X; Z/k)$. The existence of a fundamental Z/k-homology class provides a "Hurewicz homomorphism" $\Omega_*(X; Z/k) \xrightarrow{h_k} H_*(X; Z/k)$ given by considering a Z/k-manifold in X as a Z/k cycle in X. There is a long exact sequence and a commutative ladder

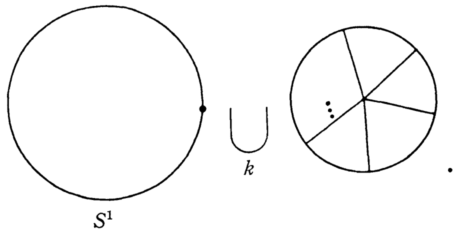
$$\begin{array}{lccccccccc}
(1.0) & \cdots \longrightarrow & \Omega_*(X) & \xrightarrow{k} & \Omega_*(X) & \xrightarrow{r} & \Omega_*(X; Z/k) & \xrightarrow{\delta} & \Omega_{*-1}(X) & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h} & \\
 & \longrightarrow & H_*(X) & \xrightarrow{k} & H_*(X) & \xrightarrow{r} & H_*(X; Z/k) & \longrightarrow & H_{*-1}(X) & \longrightarrow \cdots .
\end{array}$$

Reduction, "r", is obtained by considering a closed manifold as Z/k-manifold with empty Bockstein.

From the long exact sequence, and transversality, we see that $\Omega_*(X; Z/k)$ is naturally isomorphic to $\tilde{\Omega}_{*+1}(X^+ \wedge T_k)$, where T_k is the Moore space, $S^1 \cup_k D^2$. This last group agrees with Anderson's general homotopy theoretic definition of any homology-theory with coefficients. (*Idea of proof*, due to Tom Dieck.)

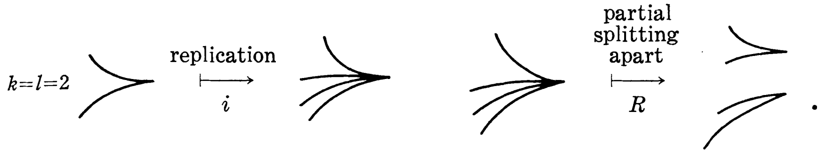
Let $S_k \hookrightarrow T_k$ be

$$\{1\} \cup \operatorname{cone}\left(\left\{e\frac{2\pi i l}{k}\right\}_{l=0,\ldots,k-1}\right) \subset S^1 \cup_{\times k} D^2$$



Then S_k has a normal bundle, everywhere except at 0, where k lines come together. Two stage transversality, first with respect to 0 and then with respect to S_k in T_k, produces a Z/k-manifold from any manifold mapping into T_k.

Corresponding to the maps $i: Z/k \hookrightarrow Z/k\cdot l$ and $Z/k\cdot l \xrightarrow[r]{} Z/k$, we have maps $\Omega_*(Z/k) \xrightarrow[R]{} \Omega_*(Z/k\cdot l)$ and $\Omega_*(Z/k\cdot l) \xrightarrow{r_*} \Omega_*(Z/k)$. Geometrically they are given by



Thus R is formed by gluing together l copies of the Z/k-manifold along the Bockstein and r_* is formed by opening up the $k\cdot l$ sheets into l copies of k-sheets (here we use the numbering). We may glue two Z/k-manifolds with boundary together along a common submanifold of the boundary of each. (For this we need the labelings to know how the sheets fit together.)

Using the maps R, define $\Omega_*(X; Z_{p^\infty}) = \varinjlim \{\Omega_*(X; Z/p^n), R\}$. A representative for a Z_{p^∞} bordism element is a Z/p^n element. It carries a fundamental class in Z_{p^∞} homology. Thus we have a "Hurewicz homomorphism" $\Omega_*(X; Z_{p^\infty}) \to H_*(X; Z_{p^\infty})$.

If $\bar{M}^n$ is a manifold with boundary let the signature of $\bar{M}$, $I(\bar{M})$, in Z be the signature of the intersection pairing $H_{n/2}(\bar{M}; Q) \otimes H_{n/2}(\bar{M}; Q) \to Q$ if $n \equiv 0 \pmod 4$ and zero otherwise.

PROPOSITION 1.1. *If n is even, $I(\bar{M}) \equiv \chi(\bar{M}) + \chi_s(\partial\bar{M}) \pmod 2$, where $\chi(\bar{M})$ is the Euler characteristic of $\bar{M}$ and $\chi_s(\partial\bar{M})$ is the rational Euler semi-characteristic of $\partial\bar{M}$.*

Proof. The intersection pairing $H_{n/2}(\bar{M}; Q) \otimes H_{n/2}(\bar{M}; Q) \to Q$ is given by $x \otimes y \to \langle PD(x), i_*y\rangle$ where $i_*: H_{n/2}(\bar{M}; Q) \to H_{n/2}(\bar{M}, \partial\bar{M}; Q)$. Thus the pairing is singular, and its radical is image $j_*: H_{n/2}(\partial\bar{M}; Q) \to H_{n/2}(\bar{M}; Q)$. Thus $I(\bar{M}) \equiv \dim(\operatorname{coker} j_*) \pmod 2$. Using the long exact sequence

$$0 \longrightarrow \operatorname{coker} j_* \longrightarrow H_{n/2}(\bar{M}, \partial\bar{M}; Q) \longrightarrow H_{n/2-1}(\partial\bar{M}; Q) \longrightarrow \cdots$$
$$\longrightarrow H_1(\bar{M}) \longrightarrow H_1(\bar{M}, \partial\bar{M}) \longrightarrow H_0(\partial\bar{M}) \longrightarrow H_0(\bar{M}) \longrightarrow H_0(\bar{M}, \partial\bar{M})$$

and Poincaré duality, we see that

$$\dim(\operatorname{coker} j_*) \equiv \textstyle\sum_{i=0}^{n} \dim\big(H_i(\bar{M}; Q)\big) + \sum_{i=0}^{n/2-1} \dim\big(H_i(\partial\bar{M}; Q)\big) \pmod 2 .$$

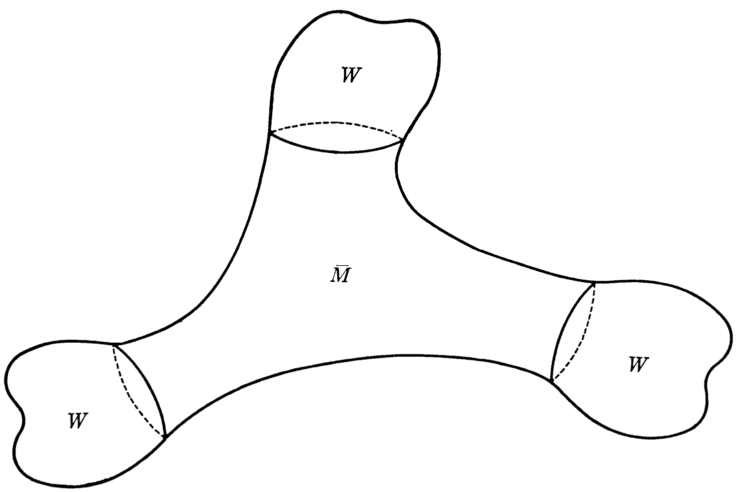
This first term is $\chi(\bar{M})$, and the second is $\chi_s(\partial\bar{M})$.

If $M_k^n = (\bar{M}^n, \delta M^n, \varphi)$ is a Z/k-manifold, then its signature, $\bar{I}(M_k)$, is the signature of $\bar{M}$. It is an integer. We denote its residue modulo k by $I(M_k)$.

THEOREM 1.2. (*Novikov's additivity theorem, see* [N].) *If M_1^n and M_2^n are oriented manifolds and $\varphi : A^{n-1} \to B^{n-1}$ is an orientation reversing isomorphism where $A^{n-1} \hookrightarrow \partial M_1$ and $B^{n-1} \hookrightarrow \partial M_2$ are submanifolds with empty boundary then $I(M_1 \cup_\varphi M_2) = I(M_1) + (M_2)$.*

PROPOSITION 1.3. *If M_k^n bounds a Z/k-manifold then* $I(\bar{M}_k^n) \equiv 0 \pmod k$.

Proof. The proof follows easily from 1.2 and the following picture:



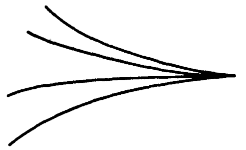
This shows that the signature reduced modulo k is a Z/k-bordism invariant. From the definitions, we see that the following diagrams commute

$$\begin{array}{ccc} \Omega_*(Z/k) & \xrightarrow{I} & Z/k \\ \downarrow R & & \downarrow i \\ \Omega_*(Z/k\cdot l) & \xrightarrow{I} & Z/k\cdot l \end{array} \qquad \begin{array}{ccc} \Omega_*(Z/k\cdot l) & \xrightarrow{I} & Z/k\cdot l \\ \downarrow r_* & & \downarrow r \\ \Omega_*(Z/k) & \xrightarrow{I} & Z/k \end{array} .$$

Thus I induces a map $\Omega_*(Z_{p^\infty}) \to Z_{p^\infty}$.

We now give a geometric definition of the tangent bundle of a differentiable Z/k-manifold. Embed $\delta M^k \hookrightarrow R^N$ for some large N. Embed $c(k)$ in

R^2 so that all lines are tangent at the cone point, i.e., in a cusp-like manner.



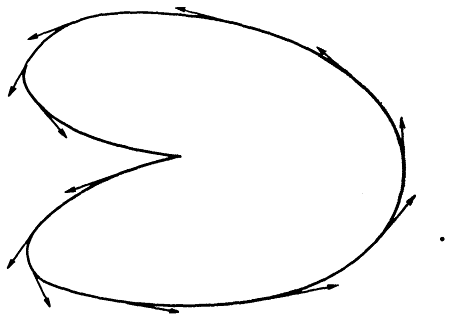
The product of these two embeddings, gives one of $\delta M \times c(k) \hookrightarrow R^{N+2}$. Its boundary is $\delta M \times k = \partial \bar{M}$. Extend this embedding to one of $\bar{M}$. Together these embeddings define one of $M_k \hookrightarrow R^{N+2}$. The tangent bundle is the set of tangent planes with the natural topology. It is well-defined since all planes over the Bockstein agree. We denote this bundle τ_M. Taking perpendicular planes gives ν_M.

Another description of the tangent bundle which works in the pl case is to take $\tau_{\bar{M}}$ and use $d(\varphi_{i,j})$ to identify $\tau_{\bar{M}} | \delta M_i$, with $\tau_{\bar{M}} | \delta M_j$, $\varphi_{i,j}$ being the isomorphism between the i^{th} and j^{th} copy of δM. For any stable characteristic class α and for any closed Z/k-manifold M, we may form the characteristic number $\langle \alpha_{\nu_M}, [M] \rangle \in Z/k$. Since $\tau_{W_k} | \partial W_k = \tau_{\partial W_k} \oplus \varepsilon^1$ for any Z/k-manifold with boundary, we see that the above characteristic numbers are Z/k-cobordism invariants.

If M_2 is a $Z/2$-manifold with non-empty Bockstein, then M has the structure of a non-orientable manifold. Let T_M be its non-orientable tangent bundle. $w_1(T_M) : M \rightarrow S^1$ is dual to δM. We have the equation

$$T_M \oplus w_1^*(\eta) = \tau_M \oplus \varepsilon^1 \tag{1.4}$$

where $\eta \xrightarrow{\pi} S^1$ is the non-trivial line bundle. To see this equation note that $T_{M_2} | (M_2 - \delta M) = \tau_{M_2} | (M_2 - \delta M)$. Thus the difference element is in $\delta M \times I / (\delta M \times \partial I)$. Since the "difference" is constant along δM, the "difference" comes from some bundle over S^1 via w_1. From the following picture one sees that this bundle must be non-trivial:



The product of two Z/k-manifolds $M_k^m \times N_k^n$ is not immediately a Z/k-manifold. It has the correct structure everywhere except along $\delta M \times \delta N$. We can cut out a neighborhood $\delta M \times c(k) \times \delta N \times c(k)$ and replace it by $\delta M \times \delta N \times W_k^2$, where W_k^2 is a Z/k-manifold with boundary $k*k = \big(c(k) \times k\big) \cup \big(-k \times c(k)\big)$. Such a manifold exists since $\Omega_1(Z/k) = 0$. The product is well-defined up to bordism since $\Omega_2(Z/k) = 0$. Thus it forms a multiplication $\Omega_*(Z/k) \otimes \Omega_*(Z/k) \to \Omega_*(Z/k)$. We denote the product of M_k and N_k by $M_k \otimes N_k$ if both M and N are Z/k-manifolds with non-empty Bockstein. If one of them is a closed manifold, the product is usual cartesian product and we denote it by $M \times N$.

The multiplication is associative since $(M_k \otimes N_k) \otimes P_k$ and $M_k \otimes (N_k \otimes P_k)$ agree everywhere except near $\delta M \times \delta N \times \delta P$, where the difference is in the transverse direction. Thus the obstruction to associativity is an element in $\Omega_3(Z/k) = 0$.

There is a natural map $M_k \otimes N_k \xrightarrow{\rho} M_k \times N_k$ which is the identity off of a neighborhood of $\delta M \times \delta N$. Near this submanifold it is

$$\delta M \times \delta N \times W_k \xrightarrow{1 \times 1 \times \pi} \delta M \times \delta N \times \big(c(k) \times c(k)\big) ,$$

where π is some map $W_k \to c(k) \times c(k)$ extending the Id on $k*k$. Also $\rho_*[M_k^m \otimes N_k^n] = [M_k^m] \otimes [N_k^n]$, thus ρ is an orientation preserving homeomorphism almost everywhere.

PROPOSITION 1.5. *$\tau_{M_k \otimes N_k}$ is stably equivalent to $\rho^*(\tau_{M_k} \times \tau_{N_k}) \oplus \pi^*\zeta$, where $\zeta \to W_k^2/\{\partial W\}$ is a vector bundle and $\pi : M \otimes N \to \delta M \times \delta N \times W_k/\{\partial\} \to W_k/\{\partial\}$ is the natural map.*

Proof. Over the complement of $\delta M \times \delta N \times W_k$, $\tau_{M\otimes N}$ and $\tau_M \times \tau_N$ are equal and ρ is the identity. Thus we have a natural difference element defined by an isomorphism outside a neighborhood of $\delta M \times \delta N$. Since the isomorphism is constant along $\delta M \times \delta N$, the proposition follows.

Note. If k is odd, then the bundle ζ is 0, but if k is even it is non-zero. Fortunately, this error term is only a psychological barrier. It sends no shock waves through our calculations.

2. Cohomology classes and homomorphisms

In this section we will prove that an integral cohomology class in a complex of "finite type" is just a pair of compatible homomorphisms $H_*(X; Q) \to Q$ and $H_*(X; Q/Z) \to Q/Z$. More precisely we have the following:

THEOREM 2.1. *If X is a* CW *complex with finite skeletons then $H^i(X; Z)$ is naturally isomorphic to the group $\mathcal{C}(X)$, of commutative diagrams*

$$\mathcal{C}(X) = \left\{ \begin{array}{ccc} H_i(X; Q) & \longrightarrow & Q \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ H_i(X; Q/Z) & \longrightarrow & Q/Z \end{array} \right\} .$$

The natural isomorphism is given by evaluation.

There is an analogous local result.

THEOREM 2.2. *If X is a* CW *complex with finite skeletons, $H^i(X; Z_{(p)})$ is naturally isomorphic to the group $\mathcal{C}_{(p)}(X)$, of commutative diagrams*

$$\mathcal{C}_{(p)}(X) = \left\{ \begin{array}{ccc} H_i(X; Q) & \longrightarrow & Q \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ H_i(X; Z_{p^\infty}) & \longrightarrow & Z_{p^\infty} \end{array} \right\} .$$

The isomorphism is given by evaluation.

Theorem 2.2 should be viewed as the localization of Theorem 2.1 at p. In fact, it is proved by tensoring 2.1 with $Z_{(p)}$. Theorem 2.1 is a reformulation of Pontrjagin duality. The theorem is proved by constructing an isomorphism between $\mathcal{C}(X)$ and $\mathrm{Hom}_c\big(H^i(X; S^1), S^1\big)$. By considering homology with real coefficients, R, we see that a homomorphism $H_i(X; Q/Z) \to Q/Z$ occurring in a diagram of $\mathcal{C}(X)$ determines a continuous homomorphism $H_i(X; S^1) \to S^1$. Conversely, given a continuous homomorphism $H_i(X; S^1) \to S^1$ we may restrict to the torsion subgroups and obtain a homomorphism $H_i(X; Q/Z) \to Q/Z$. We may also lift the continuous homomorphisms to the universal covers to obtain a homomorphism $H_i(X; R) \to R$ which preserves the integral lattice. This in turn upon tensoring with Q gives a homomorphism $H_i(X; Q) \to Q$.

Once we have the isomorphism between $\mathcal{C}(X)$ and $\mathrm{Hom}_c\big(H_i(X; S^1), S^1\big)$ we complete the following diagram:

$$\begin{tikzcd}[column sep=small] & \left\{ \begin{array}{ccc} H_i(X; Q) & \longrightarrow & Q \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ H_i(X; Q/Z) & \longrightarrow & Q/Z \end{array} \right\} \arrow[d, "\cong"] \\ H^i(X; Z) \arrow[ur, dashed, "\mathrm{Eval}"] \arrow[dr, dashed, "\mathrm{Eval}"'] & \\ & \mathrm{Hom}_c\big(H_i(X; S^1), S^1\big) . \end{tikzcd}$$

Pontrjagin duality tells us that the lower evaluation homomorphism is an isomorphism, see [P].

Before beginning the proof proper, we recall some definitions and facts.

Definition 2.1.

$$Z_{(p)} = \{a/b \in Q \mid (b, p) = 1\}$$

$$Z_{(p)} = \varinjlim_{(n,p)=1} (Z \xrightarrow{n} Z)$$

$$Z_{p^\infty} = \varinjlim \{Z/p^n, \text{inclusion}\}\ .$$

There is a natural isomorphism $Q/Z_{(p)} \xrightarrow[\pi]{\cong} Z_{p^\infty}$. It is defined by $a/bp^n \mapsto [a]\cdot[b]^{-1} \in Z/p^n \hookrightarrow Z_{p^\infty}$. This map $Q \to Z_{p^\infty}$ is easily seen to have kernel $Z_{(p)}$. $Q/Z \otimes Z_{(p)} \cong Z_{p^\infty}$, and $Q/Z \hookrightarrow S^1$ is the subgroup of torsion elements.

Theorem 2.1 $\Rightarrow$ *Theorem* 2.2. Since $Z_{(p)}$ is a direct limit and homology commutes with direct limits, examining the appropriate chain complexes shows

$$H^i(X, Z_{(p)}) \cong H^i(X, Z) \otimes Z_{(p)} \qquad \text{for } X \text{ of finite type}$$

and

$$H_i(X, Z_{p^\infty}) \cong H_i(X; Q/Z \otimes Z_{(p)}) = H_i(X; Q/Z) \otimes Z_{(p)} \qquad \text{for all } X\ .$$

Let $\mathcal{C}_{(p)}$ equal the group of commutative diagrams

$$\left\{\begin{tikzcd} H_i(X; Q) \arrow[r] \arrow[d, "\pi_*"] & Q \arrow[d, "\pi"] \\ H_i(X; Z_{p^\infty}) \arrow[r] & Z_{p^\infty} \end{tikzcd}\right\}.$$

Then we have the following commutative diagram

$$\begin{tikzcd} & \mathcal{C} \otimes Z_{(p)} \arrow[dd, "\eta"] \\ H^i(X; Z_{(p)}) \arrow[ur, "\cong", "\mathrm{ev}"'] \arrow[dr, "\mathrm{ev}"'] & \\ & \mathcal{C}_{(p)} \end{tikzcd}$$

where

$$\left(\begin{tikzcd} H_i(X; Q) \arrow[r, "\alpha_0"] \arrow[d] & Q \arrow[d] \\ H_i(X; Q/Z) \arrow[r, "\alpha"] & Q/Z \end{tikzcd}\right) \otimes a/b \xrightarrow{\eta} \left(\begin{tikzcd} H_i(X; Q) \arrow[r, "a/b\cdot\alpha_0"] \arrow[d] & Q \arrow[d] \\ H_i(X; Q/Z) \otimes Z_{(p)} \arrow[r, "\alpha\otimes a/b"] & Q/Z \otimes Z_{(p)} \end{tikzcd}\right).$$

The upper isomorphism is given by 2.1.

If $\eta((\alpha_0, \alpha) \otimes a/b)$ is 0, then α_0 is 0 and $\alpha \otimes a/b$ is 0. Since $(b, p) = 1$, this implies $H_i(X; Q/Z) \xrightarrow{a\cdot\alpha} Q/Z \xrightarrow{\pi} Z_{p^\infty}$ is 0. Since X is a complex of finite type and α agrees with 0 on the image of rational homology, some large multiple $\lambda\cdot\alpha$ is 0. The previous statement says λ may be chosen prime to

p. Thus

$$(\alpha_0, \alpha) \otimes a/b = (\alpha_0, \alpha)\cdot a\lambda \otimes 1/\lambda b = (0, 0) \otimes 1/\lambda b = 0 .$$

This shows η is 1–1.

Given (β_0, β) in $\mathcal{C}_{(p)}$ we can choose λ prime to p so that $\lambda\beta_0$ preserves the integral homology. Then $\lambda\beta_0$ determines a partial homomorphism $H_i(X; Q/Z) \xrightarrow{\alpha} Q/Z$, that is on the image of rational homology. This partial homomorphism is compatible with the partial homomorphism defined by $\lambda\beta$ on the image of $H_i(X; Z_{p^\infty})$ (where we regard $Z_{p^\infty} \subset Q/Z$). The subgroup generated by these images is a direct summand so we can construct a complete homomorphism $H_i(X; Q/Z) \to Q/Z$ compatible with $(\lambda\beta_0, \lambda\beta)$. Thus η is onto since λ is prime to p.

Thus $2.1 \Rightarrow 2.2$.

Proof of 2.1. *Step* 1. *Construction of the evaluation homomorphism,* $H^i(X; Z) \xrightarrow{\text{ev}} \mathcal{C}(X)$.

The natural isomorphisms $Q \otimes Z \xrightarrow{\cong} Q$ and $Q/Z \otimes Z \xrightarrow{\cong} Q/Z$ induce evaluation pairings $H^i(X; Z) \otimes H_i(X; G) \to G$ for $G = Q$ or Q/Z. We define *Eval*: $H^i(X; Z) \to \mathcal{C}(X)$ by

$$x \xrightarrow{\text{Eval}} \begin{array}{ccc} H_i(X; Q) & \xrightarrow{\langle x, \ \rangle} & Q \\ \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi} \\ H_i(X; Q/Z) & \xrightarrow{\langle x, \ \rangle} & Q/Z . \end{array}$$

The diagram clearly commutes, and Eval is a natural homomorphism.

According to the outline in the beginning of the section we need only construct a natural homomorphism $A_X : \mathcal{C}(X) \to \mathrm{Hom}_c\big(H_i(X; S^1), S^1\big)$ so that

(i) A_X is a monomorphism.

(ii) The following diagram commutes:

$$\begin{tikzcd} & \mathcal{C}(X) \arrow[d, "A_X"] \\ H^i(X; Z) \arrow[ur, "\text{Eval}"] \arrow[r, "{\langle x, \ \rangle}"'] & \mathrm{Hom}_c\big(H_i(X; S^1), S^1\big) . \end{tikzcd}$$

Step 2. *The construction of* A_X.

Associated to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & Q & \longrightarrow & Q/Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z & \longrightarrow & R & \longrightarrow & S^1 & \longrightarrow & 0 \end{array}$$

there is a commutative ladder of long exact sequences

$$\begin{tikzcd}
\arrow[r] & H_i(X;Z) \arrow[r] \arrow[d, equal] & H_i(X;Q) \arrow[r] \arrow[d, "\text{mono}"] & H_i(X;Q/Z) \arrow[r] \arrow[d, "i_*"] & H_{i-1}(X;Z) \arrow[r] \arrow[d, equal] & {} \\
\arrow[r] & H_i(X;Z) \arrow[r] & H_i(X;R) \arrow[r, "p_*"] & H_i(X;S^1) \arrow[r] & H_{i-1}(X;Z) \arrow[r] & {} .
\end{tikzcd}$$

From this diagram one easily proves that

(a) i_* is 1-1

(b) $\ker p_* = \text{image}\,(H_i(X;Z) \to H_i(X;R))$

(c) $(\text{image}\, i_*) + (\text{image}\, p_*) = H_i(X;S^1)$, and

(d) $(\text{image}\, i_*) \cap (\text{image}\, p_*) = \text{image}\,(H_i(X;Q) \to H_i(X;S^1))$.

From these four statements it follows easily that given $h_1 : H_i(X;R) \to R$ and $h_2 : H_i(X;Q/Z) \to Q/Z$ satisfying

(i) $h_1\,(\text{image}\, H_i(X;Z)) \hookrightarrow Z$

(ii) the composition $H_i(X;Q) \xrightarrow{\pi_*} H_i(X;Q/Z) \xrightarrow{h_2} Q/Z \overset{i}{\hookrightarrow} S^1$ equals $H_i(X;Q) \hookrightarrow H_i(X;R) \xrightarrow{h_1} R \xrightarrow{p} S^1$,

then h_1 and h_2 determine a unique map $H_i(X;S^1) \xrightarrow{h} S^1$. Since $p_* : H_i(X;R) \longrightarrow H_i(X;S^1)$ is the universal cover of a neighborhood of the identity, h will be continuous if and only if h_1 is. We now construct the homomorphism A_X.

Given a commutative diagram

$$\begin{tikzcd}
H_i(X;Q) \arrow[r, "\alpha_0"] \arrow[d, "\pi_*"] & Q \arrow[d, "\pi"] \\
H_i(X;Q/Z) \arrow[r, "\alpha"] & Q/Z ,
\end{tikzcd}$$

form $\alpha_0 \otimes 1_R : H_i(X;R) \to R$. This map is continuous since it is R linear. It sends $(\text{image}\, H_i(X;Z))$ into Z, since α_0 does.

$$p \circ (\alpha_0 \otimes 1_R) \,|\, H_i(X;Q) = p \circ \alpha : H_i(X;Q) \longrightarrow Q \longrightarrow S^1 .$$

This agrees with

$$i \circ \alpha \circ \pi_* : H_i(X;Q) \longrightarrow H_i(X;Q/Z) \longrightarrow Q/Z \hookrightarrow S^1 .$$

Thus $(\alpha_0 \otimes 1_R, \alpha)$ determines a continuous homomorphism $H_i(X;S^1) \xrightarrow{\bar\alpha} S^1$. Clearly $\bar\alpha \,|\, H_i(X;Q/Z) = \alpha$. If we define $A_X(\alpha_0, \alpha) = \bar\alpha$ this determines a homomorphism $\mathcal{C}(X) \to \operatorname{Hom}_c(H_i(X;S^1), S^1)$.

To show that A_X is a monomorphism, suppose $\bar\alpha = 0$. Then $\alpha = 0$ and $\alpha_0 : H_i(X;Q) \to Q$ must lie in the subgroup $Z \hookrightarrow Q$. But the only map $H_i(X;Q) \to Z$ is 0. Thus A_X is 1-1.

Clearly if $\alpha_0 = \langle x, \ \rangle$ on rational homology and $\alpha = \langle x, \ \rangle$ on Q/Z homology, then $A_X(\alpha_0, \alpha) = \langle x, \ \rangle$ on S^1 homology. This completes the proof of Step 2 and Theorem 2.1.

3. The index theorem at the prime 2

In this section we will discuss a natural characteristic class for orientable vector bundles. It will assign to $E \xrightarrow{\pi} B$, $\mathfrak{L}_E = 1 + l_1 + l_2 + \cdots$ with $l_i \in H^{4i}(B; Z_{(2)})$. The class is an invariant of the geometrical operation of intersecting proper submanifolds of E with B (or of intersecting bordism elements of $(D(E), S(E))$ with B where $D(E)$ and $S(E)$ are respectively the disk and sphere bundle of E).

$\mathfrak{L}_E \otimes 1_Q$ in $H^{4*}(B; Q)$ is the inverse of the Hirzebruch polynomial in the rational Pontrjagin classes of E. It measures the signature of oriented intersections with B.

$\mathfrak{L}_E$ reduced mod 2 in $H^{4*}(B; Z/2)$ is the square of the Wu class of E and measures the signatures of $Z/2$-manifold intersections with B.

$$\begin{tikzcd}[column sep=small] & \mathfrak{L}_E \in H^{4*}(B; Z_{(2)}) \arrow[dl] \arrow[dr] & \\ \begin{array}{l}\text{Square}\\ \text{of Wu}\\ \text{class}\end{array} \in H^{4*}(B; Z/2) & & \begin{array}{c}\text{inverse}\\ \text{Hirzebruch}\\ \text{class}\end{array} \in H^{4*}(B; Q)\ . \end{tikzcd}$$

The signature of a manifold and transversality extend to the topological case. (The lack of four dimensional transversality will cause no trouble.) This will allow us to extend the class $\mathfrak{L}_E$ to a natural class for all topological bundles. This is done in Section 7. We will see that this extension plays the role in topological theory which in the smooth theory is played by the integral Pontrjagin classes and the mod 2 Stiefel-Whitney classes

$$(1 + p_1 + p_2 + \cdots, 1 + w_2^2 + w_4^2 + \cdots)\ .$$

The following is a list of the germane properties of $\mathfrak{L}_E$. The purpose of this section is to construct the class and prove all the following properties.

