On the Kervaire obstruction

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The classical "Browder-Novikov" surgery problem may be stated as follows: Suppose $f: M_1 \to M$ is a normal map between n-manifolds, that is, f has degree 1 and is covered by a bundle map $\hat{f}: \nu_{M_1} \to \xi$, where ν_{M_1} is the normal bundle of M_1 and ξ some bundle over M. Then, when is f cobordant to a homotopy equivalence, where a cobordism of normal maps means a normal map onto $M \times I$? In the case when M is simply connected and $n \geqq 5$ the problem has a simple solution: If n is odd f is always cobordant to a homotopy equivalence; if $n \equiv 0(4)$ then this is true if and only if Index $(M_1) =$ Index (M); and, if $n \equiv 2(4)$, if and only if an obstruction $K(f, \hat{f}) \in \mathbf{Z}_2$ vanishes.

The purpose of this paper is to give a geometrical definition, based on immersion theory, of this obstruction, the Kervaire obstruction, and to prove a product formula which gives the obstruction for $f \times g$ in terms of invariants of f and g. The definition and formula apply when M is non-simply connected and n is any even integer but in this case the obstruction is just part of the surgery obstruction (Wall [16]). We also interpret the product formula in terms of the classifying space G/PL for PL normal maps. It implies the existence of primitive "Kervaire" classes in the $\mathbf{Z}_2$-cohomology of G/PL which induce the obstruction. We give formulae due to Brumfiel and Wall relating the various classes.

We are very grateful to both Brumfiel and Wall for allowing us to use their results and to Wall for criticism of an early version of this paper.

We work for convenience in the PL category; our results can be interpreted in either smooth or topological categories using triangulation theorems of Whitehead [17] and Kirby and Siebenmann [6].

CONTENTS

1. Definition of the Kervaire obstruction

Let

$$\begin{array}{ccc} \nu_{M_1} & \xrightarrow{\hat{f}} & \xi \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M \end{array}$$

be the given normal map, where dim $(M) = 2m$. Write

$$H_m(M_1) = H_m(M) \oplus K_m$$

where all groups have $\mathbf{Z}_2$-coefficients and

$$K_m = \ker f_*\colon H_m(M_1) \longrightarrow H_m(M) ,$$

see e.g. Wall [14; l. 3]. We will define a quadratic form Q on K_m associated to intersection pairing and then $K(f, \hat{f})$ will be the Arf invariant of this form. (A definition and basic properties of the Arf invariant are to be found in a short appendix to the paper.)

The idea is this. Represent $x \in K_m$ by a singular manifold $a\colon W^m \rightarrow M_1$ such that $f \circ a$ is bordant to zero and use general position to shift $f \circ a\colon W \rightarrow M$ to an immersion. Then use the bundle map $\hat{f}$ to pull back this immersion to an immersion of W in M_1 homotopic to a. Finally define $Q(x)$ to be the sum of the self-intersection numbers of the two immersions counted mod 2. The details are in the next three lemmas.

To represent homology classes we use the PL analogue of the Thom theorem as formulated in Conner and Floyd [3; 26]. Let $T_n(,)$ denote PL unoriented bordism, then:

LEMMA 1.1. *For any CW pair (X, A)*

$$T_*(X, A) = H_*(X, A) \otimes T_*(pt) .$$

Proof. A spectral sequence can be set up as in [3], and it collapses for the same reason: any class is represented by a PL manifold by triangulating Thom's representative.

Now we can identify K_m with $H_{m+1}(\mathbf{M}_f, M_1)$ where $\mathbf{M}_f = M \cup_f M_1 \times I$ and so any class in K_m is representable by a singular manifold (W, a) (i.e. $a\colon W \rightarrow M_1$) bordant to zero in $\mathbf{M}_f$ or equivalently such that $(W, f \circ a)$ is bordant to zero in M.

The next step was to homotope $f \circ a$ to an immersion $q\colon W \rightarrow M$ by general position (see e.g. [18; chapter 5]).

LEMMA 1.2. *q and a determine a regular homotopy class of immersions of W in M_1 homotopic to f.*

Proof. Let $n_a\colon W \to \widetilde{BPL}$ be the classifying map for the stable normal (block) bundle of a, i.e. the stable bundle $a^*(\tau_{M_1}) - \tau_W$ (cf. [7, 8] for notion of block bundle). According to Haefliger [4], regular homotopy classes of immersions of X in M_1 homotopic to a correspond bijectively with liftings of n_a in $\widetilde{BPL}_m$. Using the fact that the square $(BG; BG_m, \widetilde{BPL}; \widetilde{BPL}_m)$ has vanishing homotopy for $m > 2$ (see [9; 1.10]), they also correspond bijectively with liftings of $i \circ n_a$ in BG_m (where $i\colon \widetilde{BPL} \to BG$ is the natural map) (also Haefliger [4]).

We now use the fact that f has degree 1 which implies that ξ has reducible Thom space and hence is fibre-homotopy equivalent with the Spivak normal bundle of M [12]. Therefore $\hat{f}$ is a map of Spivak normal bundles and so f is covered by a map of Spivak tangent bundles (the associated fibre spaces to τ_{M_1} and τ_M). Therefore $(f \circ a)^*(\tau_M)$ is fibre-homotopy equivalent to $a^*(\tau_{M_1})$. Moreover the reduction of ξ determines a fibre-homotopy equivalence $\xi \to \nu_M$ up to fibre homotopy, see Wall [15], and hence the equivalence of $(f \circ a)^*(\tau_M)$ with $a^*(\tau_{M_1})$ is also determined up to fibre-homotopy; it follows that the lifting of $i \circ n_{fa}$ in BG_m determined by the immersion q, determines a lifting of $i \circ n_a$ in BG_m, as required. For the case $m \leqq 2$ one can check directly that the obstructions to immersing W in M_1 and M are the same.