(i) $\mathfrak{L}_E \in H^{4*}(B; Z_{(2)})$ is a natural characteristic class for orientable vector bundles.

(ii) $\mathfrak{L}_E \otimes 1_Q \in H^{4*}(B; Q)$ satisfies $(1 + l_1 + l_2 \cdots)\cdot(1 + L_1 + L_2 + \cdots) = 1$ where $L(E) = 1 + L_1 + L_2 \cdots$ is the Hirzebruch class of E defined by the famous polynomials in the Pontrjagin classes [M2],

$$\begin{aligned} L_1 &= p_1/3 \\ L_2 &= (7p_2 - p_1^2)/45 \\ &\ \ \vdots \qquad\quad \vdots\ . \end{aligned}$$

(iii) $\mathfrak{L}_E \in H^{4*}(B; Z/2)$ satisfies $(1 + l_1 + l_2 \cdots) = (1 + v_2^2 + v_4^2 + \cdots)$, where $V = 1 + v_1 + v_2 + \cdots$ is the Wu class defined by the Wu relation $(1 + Sq^1 + Sq^2 +$

$\cdots)(1+v_1+v_2+\cdots)=(1+w_1+w_2+\cdots)^{-1}$, [M2]. The first few polynomials are

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 + w_1^2 \\ v_3 &= w_1 w_2 \\ v_4 &= w_2^2 + w_4 + w_1 w_3 + w_1^4 \,. \end{aligned}$$

It is easy to see that the odd Wu classes are zero when E is orientable ($w_1 E = 0$).

(iv) If $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ are vector bundles, then $E \times E' \xrightarrow{\pi\times\pi'} B\times B'$ is a vector bundle and $\mathfrak{L}_{E\times E'} = \mathfrak{L}_E \otimes \mathfrak{L}_{E'}$ in $H^{4*}(B \times B'; Z_{(2)})$.

(v) If $M \xrightarrow{f} E^+$ is a map of an oriented manifold into E^+, the Thom space of E, which is transverse regular to B with inverse image V, then an orientation of E, $U_E \in H^{\dim E}(E^+)$, induces an orientation of V and the signature of $V = I(V) = \langle 1/L(\nu_M)\cdot f^*(U_E \cdot \mathfrak{L}_E), [M]\rangle$ where ν_M is the stable normal bundle of M.

(vi) If the manifold in (v) is a $Z/2$-manifold, then V is a $Z/2$-manifold and

$$\text{signature } (V) \text{ reduced mod } 2 = \langle V_M^2 \cdot f^*(U_E \cdot \mathfrak{L}_E), [M]\rangle$$

where V_M is $V(\nu_M)$.

We will see that properties (i), (ii), and (iii) characterize $\mathfrak{L}_E$. Property (iv) is a fortuitous corollary of the vanishing of the odd Wu classes. Properties (v) and (vi) should be regarded as the defining equations of $\mathfrak{L}_E$. It is these properties that we will use later to extend the characteristic class to all topological R^n-bundles in Section 7.

In this section we also give an application to Z/k-manifolds for k a power of 2. If ν_M denotes the stable normal bundle of M then we have signature $(M) = \langle \mathfrak{L}_{\nu_M}, [M]\rangle$ in Z/k. Finally, we note that $\mathfrak{L}_E$ is independent of the orientation of E and is a stable invariant, thus it has a canonical extension to non-orientable bundles.

We will construct our class in the universal example, B_{SO}. For this we will use two diagrams.

Diagram 1.

$$\begin{array}{ccc} \Omega_* & \xrightarrow{\nu} & H_*(B_{SO}) \\ {\scriptstyle\text{signature}}\big\downarrow & & \big\downarrow{\scriptstyle\langle 1/L,\ \rangle} \\ Z & \xhookrightarrow{i} & Q\,. \end{array}$$

ν is determined by classifying the stable normal bundle. The commutativity of Diagram 1 is equivalent to $\langle 1/L(\nu_M), [M]\rangle = I(M)$, the Hirzebruch index theorem.

Diagram 2.

$$\begin{array}{ccc} \Omega_* & \longrightarrow & H_*(B_{SO}) \\ {\scriptstyle\text{signature}}\downarrow & & \downarrow{\scriptstyle V^2} \\ Z & \xrightarrow{\text{reduction}} & Z/2\,. \end{array}$$

Diagram 2 commutes since $\langle V^2(\nu_M), [M]\rangle = \chi(M)_2 = (\text{signature } M)_2$.

It follows from the work of Thom and Wall [W1] that $\nu \otimes Z_{(2)}$ is onto $H_*(B_{SO}; Z_{(2)})/(\text{Torsion})$. Thus we can combine Diagrams 1 and 2 to form

Diagram 3.

$$\begin{array}{ccc} H_{4*}(B_{SO}; Z_{(2)}) & \xrightarrow{\text{“}\langle 1/L,\ \rangle\text{”}} & Z_{(2)} \\ \downarrow & & \downarrow{\scriptstyle\text{reduction}} \\ H_{4*}(B_{SO}; Z/2) & \xrightarrow[\langle V^2,\ \rangle]{} & Z/2 \end{array}$$

which commutes.

Since $\langle 1/L,\ \rangle$ takes values in $Z_{(2)}$ on the image of $\nu \otimes Z_{(2)}$ by Diagram 1 and since $\nu \otimes Z_{(2)}$ is onto, modulo torsion, "$\langle 1/L,\ \rangle$" in Diagram 3 takes values in $Z_{(2)}$ on all of $H_{4*}(B_{SO}; Z_{(2)})$. To see that the diagram commutes, use the fact that $Sq^1(V^2) = 0$ and that $H^*(B_{SO}; Z)$ has torsion only of order 2. Thus V^2 is the reduction of an integral class and vanishes when evaluated on the image of integral torsion. $H_{4*}(B_{SO}; Z_{(2)})$ is generated by T, the torsion subgroup, and $\Omega = \text{image}\,(\nu \otimes Z_{(2)})$. "$\langle 1/L,\ \rangle$" and $\langle V^2,\ \rangle$ agree on Ω by Diagrams 1 and 2. They both vanish on T since $Z_{(2)}$ is torsion free and V^2 vanishes on the image of integral torsion (being the reduction of an integral class).

Finally we can state

THEOREM 3.1. *$1/L$ and V^2 combine to define a map, $\mathfrak{L}_{(2)}$, which makes the following diagram commute*

$$\begin{array}{ccc} H_{4*}(B_{SO}; Q) & \xrightarrow{1/L} & Q \\ \downarrow{\scriptstyle\pi_*} & & \downarrow{\scriptstyle\pi} \\ H_{4*}(B_{SO}; Z_{2^\infty}) & \xrightarrow{\mathfrak{L}_{(2)}} & Z_{2^\infty}\,. \end{array}$$

Proof. Let $i\colon Z/2 \hookrightarrow Z_{2^\infty}$ and $\pi\colon Q \to Z_{2^\infty}$ be the natural maps. Then any $x \in H_{4*}(B_{SO}; Z_{2^\infty})$ may be written (not uniquely) as $\pi_*(y) + i_*(z)$ with $y \in H_{4*}(B_{SO}; Q)$ and $z \in H_{4*}(B_{SO}; Z/2)$. This follows easily from the Bockstein long exact sequences

$$\begin{array}{ccccccc} \longrightarrow H_{4*}(B_{SO}; Q) & \xrightarrow{\pi_*} & H_{4*}(B_{SO}; Z_{2^\infty}) & \xrightarrow{\beta} & H_{4*-1}(B_{SO}; Z_{(2)}) & \longrightarrow \\ \uparrow{\scriptstyle(\frac{1}{2})_*} & & \uparrow{\scriptstyle i_*} & & \| & \\ \xrightarrow{\times 2} H_{4*}(B_{SO}; Z_{(2)}) & \xrightarrow{r_*} & H_{4*}(B_{SO}; Z/2) & \xrightarrow{\beta_2} & H_{4*-1}(B_{SO}; Z_{(2)}) & \longrightarrow \end{array}$$

and the fact that $\text{im}\,\beta \subset \text{im}\,\beta_2$, i.e., that all torsion is of order 2.

Define

$$\mathcal{L}_{(2)}(\pi_*(y) + i_*(z)) = \pi(\langle 1/L, y\rangle) + i(\langle V^2, z\rangle) .$$

To show that $\mathcal{L}_{(2)}$ is well-defined, we must show that if $\pi_*(y) = i_*(z)$, then $\pi(\langle 1/L, y\rangle) = i(\langle V^2, z\rangle)$. If $\pi_*(y) = i_*(z)$, use the above ladder to construct $\bar{z}$ with $r_*\bar{z} = z$ and $\pi_*\cdot(\frac{1}{2})_*\bar{z} = \pi_*(y)$. Then $\pi\langle 1/L, y\rangle = i(\langle 1/L, \bar{z}\rangle_2)$. $\langle V^2, z\rangle = \langle V^2, r_*(\bar{z})\rangle = \langle 1/L, \bar{z}\rangle_2$. This last equation uses Diagram 3. Thus $\pi(\langle 1/L, y\rangle) = i\langle V^2, z\rangle$. By definition of $\mathcal{L}_{(2)}$ the diagram commutes.

COROLLARY 3.2. *The above diagram determines a unique* $\mathcal{L} \in H^{4*}(B_{SO}; Z_{(2)})$.

$$\mathcal{L} \otimes 1_Q = 1/L ,$$

and $\mathcal{L}$ *reduced modulo* $2 = V^2$. *Since* B_{SO} *classifies vector bundles this is equivalent to a natural cohomology class* $\mathcal{L}_E$ *associated to the vector bundle* $E \xrightarrow{\pi} B$. *Thus we have a class* $\mathcal{L}_E$ *satisfying properties* (i), (ii), *and* (iii).

Proof. The work of Section 2 provides the cohomology class using the previous theorem. $\mathcal{L} \otimes 1_Q$ and $1/L$ have the same evaluation homomorphism on $H_{4*}(B_{SO}; Q)$; thus they agree. $\mathcal{L}$ reduced modulo 2 and V^2 have the same evaluation homomorphism on $H_{4*}(B_{SO}; Z/2)$, and thus they agree. (This uses the fact that Q and $Z/2$ are fields.)

Before proving the multiplicative property of $\mathcal{L}$ we need a proposition.

PROPOSITION. $V_{2k+1} = 0$ *in* $H^{2k+1}(B_{SO}; Z/2)$.

Proof. One may make a direct calculational argument. We give a different argument using Wu's relations: If N^n is a closed n manifold and $V_{2k+1} = V_{2k+1}(\nu_N)$, then

$$V_{2k+1} \cup x = Sq^{2k+1}x \quad \text{for} \quad x \in H^{n-2k-1}(N; Z/2) .$$

If N is orientable, $Sq^1\colon H^{n-1}(N^n; Z/2) \to H^n(N^n; Z/2)$ is 0. Thus $V_{2k+1} \cup x = Sq^{2k+1}x = Sq^1Sq^{2k}x = 0$ for all $x \in H^{n-2k-1}(N; Z/2)$. Poincaré duality implies that $V_{2k+1} = 0$. Approximate B_{SO} by oriented manifolds whose normal bundles correspond to the universal bundle to show $V_{2k+1} = 0$ in $H^{2k+1}(B_{SO}; Z/2)$.

We now continue by showing $\mathcal{L}_E$ satisfies properties (iv), (v), and (vi).

Property (iv). $\mathcal{L}_{E\times E'} = \mathcal{L}_E \otimes \mathcal{L}_{E'}$. If we work in the universal example, B_{SO}, we must show

$$\oplus^* \mathcal{L} = \mathcal{L} \otimes \mathcal{L} \text{ in } H^{4*}(B_{SO} \times B_{SO}; Z_{(2)}) .$$

By the Künneth theorem $H^*(B_{SO} \times B_{SO}; Z_{(2)})$ has torsion only of order 2. Thus to show $\oplus^* \mathcal{L} = \mathcal{L} \otimes \mathcal{L}$ it suffices to show

$$\oplus^* (\mathcal{L} \otimes 1_Q) = (\mathcal{L} \otimes 1_Q) \otimes (\mathcal{L} \otimes 1_Q) \text{ and } \oplus^* (\mathcal{L})_2 = (\mathcal{L})_2 \otimes (\mathcal{L})_2 .$$

The first is a statement of the multiplicativity of the Hirzebruch L polynomials as rational classes. (It follows easily from the multiplicativity of the index.) The mod 2 equation becomes $\oplus^*(V^2_{2*}) = V^2_{2*} \otimes V^2_{2*}$. Since $(1 + w_1 + w_2 + \cdots)$ is a multiplicative class

$$(1 + v_1 + v_2 + \cdots) = (1 + Sq^1 + Sq^2 + \cdots)^{-1}(1 + w_1 + w_2 + \cdots)^{-1}$$

is also multiplicative. Thus $\oplus^* V = V \otimes V$ in B_{SO}. But the odd terms drop out by the previous proposition so that $\oplus^* V_{2*} = V_{2*} \otimes V_{2*}$. Then it follows that $\oplus^* V^2_{2*} = V^2_{2*} \otimes V^2_{2*}$.

Property (v). If $M \xrightarrow{f} E^+$ is a map of a closed oriented manifold into E^+ which is transverse regular to B with inverse image V, then an orientation of E, $U_E \in H^{\dim E}(E^+)$, induces one of V and signature $(V) = \langle L(\tau_M)\cdot f^*(U_E\cdot \mathcal{L}_E), [M]\rangle$.

Proof of (v).

$$\begin{array}{ccc} M & \xrightarrow{f} & E^+ \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ \nu_{V\hookrightarrow M} & \xrightarrow{f|} & E \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ V & \xrightarrow{f|} & B \end{array},$$

$\nu_{V\hookrightarrow M} = (f\,|\,V)^*(E)$. Thus $\tau(V) \oplus (f\,|\,V)^*E = \tau(M)\,|\,V$. By the multiplicativity of L we have

$$L(\tau(V))\cdot f^*L(E) = i^*L(\tau(M))$$

or

$$L(\tau(V)) = i^*L(\tau(M))\cdot f^*(1/L(E)) .$$

Thus

$$\text{signature } (V) = \langle L(\tau V), [V]\rangle = \langle i^*L(\tau M)\cdot f^*(1/L(E)), [V]\rangle .$$

Since $\mathcal{L}_E = 1/L(E)$ modulo torsion and signature $(V) \in Z$ we have

$$\text{signature } (V) = \langle i^*L(\tau(M))\cdot(f\,|\,V)^*(\mathcal{L}_E), [V]\rangle .$$

f^*U_E is dual to $i_*[V]$ under Poincaré duality in M, and $f\,|\,V = f\circ i$. Thus

$$\begin{aligned} \text{signature } (V) &= \langle L(\tau(M))\cdot f^*\mathcal{L}_E,\ f^*U_E \cap [M]\rangle \\ &= \langle L(\tau(M))\cdot f^*(U_E\cdot\mathcal{L}_E), [M]\rangle . \end{aligned}$$

Property (vi). If M, as above, is a $Z/2$-manifold, then V is a $Z/2$-manifold and signature $(V)_2 = \langle V^2_M\cdot f^*(U_E\cdot\mathcal{L}_E), [M]\rangle$.

Proof. As above one shows

$$\begin{aligned} \langle V^2_M, [V]\rangle &= \langle V^2_M\cdot(f\,|\,V)^*V^2(E),\ i_*[V]\rangle \\ &= \langle V^2_M\cdot f^*\mathcal{L}_E,\ i_*[V]\rangle = \langle V^2_M\cdot f^*(U_E\cdot\mathcal{L}_E), [M]\rangle . \end{aligned}$$

The proof is completed by showing $\langle V^2(\nu_V), [V]\rangle = \text{signature } (V)_2$. By Proposition 1.1, $\chi(\bar V) + \chi_s(\partial\bar V) = \text{signature } (\bar V)_2$. Here let $\bar V$ be the $Z/2$-

manifold V cut open along its Bockstein. $\chi_s(\partial \bar{V})$ vanishes mod 2 since $\partial \bar{V} = \delta V \amalg \delta V$. Thus $\chi(V)_2 = \chi(\bar{V})_2 = \text{signature } (\bar{V})_2 = \text{signature } (V)_2$. Recall that if $T(V)$ represents its tangent bundle as a non-orientable manifold $\langle V^2(T(V)^{-1}), [V]\rangle = \chi(V)_2$. From 1.4 we have that $V_M^2 = V^2(T(V)^{-1})$. Putting all this together we see that $\langle V_M^2, [V]\rangle = \text{signature } (V)_2$.

Note. Let $\mathfrak{L}_M = \mathfrak{L}_{\nu(M)}$. Then in both cases we may rewrite the above formulae as

$$\text{signature } (V) = \langle \mathfrak{L}_M \cdot f^*(U_E \cdot \mathfrak{L}_E), [M]\rangle .$$

THEOREM 3.3. *If M^n is a closed Z or $Z/2^k$-manifold, then signature* $(M) = \langle \mathfrak{L}_M, [M]\rangle$.

Proof. (This, of course, is the index theorem at the prime 2.) The case of a Z-manifold follows from the usual Hirzebruch index theorem since $\mathfrak{L}_M \otimes 1_Q = L(\tau(M))$. If M^n is a $Z/2^k$-manifold, then 1.0 and [W1] show that M^n is bordant to $r_*[W^n] + i_*[X_2^n]$ where $r\colon Z - Z/2^k$ is the usual reduction, $i\colon Z/2 \hookrightarrow Z/2^k$. W^n is a Z-manifold and X_2^n is a $Z/2$-manifold. Since the signature is a bordism invariant, as is $\langle \mathfrak{L}_M, [M]\rangle$, we need only verify the theorem for $r_*[W^n]$ and $i_*[X_2^n]$. For $r_*[W^n]$, $\langle \mathfrak{L}_W, [W]\rangle = I(W)$ by the closed manifold case. Let $Y_2^k = i(X_2) = 2^{k-1}$ copies of X_2 joined along δX_2. Then $\langle \mathfrak{L}_Y, [Y]\rangle = i\langle \mathfrak{L}_X, [X]\rangle$. To see this note that $\nu(Y_{2^k}) \mid (\text{any copy of } X_2) = \nu(X_2)$. Thus $\nu_*([Y_{2^k}]) = i_*(\nu_*([X_2]))$ in $H_*(B_{SO}; Z/2^k)$. Thus $\langle \mathfrak{L}, \nu_*[Y_{2^k}]\rangle = i\langle \mathfrak{L}, \nu_*[X_2]\rangle$. Since signature $(Y) = i(\text{signature } (X))$, if signature $(X) = \langle \mathfrak{L}_X, [X]\rangle$ then signature $(Y) = \langle \mathfrak{L}_Y, Y\rangle$ (i.e., we have reduced the problem to the $Z/2$-case). But we treated this case in property (vi). This concludes the proof of Theorem 3.3.

COROLLARY 3.4. *If $M^n \xrightarrow{f} E^+$ is a $Z/2^r$-manifold mapping into E^+ which is transverse regular to B with preimage V, then signature* $(V)_{2^r} = \langle \mathfrak{L}_M \cdot f^* U_E \cdot \mathfrak{L}_E, [M]\rangle$.

Proof. As in Property (v) we see that

$$\begin{aligned}\langle \mathfrak{L}_M \cdot f^* U_E \cdot \mathfrak{L}_E, [M]\rangle &= \langle \mathfrak{L}(\nu_M \mid V) \cdot \mathfrak{L}((f \mid V)^* E), [V]\rangle \\ &= \langle \mathfrak{L}(\nu_M \mid V \oplus (f \mid V)^* E), [V]\rangle = \langle \mathfrak{L}(\nu_V), [V]\rangle = \text{signature } (V) .\end{aligned}$$

In Section 4 we will need a multiplicative property of $\mathfrak{L}$. We prove this property here.

PROPOSITION 3.5. *Let M and N be $Z/2^r$-manifolds and $\rho\colon M \otimes N \to M \times N$ as in Section* 1. *Then* $\mathfrak{L}_{M\otimes N} = \rho^*(\mathfrak{L}_M \otimes \mathfrak{L}_N)$.

Proof. According to (1.4) $\nu_{M\otimes N} = \rho^*(\nu_M \times \nu_N) \oplus \pi^*\zeta$ where $\pi\colon M \otimes N \to W/\partial$. $\mathfrak{L}_\zeta = 1$ since $\tilde{H}^{4*}(W^2/\partial) = 0$. Thus by naturality and multiplicativity of $\mathfrak{L}$ we have

$$\begin{aligned}\mathcal{L}_{M\otimes N} &= \mathcal{L}(\nu_{M\otimes N}) = \rho^*\mathcal{L}(\nu_M \times \nu_N)\cdot\pi^*\mathcal{L}(\zeta) \\ &= \rho^*\mathcal{L}(\nu_M)\otimes\mathcal{L}(\nu_N)\cdot 1 = \rho^*(\mathcal{L}_M\otimes\mathcal{L}_N)\ .\end{aligned}$$

4. Cohomology at 2 is dual to bordism

In Section 2, we showed that $H^*(X; Z_{(2)})$ is naturally isomorphic to the group of commutative diagrams

$$\mathcal{C}_{(2)}(X) = \left\{\begin{array}{ccc} H_*(X; Q) & \longrightarrow & Q \\ \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi} \\ H_*(X; Z_{2^\infty}) & \longrightarrow & Z_{2^\infty}\end{array}\right\}$$

provided X is of finite type. In this section we will reformulate the above, still assuming that X is of finite type, in terms of homomorphisms on bordism, namely:

THEOREM 4.1. *$H^{4*}(X; Z_{(2)})$ is naturally isomorphic to the group of commutative diagrams*

$$\left\{\begin{array}{ccc} \Omega_*(X; Q) & \xrightarrow{\sigma_Q} & Q \\ \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi} \\ \Omega_*(X; Z_{2^\infty}) & \xrightarrow{\sigma_2} & Z_{2^\infty}\end{array}\right\}$$

satisfying

(i) $\sigma_Q([M^m, f]\cdot[N^n]) = \sigma_Q([M, f])\cdot I(N)$,

(ii) $\sigma_2(i_*([M^m_{2^k}, f]\otimes[N^n_{2^k}])) = \sigma_2(i_*[M^m_{2^k}, f])\cdot I(N_{2^k})$,

(iii) $\sigma_Q(M^m, f) = \sigma_2(M^m_{2^k}, f) = 0$ *if* $m \not\equiv 0(4)$,

where $i_*\colon \Omega_*(X; Z/2^k) \to \Omega_*(X; Z_{2^\infty})$ *and* $k = 1, 2, 3, \cdots$. *The equivalence is given by*

$$A = (\alpha_0 + \alpha_1 + \alpha_2 + \cdots) \longrightarrow (\sigma_{A,Q}, \sigma_{A,2})$$

where $\sigma_{A,Q}([M, f]) = \langle f^*A\cdot\mathcal{L}_M, [M]\rangle \in Q$

$$\sigma_{A,2}([M_{2^k}, f]) = \langle f^*A\cdot\mathcal{L}_M, [M_{2^k}]\rangle \in Z/2^k \hookrightarrow Z_{2^\infty}\ .$$

Because of relation (iii) we only need to study Ω_{4*}. To do this we construct homomorphisms

$$\lambda_Q\colon \Omega_{4*}(X; Q) \longrightarrow H_{4*}(X; Q)$$

and

$$\lambda_2\colon \Omega_{4*}(X; Z_{2^\infty}) \longrightarrow H_{4*}(X; Z_{2^\infty})$$

defined by

$$\lambda_Q([M, f]) = f_*(\mathcal{L}_M \cap [M])$$

and

$$\lambda_2[M_{2^k}, f] = f_*(\mathcal{L}_M \cap [M^m_{2^k}])$$

where $[M_{2^k}^m] \in H_m(M_{2^k}; Z_{2^\infty})$ and the cap product is induced by $Z_{2^\infty} \otimes Z_{(2)} \xrightarrow{\cong} Z_{2^\infty}$. These homomorphisms are not degree preserving. In fact they are perturbations of the "Hurewicz homomorphism" by lower order terms (since $\mathcal{L}_M = 1 + \cdots$). By the work of Thom and Conner-Floyd, we know that both Hurewicz maps are onto, [CF]. An easy induction argument shows that λ_Q and λ_2 are thus also onto. It is this "onto-ness" that fails at primes other than 2. Thus our analysis works only "at 2". At the odd primes there is an analogous theorem using KO-theory $(\otimes Z[\frac{1}{2}])$. See [S2], [S4].

The main point of the proof is to show that the kernels of λ_Q and λ_2 are generated by the relations given in the theorem (e.g., $\{[M^m, f]\cdot[N^n] - [M^m, f]\cdot I(N)\}$ generates kernel λ_Q). Then we have induced isomorphisms

$$\Omega_*(X; Q)/\{\text{relations}\} \xrightarrow{\cong} H_{4*}(X; Q)$$

$$\Omega_*(X; Z_{2^\infty})/\{\text{relations}\} \xrightarrow{\cong} H_{4*}(X; Z_{2^\infty}),$$

and $\mathcal{C}_{(2)}(X)$ is isomorphic to the group of commutative diagrams given in the theorem. This group, then, is isomorphic to $H^{4*}(X; Z_{(2)})$ by Theorem 2.1. The isomorphism is given by $A \mapsto \langle f^*A, (\mathcal{L}_M \cap [M])\rangle = \langle f^*A\cdot\mathcal{L}_M, [M]\rangle$. This is an outline of the proof of 4.1.

First, we note that the relation subgroups are easily seen to be in the kernels of λ_Q and λ_2 respectively. We must show that they generate the kernels.

Let $M^n \xrightarrow{f} X$ be a Q or $Z/2^k$-manifold mapping into X. Suppose inductively that $f(M^n) \hookrightarrow X^{(i)} \hookrightarrow X$. By adding elements of the "relation subgroup" to the element (M, f) we will perform a cobordism of it in $X^{(i+1)}$ until it is in $X^{(i-1)}$. Let x_σ be a point in the interior of the i-cell σ. Put f transversal to x_σ with $f^{-1}(x_\sigma) = V_\sigma^{n-i}$. Then

$$\textstyle\sum_{i \text{ cells}} \sigma \otimes [V_\sigma^{n-i}] \in C_i(X) \otimes \Omega_{n-i}(G)$$

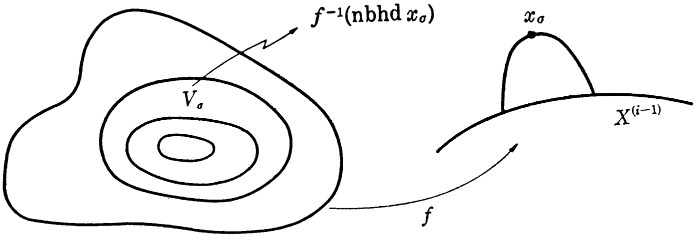
for $G = Q$ or Z_{2^∞}.

This chain is a well-defined invariant of the bordism class of (M, f) in $X^{(i)}$, and is 0 if and only if $f\colon M \to X^{(i)}$ is bordant in $X^{(i)}$ to a map into $X^{(i-1)}$. This follows easily from usual transversality theory.

LEMMA. *This chain is a cycle and its homology class in $H_i(X; \Omega_{n-i}(G))$ $(G = Q$ or $Z_{2^\infty})$ is the obstruction to cobording (M, f) in $X^{(i+1)}$ until it is in $X^{(i-1)}$.*

Proof. $\partial(\Sigma\sigma \otimes [V_\sigma^{n-i}]) = \Sigma\partial\sigma \otimes [V_\sigma^{n-i}]$ which is an obstruction chain for moving $\partial(M - \bigcup_\sigma f^{-1}(\text{neighborhood } x_\sigma))$ (which is in $X^{(i-1)}$) to 0 in $X^{(i-1)}$. But $\partial(M - \bigcup_\sigma f^{-1}(\text{nbhd}x_\sigma))$ bounds $M - \bigcup_\sigma f^{-1}(\text{nbhd}x_\sigma)$ in $X^{(i-1)}$. Thus the chain is 0.

If $\Sigma\sigma \otimes [V_\sigma^{n-i}] = -\partial(\Sigma\tau^{i+1} \otimes [W_\tau^{n-i}])$, let $h_\tau\colon S^i \to X^{(i)}$ be $\partial\tau^{i+1}$. Then add $\sum_\tau (S^i, h_\tau) \times [W_\tau^{n-i}]$ to (M, f). The former element bounds in $X^{(i+1)}$. But the chain obstructions of the sum plus (M, f) is 0. Thus $(M, f) + \sum\big((S^i, h_\tau) \times W_\tau\big)$ is bordant in $X^{(i)}$ into $X^{(i-1)}$. This completes the proof of the lemma.



Thus associated to $f\colon M \to X^{(i)}$ we have a homology class $\alpha \in H_i\big(X; \Omega_*(G)\big)$ $G = Q$ or Z_{2^∞}.

PROPOSITION 4.2. *Any* $\alpha \in H_i\big(X; \Omega_{n-i}(Z_{2^\infty})\big)$ *may be represented by a sum of products* $\alpha = \sum_r x_r \otimes \alpha_r$ *where* $x_r \in H_i(X; Z/2^r)$ *and* $\alpha_r \in \Omega_{n-i}(Z/2^r)$ *under the map*

$$H_i(X; Z/2^r) \otimes \Omega_{n-i}(Z/2^r) \longrightarrow H_i\big(X; \Omega_{n-i}(Z/2^r)\big) \longrightarrow H_i\big(X; \Omega_{n-i}(Z_{2^\infty})\big) .$$

Proof. We have a short exact sequence

$$0 \longrightarrow H_i(X) \otimes \Omega_{n-i}(Z_{2^\infty}) \longrightarrow H_i\big(X; \Omega_{n-i}(Z_{2^\infty})\big) \longrightarrow H_{i-1}(X) * \Omega_{n-i}(Z_{2^\infty}) \longrightarrow 0 .$$

We can represent any element in the tensor product term by the product of an integral homology class and a $Z/2^r$-manifold. For the *Tor* term let $x \in H_{i-1}(X)$ a class of order 2^r and $\alpha_r \in \Omega(Z_{2^\infty})$ an element of order 2^r. Then let y be a chain with $\partial y = 2^r x$. y is a $Z/2^r$ cycle and $y \otimes \alpha_r \in H_i(X; Z/2^r) \otimes \Omega_{n-i}(Z_{2^\infty})$ hits the class in $H_{i-1}(X) * \Omega_{n-i}(Z_{2^\infty})$ created by $2^r y = 0 = 2^r \alpha_r$.

The above proposition is the first one in the proof of the Z_{2^∞} case. We will present the rest of the argument in this case and leave the Q case to the reader. The Q case is analogous but simplified somewhat by the fact that Q is a field.

We have $M \xrightarrow{f} X^{(i)}$ and $\mathcal{O}(M, f) \in H_i\big(X; \Omega_{n-i}(Z_{2^\infty})\big)$, $\mathcal{O}(M, f) = \sum_r x_r \otimes \alpha_r$ where $x_r \in H_i(X; Z/2^r)$ and $\alpha_r \in \Omega_{n-i}(Z/2^r)$. Represent x_r by $V_r^i \xrightarrow{g} X^{(i)}$, and suppose $[W_{2^r}^{n-i}] = \alpha_r$.

Case 1. $(n - i) \not\equiv 0(4)$. Then form $(M, f) - \sum (V_r^i, g) \otimes W_{2^r}^{n-i}$. Since $\mathcal{O}(V_r^i, g) = g_*[V_r^i] = x_r$,

$$\mathcal{O}\big((V_r^i, g) \otimes W_{2^r}^{n-i}\big) = x_r \otimes [W_{2^r}] = x_r \otimes \alpha_r \in H_i\big(X; \Omega_{n-i}(Z_{2^\infty})\big) .$$

Thus $\mathcal{O}((M, f) - \sum (V_r^i, g) \otimes W_{2^r}^{n-i}) = 0$, and this bordism element may be deformed into $X^{(i-1)}$. Since $n - i \not\equiv 0(4)$ we have added elements in the "relation subgroup".

Case 2. $(n - i) \equiv 0(4)$. Here we must use the fact that $\lambda_2(M, f) = 0$ (which has yet to appear).

Again write $\mathcal{O}(M, f) = \sum x_r \otimes \alpha_r$ with x_r represented by $V_r^i \xrightarrow{g} X^{(i)}$. As before $\mathcal{O}((M, f) - \sum (V_r^i, g) \otimes W_r)$ is 0, and thus it may be deformed into $X^{(i-1)}$. As a result the i^{th} component of $\lambda_2((M^n, f) - \sum (V_r^i, g) \otimes W_r^{n-i}) = 0$. Since $\lambda_2(M, f) = 0$, the i^{th} component of $\sum \lambda_2((V_r^i, g) \otimes W_r^{n-i}) = 0$. This means $\sum x_r \otimes I(W_r) = 0$. Thus $\sum (V_r^i, g) \cdot I(W_r)$ may be deformed as an i-manifold into $X^{(i-1)}$. Thus $(M, f) - \sum ((V_r^i, g) \otimes W_r^{n-i}) + \sum (V_r^i, g) I(W)$ is deformable into $X^{(i-1)}$. But once again we have added an element in the relation subgroup. Since the above argument works just as well if (M, f) is replaced by a sum of manifolds of different dimension, downward induction allows us to show $((M, f) - \sum$ relations$)$ is an element in $\Omega_*(X^{(-1)}; Z_{2^\infty}) = 0$. Thus $(M, f) = (\sum$ relations) provided only that $\lambda_2(M, f) = 0$. This together with the analogous argument for λ_Q completes the proof of 4.1.

Note that this argument uses nothing about the structure of smooth bordism except the fact that $\Omega_*(X) \to H_*(X)$ has cokernel an odd torsion group.

COROLLARY 4.3. *The group $H^{4*}(X; Z_{(2)})$ is naturally isomorphic to the group of homomorphisms*

$$\left\{ \begin{array}{l} \Omega_{4*}(X; Z_{(2)}) \xrightarrow{\sigma} Z_{(2)} \\ \Omega_{4*}(X; Z/2^k) \xrightarrow{\sigma'_{2^k}} Z/2^k \quad \forall k \geqq 1 \end{array} \right\},$$

such that

(1) *σ, σ'_{2^k} are multiplicative with respect to the index,*

(2) *the σ'_{2^k} are compatible with $i\colon Z/2^k \to Z/2^{k+1}$, and*

(3) *σ and σ'_{2^k} are compatible with reduction $Z_{(2)} \to Z/2^k$.*

Proof. Since the $\{\sigma'_{2^k}\}$ are compatible with $i\colon Z/2^k \to Z/2^{k+1}$ they define a homomorphism $\Omega_{4*}(X; Z_{2^\infty}) \xrightarrow{\sigma_2} Z_{2^\infty}$ which satisfies $\sigma_2(i_*([M_{2^k}, f] \otimes N_{2^k})) = \sigma_2(i_*[M_{2^k}, f] \cdot I(N_{2^k}))$ since all σ'_{2^k} are multiplicative with respect to the index. Tensoring σ with Id_Q induces a homomorphism $\Omega_{4*}(X; Q) \xrightarrow{\sigma_Q} Q$ which is multiplicative with respect to the index. One easily sees that the compatibility of σ_2 and σ_Q with respect to $\pi\colon Q \to Z_{2^\infty}$ follows easily from the fact that σ and σ'_{2^k} are compatible with $r\colon Z_{(2)} \to Z/2^k$.

Conversely, given a commutative diagram

$$\begin{array}{ccc} \Omega_*(X; Q) & \xrightarrow{\sigma_Q} & Q \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ \Omega_*(X; Z_{2^\infty}) & \xrightarrow{\sigma_2} & Z_{2^\infty} . \end{array}$$

$\sigma_Q \,|\, (\mathrm{im}\, \Omega_*(X; Z_{(2)}))$ must lie in $Z_{(2)}$. Thus we may define $\sigma\colon \Omega_*(X; Z_{(2)}) \to Z_{(2)}$ by restriction. Similarly the σ'_{2^k} are defined by restriction. All compatibility is easily checked.

Note 1. There is a $Z/2$ version of this theorem first proved by Sullivan, see [S1], [S2], and [RS]. We could get that theorem out of this by applying our theorem to $X \wedge T_2$ where $T_2 = S^1 \cup_2 D^2$. However the original proof is simplified by the fact that $Z/2$ is a field.

Note 2. It is just for convenience that we have taken dimensions congruent to 0 modulo 4. We could work in dimensions congruent to i mod 4 and prove a similar theorem. In fact such a theorem follows by use of the suspension isomorphism.

5. Surgery on Z/n-manifolds

There are two natural geometric situations where diagrams of the type

$$(*)\qquad \begin{array}{ccc} \Omega_*(X; Q) & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \Omega_*(X; Q/Z) & \longrightarrow & Q/Z \end{array}$$

occur.

If $E \to B$ is a geometric bundle then we have a natural example of $(*)$ where X is the Thom space of E.

$$\begin{array}{ccc} \Omega_*(E^+; Q) & \longrightarrow & Q \\ \downarrow & \sigma_E & \downarrow \\ \Omega_*(E^+; Q/Z) & \longrightarrow & Q/Z . \end{array}$$

σ_E is constructed by intersecting a manifold in E^+ with the zero section and calculating the signature. Using closed manifolds we obtain the upper arrow by linearity over Q. Using Z/n-manifolds we obtain the lower arrow by to the passing direct limit over n.

In the second situation X is the universal space for surgery problems, G/pl. A diagram

$$\begin{array}{ccc} \Omega_{4*}(G/pl; Q) & \longrightarrow & Q \\ \downarrow & \sigma & \downarrow \\ \Omega_{4*}(G/pl; Q/Z) & \longrightarrow & Q/Z \end{array}$$

can be constructed using the classical surgery obstructions.

A manifold in G/pl, $M \xrightarrow{f} G/pl$ is by definition a piecewise linear R^n-bundle over M, $E \to M$, and a proper map $E \xrightarrow{\pi} R^n$ which is degree 1 on each fiber.

By transversality we obtain a "surgery problem" or "normal map" over M,

$$\pi^{-1}(0) = L \xrightarrow{f} M .$$

That is, f is a degree-one map of manifolds which is naturally covered by a bundle map

$$\nu_L \xrightarrow{\tilde{f}} F$$

where ν_L is the stable normal bundle of L and F is some bundle over M.

In the above case L is naturally framed in the total space of E so we may take F to be the stable bundle $\nu_M - E$ and $\tilde{f}$ results.

The existence of $\tilde{f}$ allows one to begin surgery on f d'après Browder and Novikov to try to make f into a homotopy equivalence.

More precisely one tries to construct a cobordism Q of L to L' and a map $Q \xrightarrow{c} M$ so that $c|L = f$ and $c|L'$ is a homotopy equivalence. c should be covered by a bundle map $\nu_Q \xrightarrow{\tilde{c}} F$. The triple $(Q, c, \tilde{c})$ is called a normal cobordism and we say in this case that f is "normally bordant" to a homotopy equivalence.

We will explain how the classical surgery theory extends to Z/n-manifolds. For example, we find that surgery to a homotopy equivalence leads to obstructions in

$$Z/n \qquad 0 \qquad Z/2 \qquad Z/2$$

for the dimensions congruent to 0, 1, 2, 3 mod 4. The $Z/2$ obstructions are interesting only when n is a power of 2.

We will describe how to compute these obstructions without doing surgery, in terms of signatures. It turns out that the theory of the Kervaire-Arf obstructions in $Z/2$ is subsumed by the "signature obstructions" in Z/n.

First consider the surgery problem for closed oriented $4k$-manifolds $L \xrightarrow{f} M$. A necessary condition for L to be cobordant to a manifold homotopy equivalent to M is that the signatures of L and M are equal. Now the difference of the signatures is just the signature of the subspace $(\operatorname{Ker} f)_{2k} =$ $(\text{kernel } f_*)_{2k}$ in $H_{2k}(L; Z)/\text{torsion}$.

Here the intersection form is even (see for example the discussion of immersed cycles below) and has determinant ± 1. Thus the difference of the signatures of L and M are congruent modulo 8, by the famous theorem on quadratic forms (see the generalization, Theorem 5.9 below).

Analysis of the surgery problem shows that f is normally bordant to a homotopy equivalence in the simply connected case if the signatures are equal. Thus the precise surgery obstruction for f is the integer

$$\sigma_f = \frac{1}{8}(\text{signature } L - \text{signature } M)\ . \tag{5.0}$$

Thus σ_f provides us with an a priori definition of the "surgery obstruction" and a homomorphism

$$\Omega_{4*}(G/pl) \xrightarrow{\sigma} Z\ .$$

To produce the map $\Omega_*(G/pl; Q/Z) \to Q/Z$, we turn to surgery on Z/n-manifolds. To do this we will use some well known facts in simply connected surgery theory which we state here for completeness.

(5.1) Let L^m and M^m be closed Z-manifolds and $f\colon L \to M$ a normal map. The theory implies the following.

(a) If $\pi_1(M) = 0$, and m is odd, then f is normally bordant over M to a homotopy equivalence.

(b) Any f is normally bordant to a surgery problem over a simply connected manifold. (This uses $\pi_1(G/pl) = 0$.)

(c) f is normally bordant over M to a map f' with $(\operatorname{Ker} f')_* = 0$ for $* \leqq [m/2] - 1$.

(d) (5.1) a, b, and c and the description of σ_f above are valid if L and M have boundaries provided $f \mid \partial L\colon \partial L \to \partial M$ is a homotopy equivalence which is held fixed.

Remark. The other interesting case for closed manifolds is dimension $4k + 2$. Here the $Z/2$ obstruction found by Kervaire is the Arf invariant [A2] of a quadratic form $(\operatorname{Ker} f_{2k+1}) \to Z/2$ found at the last step of the surgery. φ is a quadratic with respect to the skew symmetric inner product on $(\operatorname{Ker} f)_{2k+1}$ provided by intersection. The quadratic function φ is rather subtle. However, in Rourke-Sullivan [RS] there is an a priori calculation using immersed cycles.

Some of these ideas will serve us well in the Z/n-case.

We turn now to Z/n-manifolds for $n = 2^k$. Let L^{4m} and M^{4m} be $Z/2^k$-manifolds and f be a normal map between them. If $f \mid \delta L\colon \delta L \to \delta M$ is a homotopy equivalence, we open L and M up to manifolds with boundary. Here $\bar{f}\colon \bar{L} \to \bar{M}$ is a normal map of manifolds with boundary which is a homotopy equivalence on the boundaries.

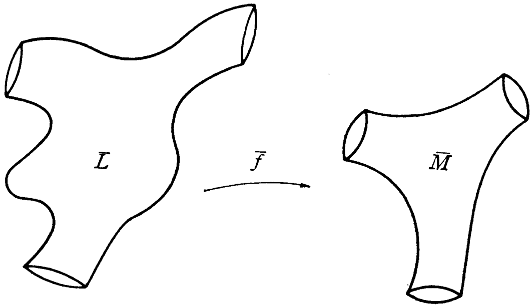


FIGURE 5.1

So we see that $\sigma(\bar{f}) = (1/8)(I(\bar{L}) - I(\bar{M}))$. However, this is the obstruction to doing surgery keeping $\bar{f}$ fixed on the boundary. We are not required to do that. We must only preserve the structure of (2^k copies of some surgery problem) on the boundary.

For example, suppose Q is the trace of some surgery of $\bar{f}$ restricted to one copy of δL in $\partial\bar{L}$ to a new homotopy equivalence. Then we can adjoin 2^k-copies of Q to $\bar{L}$.

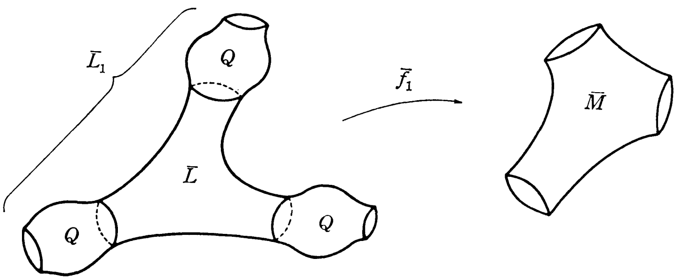


FIGURE 5.2

We find a new problem of "interior" surgery. If any one of these obstructions vanish as we vary Q we can perform the surgery on the original map to a homotopy equivalence. Now the obstruction rel boundary is

$$\text{(5.2)} \qquad \frac{1}{8}(\text{signature } \bar{L} + 2^k \text{ signature } Q - \text{signature } \bar{M}) ,$$

since signature $\bar{L}_1$ = signature $L + 2^k$ signature Q by the Novikov addition lemma.

Now the possible signatures that occur for Q are the multiples of 8.[1] We know signature $Q \equiv 0 \pmod 8$ because the form on the kernel of $Q \to \delta M$ is even with determinant ± 1. On the other hand, 8 and its multiples occur using the standard Milnor surgery problem.

[1] For dimension of Q equal to 8, 12, 16, $\cdots$.

Thus using (5.2) we see the precise surgery obstruction for the problem $L \xrightarrow{f} M$ on $Z/2^k$-manifolds, where δf is a homotopy equivalence, is

$$\frac{1}{8}(\text{signature } \bar{L} - \text{signature } \bar{M}) \text{ reduced mod } 2^k .$$

In the general case where δf is not a homotopy equivalence we proceed as follows. Construct a surgery Q of the odd dimensional problem $\delta L \xrightarrow{\delta f} \delta M$ to a homotopy equivalence. Adjoin 2^k copies of Q to $\bar{L}$ as in Figure 5.2 to obtain the relative problem

$$\bar{L}_1 \longrightarrow \bar{M}$$

with surgery obstruction

$$\frac{1}{8}(\text{signature } \bar{L} + 2^k \text{ signature } Q - \text{signature } \bar{M}) .$$

Now the above remark generalizes: Signature Q has a well-defined residue class modulo 8 and every integer in the residue class can occur if the dimension of L is greater than four.

The first part follows by putting two surgeries of δf to a homotopy equivalence together,

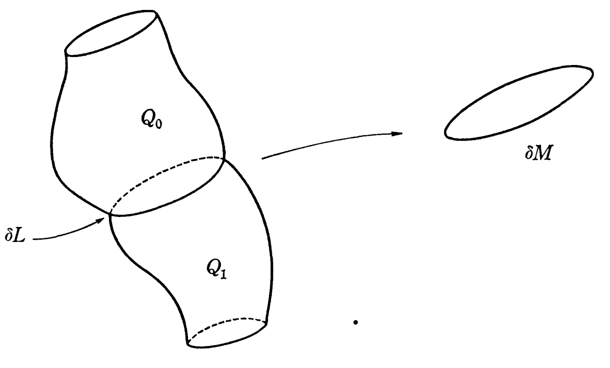


FIGURE 5.3

Then (signature Q_0 − signature Q_1) equals signature $(Q_0 \cup_{\delta L} Q_1)$ (by Novikov again) which is congruent to zero mod 8 as above.

The second part follows again by using the standard Milnor problem.

Definition. Let $\Theta(\delta f)$ be the residue class modulo 8 of any surgery of δf to a homotopy equivalence.

Now we can write a formula for the precise surgery obstruction in the general surgery problem on $Z/2^k$-manifolds in dimension $4n$, $L \xrightarrow{f} M$, namely

(5.3) $$\sigma_f = \frac{1}{8}(\text{signature } \bar{L} + 2^k\Theta(\delta f) - \text{signature } \bar{M}) ,$$

in $Z/2^k$.

PROPOSITION. (a) *If σ_f is zero in $Z/2^k$, then f is normally bordant over M to homotopy equivalence when $\pi_1 M = \pi_1 \delta M = 0$, and the dimension of M is $4m$, $m > 1$.*

(b) *If f is normally cobordant to any homotopy equivalence then σ_f is zero.*

Proof. The first part was proved in the discussion above.

For the second part, let $(Q, \delta Q)$ denote the trace of any surgery of f to a homotopy equivalence. Then open up Q along its Bockstein and consider $\bar{Q} \to \bar{M}$.

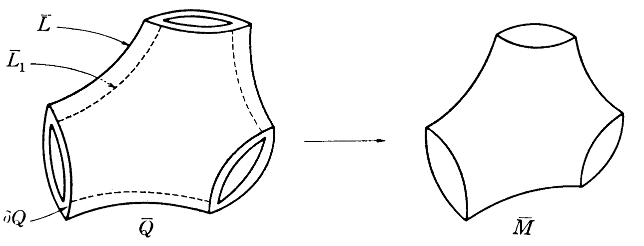


FIGURE 5.4

Here $\bar{L}_1 \to \bar{M}$ is a homotopy of manifolds with boundary. Then

$$\begin{aligned} 0 &= \text{signature } \partial\bar{Q} \\ &= \text{signature } (\bar{L} \cup 2^k\delta Q \cup -\bar{L}_1) \\ &= \text{signature } (\bar{L} + 2^k \text{ signature } \delta Q - \text{signature } \bar{L}_1) . \end{aligned}$$

Since signature δQ lies in the residue class mod 8 $\Theta(\delta f)$ and signature $\bar{L}_1 =$ signature M we have the desired result

$$\frac{1}{8}(\text{signature } \bar{L} + 2^k\Theta(\delta f) - \text{signature } M) = 0$$

in $Z/2^k$.

This shows that our understanding of the surgery obstruction on $Z/2^k$-manifolds of dimension $4n$ depends on our understanding of the invariant $\Theta(\delta f)$.

Before continuing with the Θ invariant we describe surgery in the other dimensions. In dimension $4k + 1$, if we can do surgery on the Bockstein (5.1)a and (5.1)d then show that we can produce a normal cobordism to a homotopy equivalence. However since some multiple of each Bockstein bounds, each has 0 index. Thus, in the simply connected case, surgery on the Bockstein is possible. This shows there is no obstruction in dimension $4k + 1$.

In dimension $4k+2$, we can do surgery on the Bockstein to a homotopy equivalence by (5.1)b, and we are left with the Kervaire invariant over the interiors. We can change the invariant (as in the index case) by multiples of n (over Z/n-manifolds). Since the Kervaire obstruction is of order 2, if n is odd we can always kill it by changing the Bockstein. However, if n is a power of two the obstruction is immutable. Thus $f\colon L_n^{4m+2} \to M_n^{4m+2}$ has $\sigma(f) \in Z/2 \otimes Z/n$. From the point of view of Z/n-manifolds, Bocksteins, Θ invariants, etc., we have no a priori definition of this invariant. However, Rourke and Sullivan's a priori calculation works for any $Z/2^k$-manifold by opening it up to a $Z/2$-manifold and then considering it as a non-orientable manifold.

In dimension $4k+3$, then, the only problem is on the Bockstein. The fact that n times the Bockstein bounds a surgery problem implies that n times the invariant is 0. So if n is odd, we have no invariant, whereas if n is 2^k, we have a $Z/2$ invariant,

$$\sigma(f\colon L^{4m+3} \longrightarrow M^{4m+3}) = \text{Kervaire obstruction } (f \,|\, \delta L) \in Z/2 \,.$$

This completes the description of the surgery obstructions.

We turn again to the term $\Theta(\delta f)$ in the surgery formula

$$\sigma_f = \frac{1}{8}(\text{signature } \bar{L} + 2^k \Theta \delta f - \text{signature } \bar{M}) \,.$$

Now $\Theta(\delta f)$ is a rather subtle invariant. It is not a cobordism invariant (but rather an invariant of a cobordism). We will see below that its theory encompasses that of the Kervaire obstruction which is well known to be elusive.

Thus from now on we concentrate on closed $(4n-1)$-manifolds, and surgery problems

$$M^{4n-1} \xrightarrow{\;g\;} N^{4n-1}$$

in order to study $\Theta(\delta f)$ above.

Our aim is to give an a priori geometric computation of the invariant

$$\Theta(g) \in Z/8 \,.$$

We begin.

The interesting part of the homology of a $(4n-1)$-manifold M is the torsion subgroup $T_M \subseteqq H_{2n-1}(M, Z)$. The linking number of torsion cycles x and x' is defined in the rationals, mod 1, and we have by Poincaré duality a perfect dual pairing, $(x, x') \to l(x, x')$,

$$T_M \otimes T_M \xrightarrow{\;l\;} Q/Z \,.$$

Since we are in dimension $4n-1$, l is symmetric.

Our analysis of $\Theta(g) \in Z/8$ depends on geometrically enhancing the linking on the (non-singular) subspace

$$T_g = \text{kernel } g_* \subseteqq T_M$$

to a *quadratic form*

$$T_g \xrightarrow{q} Q/Z .$$

q satisfies the equations

$$(5.4) \qquad \begin{array}{ll} (1) & q(x+y) = q(x) + q(y) + l(x, y) . \\ (2) & q(\lambda x) = \lambda^2 q(x) . \end{array}$$

We describe q.

First of all the definition of q is canonical on the odd torsion of T_g. These self-linking numbers $l(x, x)$ lie in the odd torsion subgroup of Q/Z where division by 2 is possible and unique. So we define

$$(5.5) \qquad \begin{cases} q(x) \text{ to be the unique solution of } 2q(x) = l(x, x) \\ \text{and } n^2 q(x) = 0 \text{ where } n \text{ is the order of } x \text{ (which is odd)} . \end{cases}$$

On the critical 2-torsion subgroup of T_g we use the fortuitous fact from cobordism theory that homology classes of order a power of two can be represented by a manifold.

So represent $x \in (T_g)_2$ by an embedded submanifold V of M. The embedding is produced by general position. Let C be a chain in M whose boundary is λ copies of V and assume away from its boundary C intersects V transversally. We can think of C as a Z/λ-manifold in M which intersects its own Bockstein transversally.

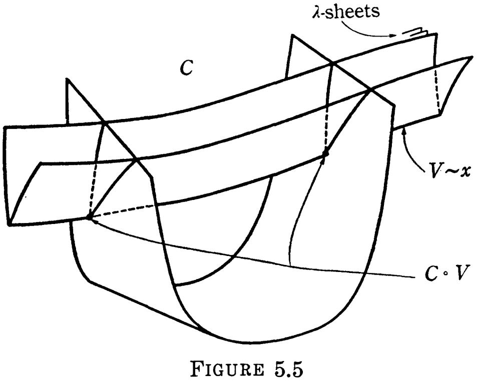


FIGURE 5.5

We obtain a simple picture of the self-linking numbers. If $C \cdot V$ is the algebraic intersection number of C and V then

$$l(x, x) = C \cdot V/\lambda \text{ in } Q/Z .$$

$C \cdot V/\lambda$ is only defined mod 1 because twisting C in a neighborhood of V changes $C \cdot V$ by multiples of λ.

However, suppose V is endowed with the extra structure of a nowhere-zero normal field F. For the pushed-off cycle V' described by the endpoints of F, $C \cdot V'$ is a well-defined integer.

Thus we may divide the self-linking $l(x, x)$ by 2 using V' and define our quadratic function q by

$$q(x) = C \cdot V'/2\lambda .$$

PROPOSITION 5.6. *If x is represented by (V, F), y is represented by (W, G) and $x + y$ is represented by $(V \cup W, F \cup G)$ where V and its normal field do not intersect W and its normal field, then*

$$q(x + y) = q(x) + q(y) + l(x, y) ,$$

and

$$2q(x) = l(x, x) .$$

Proof. Let D be a chain whose boundary is λW. Then

$$\begin{aligned} q(x + y) &= \frac{1}{2\lambda}\big((C + D)\cdot(V' + W')\big) \\ &= \frac{1}{2\lambda}C \cdot V' + \frac{1}{2\lambda}D \cdot W' + \frac{1}{2\lambda}(D \cdot V' + C \cdot W') \\ &= q(x) + q(y) + l(x, y) \text{ in } Q/Z . \end{aligned}$$

To see that $1/2\lambda(D \cdot V' + C \cdot W') = l(x, y)$ notice that $C \cdot D$ is a one d-chain, and

$$\begin{aligned} 0 = \partial(C \cdot D) &= C \cdot \partial D - (-1^{\dim C})\partial C \cdot D \\ &= C \cdot \lambda W - \lambda V \cdot D = \lambda(C \cdot W - D \cdot V) . \end{aligned}$$

Thus $C \cdot W = D \cdot V$ as integers. By the hypothesis that the field connecting W and W' misses V and vice versa, we see that $C \cdot W = C \cdot W'$ and $D \cdot V = D \cdot V'$. This shows that $C \cdot W' = D \cdot V'$ as integers, so that

$$\frac{1}{2\lambda}(D \cdot V' + C \cdot W') = \frac{1}{\lambda}(C \cdot W') = l(x, y) \in Q/Z .$$

Now the definition of q is independent of the choice of C since x is a torsion class. To see this let C' be another chain with $\partial C' = \lambda x$. $\partial\big(C \cup (-C')\big) = \lambda x - \lambda x = 0$. So $C \cup -C'$ is an integral class. But the intersection form on integral classes vanishes on torsion. As a result $(C \cup -C') \cdot x = 0$. Thus $(C \cup -C') \cdot x = C \cdot x - C' \cdot x = 0$.

Also q only depends on the immersion class of (V, F). To see this note that the different V's are obtained by twisting around V.

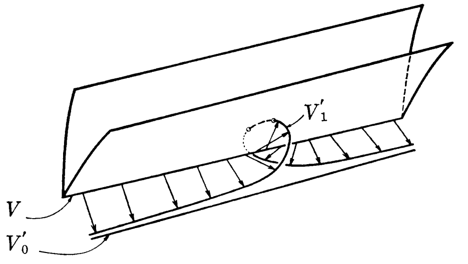


FIGURE 5.6

Each twist changes $C \cdot V'/2\lambda$ by 1/2 in Q/Z. However varying (V, F) by an immersion always changes V' by an even number of twists as the following pictorial experiment with a belt readily shows in dimension 3.

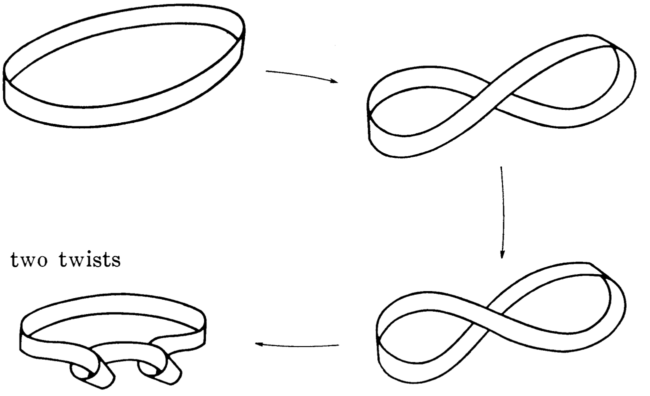


FIGURE 5.7

More generally, suppose M^{4m-1} lies in the boundary of Q^{4m} and (V, F) bounds the immersed manifold with normal field (W, G). Let z be the homology class of the cycle $Z = \lambda W - C$ in Q and d the algebraic number of double prints of W. Then

PROPOSITION 5.7.

$$-\lambda(C \cdot V') + 2d\lambda^2 = z \cdot z .$$

Proof. Push W off along G to obtain W'. Add a collar to Q and the cycle $Z' = \lambda(W' + V' \times I) - C' \times 1$.