Let $t\colon W \to M_1$ be an immersion in the class given by Lemma 1.2 and let $\sharp(q)$ and $\sharp(t)$ denote the number of self-intersections of q and t counted mod 2, which are invariants of regular homotopy.

LEMMA 1.3. *$\sharp(q) + \sharp(t) \in \mathbf{Z}_2$ is an invariant of the homology class of x.*

Proof. We first observe that introducing one new double point in q leaves $\sharp(q) + \sharp(t)$ unchanged. This is because we can assume that f, having degree 1, is a homeomorphism on the inverse image of a disc in M, and we can introduce the new double point inside this disc. Then since Lemma 1.2 relativises in a natural way we can assume t is obtained by pulling back the immersion in the disc and then in the complement. Therefore the new double point induces one new double point in t.

Next it is easy to prove that homotopic immersions in the middle dimension differ by a sequence of regular homotopies and introductions or deletions of double points. (This follows from the obstruction theory for immersions and an interpretation of the obstructions in terms of double points (cf. Wall [14, 16]); a more geometrical proof can be given as follows: Put the homotopy in general position then it becomes a regular homotopy except at a finite number of singular points each of which has the form of a cone on an immersed sphere and can be interpreted as introducing and deleting double points.) So our first observation shows invariance under the choice of immersion q homotopic to $f \circ a$.

Finally we have to check invariance under our initial choice a of representative for x. Let (W', a') be another representative (with $f \circ a'$ bordant to zero). Then $b = a \cup a'$ represents zero and $f \circ b$ is bordant to zero by bordism (V, c) say. Now consider the normal map $f \times 1: M_1 \times I \to M \times I$. By surgery (below middle dimension) on the interior we can replace this, rel boundary, by a normal map $\phi: \bar{M}_1 \to M \times I$ which is $(m-1)$-connected (see Wall [16; § 1]) and so by Lemma 1.1 $T_{m+1}(\mathbf{M}_\phi, \bar{M}_1) \cong H_{m+1}(\mathbf{M}_\phi, \bar{M}_1)$, (where $\mathbf{M}_\phi$ is the mapping cylinder of ϕ). Now consider the relative bordism element $\alpha \in T_{m+1}(\mathbf{M}_\phi, \bar{M}_1)$ defined by $(W \cup W', b)$ and (V, c). We claim α represents zero in $H_{m+1}(\mathbf{M}_\phi, \bar{M}_1)$; this is because, as remarked earlier, $K_m(\phi) = H_{m+1}(\mathbf{M}_\phi, \bar{M}_1)$ and the isomorphism is realised by the boundary map (by an easy duality argument, see e.g. Wall [14]); but $(W \cup W', b)$ does represent zero in $K_m(\phi)$ by choice. Hence α is zero in $T_{m+1}(\mathbf{M}_\phi, \bar{M}_1)$, or equivalently c is bordant to $f \circ b_1$ say, where $b_1: V_1 \to \bar{M}_1$ extends b. Now immerse V_1 properly in $M \times I$ homotopic to $f \circ b_1$ (the existence of such an immersion follows easily from obstruction theory, or more geometrically, use general position to get an immersion except at a finite number of singular points and then pipe these into a collar on the boundaries which is then deleted, cf. [18; ch. 6]). Let this immersion be q_1. Then by Lemma 1.2, applied twice, (the second time in its relative version) q_1 pulls back to a proper immersion $t_1 \simeq b_1$ of V_1 in $\bar{M}_1$. Finally denote $t = t_1 | W$, $t' = t_1 | W'$, etc. Then $\#(t) + \#(t') = 0$ since $t \cup t'$ bounds an immersion in $\bar{M}_1$ (by general position the singularities can be taken to be double lines and circles, each double line ends in two double points of $t \cup t'$) and similarly $\#(q) + \#(q') = 0$. Therefore $\#(q) + \#(t) = \#(q') + \#(t')$ as required.

By Lemma 1.3 we can define $Q(x) = \#(t) + \#(q)$ for $x \in K_m$.

THEOREM 1.4. (i)

$$Q(x + y) = Q(x) + Q(y) + x \cdot y \qquad (\text{mod } 2)$$

where $x \cdot y$ denotes $\mathbf{Z}_2$-intersection number.

(ii) Arf (Q) *vanishes if $(f, \hat{f})$ bounds. That is if it extends to a normal map*

$$\begin{tikzcd} E(\nu_{N_1}) \arrow[r, "\hat{f}"] \arrow[d] & E(\xi') \arrow[d] \\ (N_1, M_1) \arrow[r, "f"] & (N, M) \end{tikzcd}$$

where $\partial N_1 = M_1$, $\partial N = M$, and $\xi' | M = \xi$.

Theorem 1.4 (i) says that $Q(x)$ is a quadratic form, associated to the intersection pairing, and enables us to define its Arf invariant. Notice that it implies $x \cdot x = 0$ for $x \in K_m$. Part (ii) implies that Arf (Q) depends only on the normal cobordism class of $(f, \hat{f})$ and hence is a surgery obstruction.