Examining the figure below shows the two contributions to $z \cdot z = Z \cdot Z'$ are $\lambda(C \cdot V)$ and $2d\lambda^2$. (At this point we use the fact that V is embedded to gain the 2 of the formula.)

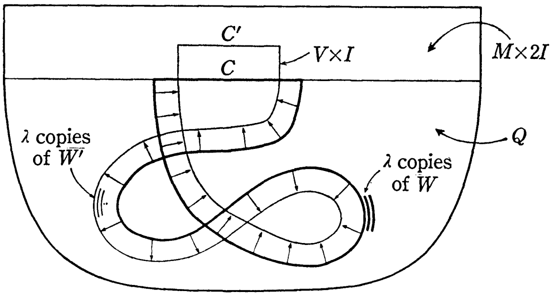


FIGURE 5.8

COROLLARY. *$q(x)$ remains unchanged if we vary our cycle with normal field by an immersed cobordism in $M \times I$.*

Proof. If $(V_0, F_0) \cup -(V_1, F_1)$ bounds (W, G) in $M \times I$, then the above proposition shows

$$-\lambda\big((C_0 - C_1)\cdot(V_0 - V_1)\big) + 2d\lambda^2 = 0$$

since there are only zero self intersections in the manifold $M \times I$. Dividing by $2\lambda^2$ gives

$$-\frac{C_0\cdot V_0}{2\lambda} - \frac{C_1\cdot V_1}{2\lambda} + d = 0$$

since the cross terms vanish.

$C_0\cdot V_0$ and $C_1\cdot V_1$, in the above equations, are calculated in $\partial(M \times I) = M \times \{1\} - M \times \{0\}$. Thus if we calculate both $C_0\cdot V_0$ and $C_1\cdot V_1$ using the original orientation of M, we see that

$$\frac{C_0\cdot V_0}{2\lambda} - \frac{C_1\cdot V_1}{2\lambda} + d = 0 .$$

Now we will see how our surgery problem $M^{4n-1} \xrightarrow{g} N^{4n-1}$ gives us the normal fields used to define the quadratic form

$$T_g \xrightarrow{q} Q/Z .$$

By the definition of a surgery problem, g is covered by a bundle map $\nu_M \xrightarrow{\tilde{g}} E$ where E is some bundle over N and ν_M is the stable normal bundle of M. Now an immersion class with normal field in M in homotopy terms is nothing more than a map $V^{2n-1} \xrightarrow{f} M^{4n-1}$ and the choice of a $(2n-1)$-bundle γ^{2n-1} in the class of the "potential stabilized normal bundle",

$$f^*\mathcal{T}_M - \mathcal{T}_V = \nu_V - f^*\nu_M .$$

This choice means choose γ and a stable isomorphism

$$\gamma^{2n-1} \sim (\nu_V - f^*\nu_M) .$$

The equivalence of these notions follows from immersion theory.

So choose $V \xrightarrow{f} M$ representing x in kernel $g_* \cap$ 2-torsion $H_{2n-1}M$ so that the composition $V \xrightarrow{f} M \xrightarrow{g} N$ bounds $Q \xrightarrow{b} N$. This is done by examining the bordism of the mapping cylinder of $M \xrightarrow{g} N$ rel M.

Now *choose immersion data for V with a normal field that "extends over Q"*. That is, a $(2n-1)$-reduction of the stable bundle $\nu_Q - b^*E$.

Note. (i) $\nu_Q - b^*E$ is isomorphic, after restriction to $\partial Q = V$, to $\nu_V - f^*\nu_M$ using $\tilde{g}$.

(ii) Q is homologically $2n-1$-dimensional so $(2n-1)$-reductions are possible.

We claim that the resulting $q(x)$ is independent of choices and we have our desired quadratic function.

Before discussing this point we will explain how $q(x)$ is used. From [S3] we have the following

THEOREM 5.8. *Let $M^{4n-1} \xrightarrow{g} N^{4n-1}$ be a surgery problem of $(4n-1)$-manifolds. Let T_g denote the torsion subgroup of the kernel of g_* in dimension $2n-1$ with the quadratic function*

$$T_g \xrightarrow{q} Q/Z$$

defined by the cycles with normal fields produced by g.

Form the Gaussian sum

$$\sum_{x \in T_g} e^{2\pi i q(x)} .$$

Then the argument of this complex number lies in $Z/8 \subset S^1$ and determines the surgery invariant $\Theta(g)$. In fact, we have

$$\sqrt{\text{order } T_g}\, e^{(i\pi/4)\Theta(g)} = \sum_{x \in T_g} e^{2\pi i q(x)} .$$

We will sketch the proof. The validity of the formula and the invariance of our definition of q from the choices make use of the trace of the surgery of g to a homotopy equivalence.

I: Proof that q is well-defined.

Write the trace of the surgery Q as $Q_0 \bigcup_\partial Q_1$ where Q_0 is obtained from M by attaching handles of index less than or equal to $2n-1$; Q_1 is obtained by attaching $(2n-1)$ bundles to $\partial Q_0 - M$; and $q \mid (\partial Q - M)$ is a homotopy equivalence.

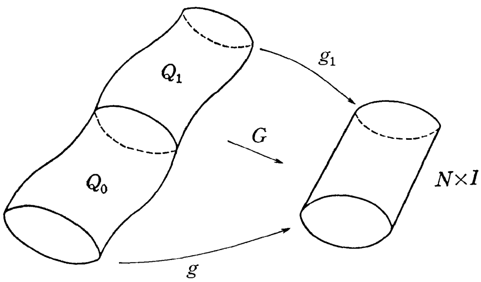


FIGURE 5.9a

Let $g_1 = G|Q_0 \cap Q_1$. The inclusions into Q_0 induce an isomorphism

$$T_g \sim T_{g_1} .$$

Choose submanifolds with normal fields (V_i, F_i) and chains C_i by the above procedure for the computation of q on T_g.

We can assume by general position that this geometric material avoids the spheres that are successively removed by surgery from M to get $M_1 = Q_0 \cap Q_1$. Thus this geometric material exists naturally in M_1 where it serves for the computation of q on T_{g_1}. Thus q on T_g and q on T_{g_1} agree.

Note. Q_0 has an orientation $\mathcal{O}_{Q_0}$ which when restricted to the boundary gives $\mathcal{O}_{M_1}$ and $-\mathcal{O}_M$. It is when we use $\mathcal{O}_{M_1}$ and $+\mathcal{O}_M$ that the linking pairings and quadratic functions agree.

This proves we need only show that q is well-defined for a map like g_1 which is $(2n - 1)$-connected.

If W_{g_1} denotes the mapping cylinder of g_1, the connectivity implies the $2n^{\text{th}}$ bordism group of (W_{g_1}, M_1) is isomorphic to the $2n^{\text{th}}$-homology group.

So let $V_i \xrightarrow{f_i} M_1$, $X_i \xrightarrow{F_i} N$, $i = 0, 1$, be two choices of submanifolds in M_1 with bounding manifolds in N for the computation of $q(x)$, where $x \in T_{g_1} \simeq H_{2n}(W_{g_1}, M_1)$. By the bordism remark above there is a $(2n + 1)$-manifold R in W_{g_1} whose boundary is $-X_0 \cup_{V_0} S \cup_{V_1} X_1$ where S is a bordism in M_1.

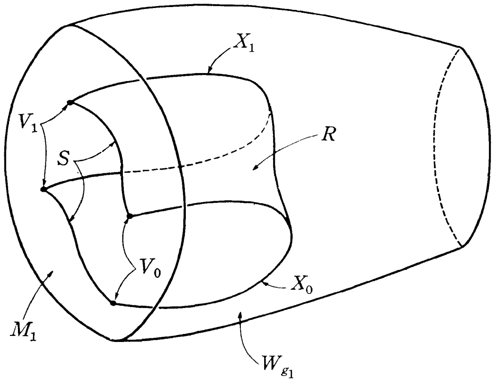


FIGURE 5.9b

Choose independently the bundle data on X_i for the separate computations of $q(x)$.

Now we can assume the data chosen over X_0 extend over S since S/V_0 has homological dimension less than $2n$.

We claim that *the bundle data induced from $X_0 \cup_{V_0} S$ on V_1 agree with that coming from X_1.* This is the crux of the argument.

Assuming this for the moment, the proof is completed as follows. Use the data on S to immerse S and apply the corollary of Proposition 5.1.

Now we turn to the claim.

Let γ denote the stable bundle $\nu_{\partial R} - E$ over the closed $2n$-manifold ∂R. We can assume we have a $2n$-bundle representing γ, with non-zero sections over $X_0 \cup_{V_0} S$ and X_1. The agreement of these sections over V_1 is the point in question and this disparity may be easily identified with the mod 2 obstruction to cross section for γ, $w_{2n}\gamma$.

Now the Thom space of γ, $T\gamma$, is S dual to the Thom space of E over ∂R since $\gamma = \nu_{\partial R} - E$, [A1]. Thus $w_{2n}\gamma = Sq^{2n}$ (Thom class of γ) can be calculated from χSq^{2n} (Thom class of E).

Now E over N^{4n-1} is fiber homotopy equivalent to the normal bundle of N^{4n-1}. Then χSq^{2n} (Thom class of E over N) is just the $2n^{\text{th}}$ Wu class of N^{4n-1} *which is zero.*

II: The relationship between the quadratic function $T_g \to Q/Z$ and the invariant $\Theta(g)$.

The relationship between $q\colon T_g \to Q/Z$ and the Θ-invariant $\Theta(g)$ is a consequence of the following theorem of analytic number theory.

THEOREM 5.9. *Let V be a rational vector space with a non-singular quadratic form $q\colon V \to Q$ which is integral on a lattice $L \subset V$, i.e., a subgroup of maximal rank. Let L^* denote the dual lattice,*

$$y \in L^* \text{ if and only if } (y \cdot l) \in Z \;\; \forall\, l \in L, \text{ where } y \cdot l = q(y + l) - q(y) - q(l)\,.$$

Then

$$\sqrt{\text{order } L^*/L}\; e^{(i\pi/4)\text{signature } q} = \sum_{x \in L^*/L} e^{2\pi i q(x)}\,.$$

Notice that the left hand side has length $\sqrt{\text{order } L^*/L}$ and argument $\in Z/8 \hookrightarrow S^1$ equal to signature of q mod 8.

This theorem is proved in [S3].

We still have some geometric-homological work to do to be able to apply this theorem.

Historical Note. The use of Gaussian sums in a surgery problem was first made by Ed Brown. He was stimulated by Paul Monsky to write the Arf invariant of a form φ on a $Z/2$-vector space V as

$$A\varphi = \operatorname{Arg}\left(\textstyle\sum_{x \in V} (-1)^{\varphi(x)}\right).$$

On the other hand, the second author was aware of the "analogous relation"

$$\alpha e^{i(\pi\tau/4)} = \int_{R^n} e^{2\pi i q(x)}$$

where α is a constant of volume and τ is the signature of the quadratic form q in the real vector space R^n.

The formula of the theorem resulted as a common generalization of these two situations. There is even a short heuristic proof using the Fourier development of $e^{2\pi i q(x)}$. This proof was only "heuristic" because the integral above is improper. Paul Monsky then showed the second author how the difficulty could be avoided using the Poisson summation formula.

Milgram, working independently in California, found the same formula almost simultaneously. In fact, he claims the result can be adapted from the classical literature.

So consider again the trace of the surgery to a homotopy equivalence.

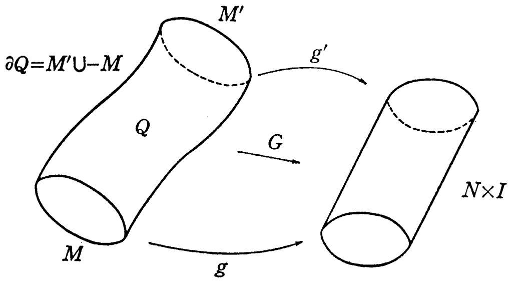


FIGURE 5.10

Let K, T, F denote respectively the kernels of the various homology maps, their torsion subgroups and their torsion-free quotients.

Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & T_g \arrow[d, equal] & \\
 & \cdots \arrow[r] & T_{2n}(Q, \partial Q) \arrow[r, "f"] \arrow[d] & T_{2n-1}\partial Q \arrow[r, "f^*"] \arrow[d] & {} \\
\cdots \arrow[r] & K_{2n}Q \arrow[r] \arrow[d] & K_{2n}(Q, \partial Q) \arrow[r] \arrow[d] & K_{2n-1}\partial Q \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r, "h"] & F_{2n}Q \arrow[r, "i"] & F_{2n}(Q, \partial Q) \arrow[r, "h^*"] \arrow[d] & F_{2n-1}\partial Q \arrow[r] & \cdots \\
 & & 0 . & &
\end{tikzcd}$$

Since G has degree one the middle row is exact being a natural summand of the full homology exact sequence of $(Q, \partial Q)$. The natural complement is the corresponding sequence for $N \times I$.

The bottom and top rows can be regarded as chain complexes.

From this we see

(i) The signature of Q can be computed from the pairing in $K_{2n}Q$ or $F_{2n}(Q)$.

(ii) The deviation from exactness in the upper sequence at the T_g point is naturally isomorphic by the "Bockstein map" to the deviation in the lower sequence at the $F_{2n}(Q, \partial Q)$ point.

Thus we may write the short exact sequence

$$(*) \qquad 0 \longrightarrow F_{2n}(Q)/\mathrm{Ker}\, i \xrightarrow{i} \mathrm{Ker}\, h^* \longrightarrow \mathrm{Ker}\, f^*/\mathrm{im}\, f \longrightarrow 0 .$$

From here the proof will proceed by showing

(i) $\mathrm{Ker}\, h^* \otimes Q$ has a non-singular quadratic form $\bar{q}$ which is integral on the lattice $F_{2n}(Q)/\mathrm{Ker}\, i$ and $\mathrm{Ker}\, h^*$ is the dual lattice.

(ii) $\bar{q}$ induces the same form on $\mathrm{Ker}\, f^*/\mathrm{im}\, f$ as $q|\mathrm{Ker}\, f^*$ where we use the original orientation of M to calculate q (not the one induced on M as the boundary of Q).

(iii) $\Theta(g) =$ signature $(\mathrm{Ker}\, h^* \otimes Q)$.

(iv) $\mathrm{Arg}\left(\sum_{x \in T_g} e^{2\pi i q(x)}\right) = \mathrm{Arg}\left(\sum_{x \in \mathrm{Ker} f^*/\mathrm{im} f} e^{2\pi i q(x)}\right)$.

From the above, $(*)$, and Theorem 5.9 we see

$$\begin{aligned} e^{(i\pi/4)(\Theta(g))} &= \mathrm{Arg}\left(\sum\nolimits_{x \in \mathrm{Ker} f^*/\mathrm{im} f} e^{2\pi i \bar{q}(x)}\right) \\ &= \mathrm{Arg}\left(\sum\nolimits_{x \in \mathrm{Ker} f^*/\mathrm{im} f} e^{2\pi i q(x)}\right) = \mathrm{Arg}\left(\sum\nolimits_{x \in T_g} e^{2\pi i q(x)}\right) . \end{aligned}$$

Thus $\sqrt{\text{order } T_g}\, e^{(i\pi/4)(\Theta(g))} = \sum_{x \in T_g} e^{2\pi i q(x)}$, or $\mathrm{Arg}\left(\sum_{x \in T_g} e^{2\pi i q(x)}\right) \in Z/8 \hookrightarrow S^1$ is equal to $\Theta(g)$.

(i) $F_{2n}(Q, \partial Q)$ is dually paired to $F_{2n}Q$ by intersection. Thus by restricting this pairing to $\operatorname{Ker} h^*$ we have $\operatorname{Ker} h^* \otimes F_{2n} \to Z$. We wish to show $\operatorname{Ker} h^*$ annihilates $\operatorname{Ker} i$. So let $\alpha \in \operatorname{Ker} h^*$ and $\beta \in \operatorname{Ker} i$. $\alpha \in \operatorname{Ker} h^*$ means that $\partial\alpha$ is a torsion element. Thus $\lambda\alpha$ may be regarded as a class in interior Q. $\beta \in \operatorname{Ker} i$ means some multiple $\lambda'\beta$ may be pushed to the boundary of W. Thus $\lambda\alpha \cdot \lambda'\beta = 0$. This implies $\alpha \cdot \beta = 0$. Thus we have a pairing $\operatorname{Ker} h^* \otimes F_{2n}(Q)/\operatorname{Ker} i \to Z$. To show that this pairing is non-singular note that

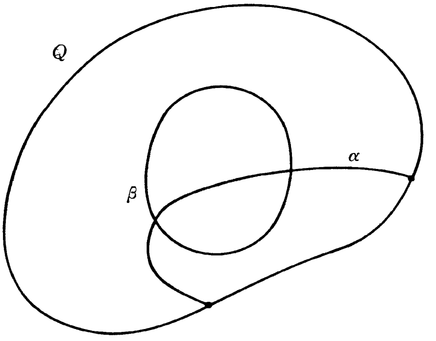


FIGURE 5.11

$\operatorname{Ker} h^* \hookrightarrow F_{2n}(Q, \partial Q)$ is a direct summand. Thus a homomorphism on it extends to $F_{2n}(Q, \partial Q)$ and so comes from $F_{2n}(Q)$. This provides an element in $F_{2n}(Q)/\operatorname{Ker} i$ hitting any homomorphism. Summarizing we see that intersection induces an isomorphism of $(\operatorname{Ker} h)^*$ with $\left(F_{2n}(Q)/\operatorname{Ker} i\right)^*$. $(\operatorname{Ker} h)^* \otimes Q = V$ has a natural quadratic function defined as follows. If $x \in (\operatorname{Ker} h)^*$ then some multiple $\lambda x = i(a)$. Define $\bar{q}(x) = x \cdot a/2\lambda$. $\bar{q}$ is actually integral on $F_{2n}(Q)/\operatorname{Ker} i \hookrightarrow (\operatorname{Ker} h)^* \otimes Q$ since $x \cdot x \in 2Z$ for $x \in \operatorname{Ker}(G)_{2n} = K_{2n}(Q)$, and as we have seen $(\operatorname{Ker} h)^*$ is the dual lattice under $\bar{q}$.

(ii) $\bar{q}$ defines a quadratic function q' on the cokernel which by (*) is $\operatorname{Ker} f^*/\operatorname{im} f$. It is given by $q'(x + \operatorname{im} f) = \bar{q}(y)$ where $\partial y = x$. Of course, $\bar{q}(y) = z^2/2\lambda^2$ where $z \in K_{2n}(Q)$ and $z \mapsto \lambda y$.

There is, however, another definition of a quadratic form of $\operatorname{Ker} f^*/\operatorname{im} f$. It is obtained by restricting q on T_g. To get a form on the quotient we must show that $q \mid (\operatorname{im} f) = 0$. But this will follow from the more general fact that on $\operatorname{Ker} f^*$, $q = q'$. To prove this let $x \in \operatorname{Ker} f^*$. Choose $y \in K_{2n}(Q, \partial Q)$ with $\partial y = x$. If x is 2-torsion, pick y also 2-torsion and represent y by an immersed manifold W, with a normal field (which exists since W is of homological dimension $2n - 1$). Let C be a chain in M with $\partial C = \lambda x$. Then we may use $\partial W \hookrightarrow M$ with its normal field and C to compute $q(x)$, since $(W \to Q \to N \times I$, normal field) when restricted to the boundary gives a permitted normal field for $\partial W \subset \partial Q$.

By Proposition 5.7

$$q(x) = C\cdot V/2\lambda = \frac{-z\cdot z}{2\lambda^2} \quad \text{where } z \in K_{2n}(Q) \text{ hits } \lambda y$$

and where we use the orientation on M coming from that on Q. Thus with this orientation $q' = -q$. However, this orientation is the negative of the original orientation on M. Using it changes the sign of q. Thus $q' = q$ on 2-torsion. On the odd torsion we need only know that the induced linking pairings agree (again we use the orientation on Q to calculate one linking pairing and the original orientation on M to calculate the other). This follows from a cruder analogue of the above in which (W, normal field) is replaced by chains, X and X', in Q with boundaries x and x' respectively.

(iii) $\Theta(g) \equiv$ signature (Q) mod 8. Since $\operatorname{Ker} i/\operatorname{im} h$ and $\operatorname{Ker} h^*/\operatorname{im} i$ are torsion

$$\big(F_{2n}(Q)/\operatorname{im} h\big) \otimes Q = \big(F_{2n}(Q)/\operatorname{Ker} i\big) \otimes Q = (\operatorname{Ker} h^*) \otimes Q\ .$$

Clearly the intersection forms on all the groups agree. Thus they all have the same signature. Signature (Q) = signature $\big(F_{2n}(Q)/\operatorname{im} h\big) \otimes Q$ since $\operatorname{im} h \hookrightarrow$ radical of the intersection pairing on $F_{2n}(Q)$.

(iv) LEMMA. *Let T be a finite quadratic space, coming from a quadratic on a rational vector space as in Theorem* 5.9. *Let $A \hookrightarrow T$ with $q\,|\,A = 0$. Form $A \xrightarrow{i} T \xrightarrow{i^*} A^*$, and let $H = \operatorname{Ker} i^*/\operatorname{im} i$. Then H has a natural quadratic function q_H which is non-singular and*

$$\operatorname{Arg}\Big(\sum\nolimits_{x\in H} e^{2\pi i q_H(x)}\Big) = \operatorname{Arg}\Big(\sum\nolimits_{x\in T} e^{2\pi i q(x)}\Big)\ .$$

Proof. We have already shown how to define q_H. It is non-singular since $A \subseteqq$ annihilator (annihilator (A)) is an isomorphism. This is a counting argument using the fact that $i^*\colon T \to A^*$ is onto since Q/Z is injective. Call (the annihilater of A) $= \operatorname{Ker} i^*$, $\mathfrak{A}$. Let $(V, \bar{q})$ be the quadratic form on a rational vector space, L a lattice on which q is integral with $(L^*/L, \bar{q}) \cong (T, q)$. Form

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & L^* \arrow[r, "\alpha"] & T \arrow[r] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[u, equal] & \mathcal{K} \arrow[r] \arrow[u, hook] & \mathfrak{A} \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[u, equal] & K \arrow[r] \arrow[u, hook] & A \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

where $K = \alpha^{-1}(A)$, $\mathcal{K} = \alpha^{-1}(\mathfrak{A})$.

Claim. (i) $\bar{q}\,|\,K$ is integral

(ii) $\mathcal{K} = K^*$

(iii) $\mathcal{K}/K = \mathfrak{A}/A = H$.

(i) If $k \in K$, then $\bar{q}(k)$ is in Z since $\bar{q}\big([k]\big) = 0 \in Q/Z$. This follows from the fact that $[k] \in A$ where q vanishes.

(ii) $x \cdot k \in Z$ for all $k \in K$ if and only if $[x] \cdot a = 0$ for all $a \in A$. Here $[x] \in T$. $[x] \cdot a = 0$ for all $a \in A$ if and only if $[x] \in \mathcal{A}$ if and only if $x \in \mathcal{K}$.

(iii) is clear.

Since $K \hookrightarrow \mathcal{K} \hookrightarrow V$ are dual lattices under $\bar{q}$ as are $L \hookrightarrow L^* \hookrightarrow V$, Theorem 5.9 implies

$$\operatorname{Arg}\left(\sum_{x \in H} e^{2\pi i q(x)}\right) = e^{(i\pi/4)\operatorname{sign}(V,\bar{q})} = \operatorname{Arg}\left(\sum_{x \in T} e^{2\pi i q(x)}\right) .$$

We finish this section with some computational examples of Theorem 5.9.

Let $T = Z/p$, p prime, with generator u and $q(u) = \lambda/p$. Let $\xi = e^{2\pi i/p}$. Then

$$\sum_{x \in T} e^{2\pi i q(x)} = 1 + \sum_{1 \geqq x \geqq -1} \xi^{\lambda x^2} .$$

Let $\left(\dfrac{y}{p}\right) = +1$, or -1 depending on whether or not y is a square mod p. Then, since all $y \neq 0$ which are squares have 2 square roots, we have

$$\begin{aligned} \sum_{x \in T} e^{2\pi i q(x)} &= 1 + 2\sum_{1 \leqq y \leqq p-1} \left(\frac{1 + \left(\dfrac{y}{p}\right)}{2}\right) \xi^{\lambda y} \\ &= 1 + \sum_{1 \leqq y \leqq p-1} \xi^{\lambda y} + \sum_{1 \leqq y \leqq p-1} \left(\frac{y^{\lambda^{-1}}}{p}\right) \xi^{y} \\ &= 1 - 1 + \left(\frac{\lambda^{-1}}{p}\right) \sum_{1 \leqq y \leqq p-1} \left(\frac{y}{p}\right) \xi^{y} . \end{aligned}$$

By a theorem in Gaussian sums, [BS], page 349, the last summation is

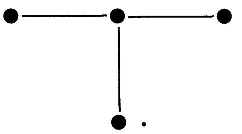
$$\left(\frac{\lambda^{-1}}{p}\right) \cdot \begin{cases} \sqrt{p} & p \equiv 1(4) \\ i\sqrt{p} & p \equiv 3(4) . \end{cases}$$

Thus $\operatorname{Arg} \sum_{x \in T} e^{2\pi i q(x)}$ is 4 or 8 for $p \equiv 1(4)$ and 2 or 6 for $p \equiv 3(4)$. It is true in general that the odd torsion only contributes arguments of order 2 or 4.

As another example, let

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix},$$

i.e., the matrix associated to



Then $\det(A) = 4$ and signature $(A) = 4$. Let $A\colon \bigoplus_4 Z \to \bigoplus_4 Z^*$ where x_1, x_2, x_3, and x_4 are the natural generators of $\bigoplus_4 Z$. $2(x_1^* - x_3^*) = A(x_1 - x_3)$ and $2(x_3^* - x_4^*) = A(x_3 - x_4)$. Thus $T = \operatorname{coker} A$ is $Z/2 \bigoplus Z/2$ generated by $(x_1^* - x_3^*)$ and $(x_3^* - x_4^*)$.

$$q(x_1^* - x_3^*) = \frac{\langle (x_1^* - x_3^*),\, A^{-1}2(x_1^* - x_3^*)\rangle}{4}$$
$$= \frac{\langle (x_1^* - x_3^*),\, (x_1 - x_3)\rangle}{4} = \frac{1+1}{4} = \frac{1}{2} \in Q/Z\,.$$

Likewise $q(x_3^* - x_4^*) = q(x_1^* - x_4^*) = 1/2$. Thus

$$\textstyle\sum_{x\in T} e^{2\pi i q(x)} = 1 + 3e^{\pi i} = -2 = \sqrt{4}\, e^{(i\pi/4)4}\,.$$

This is the space that arises in (Kervaire problem) $\otimes$ $(Z/2, 1/2)$, i.e., it is the signature 4 problem that relates Kervaire obstructions to $Z/2$ signature problems.

6. Product formulae

In this section we will study product formulae for surgery problems of $Z/2^r$-manifolds. First we consider the surgery obstruction for the product of a surgery problem with a manifold, denoted $(M \xrightarrow{f} N) \otimes L$. Then we pass to the general product. As a model for this result, consider the $Z/2$-manifold formula for the Kervaire obstruction. If $l + m \equiv 2(4)$, i.e., we are in the "Kervaire dimension",

$$\sigma\big((M^m \xrightarrow{f} N^m) \otimes L^l\big) = \sigma(M \xrightarrow{f} N)\cdot(\text{signature mod 2 of } L)$$

[RS], [S1]. Our result are analogous but more complicated in the index case $\big(l + m \equiv 0(4)\big)$. Let M^m, N^m, and L^l be $Z/2^r$-manifolds with $l + m \equiv 0(4)$. Then we have the following basic formula

$$\sigma\big((M^m \xrightarrow{f} N^m) \otimes L^l\big) = \sigma(M \xrightarrow{f} N)\cdot i(L)$$

where i is an invariant of the homological structure of L. There are four possibilities for i depending on the four possible pairs of dimensions $0 + 0 \equiv 0(4)$, $1 + 3 \equiv 0(4)$, $2 + 2 \equiv 0(4)$, and $3 + 1 \equiv 0(4)$. The values of $i(L^l)$ are described by the following theorem.