Proof of (i). Let (W_1, q_1, t_1) represent x and (W_2, q_2, t_2) represent y. (q_i and t_i are the immersions used in defining Q.) Then $(W_1 \cup W_2, q_1 \cup q_2, t_1 \cup t_2)$ represents $x + y$. But

$$\sharp(t_1 \cup t_2) = \sharp(t_1) + \sharp(t_2) + x \cdot y$$

and

$$\sharp(q_1 \cup q_2) = \sharp(q_1) + \sharp(q_2)$$

(since q_1 and q_2 represent the zero homology class, there is no cross term in the second formula). The result is now clear.

Proof of (ii). We show that there is a subspace $U \subset K_m$ of half the dimension which is self-orthogonal ($x \cdot y = 0$ for all $x, y \in U$) and such that $Q(x) = 0$ for all $x \in U$. The result then follows easily (see appendix).

Define U to be image of ∂ in the exact sequence of kernels (cf. Wall [14]).

$$K_{m+1}(N_1, M_1) \xrightarrow{\partial} K_m(M_1) \xrightarrow{i} K_m(N_1) .$$

Then U has dimension half the dimension of $K_m(M_1)$ since ∂ and i are dual maps. $x \cdot y = 0$ for all $x, y \in U$ since they bound in N_1, and it remains to prove that $Q(x) = 0$ for $x \in U$.

By Lemma 1.1 choose a representative (W, a) for x, which is bordant to zero in N_1, and with $f \circ a$ bordant to zero in M. Let the first bordism be (V, b). Then, following the proof of Lemma 1.3, shift $(V, f \circ b)$ to a proper immersion q_1 and pull back to a proper immersion t_1 of V in N_1. Then $\sharp(q) = \sharp(t) = 0$ where $q = q_1 | W$, $t = t_1 | W$ and hence $Q(x) = 0$, as required.

Remark. We can extend our definition to the case when M is a Poincaré complex, ξ its Spivak normal bundle and $\hat{f}$ a fibre map. Note that the only property of $\hat{f}$ we have used (in the proof of Lemma 1.2) is that it is a map of Spivak normal bundles, so the last two extensions require no comment. We need to define an "immersion" of one Poincaré complex in another:

Definition. If X and Y are Poincaré complexes of nominal dimensions n and p respectively, then an *immersion* of Y in X is a map $f\colon Y \to X$ together with a lifting in BG_{n-p} of the classifying map $n_f\colon Y \to BG$ for the stable normal bundle of f.

One then has a similar obstruction theory for the existence of immersions as in the PL or smooth cases. Indeed, by Haefliger [4], if X and Y are PL manifolds and $n - p > 2$ then immersions as Poincaré complexes correspond to immersions as PL manifolds.

Using this definition of immersion one can do the constructions for defining Q but one needs to define the "self-intersection" number of an immersion of one Poincaré complex in another in the middle dimension. Without a suitable embedding theorem this seems difficult in general; however, in our case we can insist that q has "zero self-intersection number" by arranging that it bounds an immersion in $M \times I$. One then defines $Q(x) = \#(t)$ and the properties are proved as before.

2. The product formula

Let $(g, \hat{g})\colon P_1 \to P$ be a second normal map of even dimensional closed manifolds. Then $(f \times g, \hat{f} \times \hat{g})\colon P_1 \times M_1 \to P \times M$ is a normal map, and we have:

THEOREM 2.1.

$$K(f \times g, \hat{f} \times \hat{g}) = K(f, \hat{f})\chi(P) + K(g, \hat{g})\chi(M)$$

where $\chi(\ )$ *denotes* mod 2 *Euler characteristic.*

Remark. The Kervaire obstruction of a normal map which is the product of odd dimensional normal maps is zero (see § 4 for a proof), thus we can compute $K(f \times g, \hat{f} \times \hat{g})$ in all cases.

Proof. Let p be dimension of P and write $p = 2q$. By surgery below the middle dimension we may assume that K_m and K_q $\big(=\ker (g_*)_q\big)$ are the only non-zero kernels of f_* and g_*. Denote by K_{q+m} the middle-dimensional kernel of $(f \times g)_*$ and one then has an orthogonal splitting:

$$K_{q+m} \cong K_q \otimes K_m \oplus K_q \otimes H_m(M) \oplus H_q(P) \otimes K_m$$

("orthogonal" refers to the intersection pairing) and one can compute $K(f \times g) = \operatorname{Arf}(Q \,|\, K_{q+m})$ by adding contributions from each of the summands.

The first observation is that we can extend the *definition* of Q to all of $H_m(M_1)$ and $H_q(P_1)$, in a non-unique way, by choosing any representatives for homology and using Lemma 1.2, which still applies. Then for $x \otimes y \in K_{q+m}$ we have

$$Q(x \otimes y) = Q(x)(y \cdot y) + Q(y)(x \cdot x) \tag{1}$$

$$(x \otimes y) \cdot (x' \otimes y') = (x \cdot x')(y \cdot y') \ . \tag{2}$$

The second formula is of course true in general. To check the first formula, we can use a representative for $x \otimes y$ which is the product of representatives used for x and y (since one of x or y is in K_m or K_q its representative is bordant to zero in N or M and hence the product is bordant to zero in $M \times N$).

Then we can use the products $q_1 \times q_2$, $t_1 \times t_2$ of the immersions used in defining $Q(x)$ and $Q(y)$ and assume $\sharp(q_1) = \sharp(q_2) = 0$ by the observation at the start of proof of Lemma 1.3. The formula then follows from the observation that

$$\sharp(t_1 \times t_2) = \sharp(t_1)(y\cdot y) + \sharp(t_2)(x\cdot x)$$

which is seen by shifting one sheet of each of the double manifolds of $t_1 \times t_2$ into general position with respect to the other. (More precisely let a be a double point of t_1 then we have the double manifold $a \times t_2(W_2)$ in im $t_1 \times t_2$ whose neighbourhood can be identified with $X \times t_2(W_2)$ where X is a pair of q-planes in R^{2q} intersecting transversally at one point. Now consider a general position shift of $t_2(W_2)$ and apply its product with the identity to one of the q-planes $\times t_2(W_2)$. Then this reduces the intersection to μ points where $\mu =$ the number of intersections of $t_2(W_2)$ with itself in general position, which is of course equal to $y\cdot y$ mod 2.)

We now proceed to the computation of the Arf invariant:

Part A $\qquad \operatorname{Arf}(Q \mid K_q \otimes K_m) = 0\,.$

Recall from the appendix that each of K_q and K_m may be split into a direct sum of copies of $\mathbf{H}^{(0)}$ and $\mathbf{H}^{(1)}$ and that the Arf invariant is the number of $\mathbf{H}^{(1)}$'s in the summation (mod 2). (The $\mathbf{H}^{(i)}$ are particular 2-dimensional spaces with quadratic forms, which are defined in the appendix.)

Using the fact that $x\cdot x = 0$ and $y\cdot y = 0$ for $x \in K_m$, $y \in K_q$ respectively (remark below Theorem 1.4), it follows easily from formula (1) that $K_q \otimes K_m$ is a direct sum of copies of $\mathbf{H}^{(0)}$ only and so has zero Arf invariant.

Part B $\qquad \operatorname{Arf}\big(Q \mid K_q \otimes H_m(M)\big) = \chi(M) \operatorname{Arf}(Q \mid K_q)\,.$

Forget the value of Q on $H_m(M) \subset H_m(M_1)$ and write this summand as a direct sum of copies of $\mathbf{H}$ and U, where $\mathbf{H}$ is the 2-dimensional vector space with pairing given by the matrix in the appendix and U is the 1-dimensional space, basis vector x and $x\cdot x = 1$.

Using formulae (1) and (2) the reader may readily check the following identities (observe that $Q(x)$ is irrelevant for $x \in H_m(M)$ since $y\cdot y = 0$ for $y \in K_q$):

$$\mathbf{H}^{(0)} \otimes \mathbf{H} \cong \mathbf{H}^{(1)} \otimes \mathbf{H} \cong \mathbf{H}^{(0)} \oplus \mathbf{H}^{(0)}\,,$$

$$\mathbf{H}^{(0)} \otimes U = \mathbf{H}^{(0)}$$

and

$$\mathbf{H}^{(1)} \otimes U = \mathbf{H}^{(1)}\,.$$

It follows that $\operatorname{Arf}\big(Q \mid K_q \otimes H_m(M)\big) = \alpha \operatorname{Arf}(Q \mid K_q)$, where $\alpha =$ number of summands U in $H_m(M)$ (mod 2). But α equals the dimension of $H_m(M)$

(mod 2), since $\dim \mathbf{H} = 2$, which in turn is equal to $\chi(M)$ (mod 2) by duality.

Part C $$\operatorname{Arf}\left(Q \mid K_m \otimes H_q(P)\right) = \chi(P) \operatorname{Arf}(Q \mid K_m).$$

The proof is identical to that of Part B.

3. The classifying space for normal maps

We will define a bundle theory with Whitney sums and appeal to results of [10] to find a countable CW-complex G/PL (with H-space structure) which classifies it.

The bundle theory. A $\widetilde{G}_q/\widetilde{PL}_q$-bundle over a PL cell complex K is a pair (ξ^q, t) where ξ^q is a PL block bundle with base K and $t: E(\xi^q) \to K \times I^q$ is a block homotopy trivialisation, cf. [11; § 2], Casson [2].

Two such (ξ^q, t) (η^r, s) are *stably equivalent* if there is a stable isomorphism $h: E(\xi^q \oplus \varepsilon^{r+u}) \to E(\eta^r \oplus \varepsilon^{q+u})$ such that $t \oplus$ id. is block homotopic to $s \oplus$ id.

Whitney sum is defined (on stable equivalence classes) as $(\xi \oplus \eta, t \oplus s)$ and induced bundles are defined as $(f^*\xi, t \circ \hat{f})$ where $\hat{f}: E(f^*\xi) \to E(\xi)$ is the natural bundle map.

Let $G/PL[K]$ denote the set of stable equivalence classes. It is an abelian group under Whitney sum and becomes a contravariant functor on the category of polyhedra via the induced bundle (see [10, 11] for details and also for extension to the category of arbitrary CW-complexes).

THEOREM 3.1. *There is a countable CW-complex with H-space structure, G/PL, such that $[\quad, G/PL]$ and $G/PL[\ \ ]$ are naturally equivalent functors.*

Proof. The Δ-set $\widetilde{G}_q/\widetilde{PL}_q$ classifies $\widetilde{G}_q/\widetilde{PL}_q$-bundles (see [10; § 3]); stabilizing, $\widetilde{G}/\widetilde{PL}$ classifies G/PL-bundles. For the usual reasons we may replace $\widetilde{G}/\widetilde{PL}$ by a countable CW-complex G/PL (cf. [7; 2.6 et seq.]). Now $\widetilde{G}/\widetilde{PL}$ has a multiplication as a Δ-set, which endows G/PL with an H-space structure corresponding to Whitney sum.