THEOREM 6.1. *Case* (0). *If $l \equiv 0(4)$, then $i(L)$ is the signature of L in $Z/2^r$.*

Case (1). *If $l \equiv 1(4)$, and, if in addition, L is a closed Z-manifold, $i(L)$ is the rank* mod 2 *of $\big($torsion $H_{2k}(L; Z)\big) \otimes Z/2$. $\big($Here the product $\sigma(M \xrightarrow{f} N)\cdot i(L)$ is formed in $Z/2$ and then considered as an element in $Z/2^r$.$\big)$*

Denote this invariant of manifolds $d(L)$ after de Rham who first considered it. For the general case in this dimension see (1′) *below.*

Case (2). *If $l \equiv 2(4)$, then $i(L)$ is just the de Rham invariant of the Bockstein of L, $d(\delta L)$.*

Case (3). *If $l \equiv 3(4)$, then $i(L) = 0$.*

To treat Case (1) in full generality we must extend the de Rham invariant to $Z/2^r$-manifolds (in analogy with the way the index is generalized to $Z/2^r$-manifolds to treat Case (0)). Our generalization of the de Rham invariant is as follows. Let W^{4k+1} be an orientable manifold with boundary. The signature of ∂W is 0 so we may choose $\mathfrak{A} \hookrightarrow H_{2k}(\partial W)$ a maximal torsion free self-annihilating subspace with 1/2 the total rank, see [M3]. Define $T_{W,\mathfrak{A}}$ to be the following torsion group

$$T_{W,\mathfrak{A}} = \text{torsion } \{H_{2k}(W)/i_*(\mathfrak{A} + \text{torsion } H_{2k}(\partial W))\}$$

where $i\colon \partial W \hookrightarrow W$.

Then there is an extension of geometric linking which gives a non-singular skew-symmetric pairing

$$T_{W,\mathfrak{A}} \otimes T_{W,\mathfrak{A}} \longrightarrow Q/Z .$$

We define $d(W, \mathfrak{A})$ to be the de Rham invariant of this pairing, i.e., $d(W, \mathfrak{A}) = rk_{Z/2}(T_{W,\mathfrak{A}} \otimes Z/2)$. If L^{4l+1} is a $Z/2^r$-manifold choose any maximal self-annihilating subspace $D \subset H_{2l}(\delta L)$. (Such a subspace D will have rank equal to 1/2 the rank of $H_{2l}(\delta L)$ since $I(\delta L) = 0$.) Open L up to form $\bar{L}$ a manifold with boundary. $\partial \bar{L} = \coprod_{2^r} \delta L$. The invariant $d(\bar{L}, \underbrace{D \oplus \cdots \oplus D}_{2^r}) \in Z/2$ is an invariant of L independent of D.

Case (1′). $i(L) = d(\bar{L}, D \oplus \cdots \oplus D) \in Z/2$.

Let us begin our proofs of the cases in (6.1.)

Case 3. We can assume (after a cobordism of range and domain) that the manifolds in question have simply connected Bockstein and when opened up are simply connected.

Then, since $\dim N \equiv 1 \pmod 4$ we may perform surgery on f to make it a homotopy equivalence. $f \otimes$ (Identity L) is also a homotopy equivalence and the surgery obstruction of the product is zero. Since $i(L) = 0$ in this case, this proves Case 3.

In Cases (0) and (2) we may assume by the statements 5.1(a), (b), and (c) that $M^{2m} \xrightarrow{f} N^{2m}$ is a surgery problem over closed manifolds, and is $(m-1)$-connected.

In Case (1) we can assume $\delta M \xrightarrow{\delta f} \delta N$ has this form.

Our first task is to compute the Θ invariant of:
(i) Cases (0) and (2),

$$h = I_{\delta L} \times f\ ,$$

$$\delta L \times M \xrightarrow{h} \delta L \times N\,,$$

(ii) Case (1),

$$h = L_L \times \delta f\ ,$$

$$L \times \delta M \xrightarrow{h} L \times \delta N\ .$$

These problems are all of the form

$$Q \times M^{2m} \xrightarrow{h} Q \times N^{2m}\ ,$$
$$h = I \times f$$

where Q, M, and N are closed and oriented, and $\operatorname{Ker} f$ has torsion free homology concentrated in dimension m.

To compute Θ for such a problem we need to know the quadratic function associated to the linking form in torsion $\big(\operatorname{Ker}(I \times f)\big)$. Since $\operatorname{Ker} f_i$ is free and 0 unless $i = m$, the group T_h is equal to

$$\text{torsion}\ \ (H_l Q) \otimes \operatorname{Ker} f_m\ ,$$

where $\dim Q = 2l + 1$, $\dim M = 2m$.

The linking pairing on this subspace can be determined homologically and satisfies the formula

$$l(x \otimes y,\, x' \otimes y') = l(x,\, x')(y \cdot y') \tag{6.2}$$

where $y \cdot y'$ denotes the intersection pairing in $H_m(M)$. Thus we know the quadratic function q on the odd torsion subgroup of T_h by a formal homological calculation. A more complicated formula (6.3 below) derived geometrically is needed to calculate q on the 2-torsion.

Let $x \otimes y$ be a generator of the 2-torsion subgroup of T_h. We will see how to use f to represent y (or some odd multiple) by an immersed manifold U in M. We will also represent x by an embedded submanifold V^l of Q^{2l+1} with nowhere-zero normal field. A map $V^l \to Q$ can be found by general bordism theory which says $\Omega_*(Q) \xrightarrow{h} H_*(Q)$ has an odd-torsion cokernel. Since x is a 2-torsion class there must be some $V \xrightarrow{f} Q$ representing x. $V^l \to Q^{2l+1}$ is embedded and has a nowhere-zero normal field by general position. We represent y (or some odd multiple) by $U \to M$ so that $f(U)$ bounds W^{m+1} in

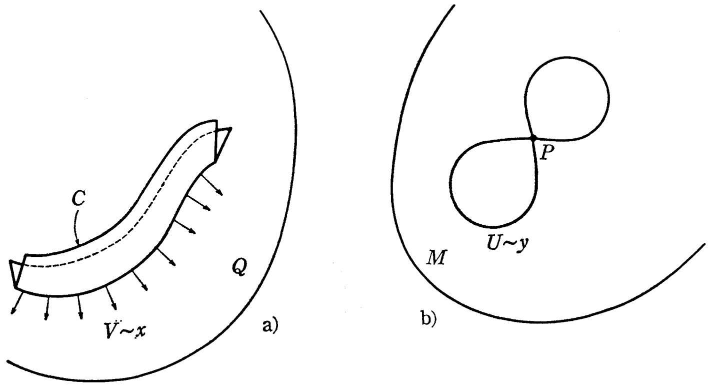


FIGURE 6.1

N. This again follows from the bordism theory as above. As in Section 5 we use the bundle information in the normal map to reduce $f^*\nu_M - \nu_U$ to an m bundle. This provides an immersion of U^m in M^{2m}.

Let d_U denote the algebraic number of double points of U in M for m even and the actual number of double points of U in M for m odd. Let χ_V denote the Euler number of the normal subbundle to V perpendicular to the field. Let l_V denote the algebraic number of points in the intersection of V', the cycle described by the endpoints of the normal field vectors to V, and C, a chain whose boundary is λV where $\lambda =$ order x. $l_V/\lambda = l(x, x)$ by definition.

One of the main geometric points of the paper is the following:

Local product formula

$$q(x \otimes y) = \frac{\big(l_V + (-1)^m l_V + \lambda\chi_V\big)d_U}{2\lambda}. \tag{6.3}$$

Proof. Recall V is embedded in Q with a chosen normal field. Also, the immersion data (in terms of a bundle reduction) for U in M has been chosen so that it "extends" in the sense of Section 5 over some manifold W in N whose boundary is U. Thus $V \times U$ has a natural immersion in $Q \times M$ with normal field and the bundle data for this clearly "extends" over the manifold $V \times W$ in $Q \times N$. By the recipe of Section 5 we have a suitable immersed cycle with normal field for computing the quadratic function on $x \otimes y$.

We assume that the leaves of C come into V along the negative of the field (Figure 6.1). We will use a small isotopy of C in Q which moves V a small distance "down" along the field.

Think of $C \times U$ as fibered over U with fibers C. Near each double point P of U push the C fibers down slightly using the isotopy of C in a damped

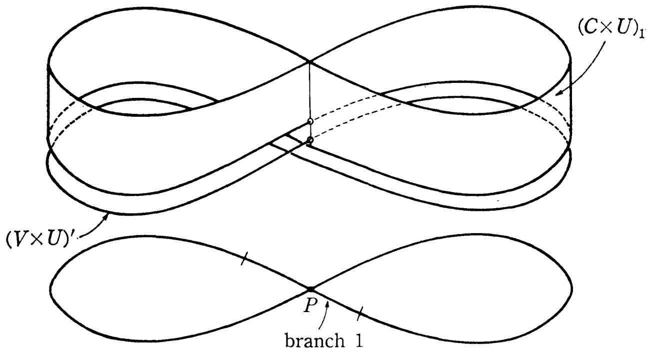


FIGURE 6.2

manner over a small neighborhood of P in branch 1.

We obtain a new chain $(C \times U)_1$ whose boundary is now embedded and fibers over $U \subset M$ (Figure 6.2). We want to calculate the generic intersection of this embedded cycle $(V \times U)'$ pushed down slightly along the normal field with $(C \times U)_1$.

Now U has a non-zero normal field in M because the normal bundle to U extends over the manifold W. Let U_ε denote a generic translate of U along this field (Figure 6.3). Push $(C \times U)_1$ fiberwise to a new chain $(C \times U)_\varepsilon$ lying over U_ε in the same way $(C \times U)_1$ lies over U. Now $(V \times U)'$ intersects $(C \times U)_\varepsilon$ only over points like P_+ and P_- in Figure 6.3.

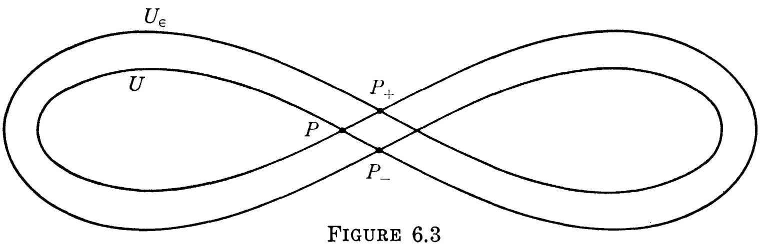


FIGURE 6.3

Over P_+ and P_- we have respectively

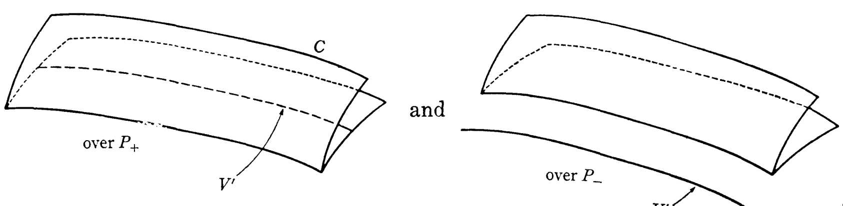


FIGURE 6.4

The intersection is generic over P_- with l_V intersection points. Over P_+ we move the fiber V' of $(V \times U)'$ to intersect C transversally, and we have

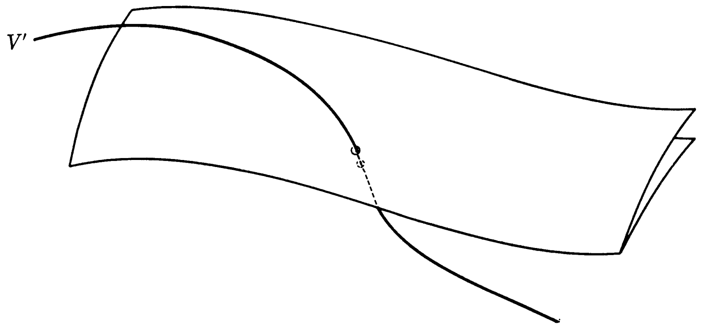


FIGURE 6.5

$l_V + \lambda\chi_V$ intersection points. The signs associated to P_+ and P_- by the intersection of U_ε with U are respectively $+$ and $-$ if m is odd. If m is even each has the sign of the double point P. So we see that the contribution over P to the generic intersection of $(V \times U)'$ and $(C \times U)_\varepsilon$ is

$$l_V + \lambda\chi_V + (-1)^m l_V .$$

Adding over all double points we obtain the desired formula

$$q(x \otimes y) = \frac{(l_V + (-1)^m l_V + \lambda\chi_V) d_U}{2\lambda} \in Q/Z .$$

Proof of product formulae. We can apply the local geometric product formula to deduce the surgery formulae above:

Case 0. In case m is even, then l is odd. Thus χ_V is zero by skew-symmetry, ν being an orientable l bundle over an l-manifold. d_U is clearly $(1/2)(y \cdot y)$, and l_V/λ is the self-linking number $l(x, x)$. Thus (6.3) becomes

$$q(x \otimes y) = \frac{1}{2}(y \cdot y) \cdot l(x, x) \in Q/Z .$$

So we find ourselves in the situation of the algebraic Proposition 6.7 which tells us the Θ invariant of $T_h \xrightarrow{q} Q/Z$ associated to the surgery problem $\delta L \times M \xrightarrow{h} \delta L \times N$ is zero.

By formula 5.3 we find that surgery obstruction for $(M \xrightarrow{f} N) \otimes L$, m even, is just

$$\frac{1}{8}(\text{signature } M \times L - \text{signature } N \times L)$$

or

$$\sigma(M \xrightarrow{f} N)\cdot \text{signature } L\ .$$

So $i(L)$ is signature L and this proves Case 0.

When m is odd $y \xrightarrow{\varphi} d_U$ (mod 2) defines a quadratic function on $K_f =$ (kernel $f_*)_l$ with values in $Z/2$ associated to the intersection form on K_f. The proof that d_U as constructed by our procedure depends only on $y \in K_f$ is exactly like the proof of Section 5 about the quadratic function q. The fact that φ is quadratic and associated to intersection follows immediately from the figure.

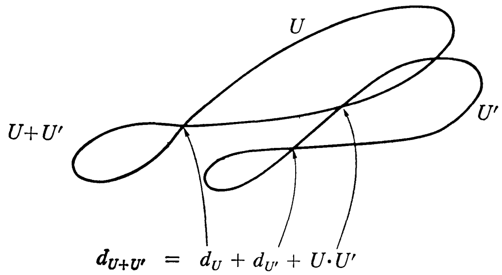


FIGURE 6.6

This discussion is more detailed in [RS] where it is shown that the Arf invariant of φ is the Kervaire surgery obstruction of $M \xrightarrow{f} N$. Again the discussion of Section 5 can be adapted to show this.

Now we turn to the Euler number term in the formula χ_V which bears a rather pretty relationship to the self-linking number of homology class x determined by V.

Suppose we have $V^{2k} \subset \operatorname{int} M^{4k+1}$ and a chain C whose boundary is $\lambda V \cup A$ where $A \subset \partial M^{4k+1}$. We assume that C intersects V transversally in l_V points and that in ∂M, $A\cdot A = 0$. Then it follows that

$$\varepsilon = \lambda V' + l_V\cdot S + A$$

is homologous to zero in the complement of V. Here V' is again the pushed-off cycle defined by the normal field and S is the boundary of an oriented normal disk to V. Thus ε as a cycle in the normal sphere bundle to $V \cup \partial M$ must have zero self-intersection. (Make the part of C outside the normal disk bundle to V transversal to itself and examine the boundary of this one cycle.)

Computing this intersection gives

$$\lambda^2(V'\cdot V') + 2\lambda l_V(V'\cdot S) + (l_V)^2(S\cdot S) + [A]\cdot[A] = 0$$

or,

$$\lambda^2\chi_V + 2\lambda l_V = 0$$

or,

$$\frac{1}{2}\chi_V = -l_V/\lambda\,. \tag{6.4}$$

Applying this to the case where $A = \varnothing$ and $\partial C = \lambda V$ we see that χ_V is even or odd depending on whether $l(x, x)$ is zero or not in $Z/2 \subseteqq Q/Z$.

So formula (6.3) becomes

$$q(x \otimes y) = l(x, x)\cdot\varphi(y)$$

where on the right we multiply in $Z/2$ and regard the result in Q/Z. Thus we find ourselves in the algebraic situation of Proposition 6.8 below which computes the Θ invariant of $K_f \otimes T_{\partial L} \xrightarrow{q} Q/Z$ to be

$$4(\mathrm{Arf}\,\varphi)\cdot(\mathrm{rank}\; T_{\partial L} \otimes Z/2) \in Z/8\,.$$

From this we may prove Cases (1) and (2).

Case (1). $\Theta(L^{4l+1} \times (\partial M \xrightarrow{\partial f} \partial N)) = 4d(L)\cdot\sigma(\partial f)$. Of course $\sigma(\partial f) = \sigma(f)$ by definition in this case. Since signature of $L \times \bar{M} =$ signature of $L \times \bar{N} = 0$, formula (5.3) tells us $\sigma(L \times (M \xrightarrow{f} N)) = d(L)\cdot\sigma(f) \in Z/2 \hookrightarrow Z/2^k$. Thus $i(L) = d(L)$.

Case (2). $\Theta(\partial L \times (M^{4m+2} \xrightarrow{f} N^{4m+2})) = 4d(\partial L)\cdot\sigma(f)$. Again since signature $\bar{L} \times M =$ signature $\bar{L} \times N$, (5.3) yields $\sigma(L^{4l+2} \otimes (M \xrightarrow{f} N)) = d(\partial L)\cdot\sigma(f) \in Z/2 \hookrightarrow Z/2^k$. Thus $i(L^{4l+2}) = d(\partial L)$ in this case.

Case (1′). Before beginning the proof of (1′) proper we study the generalized linking pairing mentioned in the introduction to this section. Let W^{4k+1} be an oriented manifold and let $\mathfrak{A} \subset H_{2k}(\partial W; Z)/\mathrm{Torsion}$ be a maximal self-annihilating subspace. Its rank is 1/2 the rank of $H_{2k}(\partial W)$. For the existence of such a subspace see [M3]. Let

$$T_{W,\mathfrak{A}} = \mathrm{torsion}\,[H_{2k}(W)/i_*(\mathfrak{A} + \mathrm{Torsion}\; H_{2k}(\partial W))]$$

where i_* is the map induced by inclusion. Let $x \in T_{W,\mathfrak{A}}$. Represent x by a cycle $X^{2k} \hookrightarrow W$. Some multiple of X, λX, is homologous to $A \in \mathfrak{A} +$ Torsion $H_{2k}(\partial W)$. Let C^{2k+1} be a chain whose boundary is $\lambda X - A$, where as before the λ sheets come into X in a cusp-like manner along a normal field. For any $y \in T_{W,\mathfrak{A}}$, let Y be a cycle representing it. Define $l(x, y) = (1/\lambda)C\cdot Y$ where Y is shifted to miss X.

LEMMA. *l: $T_{W,\mathfrak{A}} \oplus T_{W,\mathfrak{A}} \rightarrow Q/Z$ is a well-defined, skew-symmetric, non-singular pairing.*

Proof. Let C' be another chain with boundary $\lambda X - A'$ for $A' \in \mathfrak{A} +$ Torsion $H_{2k}(\partial W)$. Then $C \cup - C'$ is a chain with boundary $A' - A$. Since some multiple of Y, μY, is homologous to $A'' \in \mathfrak{A} +$ Torsion, we see that

$$(C \cup - C') \cdot Y = \frac{1}{\mu}(A' - A) \cdot A'' = 0 ,$$

since $(\mathfrak{A} + \text{Torsion } H_{2k}(\partial W))$ is self-annihilating. Thus we see that $C \cdot Y = C' \cdot Y$. Thus $(1/\lambda) C \cdot Y = (1/\lambda) C' \cdot Y$. If we shift Y around we change $(1/\lambda) C \cdot Y$ only by integers. The proof of this is the same as in the closed manifold case. Thus $l(x, y) \in Q/Z$ is well-defined. To calculate $l(y, x)$ we must take D with

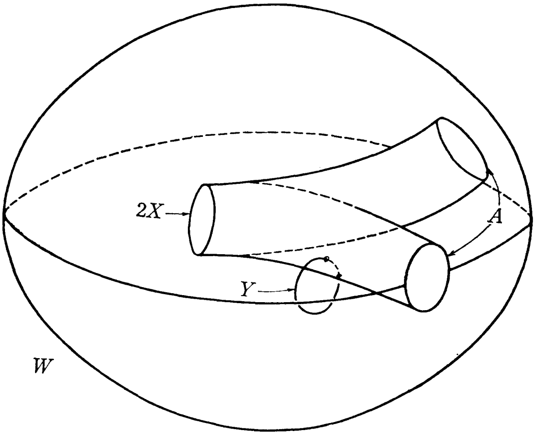


FIGURE 6.7

$\partial D = \mu Y - A''$, $A'' \in \mathfrak{A} +$ Torsion $H_{2k}(\partial W)$ and form $(1/\mu) D \cdot X$. $C \cdot D$ is a one-complex whose boundary is

$$\lambda X \cdot D + C \cdot \mu Y + A \cdot A'' = \lambda X \cdot D + C \cdot \mu Y .$$

Thus since $X \cdot D = D \cdot X$ and $\partial(C \cdot D) = 0$, we have

$$0 = \lambda D \cdot X + \mu C \cdot Y$$

or

$$0 = \frac{1}{\mu} D \cdot X + \frac{1}{\lambda} C \cdot Y$$

or

$$l(y, x) + l(x, y) = 0 ,$$

i.e., l is skew-symmetric. We will not need the fact that l is non-singular. This, however, will *follow* from the calculation of the surgery kernel in Case (1′). For these reasons we do not include an algebraic proof.

Now we consider Case (1′) for $Z/2$-manifolds. We have $f: M^{4k+3} \to N^{4k+3}$ a surgery problem between $Z/2$-manifolds and L^{4l+1} a $Z/2$-manifold. We wish to calculate $\sigma(f \otimes 1_L: M \otimes L \to N \otimes L)$.

We first form a nice representative $Z/2$-surgery problem in dimension $4k+3$ whose surgery obstruction, i.e., the Kervaire invariant along the Bockstein, is non-zero. Let $K^{4k+2} \xrightarrow{k} S^{4k+2}$ be the usual Kervaire surgery problem. We claim that this problem admits an orientation reversing involution. Namely, on

$$K^{4k+2} = \tau(S^{2k+1}) \mathop{\#}_{\text{plumb}} \tau(S^{2k+1}) \cup c(\text{boundary})$$

we exchange the two copies of $\tau(S^{2k+1})$ and then extend by coning over the boundary. On S^{4k+2} we use the antipodal map. One checks easily that these involutions are of degree -1 and that the normal map may be taken compatible with respect to them.

Let $X^{4k+3} \xrightarrow{f_\tau} Y^{4k+3}$ be the $Z/2$-surgery problem $K^{4k+2} \times I/\tau \xrightarrow{f \times I} S^{4k+2} \times I/\tau'$ where τ and τ' are the involutions just described. Clearly $\sigma(f_\tau) =$ (the Kervaire invariant of $\tau_\tau \mid \delta X) = 1 \in Z/2$. If $f\colon M^{4k+3} \to N^{4k+3}$ is a general surgery problem between $Z/2$-manifolds, and L^{4l+1} is a $Z/2$-manifold, then $f \otimes 1_L\colon M \otimes L \to N \otimes L$ is homotopic to a homotopy equivalence if f is. If $\sigma(\delta f) = 1$, then $M \amalg X \xrightarrow{f \amalg f_\tau} N \amalg Y$ has obstruction 0. Thus $\sigma(M \otimes L \xrightarrow{f \otimes 1} N \otimes L) = \sigma(X \otimes L \to Y \otimes L)$. Thus we see that in general $\sigma(M \otimes L \xrightarrow{f \otimes 1} N \otimes L) = \sigma(f \mid \delta M) \cdot \sigma(X \otimes L \to Y \otimes L)$, and hence if suffices to consider the latter problem. Of course $4\sigma(X \otimes L \xrightarrow{f_\tau \otimes 1} Y \otimes L) = \Theta\big(f_\tau \otimes 1\colon \delta(X \otimes L) \to \delta(Y \otimes L)\big)$ in $Z/8$.

One sees easily from the definition that

$$\delta(X \otimes L) = (K \times \bar{L})/\tau \times \mathrm{id}_{\delta L}\,,$$

$$\delta(Y \otimes L) = (S \times \bar{L})/\tau' \times \mathrm{id}_{\delta L}\,,$$

and

$$(f_\tau \otimes 1_L)\,|_\delta = k \times \mathrm{id}_{\bar{L}}\,.$$

Here $\bar{L}$ is the $Z/2$-manifold L opened up along its Bockstein; $\mathrm{id}_{\delta L}\colon \delta L_1 \to \delta L_2$ is the identification of the two copies of δL in $\delta \bar{L}$. Notice that $\tau \times \mathrm{id}_{\delta L}$ and $\tau' \times \mathrm{id}_{\delta L}$ are orientation reversing and thus $K \times \bar{L}/\tau \times \mathrm{id}_{\delta L}$ and $S \times \bar{L}/\tau' \times \mathrm{id}_{\delta L}$ are oriented manifolds.

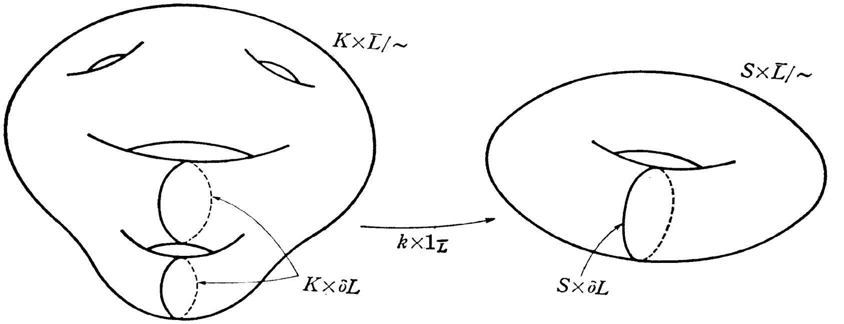


FIGURE 6.8

We will do surgery on the codimension one manifold $K \times \delta L$ until it maps by a homotopy equivalence to $S \times \delta L$. After doing this surgery we will have a new surgery problem. The torsion of its kernel in dimension $(2k + 1) + 2l$ will be $H_{2k+1}(K) \otimes T_{\bar{L}, D \oplus D}$ where $D \subset H_{2l}(\delta L)$ is a maximal self-annihilating subspace. The linking pairing on this torsion kernel will be given by

$$\bar{l}(x \otimes y, z \otimes w) = (x \cdot z) \cdot l(y, w)$$

where $x \cdot z$ is intersection on $H_{2k+1}(K)$ and $l(y, w)$ is the linking pairing defined in the previous lemma. We will also show that the quadratic form on this torsion group is given by $q(x \otimes y) = \mu(x) \cdot l(y, y)$ where μ is the Kervaire quadratic form on $H_{2k+1}(K)$. Thus by the algebraic Proposition 6.8 below we see that Θ of this surgery problem is

$$4 \cdot \sigma(K \longrightarrow S) \cdot d(T_{\bar{L}, D \oplus D}) = 4 \cdot d(T_{\bar{L}, D \oplus D}) .$$

Case (1′) is then completed by showing that Θ of this problem equals Θ of the original problem. This is proved by showing that the signature of the trace of the surgeries between them is 0. Thus $i(L) = d(\bar{L}, D \oplus D)$. This is an outline of the proof of Case (1′).

We wish to do surgery on $K \times \delta L \xrightarrow{k \times 1} S \times \delta L$ in a special manner. Namely we will inductively do surgery to produce $W \xrightarrow{F} S \times \delta L$ with $\partial_- W = K \times \delta L$ and $F | \partial_+ W$ more and more highly connected such that

(1) $\mathrm{Ker}_* F \cong H_{2k+1}(K) \otimes M$,

(2) $\mathrm{Ker}_* (F | \partial_+ W) \cong H_{2k+1}(K) \otimes M_+$ for appropriate graded groups M and M_+, and such that

(3) The inclusions $\partial_- W \rightarrow W$ and $\partial_+ W \rightarrow W$ preserve these tensor product splittings.

Suppose we have produced such a W with $F | \partial_+ W$ $(2k + 1 + i)$-connected. Then do surgery on classes representing

$$\{x \otimes \alpha_1, y \otimes \alpha_1, \cdots, x \otimes \alpha_n, y \otimes \alpha_n, x \otimes \beta_1, y \otimes \beta_1, \cdots, x \otimes \beta_s, y \otimes \beta_s\}$$

where x and y are the usual generators for $H_{2k+1}(K)$, $\{\alpha_1, \cdots, \alpha_n\}$ is a basis for M_+/Tor in dimension $i + 1$ and $\{\beta_1, \cdots, \beta_s\}$ generate $\mathrm{Tor}\, M_+$ in dimension $i + 1$. These surgeries enlarge the normal bordism W, so that it still satisfies (1), (2), and (3) above and in addition $F | \partial_+ W$ is $(2k + 1 + i + 1)$-connected.

We do this for $i<2l-1$. Thus we may assume

$$(\mathrm{Ker}\, F\,|\,\partial_+ W)_* = \begin{cases} 0 & * \leqq 2k+2l-1 \\ H_{2k+1}(K)\otimes H_{2l}(\delta L) & * = 2k+2l+1 \end{cases}$$

and that $\mathrm{Tor}\, K_{2k+2l}(F\,|\,\partial_+ W) = H_{2k+1}(K)\otimes \mathrm{Tor}\, H_{2l-1}(\delta L)$. Now we do the same type of surgery as above. This produces a new normal bordism

$$W \xrightarrow{F} S\times\delta L\times I$$

satisfying (1), (2), and (3) above with

$$\mathrm{Ker}\,(F\,|\,\partial_+ W)_* = \begin{cases} 0 & * \neq 2k+2l+1 \\ H_{2k+1}(K)\otimes[(H_{2l}(\delta L)/\mathrm{Tor})\oplus F] & * = 2k+2l+1 \end{cases}$$

where F is a free abelian group.

If we did surgery in dimension $2k+1+2l-1$ on

$$\{x\otimes\alpha_1,\ y\otimes\alpha_1,\ \cdots,\ x\otimes\alpha_n,\ y\otimes\alpha_n,\ x\otimes\beta_1,\ y\otimes\beta_1,\ \cdots,\ x\otimes\beta_s,\ y\otimes\beta_s\}$$

where $\{\alpha_1,\cdots,\alpha_n\}$ are a basis for $(M_+)_{2l-1}/\mathrm{Torsion}$ and $\{\beta_1,\cdots,\beta_s\}$ generate Torsion $(M_+)_{2l-1}$, then F has two generators b_i and b_i' for each β_i. $x\otimes b_i$ (or $y\otimes b_i$) as a class in $H_{2k+1+2l}(\partial_+ W)$ and then included into $H_{2k+1+2l}(W)$ has the same image as the inclusion of $x\otimes\beta_i^*$ (or $y\otimes\beta_i^*)\in H_{2k+1}(K)\otimes H_{2l}(\delta L)$ into $H_{2k+1+2l}(W)$. Here $\beta_i^*\in\mathrm{Torsion}\, H_{2l}(\delta L)$ is the dual element to β_i. The class $x\otimes b_i'$ (or $y\otimes b_i'$) is given by the following cycle.

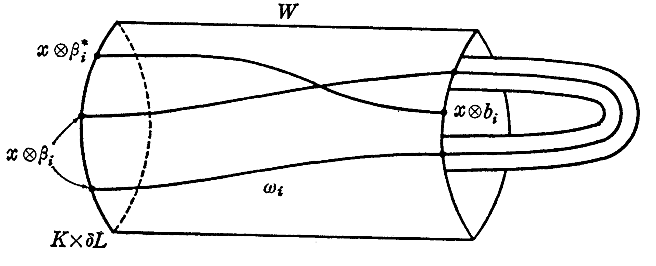


FIGURE 6.9

Let λ_i be the order of $(x\otimes\beta_i)$, h_i the handle added along the sphere representing the class of $x\otimes\beta_i$ in $\partial_+ W$, C_i a homology in $\partial_- W$ from $\lambda_i(x\otimes\beta_i)$ to 0, and ω_i a homology from $x\otimes\beta_i$ in $\partial_- W$ to $x\otimes\beta_i$ in $\partial_+ W$. Then $x\otimes b_i'$ is represented by $\lambda_i h_i\cup\lambda_i\omega_i\cup C_i$. Thus the intersection pairing on $\mathrm{Ker}\,(F\,|\,\partial_+ W) = H_{2k+1}(K)\otimes[(H_{2l}(\delta L)/\mathrm{Tor})\oplus F]$ is the usual intersection pairing on $H_{2k+1}(K)$ tensored with

$$\begin{array}{ccc} H_{2l}(\delta L)/\mathrm{Tor} & b_i\text{'s} & b_i'\text{'s} \end{array}$$
$$\left(\begin{array}{c|c|c} \begin{array}{c}\text{intersection}\\ \text{in } H_{2l}(\delta L)\end{array} & 0 & * \\ \hline 0 & 0 & I \\ \hline * & I & * \end{array}\right).$$

By changing the b_i''s we may make this matrix

$$\left(\begin{array}{c|c|c} \begin{array}{c}\text{intersection}\\ \text{in } H_{2l}(\delta L)\end{array} & 0 & 0 \\ \hline 0 & 0 & I \\ \hline 0 & I & 0 \end{array}\right).$$

Now we do surgery in dimension $2k + 1 + 2l$. Let $D \subset H_{2l}(\delta L)$ be a maximal self-annihilating subspace and $(d_1, \cdots, d_k)$ a basis for D. We do surgery on spheres representing

$$\{x \otimes d_1, y \otimes d_1, \cdots, x \otimes d_k, y \otimes d_k, x \otimes b_1, y \otimes b_1, \cdots, x \otimes b_s, y \otimes b_s\} .$$

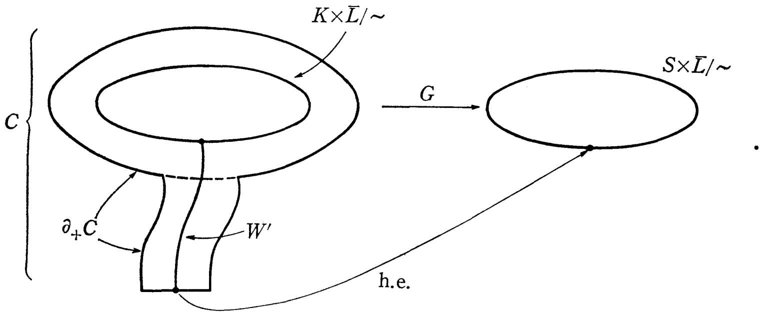
We must first show that this is possible, i.e., that the pairwise intersection numbers of these classes are all zero and that the self-intersection form vanishes on them. Since all of the classes are homologous in W to classes in $\partial_- W = K \times \delta L$ it is sufficient to make the calculations on their images in $\partial_- W$ (see [RS] or for example Proposition 5.7 and its corollary). But $\partial_- W = K \times \delta L$, thus the intersection form is intersection in K tensor with intersection in δL, and by [RS] the self-intersection form in $K \times \delta L$ is $\mu \otimes \cdot$ where μ is the self-intersection form in K and $\cdot$ is intersection in δL. Now $x \otimes d_i$ in $\partial_- W$ is $x \otimes d_i$ whereas $x \otimes b_i$ is $x \otimes \beta_i^*$. Since $(D + \text{Torsion})\cdot(D + \text{Torsion}) = 0$, the intersection and self-intersection forms are 0 on the subspace. Thus we may perform surgery on these classes.

Since we are doing surgery on a "subkernel" in Wall's notation, i.e., a subspace of 1/2 the rank of $\operatorname{Ker}(F \mid \partial_+ W)$, the result enlarges W to

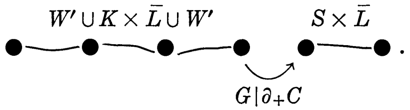
$$W' \xrightarrow{F'} S \times \delta L \times I$$

satisfying (1), (2), and (3) above and with $F \mid \partial_- W = k \times \delta f \colon K \times \delta L \to S \times \delta L$, and $F \mid \partial_+ W$ a homotopy equivalence.

Now we form $(K \times \bar{L}/\sim) \times I \cup W' \times I$.



Using the Meyer-Vietoris sequence it is easy to calculate $\mathrm{Ker}(G|\partial_+C)$. It is the same as the kernel of



Since all of the kernel groups of the various pieces split as tensor products of $H_{2k+1}(K)$ with some group, so does the kernel of G. $\mathrm{Ker}(G|\partial_+C) \cong H_{2k+1}(K) \otimes (M_G)_*$. M_G is calculated from the following sequence:

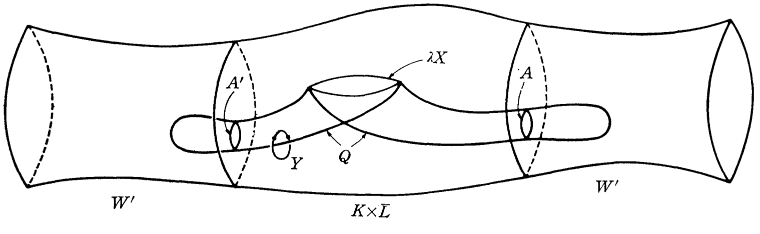
$$\longrightarrow H_*(\delta L) \oplus H_*(\delta L) \longrightarrow H_*(\bar{L}) \oplus M_{W'} \oplus M_{W'} \longrightarrow M_G \longrightarrow .$$

From this we see that

$$(M_G)_{2l} \cong [H_{2l}(\bar{L})/\mathrm{im}\, i_*(D + \mathrm{Tor}\, H_{2l}(\delta L) \oplus D + \mathrm{Tor}\, H_{2l}(\delta L))] \oplus \text{Free abelian group} .$$

Thus Torsion $H_{2k+2l+1}(\mathrm{Ker}\, G|\partial_+C) \cong H_{2k+1}(K) \otimes T_{\bar{L},D\oplus D}$.

Next we must show that the linking pairing and quadratic form on this torsion group are as claimed. Given a and b in $T_{\bar{L},D\oplus D}$ and x and y in $H_{2k+1}(K)$, represent $x \otimes a$ and $y \otimes b$ by disjoint cycles X and Y in $K \times \bar{L}$. Some multiple of X, λX, is homologous to $i_{1*}(A) + i_{2*}(A')$ where A and A' represent homology classes in $H_{2k+1}(K) \otimes (D + \text{Torsion}\, H_{2l}(\delta L))$ and i_1, i_2 are the two inclusions of $K \times \delta L$ into $K \times \bar{L}$. Both A and A' bound in W. Let the union of these three homologies be Q.



By definition $l(x \otimes a, y \otimes b) = (1/\lambda) Q \cdot Y$. This also equals $(1/\lambda) D \cdot Y$ where

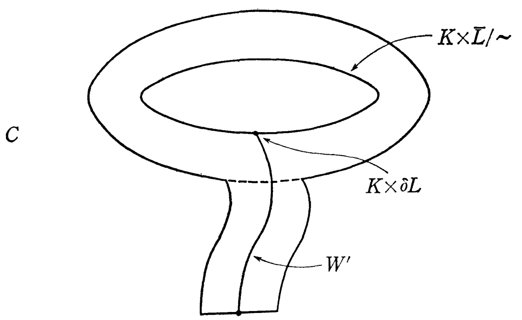
D is the homology of λX to $i_{1*}(A) + i_{2*}(A')$. Since $K \times \bar{L}$ is a product, this intersection is $(x \cdot y) \cdot l(a, b)$ where $x \cdot y$ is intersection K and $l(a, b)$ is the linking on $T_{(\bar{L}, D \oplus D)}$.

Note. One copy of W' is attached by the identity to $K \times \delta L_1$. The other copy of W' is attached to $K \times \delta L_2$ by $(\tau \times \mathrm{id})$. Thus to know that A' bounds in W' we must know that $(\tau \times \mathrm{id})_* A'$ bounds. But by the way we did surgery to create W' if $x \otimes a$ bounds in W so does $\tau(x \otimes a)$.

To calculate $q(x \otimes a)$ we must take Q and intersect it with X' where X' is a pushed-off copy of X using normal fields coming from the surgery problem. Since $K \times \bar{L} \to S \times \bar{L}$ is a product surgery problem $x \otimes a$ is represented by a product $S^{2k+1} \times M^{2l} \to K \times \bar{L}$ and the normal field is obtained by using the μ-form on K to immerse S^{2k+1} and then crossing this with any normal field on M^{2l}. Now the same argument as is used to derive equations 6.3 and 6.4 shows that $q(x \otimes a) = \mu(x) \cdot l(a, a)$. This completes the calculation of the kernel of the new surgery problem. Proposition 6.8 below then shows that for this problem

$$\Theta = 4 \cdot \sigma(K \to L) \cdot d(T_{(\bar{L}, D \oplus D)}) = 4d(T_{(\bar{L}, D \oplus D)}) .$$

We need only show that Θ of this problem equals Θ of the original problem, i.e., that the bordism



has 0 index.

Since the homology of $K \times \bar{L}$, $K \times \delta L$, and W' all admit involutions given by interchanging the generators of $H_{2k+1}(K)$, and the inclusions are equivariant with respect to these involutions, we see that the homology of C admits such an involution (though C does not). This involution changes the sign of the intersection pairing. Thus the index of C is 0. This completes the proof of Case (1′) for $Z/2$-manifolds.

To prove (1′) for $Z/2^r$-manifolds we observe that $\sigma(f \otimes 1_L \colon M \otimes L \to N \otimes L)$ is an element of order 2 in Q/Z and equal to $\sigma(\bar{f} \otimes 1_{\bar{L}} \colon \bar{M} \otimes \bar{L} \to \bar{N} \otimes \bar{L})$

where in this case $^-$ means open up to a $Z/2$-manifold. Thus $\sigma(f \otimes 1_L) = \sigma(\bar{f} \mid \delta M) \cdot i(\bar{L})$.

$$\sigma(\bar{f} \mid \delta M) = \sigma(f \mid \delta M) \text{ and } i(\bar{L}) = d(T_{(L, \underbrace{D \oplus \cdots \oplus D}_{2r})}) .$$

This proves Case (1′) in general.

General product formula. Let $M^n \xrightarrow{f} N$ and $L^l \xrightarrow{g} Q$ be two surgery problems of $Z/2^k$-manifolds whose product

$$M \otimes L \xrightarrow{f \otimes g} N \otimes Q$$

is of total dimension $4i$, i.e., $n + l \equiv 0(4)$.

We can use the above work to show

THEOREM 6.5.

$$\sigma(f \otimes g) = i(Q)\sigma(f) + i(N)\sigma(g) + 8\sigma(f)\sigma(g) .$$

First we note a few geometric facts about surgery obstructions.

(i) Any surgery problem $M \xrightarrow{f} N$ is cobordant to a disjoint sum

$$f \sim h + s$$

where h is a homotopy equivalence and s is a standard problem.

If $n \equiv 0(4)$ and $n > 4$, we may take s to be the multiple of the problem m corresponding to the Milnor manifold mapping to the sphere.

If $n \equiv 1(4)$ s is trivial.

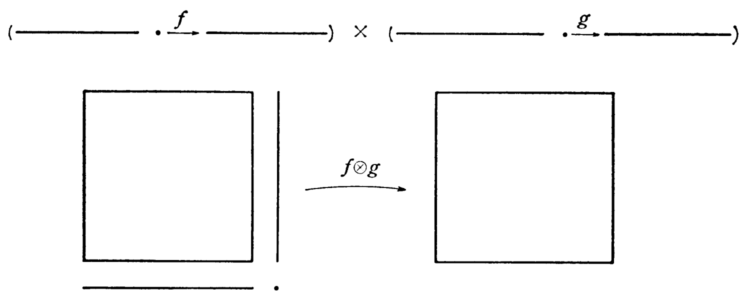
If $n \equiv 2(4)$ s is the problem k corresponding to the Kervaire manifold K^n mapping to the sphere, S^n, or trivial.

If $n \equiv 3(4)$, in the $Z/2$-manifold case, s is $f_\tau \colon X \to Y$ as above. In the $Z/2^r$ case we replicate this problem 2^{r-1} times and identify along Bocksteins to obtain k''. These descriptions follow from simply connected surgery theory.

Given $f \colon M^n \to N$ we may assume $\pi_1(\delta N) = \pi_1(\bar{N}) = 0$. Once we have this, we can do surgery on $\delta f \mid \colon \delta M \to \delta N$ to make it $[(n-1)/2] - 1$-connected. If $n \neq 3$ (4), we may complete this to make δf a homotopy equivalence. Once we have this, we perform surgery on $\bar{M} \xrightarrow{f} \bar{N}$ rel boundary to make it $[n/2] - 1$-connected. If n is odd we may complete this surgery to a homotopy equivalence. If n is even we may do surgery until the kernel is that of the standard problem. But these kernel elements may be represented by embedded spheres, and this enables us to write $\bar{M} = \bar{N}' \# K$ or $\bar{N}' \# W$ where $\bar{N}' \to \bar{N}$ is a homotopy equivalence. This proves all cases except (3). Case (3) is proved by forming a connected sum with the correct standard problem f_τ and then doing surgery to a homotopy equivalence.

(ii) The surgery obstruction for $M \xrightarrow{f} N$ is equal to that of the composition surgery problem $M \xrightarrow{f} N \xrightarrow{h} Q$ where h is an arbitrary homotopy equivalence. This is clear from any definition of the surgery obstruction.

These two facts allow us to decompose the general product into sums of products of homotopy equivalences and standard problems.



Thus

$$\begin{aligned}
f \otimes g &\sim (h + m) \otimes (h' + m') \\
&= h \otimes h' + m \times h' + h \times m' + m \times m' && \text{if } \dim f \equiv 0 (\bmod 4), \text{ or} \\
&\sim h \otimes (h' + \varepsilon f_\tau) \\
&= h \otimes h' + h \otimes \varepsilon f_\tau && \text{if } \dim f \equiv 1 (\bmod 4), \text{ or} \\
&\sim (h + \varepsilon k) \otimes (h' + \varepsilon' k) \\
&= h \otimes h' + \varepsilon k \times h' + h \times \varepsilon' k + \varepsilon k \times \varepsilon' k && \text{if } \dim f = 2 (\bmod 4)\ ,
\end{aligned}$$

where h and h' are homotopy equivalences m and m' are multiples of the Milnor surgery problem, and ε and ε' are 0 or 1.

(a) $\sigma(h \otimes h') = 0$ since $h \otimes h'$ is a homotopy equivalence.
(b) $\sigma(m \times h') = \sigma(m) \cdot i(\text{range } h')$.
(c) $\sigma(h \times m') = \sigma(m') \cdot i(\text{range } h)$.
(d) $\sigma(h \times \varepsilon' k) = i(\delta(\text{range } h)) \cdot \varepsilon'$.
(e) $\sigma(\varepsilon k \times h') = i(\delta(\text{range } h')) \cdot \varepsilon$.
(f) $\sigma(h \otimes k'') = i(\text{range } h) \cdot \sigma(k'')$.

The range of h is N and the range of h' is Q. Thus establishing (a) through (f) will prove the theorem. These are all proved in the same manner by factoring the map through a homotopy equivalence and then using remark (ii). For example $h \times k' = (h \times \text{Id}_{\text{range} k'}) \circ (\text{Id}_{\text{domain} h} \times k')$; and $h \times \text{Id}_{\text{range} k'}$ is a homotopy equivalence, so $\sigma(h \times k') = \sigma(\text{Id}_{\text{domain} h} \times k') = i(\delta(\text{domain } h)) \cdot \sigma(k')$. In each case we get a formula just as we want, except that in place of $i(\text{range } h \text{ or } h')$ we have $i(\text{domain } h \text{ or } h')$. However, since h is a homotopy equivalence $i(\text{domain } h) = i(\text{range } h)$. The reason for using $i(\text{range } h \text{ or } h')$ in the formula is that range h or $h' = N$ or Q but the domains are usually more accessible in the terms of the original problem.

We now consider terms of the form $i(M^m \otimes N^n)$.

PROPOSITION 6.6.

$$i(M^m \otimes N^n) = \begin{cases} i(M^m)\cdot i(N^n) & m \equiv 0(4) \text{ or } n \equiv 0(4) \\ 0 & \text{otherwise} . \end{cases}$$

We divide the proof into 4 cases according to the value of $m + n (\mathrm{mod}\, 4)$.

Proof. $m + n \equiv 0(4)$: Let $W^8 \xrightarrow{f} S^8$ be the standard surgery problem of dimension 8 and obstruction 1. Then

$$\begin{aligned} i(M \otimes N) &= \sigma\big(W \times (M \otimes N) \xrightarrow{f\times 1} S \times (M \otimes N)\big) \\ &= \sigma\big((W \times M) \otimes N) \xrightarrow{(f\times 1)\otimes 1} ((S \times M) \otimes N\big) \\ &= 0 \text{ unless } m \equiv 0(4) \text{ when it is } I(M)\cdot I(N) . \end{aligned}$$

$m + n \equiv 1(4)$: Let $f_\tau: (K \times I)/\sim \longrightarrow (S \times I)/\sim$ be as above. Then

$$i(M \otimes N) = \sigma\big(f_\tau \otimes (M \otimes N)\big) = \begin{cases} I(M)\cdot i(N) & m \equiv 0 \\ i(M)\cdot I(N) & \quad\;\; 1 \\ 0 & \quad\;\; 2 \\ 0 & \quad\;\; 3(4) . \end{cases}$$

$m + n \equiv 2$: Let $k: K \longrightarrow S$. Then

$$\sigma\big(k \times (M \otimes N)\big) = \begin{cases} I(M)\cdot i(N) & m \equiv 0 \\ 0 & \quad\;\; 1 \\ i(M)\cdot I(N) & \quad\;\; 2 \\ 0 & \quad\;\; 3 . \end{cases}$$

$m + n \equiv 3$: $i(M \otimes N) = 0$ since i of a $4k + 3$-manifold is 0.

This proves 6.6. As a corollary we have the product formula for the signature $I(M \otimes N) = I(M)\cdot I(N)$. This fact is surprisingly hard to prove. In fact we know no direct proof of it.

We now develop the algebra necessary to prove the results quoted above. We begin with a few definitions. A finite linking space (K, l) is a finite abelian group K together with $l: K \otimes K \longrightarrow Q/Z$ a non-singular map. In addition l is required to be either symmetric or skew symmetric. The de Rham invariant, $d(K, l) \in Z/2$ is defined in the skew symmetric case. It is the rank mod 2 of $K \otimes Z/2$. Thus if L^{4l+1} is a closed orientable manifold, then $d(L) = d(\text{Torsion } H_{2l}(L)$, natural linking).

A quadratic space, (K, q), is a finite abelian group and $q: K \longrightarrow Q/Z$ so that $q(x + x') - q(x) - q(x')$ defines a linking pairing on the space. We say that (K, q)

refines (K, l) if $l(x, x') = q(x + x') - q(x) - q(x')$. The Θ invariant of (K, q) is $1/2\pi(\mathrm{Arg}(\sum_{x \in K} e^{2\pi i q(x)})) \in Z/8$. Thus $\Theta(M^{4m+3} \xrightarrow{f} N) = \Theta(\mathrm{Torsion}(\mathrm{Ker}\, f)_{2m+1}$, associated form$)$. An inner product space $(L, \cdot)$ is a free abelian group L, and a non-singular pairing $L \otimes L \to Z$. The pairing is required to be symmetric or skew symmetric. If it is symmetric and satisfies the additional property that $y \cdot y$ is even we say the pairing is even.

Note. A pairing is even if and only if there is a quadratic function $q: L \to Z$ so that $y \cdot y' = q(y + y') - q(y) - q(y')$. In the case of skew symmetric pairings, $\varphi: L \to Z/2$ is an associated quadratic form if $\varphi(y + y') - \varphi(y) - \varphi(y') \equiv y \cdot y' \bmod 2$. Such a triple $(L, \cdot, \varphi)$ has an Arf invariant in $Z/2$, see [RS].

PROPOSITION 6.7. *Let $(L, \cdot)$ be an even symmetric inner product space and (K, l) a symmetric linking space. Then $K \otimes L$ has a natural product linking defined by*

$$\bar{l}(x \otimes y, x' \otimes y') = l(x, x')(y \cdot y') \in Q/Z \quad (\textit{extended by bilinearity}) .$$

Associated to this linking there is a quadratic function determined by $q(x \otimes y) = l(x, x) \cdot (1/2)(y, y) \in Q/Z$. $(K \otimes L, q)$ is a quadratic space and $\Theta(K \otimes L, q) = 0$.

PROPOSITION 6.8. *Let $(L, \cdot, \varphi)$ be a skew symmetric inner product space together with an associated quadratic form. Let (K, l) be a skew symmetric linking space. $K \otimes L$ has a natural product linking given by*

$$\bar{l}(x \otimes y, x' \otimes y') = l(x, x')(y \cdot y') \in Q/Z \quad (\textit{extended by bilinearity}) .$$

There is an associated quadratic form determined by $q(x \otimes y) = l(x, x) \cdot \varphi(y) \in Z/2 \hookrightarrow Q/Z$. $(K \otimes L, q)$ is a quadratic space and $\Theta(K \otimes L, q) = 4d(K, l) \cdot \sigma(L, \cdot, \varphi)$ where $\sigma(L, \cdot, \varphi)$ is its Arf invariant.

Proof of 6.7. It is clear that the linking pairing, $\bar{l}$, given in the statement of 6.7 is non-singular and symmetric since both l and the inner product are. Given the linking, $\bar{l}$, and the values of q on a generating set, there is at most one quadratic form associated to $\bar{l}$ with the given values. We show such a q exists by the following equational definition:

$$q\left(\sum_i x_i \otimes y_i\right) = \sum_i l(x_i, x_i) \cdot \frac{1}{2} y_i \cdot y_i + \sum_{i<j} l(x_i, x_j)(y_i \cdot y_j) .$$

This proves $(K \otimes L, q)$ is a quadratic space.

One way to define linking spaces is as follows. Let $(V, \cdot)$ be a rational symmetric inner product space, and $R \hookrightarrow V$ an integral lattice (i.e., $r \cdot r' \in Z$ if $r, r' \in R$) of maximal rank. Let R^* be its dual, $R^* = \{x \in V \mid x \cdot R \in Z\}$.

Define $l\colon R^*/R \otimes R^*/R \to Q/Z$ by $l(x + R, x' + R) = x \cdot x' \in Q/Z$. R^*/R is finite since R is of maximal rank and $\cdot$ is nonsingular. l is checked to be well-defined, symmetric, and non-singular.

Wall has shown all linking spaces arise this way, up to isomorphism [W2]. Thus let $(V, \cdot, R)$ be such that $(R^*/R, l)$ is isomorphic to (K, l). Furthermore $(V \otimes L, \cdot)$ is a non-singular rational inner product space. In this space $(R \otimes L, \cdot)$ is an *even* lattice of maximal rank with dual $R^* \otimes L$. We have a short exact sequence

$$0 \longrightarrow R \otimes L \longrightarrow R^* \otimes L \longrightarrow K \otimes L \longrightarrow 0 .$$

Since $R \otimes L$ is an even lattice the inner product in $V \otimes L$ determines a quadratic function on $K \otimes L$, as in 5.3. $q'(x \otimes y) = 1/2 l(x, x) y \cdot y \in Q/Z$. So that $q' = q$. Thus by 5.9 $\Theta(K \otimes L, q) = I(V \otimes L) \bmod 8$. Since L is an even symmetric form $I(L) \equiv 0(8)$. Thus $I(V \otimes L) = I(V) \cdot I(L) \equiv 0(8)$. This proves 6.7.

Proof of 6.8. To prove 6.8 we classify skew linking spaces and skew inner product spaces with quadratic forms.

Classification of skew inner product spaces. Let $H = \left(Z \oplus Z, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. Then if $(L, \cdot)$ is a skew inner product space $(L, \cdot) \cong \bigoplus H$. To see this take a basis vector, y. By nonsingularity there exists y' so that $y \cdot y' = 1$. Since $y \cdot y = 0 = y' \cdot y'$, y and y' span a direct summand isomorphic to H. Extend y, y' to a basis $y, y', z_1, \cdots, z_k$. Let $z_i' = z_i + (y' \cdot z_i) y - (y \cdot z_i) y'$. Clearly $y, y', z_1', \cdots, z_k'$ is a basis and $y \cdot z_i' = y' \cdot z_i' = 0$. This splits $(L, \cdot) \cong H \oplus (L', \cdot)$, and induction completes the classification. If there is an associated quadratic function φ it automatically splits, also. On H itself there are two possibilities for φ, up to isomorphism. Let y and y' be generators then $\varphi(y) = \varphi(y') = 1$ or $\varphi(y) = \varphi(y') = 0$ are the two forms. (If $\varphi(y) = 1$ and $\varphi(y') = 0$ replace y by $y + y'$ and $\varphi(y + y') = 0$.) Call the first space H^1 and the second H^0.

Classification of skew symmetric inner product spaces with associated quadratic forms. $(L, \cdot, q) \cong \bigoplus H^1 \bigoplus H^0$. The number of H^1 terms mod 2 is invariant and in fact is the Arf invariant (see [RS]).

Skew symmetric linking spaces are classified in an analogous manner. The results, due to de Rham are $(K, l) \simeq \bigoplus_k T_k \bigoplus T'_{2^k} \oplus \varepsilon$ where $T_k = Z/k \oplus Z/k$ generated by x and x' with $l(x, x) = l(x', x') = 0$, and $l(x, x') = 1/k$. $T'_{2^k} = Z/2^k \oplus Z/2^k$ generated by x and x' with $l(x, x) = 1/2$, $l(x', x') = 0$, and $l(x, x') = 1/2^k$. ε denotes 0 or 1 copies of $Z/2$ with $l(x,x) = 1/2$. The de Rham invariant of such a form is, of course, ε. This completes the classification, and we return to the proof of 6.8 proper.

Again $\bar{l}$ given in the statement of 6.8 is easily seen to be a symmetric nonsingular linking pairing. q is proved to extend (it has at most 1 extension as in 6.7) by the formula

$$q\left(\sum_i x_i \otimes y_i\right) = \sum_i l(x_i, x_i)\cdot\varphi(y_i) + \sum_{i<j} l(x_i, x_j)(y_i, y_j)$$

where the first term is in $Z/2 \hookrightarrow Q/Z$. Thus $(K \otimes L, \bar{l})$ is a quadratic space.

To calculate $\Theta(K \otimes L, q)$ we will assume (K, l) and $(L, \cdot, \varphi)$ are indecomposable elements in the classification schemes given above.

(1) $K = T_k$ and $L = H^0$. Then $K \otimes L = \bigoplus_4 Z/k$ generated by $x \otimes y$, $x \otimes y'$, $x' \otimes y$, $x' \otimes y'$.

$$\begin{aligned} \bar{l}(x \otimes y, x \otimes y') &= \bar{l}(x \otimes y, x' \otimes y) \\ &= \bar{l}(x' \otimes y', x \otimes y') = \bar{l}(x' \otimes y', x' \otimes y) = 0\,. \end{aligned}$$

Thus the subspace generated by $\{x \otimes y, x' \otimes y'\}$ is orthogonal to that generated by $\{x \otimes y', x' \otimes y\}$. Since $\bar{l}(x \otimes y, x' \otimes y') = 1/k = -\bar{l}(x' \otimes y, x \otimes y')$ and $q(z) = 0$ for all four given generators, we have split $(K \otimes L, q)$ into $(A, q) \oplus (A, -q)$. Since Θ is additive, this implies $\Theta(K \otimes L, q) = 0$.