This multiplication is defined as follows. Let $h_i: \Delta^k \times I^{q_i} \circlearrowleft$ be two equivalences $i = 1, 2$. Define their product $h_1 \cdot h_2: \Delta^k \times I^{q_1+q_2} \circlearrowleft$ to be the composition

$$\Delta^k \times I^{q_1+q_2} \xrightarrow{\text{diag}} (\Delta^k)^2 \times I^{q_1} \times I^{q_2} \xrightarrow{h_1 \times h_2}$$
$$(\Delta^k)^2 \times I^{q_1+q_2} \xrightarrow{r \times \text{id}} \Delta^k \times I^{q_1+q_2}$$

where $r: (\Delta^k)^2 \to \Delta^k$ is a retraction on the diagonal.

Connection with normal maps. Suppose $(f, \hat{f}): M_1 \to M$ is a normal map where M is any manifold (possibly with boundary). Then as remarked earlier, $T(\xi)$ is reducible, indeed the canonical reduction of $T(\nu_{M_1})$ composed with $T(\hat{f})$ determines a reduction, say $g: D^{n+N} \to T(\xi)$. Then the uniqueness theorem of

Spivak [12] implies that ξ is block homotopy equivalent to ν_M (as stable bundles) or equivalently that $\xi - \nu_M$ is block homotopy trivial. In fact, the proof of Spivak's Theorem given in Wall [15; 3.5] shows that there is a block homotopy equivalence $q\colon \xi \to \nu_M$ determined up to block homotopy by the criterion that $T(q)$ commutes up to homotopy with g and the canonical reduction p of $T(\nu_M)$:

$$\begin{tikzcd} & D^{n+N} \arrow[dl, "g"'] \arrow[dr, "p"] & \\ T(\xi) \arrow[rr, "T(q)"'] & & T(\nu_M) \end{tikzcd} .$$

This means that there is a well-defined block homotopy trivialisation t of $\xi - \nu_M$. The pair $(\xi - \nu_M, t)$ is the G/PL-bundle (base M) associated to the normal map $(f, \hat{f})$.

Conversely a G/PL-bundle (η, t) with base M determines a block homotopy equivalence q between $\xi = \eta + \nu_M$ and ν_M, and hence, on composing p with $T(q^{-1})$, a reduction $g\colon D^{n+N} \to T(\xi)$. By [8] we can assume that g is transverse regular to $M \subset T(\xi)$ and this gives us a manifold $M_1 \subset D^{n+N}$, a degree 1 map $g|\colon M_1 \to M$ covered by a bundle map $g|\colon \nu_{M_1} \to \xi$. In other words, a normal map. It is readily verified that these two constructions are inverse on cobordism and equivalence classes respectively and we have proved:

THEOREM 3.2. *There is a bijection between normal cobordism classes of normal maps onto M and the set of homotopy classes $[M, G/PL]$.*

4. Construction of the Kervaire classes

Let $f\colon M^{2m} \to G/PL$ be a map, where M is closed, then by Theorem 3.2 it makes sense to talk of the Kervaire obstruction $K(f)$. Next let $Z(\ )$ be a *genus*, that is $Z(M) = 1 + Z_1 + Z_2 + \cdots \in H^*(M)$ is a $\mathbf{Z}_2$-cohomology class defined for each closed manifold with two properties

(1) $$Z(M \times N) = Z(M) \otimes Z(N)$$

(2) $$\langle Z(M), [M]\rangle = \chi(M) .$$

Examples of genera are $W(\ )$ the total Stiefel-Whitney class and $V^2(\ )$ the square of the Wu class (see next section).

THEOREM 4.1. (*The Kervaire class for unoriented manifolds.*) *There is a unique primitive $\mathbf{Z}_2$-cohomology class*

$$k^Z = k_2^Z + k_4^Z + k_6^Z + \cdots \in H^*(G/PL)$$

such that for any map $f\colon M \to G/PL$, where M is closed and even dimensional, we have

$$K(f) = \langle f^*k^Z \smile Z(M), [M]\rangle .$$

Proof. Construction of k^Z. k^Z is constructed by induction. Suppose that $k^Z_{(2r)} = k^Z_2 + k^Z_4 + \cdots + k^Z_{2r}$ has been constructed and that $K(f)$ is given by the above formula, for $m \leqq r$ where $\dim M = 2m$, with $k^Z_{(2r)}$ replacing k^Z. We have to construct k_{2r+2}. Now by the proof of Lemma 1.1 $T_{2r+2}(G/PL) \to H_{2r+2}(G/PL)$ is onto with kernel generated by decomposable elements, where a singular manifold (W, f) *is decomposable* if $W = M \times N$ where $\dim N > 0$ and $f = g \circ \pi_1$ where $g: M \to X$ (see [3]). So it suffices to define a homomorphism $\lambda: T_{2r+2}(G/PL) \to \mathbf{Z}_2$ which vanishes on decomposable elements. Let $f: M^{2r+2} \to G/PL$ be any singular manifold. Define

$$\lambda(f) = K(f) + \langle f^* k^Z_{(2r)} \smile Z(M), [M] \rangle .$$

It is clear that λ is a homomorphism and it remains to check

(1) λ vanishes on decomposables

(2) $k^Z_{(2r+2)} = k^Z_{(2r)} + k^Z_{2r+2}$ has the inductive property.

(2) is immediate from the definition of λ and the fact that

$$\begin{aligned} \langle f^* k^Z_{2r+2} \smile Z(M), [M] \rangle &= \langle f^* k^Z_{2r+2}\,,\ [M] \rangle \\ &= \langle k^Z_{2r+2}\,,\ f[M] \rangle \\ &= \lambda(f) && \text{by definition.} \end{aligned}$$

To prove (1) we need

LEMMA 4.2. *Suppose $f: M \to G/PL$ corresponds (via 3.2) to the normal map $(f_1, \hat{f}_1): M_1 \to M$, then $f \circ \pi_1: M \times N \to G/PL$ corresponds to the normal map $(f_1, \hat{f}_1) \times \mathrm{id}: M_1 \times N \to M \times N$.*

The proof is entirely straightforward and will be left to the reader.

Proof of (1). *Case* A. $\dim M$ and $\dim N$ are both even, $\dim N > 0$, $(\dim M + \dim N = 2r + 2)$.

$$\begin{aligned} &\langle (f \circ \pi_1)^* k^Z_{(2r)} \smile Z(M \times N)\,,\ [M \times N] \rangle \\ &= \langle f^* k^Z_{(2r)} \otimes 1 \smile Z(M) \otimes Z(N)\,,\ [M] \otimes [N] \rangle \\ &= K(f) \cdot \langle Z(N)\,,\ [N] \rangle && \text{by induction} \\ &= K(f)\chi(N) \\ &= K\big((f_1, \hat{f}_1) \times \mathrm{id}\big) && \text{by the product formula} \\ &= K(f \circ \pi_1) && \text{by Lemma 4.2.} \end{aligned}$$

Case B. $\dim M$ and $\dim N$ are both odd. A similar computation as above shows that

$$\begin{aligned} &\langle (f \circ \pi_1)^* k^Z_{(2r)} \smile Z(M \times N)\,,\ [M \times N] \rangle \\ &= \langle f^* k^Z_{(2r)} \smile Z(M)\,,\ [M] \rangle \chi(N) \\ &= 0 \end{aligned}$$

since dim N is odd. The result then follows from Lemma 4.2 and the following

LEMMA 4.3. *The Kervaire obstruction of a normal map, which is the product of two normal maps of odd dimensional manifolds, is zero.*

Proof. In the odd dimensional case one can perform surgery to kill all of $K_*(f)$ (we are using $\mathbf{Z}_2$-coefficients!). This follows by similar methods to the easy part of the proof Kervaire and Milnor [5] (killing the free part of the middle-dimensional kernel): First kill the below-middle-dimensional kernels with integer coefficients. Then apply the diagram on p. 515 of [5] with $\mathbf{Z}_2$-coefficients. To kill a class we need: (1) to represent by an imbedded sphere with trivial normal bundle; (2) primitivity. (2) is automatic with $\mathbf{Z}_2$-coefficients while (1) follows from the relative Hurewicz theorem.

Uniqueness and primitivity. This completes the construction of k^Z. Uniqueness is proved by induction. For the induction step, observe that the definition of λ, and hence of k^Z_{2r+2}, was forced on us, if the formula was to work. To prove primitivity, let $m: G/PL \times G/PL \to G/PL$ be the multiplication and suppose that $m^*(k^Z) = 1 \otimes k^Z + k^Z \otimes 1 + \alpha$. We have to show $\alpha = 0$. This will follow from the product formula. We need an extension of Lemma 4.2, again straightforward:

LEMMA 4.4. *Suppose $(f_1, \hat{f}_1): M_1 \to M$ and $(g_1, \hat{g}_1): N_1 \to N$ are normal maps and f, g the corresponding maps to G/PL. Then $m \circ (f \times g): M \times N \to G/PL$ corresponds to the normal map $(f_1, \hat{f}_1) \times (g_1, \hat{g}_1): M_1 \times N_1 \to M \times N$.*

We can now compute $K(f \times g)$ in two ways. First using the product formula and second using the class k^Z. We have:

$$K(f \times g) = \chi(M)K(g) + \chi(N)K(f) \tag{1}$$

and

$$\begin{aligned} K(f \times g) &= \langle (m \circ (f \times g))^* k^Z \smile Z(M \times N)\ ,\ [M \times N] \rangle \\ &= \langle (f \times g)^*(1 \otimes k^Z + k^Z \otimes 1 + \alpha) \smile Z(M \times N)\ ,\ [M \times N] \rangle \\ &= \langle (f \times g)^* \alpha \smile Z(M \times N), [M \times N] \rangle + K(f \times g) \end{aligned} \tag{2}$$

by the usual computation. It follows that the first term is zero. Since this is true for all manifolds M and N and all maps f, g, it follows easily that $\alpha = 0$, as required.

Remark. This is the first time we have used the full two term product formula. Previously we have only used it when one of the terms was zero.

COROLLARY 4.5. *$K: [M, G/PL] \to \mathbf{Z}_2$ is a homomorphism.*

This follows from the primitivity of k^Z.

We now define a $\mathbf{Z}_2$-*manifold* to be a manifold M in which $W_1(M)$ is the reduction of an integral class or, equivalently, $W_1^2 = 0$.

THEOREM 4.6. (*The Kervaire class for* $\mathbf{Z}_2$-*manifolds.*) *There is a unique primitive* $\mathbf{Z}_2$-*cohomology class*

$$\tilde{k}^Z = \tilde{k}_2^Z + \tilde{k}_6^Z + \tilde{k}_{10}^Z + \cdots \in H^*(G/PL)$$

such that for any map $f: M \to G/PL$, *where M is a closed* $\mathbf{Z}_2$-*manifold of dimension* $\equiv 2(4)$, *we have*

$$K(f) = \langle f^* k^Z \smile Z(M) \, , \; [M] \rangle \, .$$

Proof. The proof is very similar to that of Theorem 4.1. First let $T_*(\ ;\mathbf{Z}_2)$ denote bordism of $\mathbf{Z}_2$-manifolds then we again have $T_*(G/PL;\mathbf{Z}_2) \to H_*(G/PL)$ onto with kernel generated by decomposables. (The proof of this is the same, except that one uses the fact that any homology class is represented by a $\mathbf{Z}_2$-manifold, essentially due to Thom.)