(2) $K = T_k$ and $L = H^1$. The quadratic function takes on exactly the same values as above so that $\Theta(K \otimes L, q) = 0$.

(3) $K = T'_{2^k}$ and $L = H^0$. $K \otimes L \cong \bigoplus_4 Z/2^k$ generated by $x \otimes y$, $x \otimes y'$, $x' \otimes y$, and $x' \otimes y'$. Using the basis $a_1 = x' \otimes y$, $a_2 = x \otimes y' + 2^{k-1}x' \otimes y'$, $a_3 = x \otimes y$, and $a_4 = x' \otimes y'$, we decompose $(K \otimes L, q)$ into $(A, q) \oplus (A, -q)$. This proves $\Theta = 0$.

(4a) $K = T'_{2^k}$, $L = H^1$, and $k > 1$.

Using the basis given in 3 we write $(K \otimes L, q) = (A, q) \oplus (A, -q)$ to show that $\Theta = 0$.

(4b) $K = T'_2$ and $L = H^1$. $T'_2 \cong (Z/2, 1/2) \oplus (Z/2, 1/2)$.

Thus $\Theta(T'_2 \otimes H^1) = 2\Theta\big((Z/2, 1/2) \otimes H^1\big)$. (4b) then follows from Case 6 where we show $\Theta\big((Z/2, 1/2) \otimes H^1\big) = 4 \in Z/8$.

(5) $K = (Z/2, 1/2)$ and $L = H^0$.

Then $K \otimes L = Z/2 \oplus Z/2$ generated by $x \otimes y$ and $x \otimes y'$. $\bar{l}(x \otimes y, x \otimes y') = 1/2$. $q(x \otimes y) = q(x \otimes y') = q(0) = 0$. $q(x \otimes y + x \otimes y') = 1/2$. Thus our Gaussian sum is $e^0 + e^0 + e^0 + e^{\pi i} = 1 + 1 + 1 - 1 = 2$. Thus $(1/2\pi)\operatorname{Arg} = 0$ and $\Theta = 0$.

(6) $K = (Z/2, 1/2)$ and $L = H^1$. Then $K \otimes L = Z/2 \oplus Z/2$ with $q(x \otimes y) = q(x \otimes y') = q(x \otimes y + x \otimes y') = 1/2$ and $q(0) = 0$. The Gaussian sum is $-1 - 1 - 1 + 1 = -2$, and $\Theta = 4$.

This proves 6.8 since in all cases except 6 either $d(K, l) = 0$ or $\sigma(K, \cdot, \varphi) = 0$. In Case (6) $d(K, l) = 1 = \sigma(L, \cdot, \varphi)$, and $\Theta = 4d(K, l)\cdot\sigma(L, \cdot, \varphi)$.

7. Extension of $\mathfrak{L}_E$ to pl and topological R^n-bundles

We may now apply Theorem 4.1 and the product formula 6.6 to produce a characteristic class for oriented pl R^n-bundles, $\pi: E \to B$. This class $\mathfrak{L}_E \in H^{4*}(B; Z_{(2)})$ will be an extension of the class defined for smooth bundles in Section 3. The general characteristic class also satisfies the following properties.

(1) $\mathfrak{L}_E$ is a natural, stable characteristic class for piecewise linear R^n-bundles.

(2) $(\mathfrak{L}_E \otimes 1_Q)\cdot L(E) = 1$ where $L(E)$ is the Hirzebruch L class.

(3) $(\mathfrak{L}_E)_2 = V^2(E)$ where V is the Wu class.

(4) $\mathfrak{L}_{E\times F} = \mathfrak{L}_E \otimes \mathfrak{L}_F$ (multiplicativity).

(5) If $f: M \to E^+$ is a map of an oriented, closed, smooth manifold into the one point compactification of E, which is pl transverse regular to B with pre-image V, then an orientation of E, $U_E \in H^{\dim E}(E^+; Z)$, induces an orientation of V, and the signature of V is

$$\langle L(\tau_M)\cdot f^*(\mathfrak{L}_E\cdot U_E), [M]\rangle = \langle \mathfrak{L}_M\cdot f^*(\mathfrak{L}_E\cdot U_E), [M]\rangle .$$

(6) If the manifold in (5) is a $Z/2^k$-manifold, then V is also and signature $(V)_{2^k} = \langle \mathfrak{L}_M\cdot f^*(\mathfrak{L}_E\cdot U_E), [M_{2^k}]\rangle \in Z_{2^\infty}$.

(7) Properties (5) and (6) hold for all pl manifolds mapping into E^+.

First we construct a class $\mathfrak{L}'_E$ for all pl R^n-bundles using properties (5) and (6) to find the homomorphisms required by Theorem 4.1. By the naturality of this construction and Corollary 3.4 (which says $\mathfrak{L}_E$ for smooth bundles satisfies (5) and (6)) we see that this class is indeed an extension of the class in Section 3. We then proceed to show that it satisfies properties (1) through (4). We then prove a pl-index theorem at 2, namely that $\langle \mathfrak{L}_V, [V]\rangle =$ signature V for V a pl Z or $Z/2^k$-manifold, where as usual $\mathfrak{L}_V$ is $\mathfrak{L}(\nu_V)$. Using this and the multiplicative property, (7) follows. Finally, once we have property (7) we prove an extension of Theorem 4.1 to pl-bordism.

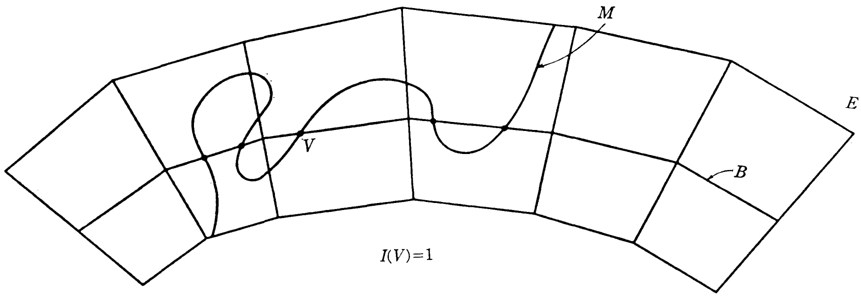
We begin the construction of $\mathfrak{L}'_E$. Let $E^n \to B$ be a pl R^n-bundle. According to Corollary 4.8, homomorphisms $\tilde{I}: \tilde{\Omega}_{4*+n}(E^+; Z_{(2)}) \to Z_{(2)}$ and $\tilde{I}_{2^k}: \tilde{\Omega}_{4*+n}(E^+; Z/2^k) \to Z_{2^k}$ satisfying:

(1) $\{\tilde{I}_{2^k}\}$ are compatible with $i: Z/2^k \hookrightarrow Z/2^{k+1}$.

(2) $\tilde{I}$ and $\tilde{I}_{2^k}$ are compatible with $r: Z_{(2)} \to Z/2^k$.

(3) $\tilde{I}$ and $\tilde{I}_{2^k}$ are multiplicative with respect to the signature given cohomology classes $\tilde{\mathfrak{L}} \in \tilde{H}^{4*+n}(E^+; Z_{(2)}) \cong H^{4*}(B; Z_{(2)})$.

We construct such homomorphisms. $\widetilde{I}$ and $\widetilde{I}_{2^k}$ are defined the same way. Assign to $f: M \to E^+$, the signature of $V = f^{-1}(B)$, after f has been moved to be transverse regular to B.



Since the bordism class of V is an invariant of that of (M, f), so is its signature. Thus $\widetilde{I}([M, f]) = I(V)$ and $\widetilde{I}_{2^k}([M, f]) = I(V_{2^k}) \in Z/2^k \hookrightarrow Z_{2^\infty}$ are functions on the bordism groups.

One sees easily that $\widetilde{I}$ and $\widetilde{I}_{2^k}$ are homomorphisms and compatible with $r: Z_{(2)} \to Z/2^k$. To see that the $\widetilde{I}_{2^k}$ are compatible with $i: Z/2^k \hookrightarrow Z/2^{k+1}$, recall that the map from $\Omega_*(X; Z/2^k) \to \Omega_*(X; Z/2^{k+1})$, is replication along the Bockstein. If we replicate a transversal map then its preimage is replicated and thus has twice the signature. We check that $\widetilde{I}$ and $\widetilde{I}_{2^k}$ are multiplicative with respect to the signature. If $[M, f]$ is a bordism element then $[M, f]\cdot[N]$ is represented by $M \times N \xrightarrow{\pi_M} M \xrightarrow{f} E^+$. If f is transverse regular with pre-image V, then $f \circ \pi_M$ is transverse regular with pre-image $V \times N$. Thus

$$\widetilde{I}([M, f]\cdot[N]) = I(V \times N) = I(V)\cdot I(N) = \widetilde{I}(M, f)\cdot I(N) .$$

If $[M_{2^k}, f]$ is a $Z/2^k$-bordism element, then $[M_{2^k}, f] \otimes [N]$ is represented by $M \otimes N \xrightarrow{\rho} M \times N \xrightarrow{\pi_M} M \xrightarrow{f} E^+$. If f is transverse regular with pre-image V_{2^k}, then $f \circ \pi_M \circ \rho$ is transverse regular with pre-image $V \otimes N$. Thus

$$\widetilde{I}_{2^k}([M_{2^k}, f] \otimes [N_{2^k}]) = I(V \otimes N) = I(V)\cdot I(N) = \widetilde{I}_{2^k}([M_{2^k}, f])\cdot I(N)$$

by Theorem 6.6. This shows $\widetilde{I}$ and $\widetilde{I}_{2^k}$ are multiplicative with respect to the signature. Thus they define a cohomology class $\bar{\mathfrak{L}}_E \in \widetilde{H}^{4*+n}(E^+; Z_{(2)})$. Let $\mathfrak{L}'_E \in H^{4*}(B; Z_{(2)})$ satisfy $\mathfrak{L}'_E \cdot U_E = \bar{\mathfrak{L}}_E$. Thus if $f: M \to E^+$ is a smooth Z or $Z/2^k$-bordism element which is transverse regular to B with pre-image V then $I(V) = \widetilde{I}([M, f]) = \langle \mathfrak{L}_M \cdot f^*(\mathfrak{L}'_E \cdot U_E), [M]\rangle$. If E is a smooth bundle, $\mathfrak{L}_E$ satisfies the same formula. Thus by uniqueness $\mathfrak{L}'_E = \mathfrak{L}_E$ for smooth bundles. We rename the class $\mathfrak{L}_E$. We now show it satisfies properties (1) through (4).

Proof of (1). *Naturality and stability.* Naturality follows immediately from the naturality of transversality and Theorem 4.1. To prove stability note that

(1) $(E^n \oplus \varepsilon^1)^+ \cong E^+ \wedge S^1$, and under that identification;

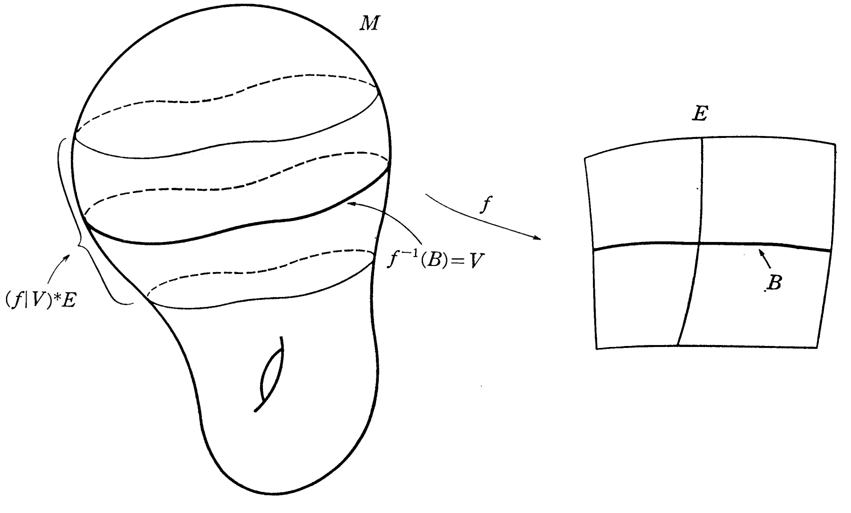
(2) $U_{E\oplus\varepsilon^1} = \sigma^* U_E$ where σ^* is suspension isomorphism.

(3) Suspension in bordism is given by crossing with S^1.

From these facts it is clear that both the homomorphism and the formula commute with suspension.

Proof of (2). $(\mathfrak{L}_E \otimes 1_Q)\cdot L(E) = 1$. This follows immediately from the corresponding property for $\mathfrak{L}_E$ restricted to smooth bundles. In fact this is the original definition of Thom of the rational characteristic classes for pl-theory.

Proof of (3). $(\mathfrak{L}_E)_2 = V_E^2$. If $M_2 \xrightarrow{f} E^+$ is a transversal map of a $Z/2$-manifold into E^+ with pre-image V_2, then $\nu_V = \nu_M | V \oplus \nu_{V\hookrightarrow M} = \nu_M | V \oplus (f|V)^*E$.



Since V^2 is a multiplicative class and $\langle V_V^2, [V]\rangle = I(V)_2$ we have

$$\langle V_M^2\cdot f^*(V_M^2\cdot U_E), [M]\rangle = \langle i^*(V_M^2\cdot V_{f*E}^2), [V]\rangle\langle V_V^2, [V]\rangle = I(V)_2\ .$$

Thus V_E^2 and $\mathfrak{L}_E$ give the same homomorphism on $Z/2$-bordism. By uniqueness of the $Z/2$ theorem $(\mathfrak{L}_E)_2 = V_E^2$.

Note. We could give a completely analogous argument for the rational case.

Proof of (4). $\mathfrak{L}_{E\times F} = \mathfrak{L}_E \otimes \mathfrak{L}_F$ in $H^{4*}(B\times B'; Z_{(2)})$. Let $E \xrightarrow{\pi} B$ and $F \xrightarrow{\pi'} B'$ be pl R^n-bundles with orientations U_E and U_F. Then $E\times F \xrightarrow{\pi\times\pi'} B\times B'$ is a pl R^{2n}-bundle with orientation $U_E \otimes U_F$. $(E\times F)^+ = E^+ \wedge F^+$. We will check first that $\mathfrak{L}_{E\times F}$ and $\mathfrak{L}_E\otimes\mathfrak{L}_F$ evaluate the same on products in

$\widetilde{\Omega}_*(E^+ \wedge F^+; Q)$ and $\widetilde{\Omega}_*(E^+ \wedge F^+; Z/2^k)$. We consider the rational case first. Let $f: M \to E^+$ and $g: N \to F^+$ be transverse regular with preimages V and W. Then a representative for $[M, f] \times [N, g]$ is $M \times N \xrightarrow{f \wedge g} E^+ \wedge F^+$. It is transverse regular with preimage $V \times W$. Thus $\widetilde{I}([M, f] \otimes [N, g]) = \widetilde{I}([M, f]) \cdot \widetilde{I}([N, g])$. This implies that

$$\begin{aligned}\langle \mathfrak{L}_{M\times N} \cdot (f \wedge g)^*(\mathfrak{L}_{E\times F} \cdot U_{E\times F}), [M \times N]\rangle \\ = \langle \mathfrak{L}_M \cdot f^*(\mathfrak{L}_E \cdot U_E), [M]\rangle \cdot \langle \mathfrak{L}_N \cdot f^*(\mathfrak{L}_F \cdot U_F), [N]\rangle .\end{aligned}$$

The first term is equal to (using property 4 for the smooth $\mathfrak{L}_M$ class)

$$\begin{aligned}\langle \mathfrak{L}_M \otimes \mathfrak{L}_N \cdot (f \wedge g)^*(\mathfrak{L}_{E\times F} \cdot U_E \otimes U_F), [M] \otimes [N]\rangle \\ = \langle \mathfrak{L}_M \otimes \mathfrak{L}_N \cdot (f \times g)^* \mathfrak{L}_{E\times F} \cdot f^* U_E \otimes g^* U_F, [M] \otimes [N]\rangle\end{aligned}$$

whereas the second term is equal to

$$\langle \mathfrak{L}_M \otimes \mathfrak{L}_N \cdot (f \times g)^*(\mathfrak{L}_E \otimes \mathfrak{L}_F) \cdot f^* U_E \otimes g^* U_F, [M] \otimes [N]\rangle .$$

Thus $\mathfrak{L}_E \otimes \mathfrak{L}_F$ and $\mathfrak{L}_{E\times F}$ evaluate the same on rational products. We now check that they are the same on $Z/2^k$-products. $([M_{2^k}, f] \otimes [N_{2^k}, g])$ is represented by $M \otimes N \xrightarrow{\rho} M \times N \xrightarrow{f \wedge g} E^+ \wedge F^+$. This is transverse regular with pre-image $V \otimes W$ provided f and g are transverse regular with pre-image V and W respectively. Thus

$$\widetilde{I}_{2^k}([M_{2^k}, f] \otimes [N_{2^k}, g]) = \widetilde{I}_{2^k}([M_{2^k}, f]) \cdot \widetilde{I}_{2^k}([N_{2^k}, g]) .$$

The argument above then shows that $\mathfrak{L}_E \otimes \mathfrak{L}_F$ and $\mathfrak{L}_{E\times F}$ evaluate the sums of this product. Multiplicativity then is completed with the following lemma.

LEMMA 7.1. (1) $\widetilde{\Omega}_*(X \wedge Y; Q) \cong \widetilde{\Omega}_*(X; Q) \otimes_{\widetilde{\Omega}_*(pt;Q)} \widetilde{\Omega}_*(Y; Q)$.

(2) *$\widetilde{\Omega}_*(X \wedge Y; Z_{2^\infty})$ is generated by the images of $\widetilde{\Omega}_*(X; Z/2^k) \otimes \widetilde{\Omega}_*(Y; Z/2^k)$ in $\widetilde{\Omega}_*(X \wedge Y, Z/2^k)$.*

Proof. (1) Since Q is a field, the result follows easily from the isomorphism $\Omega_*(X, Q) \cong H_*(X, \Omega_*(pt, Q))$.

(2) Using the Conner-Floyd spectral sequence [CF], we see that $\widetilde{\Omega}_*(X; Z_{2^\infty}) \cong \widetilde{H}_*(X; \Omega_*(pt; Z_{2^\infty}))$. Since $\Omega_*(pt; Z_{2^\infty}) \cong \bigoplus Z_{2^\infty} \bigoplus Z/2$, it suffices to prove

(a) $\widetilde{H}_*(X; Z/2) \otimes \widetilde{H}_*(Y; Z/2) \to \widetilde{H}_*(X \wedge Y; Z/2)$ is an isomorphism, and

(b) $\bigoplus_k \widetilde{H}_*(X; Z/2^k) \otimes \widetilde{H}_*(Y; Z/2^k) \to \widetilde{H}_*(X \wedge Y; Z_{2^\infty})$ is onto.

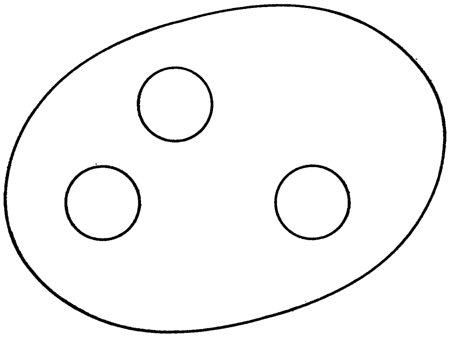
(a) follows immediately since $Z/2$ is a field. To show (b) we show that $\widetilde{H}_*(X; Z/2^k) \otimes \widetilde{H}_*(Y; Z/2^k) \to \widetilde{H}_*(X \wedge Y; Z_{2^\infty})$ is onto {elements of order 2^k}/{elements of order 2^{k-1}}. To prove this we break the chain complexes of X and Y up into elementary complexes and show that this is true in case by case analysis. (The crucial case is

$$X = \{Z \xrightarrow{x2^s} Z\} \quad Y = \{Z \xrightarrow{x2^r} Z\} \quad k \leqq r \leqq s.)$$

THEOREM 7.2. (*pl signature theorem at the prime* 2). *If V is a pl Z- or $Z/2^k$-manifold*

$$\langle \mathfrak{L}_{\nu(V)}, [V] \rangle = I(V) .$$

Proof. The case of a closed manifold is classical. Suppose V is a $Z/2^k$-manifold. Let $S^N_{2^k} = (D^N - (2^k - 1)B^N_\varepsilon, S^{N-1})$ where B^N_ε are disjoint balls in D^N.



Then $\nu(S^N) = \varepsilon^N$ and if N is sufficiently large $V_{2^k} \hookrightarrow S^N_{2^k}$ with normal bundle, the stable normal bundle of V_{2^k}. Let $\pi\colon S^N_{2^k} \to T(\nu(V))$ be the Thom-Pontrjagin collapsing map. Then

$$\begin{aligned} I(V) = \tilde{I}(S^N_{2^k}, \pi) &= \langle \mathfrak{L}_{S^N_{2^k}} \cdot \pi^*(\mathfrak{L}_{\nu(V)} \cdot U_{\nu(V)}), [S^N] \rangle \\ &= \langle 1 \cdot \mathfrak{L}_{\nu(V)} \cdot U_{\nu(V)}, [T(\nu(V))] \rangle = \langle \mathfrak{L}_{\nu(V)}, [V] \rangle . \end{aligned}$$

Note. At odd primes there is a signature formula using the canonical K-homology class constructed in [S1], [S2].

Property (7). If $f\colon M \to E^+$, M a *pl* Z- or $Z/2^k$-manifold, with f transverse regular to B and $f^{-1}(B) = V$, then $I(V) = \langle \mathfrak{L}_M \cdot f^*(\mathfrak{L}_E \cdot U_E), [M[\rangle$.

Proof. $I(V) = \langle \mathfrak{L}_{\nu_V}, [V] \rangle$. $\nu(V) = \nu_M | V \oplus (f|V)^*E$. Thus $\mathfrak{L}_{\nu_V} = i^*\mathfrak{L}_M \cdot f^*\mathfrak{L}_E$, and

$$\langle \mathfrak{L}_M \cdot f^*(\mathfrak{L}_E \cdot U_E), [M] \rangle = \langle i^*\mathfrak{L}_M \cdot f^*\mathfrak{L}_E, [V] \rangle = I(V) .$$

THEOREM 7.3. *$H^{4*}(X; Z_{(2)})$ is naturally isomorphic to the group, $\mathcal{C}^{pl}_{(2)}(X)$ of commutative diagrams*

$$\left\{ \begin{array}{ccc} \Omega^{pl}_{4*}(X; Q) & \xrightarrow{\sigma_Q} & Q \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Omega^{pl}_{4*}(X; Z_{2^\infty}) & \xrightarrow{\sigma_2} & Z_{2^\infty} \end{array} \right\}$$

such that σ_Q and σ_2 are multiplicative with respect to the index. The isomorphism is given by $A \mapsto \sigma_A$ where $\sigma_A(M, f) = \langle f^*A \cdot \mathfrak{L}_M, [M] \rangle$.

Proof. Once we have the class $\mathfrak{L}_M$ for pl manifolds, this theorem is proved exactly as Theorem 4.1 is proved. We define homomorphisms $\Omega_{4*}^{pl}(X; G) \to H_{4*}(X; G)$, by $[M, f] \mapsto f_*(\mathfrak{L}_M \cap [M])$. These are onto and the calculation of the kernels is completely analogous, that is to say, the same obstruction theory in Section 4 is valid for pl-bordism. Thus the kernels of the maps $\lambda_Q^{pl}\colon \Omega_{4*}^{pl}(X; Q) \to H_{4*}(X; Q)$ and $\lambda_2^{pl}\colon \Omega_{4*}^{pl}(X; Z_{2^\infty}) \to H_{4*}(X; Z_{2^\infty})$ are generated by the same relations.

COROLLARY 7.4. *Given homomorphisms as above and a cohomology class, α, giving these homomorphisms on smooth bordism then α also gives them on pl-bordism.*

Note. Everything in this section is valid for topological bundles and manifolds. The proofs are exactly the same except for one additional technicality, the lack of 4 dimensional transversality theorem. This is overcome by forming the product with CP^2.

8. The cohomology of G/pl and G/Top

The other natural situation in which we have homomorphisms from the bordism occurs in the classifying space for surgery problems, G/pl. The homomorphism assigns to $f\colon M^{4n} \to G/pl$ the surgery obstruction of the corresponding normal map $(f, \tilde{f})\colon (N, \nu_N) \to (M, \xi)$ obtained by transversality. As we noted in Section 5 the value of this homomorphism lies in Z or $Z/2^k$ if M is a Z or $Z/2^k$-manifold. We collect this information into a cohomology class $\mathfrak{L}$.

THEOREM 8.7. *Associated to any surgery problem $(f, \tilde{f})\colon (N, \nu_N) \to (M, \xi)$ there is a natural cohomology class $\mathfrak{L}_f \in H^{4*}(M, Z_{(2)})$ such that*

(1) $8\mathfrak{L}_f + 1 = \mathfrak{L}_\xi \cdot \mathfrak{L}_M^{-1}$

(2) $\mathfrak{L}_{f\times g} = \mathfrak{L}_f \otimes 1 + 1 \otimes \mathfrak{L}_g + 8\mathfrak{L}_f \otimes \mathfrak{L}_g$.

This class $\mathfrak{L}_f$ figures in a cohomological formula for surgery obstruction on any pl sub-manifold of M.

This natural class $\mathfrak{L}_f$ associated to any surgery problem is equivalent to a universal class $\mathfrak{L}$ in the cohomology of the classifying space G/pl. In this case this class implies very strong statements about the homotopy type of G/pl. In fact it, together with the Kervaire classes, k_{4i+2}, gives G/pl a canonical splitting into a product of Eilenberg-MacLane spaces at the prime 2 (except for the k-invariant of order 2, δSq^2, connecting π_2 and π_4). For G/Top these classes give a complete splitting at 2 [S2].

We begin our construction of the class $\mathfrak{L}_f$. The first step is to compute the de Rham invariant in terms of the Wu classes. Then we combine this computation with the k-classes introduced by the second author in [S1], [S2], [RS] for studying the Kervaire obstruction. We obtain a cohomological surgery formula for $(N^n \to M^n) \otimes L^l$ when $n + l \equiv 0(4)$ and $n \neq 0(4)$. We subtract this cohomological expression from the surgery obstruction homomorphism to obtain a homomorphism which gives the cohomology class $\mathfrak{L}_f$ by the method of Section 4.

Let M be a $Z/2^k$-manifold. Define w_M to be the characteristic class

$$V \cdot Sq^1 V = (1 + v_2 + v_4 + \cdots) \cdot Sq^1(1 + v_2 + v_4 + \cdots)$$

where v_{2i} is the $2i^{\text{th}}$ Wu class of the oriented normal bundle of M.

Recall the invariant $i(M)$ defined in Theorem 6.1.

THEOREM 8.1. *If M is a $Z/2^k$-manifold then $i(M)$ satisfies*

(a) $i(M) = \langle w_M, M\rangle$ *if* dim M $\equiv 1$ (mod 4)

(b) $2^{k-1}i(M) = \langle \delta^* w_M, M\rangle$ *in* $Z/2^k$ *if* dim $M \equiv 2$ (mod 4) *where* δ^* *is the* $Z/2 \to Z/2^k$ *Bockstein.*

Proof. (a) Open M up to a $Z/2$-manifold M'. From our definition it is clear that $i(M) = i(M')$ in $Z/2$. It is also clear that $\langle w_M, M\rangle = \langle w_{M'}, M'\rangle$. So we are reduced to the case of $Z/2$-manifolds which is proved in Lemma 8.2 below.

(b) Note first the calculation: if x is a $Z/2$-cohomology class and y is a $Z/2^k$-homology class, then $\langle x, \delta y\rangle$ in $Z/2$ and $\langle \delta^* x, y\rangle$ in $Z/2^k$ are related by the formula

$$2^{k-1}\langle x, \delta y\rangle = \langle \delta^* x, y\rangle \quad \text{in} \quad Z/2^k\,.$$

Thus

$$\begin{aligned}\langle \delta^* w_M, M\rangle &= 2^{k-1}\langle w_M, \delta M\rangle \\ &= 2^{k-1}\langle w_{\delta M}, \delta M\rangle \\ &= 2^{k-1}d(\delta M)\end{aligned}$$

again by Lemma 8.2 below. This proves (b).

LEMMA 8.2. $d(M^{4k+1}) = \langle v_{2k}(M)\cdot Sq^1 v_{2k}(M), [M]\rangle$, *for M a Z- or $Z/2$-manifold.*

Proof. We will work first with oriented $4k + 1$-manifolds with boundary and then specialize to Z- or $Z/2$-manifolds. Let $A \hookrightarrow H_{2k}(\partial M)/\text{Torsion}$ be a maximal self-annihilating subspace of dimension equal to one half the dimension of $H_{2k}(\partial M; Z)/\text{Torsion}$. Denote by $A + T \hookrightarrow H_{2k}(\partial M; Z)$ all classes with image contained in A. We have defined a skew symmetric form l on $K = \text{Torsion}\,\big(H_{2k}(M; Z)/i_*(A + T)\big)$ (see Section 6), and we will calculate here the de Rham invariant of (K, l). To do this we first calculate $l(x, x)$ for $x \in K$.