Remark. The notation $T_*(\ ;\mathbf{Z}_2)$ is intended to suggest "oriented bordism with $\mathbf{Z}_2$-coefficients". This is justified by the fact that a $\mathbf{Z}_2$-manifold M contains a codimension 1 submanifold W such that $M - W$ is orientable and orientation changes at W. This is seen as follows: Since W_1 is the reduction of an integral class it is represented by a map $w: M \to S^1$. Shift w transverse regular to $\{1\} \in S^1$ and then $W = w^{-1}(\{1\})$ is a suitable submanifold.

The construction of $\tilde{k}^Z$ now proceeds exactly as that of k^Z. The proof of (1) case B is as before; that of case A falls into two subcases (a) dim $M \equiv 2(4)$, which is exactly like the old case A, and (b) dim $M \equiv 0(4)$; in this case both parts of λ are zero by the product formula and the following lemma:

LEMMA 4.7. *If N is a* $\mathbf{Z}_2$-*manifold of dimension* $\equiv 2(4)$ *then* $\chi(N)$ *is zero* (mod 2).

The lemma is proved by a simple computation with the Stiefel-Whitney classes using the fact that $W_1^2 = 0$ (see Stong [13]). It also follows from a consideration of the "orientation submanifold" of the remark.

This completes the construction of $\tilde{k}^Z$. It is again easy to prove that $\tilde{k}^Z$ is unique, and primitivity follows by an argument similar to that used earlier.

5. Relations between the Kervaire classes

Let M be a closed manifold and denote by $W(M)$ the total Stiefel-Whitney class of M and by $V(M)$ the Wu class of M. We shall regard $V(M) = 1 + V_1(M) + V_2(M) + \cdots$ as defined by the formula

(5.1) $$V_j \smile x = \mathrm{Sq}^j(x)$$

where $x \in H^{n-j}(M)$ and $n = \dim M$.

LEMMA 5.2. *$W(\ )$ and $V^2(\ )$ are genera.*

Proof. The case of $W(\ )$ is well-known. For $V^2(\ )$, the multiplicative property follows from the equivalent definition of V (cf. [1]):

$$\mathrm{Sq}(V) = W . \tag{5.3}$$

Now by (5.1) we have $V_j(M) = 0$ for $j > n/2$ and hence

$$W_{2m}(M) = V_{2m} + \mathrm{Sq}^1(V_{2m-1}) + \cdots + \mathrm{Sq}^m(V_m) = V_m^2$$

when $n = 2m$. So that $\langle V^2(M), [M]\rangle = \chi(M)$, as required.

Now by Lemma 5.2 we can define the Kervaire classes corresponding to $W(\ )$ and $V^2(\ )$ and we have:

THEOREM 5.4. (*Brumfiel and Wall.*)

(1) $$k^{V^2} = \tilde{k}^{V^2}$$

(2) $$k^{V^2} = \chi(\mathrm{Sq})_2 . k^W$$

(3) $$\tilde{k}^{V^2} = \chi(\mathrm{Sq})_4 . \tilde{k}^W$$

(4) $$k^W = (1 + \mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^2)\tilde{k}^W$$

where χ is the canonical anti-automorphism of the Steenrod algebra.

The proof of Theorem 5.4 which we give is due to Brumfiel and Madsen. First note:

LEMMA 5.5. *$V^2(M) = 1 + V_1^2(M) + V_2^2(M) + \cdots$ has terms in even dimensions only, and, if M is a $\mathbf{Z}_2$-manifold, in dimensions $\equiv 0(4)$ only.*

Proof. The first part is obvious. For the second part it suffices to show that $V_{2l+1}^2 = 0$ for $\mathbf{Z}_2$-manifolds. But $V_{2l+1} = 0$ for oriented manifolds since

$$\begin{aligned} \mathrm{Sq}^{2l+1}(x) &= \mathrm{Sq}^1\mathrm{Sq}^{2l}(x) \\ &= V_1 \smile \mathrm{Sq}^{2l}(x) = W_1 \smile \mathrm{Sq}^{2l}(x) \qquad \text{by (5.1) and (5.3)} \end{aligned}$$

where $x \in H^{n-2l-1}(M)$.

Consequently the universal Wu class $V_{2l+1} \in H^{2l+1}(BPL)$ maps to zero in $BSPL$ and is therefore in the ideal generated by W_1. (Note $BPL = BSPL \times RP^\infty$ and W_1 generates $H^*(RP^\infty)$.) Hence $W_1^2 = 0$ implies $V_{2l+1}^2 = 0$, as required.

Proof of Theorem 5.4 (1). We will deduce from Lemma 5.5 that k^{V^2} has terms in dimensions $\equiv 2(4)$ only. It then follows that k^{V^2} has the same properties as $\tilde{k}^{V^2}$ and so they are equal by the uniqueness part of Theorem 4.6. Suppose inductively that we have shown that $k_{4j}^{V^2} = 0$ for $j < r$, and that $k_{4r}^{V^2} \neq 0$. Choose $x \in H_{4r}(G/PL)$ so that $\langle x, k_{4r}^{V^2}\rangle = 1$ and represent x by a singular $\mathbf{Z}_2$-manifold $f\colon M^{4r} \to G/PL$, then

$$\begin{aligned} K(f) &= \langle f^*k^{V^2} \smile V^2(M)\ ,\ [M]\rangle \\ &= \langle f^*k^{V^2}_{4r}\ ,\ [M]\rangle && \text{by Lemma 5.5 and induction} \\ &= 1 && \text{by choice of } f\,. \end{aligned}$$