Claim. (i) $\langle v_{2k}(\partial M), \alpha\rangle = 0$ for all $\alpha \in A + T$.
(ii) $l(x, x) = (1/2)\langle v_{2k}(M), x\rangle$ for $x \in K$.

Proof of claim. (i) $\langle v_{2k}(\partial M), \alpha\rangle = \alpha^2 \bmod 2$ since we are in an oriented $4k$-manifold, ∂M. But since A is self-annihilating and T is torsion, $\alpha^2 = 0$ for all $\alpha \in A + T$.

(ii) By (i) $\langle v_{2k}(M), i_*(\alpha)\rangle = \langle v_{2k}(\partial M), \alpha\rangle = 0$. Thus $\langle v_{2k}(M), x\rangle \in Z/2$ is well-defined for $x \in K$. Since $l(x, x)$ is of order 1 or 2, we are free to multiply by an odd integer to make the calculation. In view of this we may assume that all chains involved are actually manifolds. Let $V_x^{2k} \xhookrightarrow{f} M^{4k+1}$ represent x (V oriented closed), and C be a $2k + 1$, Z/λ-manifold with $\partial C \hookrightarrow \partial M$ a closed manifold representing some $\alpha \in A + T$, and $\delta C = V_x$. Such a manifold C exists since λx is homologous in M to $i_*(\alpha)$ for some $\alpha \in A + T$. $l(x, x) = V\cdot C/\lambda$. By 6.4 $V\cdot C/\lambda + \chi(\xi)/2 = 0$ where ξ is the complement in $\nu_{V\subset M}$ to a section. Thus we have $l(x, x) \equiv (1/2)\chi(\xi)$ modulo Z. But

$$\begin{aligned}\chi(\xi) &\equiv \langle w_{2k}(\nu_{V\subset M}), [V_x]\rangle \bmod 2\\ &= \langle v_{2k}(\nu_V - \nu_{V\subset M}), [V_x]\rangle\\ &= \langle v_{2k}(M), x\rangle .\end{aligned}$$

Thus $l(x, x) \equiv (1/2)\langle v_{2k}(M), x\rangle$ modulo Z. This proves (ii).

Now we will produce an element $y \in K$ such that $l(y, x) = l(x, x)$ for all $x \in K$. Once we have such a class the algebraic sublemma below implies that $d(K, l) = l(y, y)$.

Claim. $v_{2k}(M)$ is the reduction of an integral cohomology class $\alpha^* \in H^{2k}(M; Z)$ where α^* has the property that $\langle \alpha^*, A + T\rangle = 0$.

Proof. The following diagram is commutative:

$$\begin{tikzcd} H^*(X; Z) \arrow[r] \arrow[d, "\mathrm{red}"] & \mathrm{Hom}(H_*(X), Z) \arrow[r, "r_*"] & \mathrm{Hom}(H_*(X), Z/2) \\ H^*(X; Z/2) \arrow[rr, "\cong"] & & \mathrm{Hom}(H_*(X; Z/2), Z/2) \arrow[u, "\mathrm{red}^*"'] \end{tikzcd}$$

$v_{2k}(M)\colon H_{2k}(M) \to Z/2$ annihilates $A + T$. Thus there is a homomorphism $H_{2k}(M) \to Z$ annihilating $A + T$ and projecting to $v_{2k}(M)$ under r_*. This proves the claim.

Let $\zeta \in H_{2k}(M; Z/2)$ be the Poincaré dual of $v_{2k}(M)$. The claim above implies that there is a class $\alpha \in A + T \subset H_{2k}(M; Z)$ whose mod 2 reduction is ζ. (A homology class is in $A + T$ if and only if its intersection with every class in A is 0.) This implies that we may find a chain W^{2k+1} such that

(1) $W^{2k+1} \cap \partial M$ is an integral cycle representing α.

(2) W^{2k+1} is a relative $Z/2$-cycle, i.e., $\partial W = W \cap \partial M + 2Y^{2k}$.

(3) The relative $Z/2$-homology class of W is Poincaré dual to $v_{2k}(M) \in H^{2k}(M; Z/2)$.

It follows that Y is an integral cycle which modulo 2 is Poincaré dual to $Sq^1 v_{2k}(M)$. Let y be the integral homology class of Y. Clearly $y \in K$ since W provides a homology of $2Y$ to α.

Claim. $l(y, x) = l(x, x)$ for all $x \in K$.

Proof. $l(y,x) = (1/2)W \cdot X$ where X is a cycle representing x. By Poincaré duality $W \cdot X \equiv \langle v_{2k}(M), [X]\rangle \bmod 2$. But $(1/2)\langle v_{2k}(M), [X]\rangle = l(x, x)$ by the first claim. Thus $l(y, x) = l(x, x)$. Thus $d(K, l) = l(y, y) = \delta W \cdot W \bmod 2$. (In essence we have used the subspace A to pull $Sq^1 v_{2k}(M)$ back to relative cohomology class and then cupped this with $v_{2k}(M)$.)

Application to Z or $Z/2$-manifolds. If M^{4k+1} is closed, then we have no choice for W. It represents the dual of v_{2k} whereas ∂W is dual to $Sq^1 v_{2k}$. Thus $W \cdot \delta W = v_{2k} \cdot Sq^1 v_{2k}$. If M^{4k+1} is an opened up $Z/2$-manifold, then the subspace A is of the form $A' \oplus A'$ where $A' \hookrightarrow H_{2k}(\delta M)/\text{Torsion}$. Thus the $Z/2$-manifold with boundary, W, may be chosen so that the two copies of ∂W agree under the change of components in ∂M. Thus $W/\sim \hookrightarrow M^{4k+1}/\sim$ is a $Z/2$-cycle representing the dual of v_{2k} whereas δW represents the dual of $Sq^1 v_{2k}$. Thus $\delta W \cdot W = \langle v_{2k} Sq^1 v_{2k}, [M]\rangle$.

ALGEBRAIC SUBLEMMA. *If (K, l) is a skew symmetric, non-singular, linking space and $y \in K$ is such that $l(y, x) = l(x, x)$ then $d(K, l) = l(y, y)$.*

Proof. First notice that $2y = 0$ since $l(2y, x) = 2l(x, x) = 0$, for every x.

Case 1. $l(y, x) = 0$. Then $l(x, x) = 0$ for all $x \in K$. This clearly implies $d(K, l) = 0 = l(y, y)$.

Case 2. $l(y, y) = 1/2$. In this case y generates a direct summand in (K, l) isomorphic to $(Z/2, 1/2)$. In the perpendicular space $l(x, x) = l(y, x) = 0$. Thus Case 1 implies $d(K, l) = 0 + d(Z/2, 1/2) = 1$.

Case 3. $l(y, y) = 0$. Suppose $y = 2^k y'$ with y' not divisible by 2. By non-singularity there is an x with $o(x) = \text{order}(x) = \text{order}(y')$ and $l(y', x) = 1/o(x)$. Then (x, y') generates a direct summand of (K, l) isomorphic to $Z/o(x) \oplus Z/o(x)$. In the perpendicular space $l(y, x) = 0$ so that $d(K, l) = d\big(Z/o(x) \oplus Z/o(x)\big) + 0 = 0$.

PROPOSITION 8.3. *If M and L are two $Z/2^k$-manifolds, then*

$$w_{L\otimes M} = \rho^*(w_L \otimes \mathfrak{L}_M + \mathfrak{L}_L \otimes w_M)$$

where $\rho: L \otimes M \to L \times M$ is the natural map.

Proof. For the normal bundle we have

$$\nu_{L\otimes M} = \rho^*(\nu_L \times \nu_M) \oplus \pi^*\zeta$$

where $\pi\colon L \otimes M \to W^2/\partial$ is the projection discussed in Section 1.

Since $Sq^1\pi^* V(\zeta) = 0$ and $\pi^* V^2(\zeta) = 1$ for dimension reasons — $H^i(W^2/\partial, Z/2)$ vanishes for $i > 2$ — we have

$$\begin{aligned} w_{L\otimes M} = w(\nu_{L\otimes M}) &= \rho^*(V_L \otimes V_M)\cdot\pi^* V(\zeta)\cdot Sq^1\big(\rho^*(V_L \otimes V_M)\cdot\pi^* V(\zeta)\big) \\ &= \rho^*(V_L \otimes V_M)\cdot Sq^1\big(\rho^*(V_L \otimes V_M)\big)\cdot\pi^* V^2(\zeta) \\ &= \rho^*(V_L\cdot Sq^1 V_L \otimes V_M^2 + V_L^2 \otimes V_M\cdot Sq^1 V_M) \\ &= \rho^*(w_L \otimes \mathfrak{L}_M + \mathfrak{L}_L \otimes w_M)\ . \end{aligned}$$

We recall the work in [S1], [S2], [RS] on the Kervaire obstruction. Let us denote a degree one normal map by the map of base spaces. Thus "f" represents

$$\begin{array}{ccc} \nu_N & \xrightarrow{\tilde f} & \xi \\ \big\downarrow & & \big\downarrow \\ N_{2k} & \xrightarrow{f} & M_{2k}\ . \end{array}$$

Then there is a natural characteristic class $k_f \in H^{4*+2}(M; Z/2)$ such that $\sigma(f) = \langle k_f\cdot(\mathfrak{L}_M)_2, [M]\rangle$ for any surgery problem over a $Z/2$-manifold of dimension $4n + 2$ and such that $k_{f\times g} = k_f \otimes 1 + 1 \otimes k_g$. Naturality is with respect to the following "pull back" construction. If $(f, \tilde f)\colon (M', \nu_{M'}) \to (M, \xi)$ is a surgery problem and $g\colon L \to M$, then form $(f \times \mathrm{Id}_L, \tilde f \times \mathrm{Id}_{\nu_L})\colon (M' \times L, \nu_{M'} \times \nu_L) \to (M \times L, \xi \times \nu_L)$. Transversality then produces a surgery problem over the graph of g, $L \hookrightarrow L \times M$.

This formula also holds for surgery problems over $Z/2^k$-manifolds of dimension $4n + 2$. This is seen by opening the $Z/2^k$ problem up to a $Z/2$ problem, which does not change the surgery obstruction, or the value of the formula.

If "f" denotes $(f, \tilde f)\colon (M_{2^k}'^m, \nu_{M'}) \to (M_{2^k}^m, \xi)$ and N^n is a $Z/2^k$-manifold, let "f"$\otimes N^n$ denote the surgery problem $\big(f\otimes 1_N, \rho^*(\tilde f \times 1_{\nu_N})\colon (M'\otimes N, \rho^*(\nu_M \times \nu_N)\big) \to \big(M \otimes N, \rho^*(\xi \times \nu_N)\big)$. If "$f$" is classified by $h_f\colon M \to G/pl$, then "f" $\otimes N$ is represented by $h_f\cdot\pi\cdot\rho\colon M \otimes N \to M \times N \to M \to G/pl$. Thus the bordism class of the classifying map for "f" $\otimes N$ is $[M, h_f] \otimes [N]$, i.e., this construction on normal maps corresponds to the module structure of $\Omega_*(G/pl; Z/2^k)$ over $\Omega_*(pt; Z/2^k)$.

Let "$k_f \otimes 1$" denote $\rho^*\pi^*k_f$ in $H^*(M \otimes N, Z/2)$.

We now give a cohomology formula for the cross terms.

THEOREM 8.4. *$\sigma($"f" $\otimes N^n) = \langle \delta^*($"$k_f \otimes 1$"$\cdot w_{M\otimes N}), [M \otimes N]\rangle$ provided that $m + n \equiv 0(4)$ and $m \not\equiv 0(4)$. Here δ^* is the $Z/2 \to Z$ Bockstein.*

Proof. *Case* $m = 1$ *and* $n = 3(4)$. By Theorem 6.1 $\sigma(``f" \otimes N^n) = 0$. Now

$$\begin{aligned}\delta^*(``k_f \otimes 1" \cdot w_{M\otimes N}) &= \delta^*\big(\rho^*(k_f \otimes 1 \cdot (w_M \otimes \mathfrak{L}_N + \mathfrak{L}_M \otimes w_N))\big) \\ &= \delta^*\big(\rho^*(k_f \cdot w_M \otimes \mathfrak{L}_N + k_f \cdot \mathfrak{L}_M \otimes w_N)\big) .\end{aligned}$$

Thus the right hand side of the formula in the theorem is

$$\begin{aligned}&\langle \delta^*(k_f \cdot w_M \otimes \mathfrak{L}_N + k_f \cdot \mathfrak{L}_M \otimes w_N), [M] \otimes [N] \rangle \\ &\quad = 2^{k-1}\langle k_f \cdot w_M \otimes \mathfrak{L}_N + k_f \cdot \mathfrak{L}_M \otimes w_N, [\delta M] \otimes [N] + [M] \otimes [\delta N] \rangle .\end{aligned}$$

But since the dimension of $n \equiv 3(4)$, $\langle \mathfrak{L}_N, [N] \rangle = \langle \mathfrak{L}_N, [\delta N] \rangle = \langle w_N, N \rangle = \langle w_N, \delta N \rangle = 0$. Thus all the above terms vanish and we have equality, $0 = 0$.

Case $m = 2$ *and* $n = 2(4)$. As above we reduce the formula to

$$2^{k-1}\langle k_f \cdot w_M \otimes \mathfrak{L}_N + k_f \cdot \mathfrak{L}_M \otimes w_N, [\delta M] \otimes [N] + [M] \otimes [\delta N] \rangle .$$

By dimension considerations all terms vanish except

$$\begin{aligned}2^{k-1}\langle k_f \cdot \mathfrak{L}_M \otimes w_N, [M] \otimes [\delta N] \rangle &= 2^{k-1}\langle k_f \cdot \mathfrak{L}_M, [M] \rangle \cdot \langle w_N, [\delta N] \rangle \\ &= 2^{k-1}\sigma_f \cdot i(N) \in Z/2^k \\ &= \sigma(``f" \otimes N)\end{aligned}$$

by Case 2 of Theorem 6.1.

Case $m \equiv 3$ *and* $n = 1(4)$. Once again we reduce the formula. In this case all terms vanish for dimension reasons except

$$\begin{aligned}&2^{k-1}\langle k_f \cdot \mathfrak{L}_M \otimes w_N, [\delta M] \otimes [N] \rangle + 2^{k-1}\langle k_f \cdot w_M \otimes \mathfrak{L}_N, [M] \otimes [\delta N] \rangle \\ &\quad = 2^{k-1}\sigma(\delta f) \cdot i(N) + 2^{k-1}\langle k_f \cdot w_M, [M] \rangle \cdot \langle \mathfrak{L}_N, [\delta N] \rangle .\end{aligned}$$

Now

$$\langle L_N, [\delta N] \rangle = \langle L_{\delta N}, [\delta N] \rangle = I(\delta N) = 0 .$$

Thus the formula yields $2^{k-1} \cdot \sigma(\delta f) \cdot i(N) = \sigma(``f" \otimes N^n)$. This completes the proof of Theorem 8.4.

PROPOSITION 8.5. *If* "f" *and* N^n *are as above with* $m \equiv n \equiv 0(4)$, *then*

$$\langle \delta^*(``k_f \otimes 1" \cdot w_{M\otimes N}), [M \times N] \rangle = \langle \delta^*(k_f \cdot w_M), [M] \rangle \cdot I(N) .$$

Proof. $\langle \delta^*(\rho^* \pi^* k_f \cdot w_{M\otimes N}), [M \otimes N] \rangle = 2^{k-1}\langle k_f \cdot w_M \otimes \mathfrak{L}_N + k_f \cdot \mathfrak{L}_M \otimes w_N, [\delta M] \otimes [N] + [M] \otimes [\delta N] \rangle$. All terms vanish except $2^{k-1}\langle k_f \cdot w_M, [\delta M] \rangle \cdot \langle \mathfrak{L}_N, [N] \rangle$ which is equal to $\langle \delta^*(k_f \cdot w_M), [M] \rangle \cdot I(N)$.

Now we eliminate the cross term contribution by defining a new homomorphism

$$\sigma'(``f") = \sigma(``f") - \langle \delta^*(k_f \cdot w_M), [M] \rangle$$

provided dim "f" is 0(4). This new homomorphism has the correct multiplicative properties for the application of Section 4.

THEOREM 8.6. (a) $\sigma'(\text{“}f\text{”} \otimes N^n) = \sigma'(\text{“}f\text{”}) \cdot I(N)$ *for* $\dim \text{“}f\text{”} = m$ *and* $m + n \equiv 0(4)$.

(b) $\sigma'(i_* \text{“}f_{2^k}\text{”}) = i(\sigma'(\text{“}f_{2^k}\text{”}))$ *where* $i: Z/2^k \hookrightarrow Z/2^{k+1}$.

(c) *If* $r: Z \to Z/2^k$ *is reduction*, mod 2^k, *then* $r(\sigma(f)) = \sigma'(\text{“}r_* f\text{”})$.

Proof. (a) If $m + n \equiv 0(4)$ and $m \not\equiv 0(4)$, then $\sigma'(\text{“}f\text{”} \otimes N^n) = 0$ by Theorem 8.4. If $m \equiv n \equiv 0(4)$, then both σ and $\langle \delta^*(k_f \cdot w_M), [M] \rangle$ are multiplicative with respect to the index. Thus σ' is also.

(b) Clearly

$$\begin{aligned} i(\langle \delta^*(k_f \cdot w_M), [M_{2^k}] \rangle &= 2 \cdot 2^{k-1} \langle k_f \cdot w_M, [\delta M] \rangle \\ &= 2^k \langle k_f \cdot w_M, [\delta(i_*(M))] \rangle = \langle \delta^*(k_{i_*(f)} \cdot w_{i_*(M)}), [i_*(M)] \rangle \ . \end{aligned}$$

But $\sigma(i_* \text{“}f_{2^k}\text{”}) = i\sigma(\text{“}f_{2^k}\text{”})$. These two facts together prove (b).

(c) $r(\sigma(\text{“}f\text{”})) = \sigma(\text{“}r_* f_{2^k}\text{”})$ by definition.

Since the error term is of order 2 it vanishes on closed manifolds. So for a closed manifold $\sigma = \sigma'$. This proves (c).

THEOREM 8.7. (*The cohomological surgery formula*). *Associated to any surgery problem*, “f” *over* M^m, *there is a natural class* $\mathfrak{L}_f \in H^{4*}(M; Z_{(2)})$. (*Naturality means with respect to the transversality-pull-back construction*.) *If* $m \equiv 0(4)$ *and* M *is pl a* Z *or* $Z/2^k$-*manifold, then* $\sigma(f) = \langle \mathfrak{L}_f \cdot \mathfrak{L}_M, [M] \rangle + \langle \delta^*(k_f \cdot w_M), [M] \rangle$.

Proof. We work in the universal example G/pl. We have constructed homomorphisms $\sigma: \Omega_*(G/pl; Z_{(2)}) \to Z_{(2)}$ and $\sigma': \Omega_*(G/pl; Z/2^k) \to Z/2^k$ and checked multiplicativity and compatibility. Thus by Theorem 4.1, there is a unique class $\mathfrak{L} \in H^{4*}(G/pl; Z_{(2)})$ such that $\sigma'(M, g) = \langle g^* \mathfrak{L} \cdot \mathfrak{L}_M, [M] \rangle$. Define $\mathfrak{L}_f$ for a surgery problem “f” by classifying “f”, to get $h_f: M \to G/pl$, then set $\mathfrak{L}_f = h_f^* \mathfrak{L}$.

THEOREM 8.8. *Properties of* $\mathfrak{L}_f$:

(1) $\mathfrak{L}_{f \times g} = \mathfrak{L}_f \otimes 1 + 1 \otimes \mathfrak{L}_g + 8\mathfrak{L}_f \otimes \mathfrak{L}_g$.

(2) *If* “f” *is the problem*

$$\begin{array}{ccc} \nu_{M'} & \xrightarrow{\tilde{f}} & \xi \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & N \end{array}$$

then $8\mathfrak{L}_f + 1 = \mathfrak{L}_\xi \cdot \mathfrak{L}_N^{-1}$.

Proof. (1) We work in the universal example G/pl. Property (1) then reads $\oplus^* \mathfrak{L} = \mathfrak{L} \otimes 1 + 1 \otimes \mathfrak{L} + 8\mathfrak{L} \otimes \mathfrak{L}$.

We will use the general product formula Theorem 6.5.

$$(6.5) \quad \sigma\big((M \xrightarrow{f} N^n) \otimes (L \xrightarrow{f} Q^q)\big) = \sigma(f)\cdot i(Q) + i(N)\cdot\sigma(g) + 8\sigma(f)\sigma(g)$$

provided $n + q \equiv 0(4)$ (where the cross terms are in $Z/2 \hookrightarrow Z/2^k$). We wish to derive from this that

$$(8.9) \quad \sigma'\big((M \xrightarrow{f} N^n) \otimes (L \xrightarrow{g} Q^q)\big) = \sigma'(f)\cdot I(Q) + I(N)\cdot\sigma'(g) + 8\sigma'(f)\sigma'(g)$$

provided that $n + q \equiv 0(4)$.

Since $\sigma' - \sigma$ is of order 2, $8\sigma'(f)\cdot\sigma'(g) - 8\sigma(f)\cdot\sigma(g) = 0$. The left hand side of (8.9) minus the left hand side of (6.5) is

$$(*) \quad \begin{aligned}&\langle \delta^*(k_{f\otimes g}\cdot w_{N\otimes Q}), [N\otimes Q]\rangle \\ &\quad = 2^{k-1}\langle (k_f\otimes 1 + 1\otimes k_g)\cdot(w_N\otimes \mathfrak{L}_Q + \mathfrak{L}_N \otimes w_Q), [\delta N]\otimes[Q] + [N]\otimes[\delta Q]\rangle .\end{aligned}$$

We will evaluate this in the three cases $0 + 0$, $1 + 3$, $2 + 2$.

Case (0). $n \equiv 0 \equiv q(4)$. Then (*) becomes

$$\begin{aligned}&\langle \delta^*(k_f\cdot w_N), [N]\rangle\cdot I(Q) + I(N)\cdot\langle \delta^*(k_g\cdot w_Q), [Q]\rangle \\ &\quad = \big(\sigma'(f) - \sigma(f)\big)\cdot I(Q) + I(N)\cdot\big(\sigma'(g) - \sigma(g)\big) .\end{aligned}$$

Since $i(Q) = I(Q)$ and $i(N) = I(N)$ in this case, we have verified (8.9) for $n \equiv q \equiv 0(4)$.

Case (1). $n \equiv 1(4)$, $q \equiv 3(4)$. In this case (*) becomes

$$2^{k-1}\big(\langle w_N, [N]\rangle\cdot\langle k_g\cdot\mathfrak{L}_Q, [\delta Q]\rangle + \langle \mathfrak{L}_N, [\delta N]\rangle\cdot\langle k_g\cdot w_Q, [Q]\rangle\big) .$$

But

$$\langle \mathfrak{L}_N, [\delta N]\rangle = \langle \mathfrak{L}_{\delta N}, [\delta N]\rangle = 0 .$$

So we are left with $2^{k-1}d(N)\cdot\sigma(\delta g)$ which is equal to $i(N)\sigma(g)$ in this dimension. Since $i(Q) = I(Q) = 0$ for $q \equiv 3(4)$ and $I(N) = 0$ since $n \equiv 1$ (4) our difference is $i(N)\cdot\sigma(g)$ and this proves Case (1).

Case (2). $n \equiv 2$, $q \equiv 2$. In this case (*) becomes

$$\begin{aligned}&2^{k-1}\big(\langle w_N, [\delta N]\rangle\cdot\langle k_g\cdot\mathfrak{L}_Q, [Q]\rangle + \langle k_f\cdot\mathfrak{L}_N, [N]\rangle\cdot\langle w_Q, [\delta Q]\rangle\big) \\ &\quad = i(N)\cdot\sigma(g) + \sigma(f)\cdot i(Q)\end{aligned}$$

which is easily seen to be the difference of the right hand side of (8.9) and (6.3). This completes the proof of (8.9).

Since $\langle \mathfrak{L}_{f\otimes g}\cdot\mathfrak{L}_{N\otimes Q}, [N\otimes Q]\rangle = \sigma'(f\otimes g)$ and

$$\begin{aligned}&\langle (\mathfrak{L}_f\otimes 1 + 1\otimes\mathfrak{L}_g + 8\mathfrak{L}_f\otimes\mathfrak{L}_g)\cdot\mathfrak{L}_{N\otimes Q}, [N\otimes Q]\rangle \\ &\quad = \langle \mathfrak{L}_f\cdot\mathfrak{L}_N, [N]\rangle\cdot I(Q) + I(N)\cdot\langle \mathfrak{L}_g\cdot\mathfrak{L}_Q, [Q]\rangle \\ &\qquad + 8\langle \mathfrak{L}_f\cdot\mathfrak{L}_N, [N]\rangle\cdot\langle \mathfrak{L}_g\cdot\mathfrak{L}_Q, [Q]\rangle \\ &\quad = \sigma'(f)\cdot I(Q) + I(N)\cdot\sigma'(g) + 8\sigma'(f)\cdot\sigma'(g) ,\end{aligned}$$

we see that $\mathfrak{L}_{f\otimes g}$ and $\mathfrak{L}_f \otimes 1 + 1 \otimes \mathfrak{L}_g + 8\mathfrak{L}_f \otimes \mathfrak{L}_g$ evaluate the same on products of $Z/2^k$-manifolds in $\Omega_*(G/pl \times G/pl; Z_{2^\infty})$. The usual rational calculation of closed manifolds shows that they agree on $(N \xrightarrow{h_f} G/pl) \times (M \xrightarrow{h_g} G/pl)$ in $\Omega_*(G/pl \times G/pl; Q)$. We now apply Lemma 7.1 which says this is enough to imply $\mathfrak{L} \otimes \mathfrak{L} = \mathfrak{L} \otimes 1 + 1 \otimes \mathfrak{L} + 8\mathfrak{L} \otimes \mathfrak{L}$ in $H^*(G/pl \times G/pl; Z_{(2)})$.

(2) If $(M \xrightarrow{f} N, \nu_M \xrightarrow{\tilde{f}} \xi)$ is a normal map, we want to show that $8\mathfrak{L}_f + 1 = \mathfrak{L}_\xi \cdot \mathfrak{L}_N^{-1}$. To prove this we show $8\sigma'(f) = I(M) - I(N)$. This is clear from Section 5, and the fact that $\sigma - \sigma'$ is 2 torsion if dimension $M \equiv 0(4)$. If dimension $M \not\equiv 0(4)$ then both sides are 0. Thus

$$\begin{aligned}\langle (8\mathfrak{L}_f + 1)\cdot \mathfrak{L}_N, [N]\rangle &= \langle 8\mathfrak{L}_f \cdot \mathfrak{L}_N, [N]\rangle + \langle \mathfrak{L}_N, [N]\rangle \\ &= 8\sigma'(f) + I(N) = I(M)\ .\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle (\mathfrak{L}_\xi \cdot \mathfrak{L}_N^{-1})\cdot \mathfrak{L}_N, [N]\rangle &= \langle \mathfrak{L}_\xi, [N]\rangle = \langle \mathfrak{L}_\xi, f_*[M]\rangle \\ &= \langle f^*\mathfrak{L}_\xi, [M]\rangle = \langle \mathfrak{L}_{f^*\xi}, [M]\rangle = \langle \mathfrak{L}_{\nu_M}, [M]\rangle = I(M)\ .\end{aligned}$$

This shows $8\mathfrak{L}_f + 1$ and $\mathfrak{L}_\xi \cdot \mathfrak{L}_M^{-1}$ agree in the top dimension for all surgery problems. Thus they must be equal.

If we work in the universal example G/pl, then this formula becomes $8\mathfrak{L} + 1 = \eta^*\mathfrak{L}_{pl}$ where $\mathfrak{L}_{pl} \in H^{4*}(B_{pl}; Z_{(2)})$ is the class constructed in this section and 7, and $\eta\colon G/pl \to B_{pl}$ is the natural map. This follows easily from the fact that if $h_f\colon N \to G/pl$ classifies the above problem then $\eta \cdot h_f$ is the virtual bundle $(\xi - \nu_N)$.

Note. All this works for G/Top. We avoid the low dimensional problems by crossing with CP^2. Thus $\mathfrak{L}_f$ is a natural class for all topological surgery problems. The reason that we state the theorems in terms of pl-theory is that it is patently an outgrowth of transversality which hold for pl-theory but may fail for topological theory.

The $\mathfrak{L} \in H^{4*}(G/Top; Z_{(2)})$ provides a canonical map $G/Top \to \prod_{i \geq 1} K(Z_{(2)}, 4i)$. It together with k defines $G/Top \to \prod_{i\geq 1} K(Z/2, 4i-2) \times K(Z_{(2)}, 4i)$. This map is "localization at 2", see [S4], i.e., on π_* it is tensoring with $Z_{(2)}$. Thus it gives a *canonical* splitting of G/Top at 2 into a product of Eilenberg-MacLane spaces. The composition $G/pl \to G/Top \to \prod_{i\geq 1} K(Z/2, 4i-2) \times K(Z_{(2)}, 4i)$ is $\otimes Z_{(2)}$ on π_j for $j \neq 4$. On π_4 it is $Z \xrightarrow{\times 2} Z_{(2)}$. Thus this gives G/pl a canonical splitting into $K(Z/2, 2) \times_{\delta^* Sq^2} K(Z_{(2)}, 4) \times \prod_{i\geq 2} K(Z/2, 4i-2) \times K(Z_{(2)}, 4i)$. This is just a refinement of the argument in [S2] which proves the existence of such structures for G/pl at the prime 2.

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