This contradicts Lemma 5.6, below, which we prove at the end of the section:

LEMMA 5.6. *The Kervaire obstruction of any normal map onto a* $\mathbf{Z}_2$-*manifold of dimension* $\equiv 0(4)$ *is zero.*

Proof of Theorem 5.4 (2). Let $f\colon M \to G/PL$ be any map with M closed and even dimensional. Then we have

$$\begin{aligned} &\langle f^*k^W \smile W(M)\ ,\ [M]\rangle \\ &= \langle f^*k^W \smile \operatorname{Sq} V(M)\ ,\ [M]\rangle && \text{by (5.3)} \\ &= \langle \operatorname{Sq}\chi(\operatorname{Sq}) f^*k^W \smile \operatorname{Sq} V(M)\ ,\ [M]\rangle \\ &= \langle \operatorname{Sq}\big(f^*(\chi(\operatorname{Sq})k^W\big) \smile V(M))\ ,\ [M]\rangle \\ &= \langle V(M) \smile f^*\big(\chi(\operatorname{Sq})k^W\big) \smile V(M)\ ,\ [M]\rangle && \text{by (5.1)} \\ &= \langle f^*\big(\chi(\operatorname{Sq})k^W\big) \smile V^2(M)\ ,\ [M]\rangle \\ &= \langle f^*\big(\chi(\operatorname{Sq})k^W\big)_{2*} \smile V^2(M)\ ,\ [M]\rangle && \text{by Lemma 5.5} \\ &= \langle f^*\big(\chi(\operatorname{Sq})_{2*}k^W\big) \smile V^2(M)\ ,\ [M]\rangle\,. \end{aligned}$$

Consequently $\chi(\operatorname{Sq})_{2*}k^W$ has the same properties as k^{V^2} and they are equal by the uniqueness part of Theorem 4.1.

Proof of (3). The proof is formally identical to part (2). M is a closed $\mathbf{Z}_2$-manifold of dimension $\equiv 2(4)$, and we use the uniqueness part of Theorem 4.6.

Proof of (4). Combining (1), (2), and (3) we have

$$\chi(\operatorname{Sq})_{2*}k^W = \chi(\operatorname{Sq})_{4*}\tilde{k}^W$$

and the result follows from the identity

$$\chi(\operatorname{Sq})_{4*} = \chi(\operatorname{Sq})_{2*}(1 + \operatorname{Sq}^2 + \operatorname{Sq}^2\operatorname{Sq}^2)$$

which is a consequence of the Adem relations.

Proof of Lemma 5.6. Let $(f, \hat{f})\colon M_1 \to M$ be the given normal map where $\dim M = 2m$, and let $W \subset M$ be an orientation submanifold in the sense described in the remark in Theorem 4.6. By transversality assume $f^{-1}(W) = W_1$ is a codimension 1 submanifold of M_1 and then $(f, \hat{f})|\colon W_1 \to W$ is a normal map. Now W is orientable and so we can kill $\pi_1(W)$ by 1-dimensional surgeries. Then by simultaneously surgering the transverse pull backs in W_1 we see that this surgery extends to a cobordism of normal maps, where the definition is enlarged to include a normal map onto a cobordism. Then since we are now in the simply-connected odd dimensional case, $f\,|\,W_1\colon W_1 \to W$

is normal cobordant to a homotopy equivalence. This cobordism extends to a cobordism of the original normal map $f\colon M_1 \to M$ in the obvious way. So we can assume $f \mid W_1$ is a homotopy equivalence, and, by below-middle-dimensional surgery, that $K_m(M - W; \mathbf{Z})$ is the only non-zero kernel of $f \mid (M_1 - W_1)$. Now from the Mayer-Vietoris sequence of kernels we have $K_m(M; \mathbf{Z}) = K_m(M - W; \mathbf{Z})$ and we can define an even symmetric bilinear form B on $K_m(M; \mathbf{Z})$ by intersecting in $M_1 - W_1$ which is orientable. It is easy to check that $Q(x) = (1/2)B(x, x) \bmod 2$ and hence Q is the "$\mathbf{Z}_2$-reduction" of B. The result then follows since the Arf invariant of such a reduction is always zero. The simplest proof of this fact is to check by direct computation on a canonical form (e.g. the standard 8×8 matrix used by Milnor et al.).

Appendix. Notes on the Arf invariant

Let V be a finite dimensional $\mathbf{Z}_2$-vector space. A quadratic form Q on V consists of

(1) a non-singular bilinear pairing $b\colon V \times V \to \mathbf{Z}_2$ (we write $x \cdot y$ for $b(x, y)$)

(2) a function $Q\colon V \to \mathbf{Z}_2$

which satisfy the relation

(3) $Q(x + y) = Q(x) + Q(y) + x \cdot y$.

(Alternatively one can regard $x \cdot y$ as defined by (3).)

Observe that (3) implies $Q(0) = 0$ and $x \cdot x = 0$ for all x. Notice also that (3) shows how to compute Q given the pairing and the values of Q on a basis for V. This observation enables us to define the direct sum $V_1 \oplus V_2$ of spaces with quadratic forms by letting the pairing matrix for $V_1 \oplus V_2$ be

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$$

with respect to basis $(b_1 \mid b_2)$, where b_i is a basis for V_i, and A_i the matrix of the pairing for V_i, and defining $Q \mid V_i = Q_i$ where $V_i \subset V_1 \oplus V_2$ is the canonical embedding.

Now let $\mathbf{H}^{(0)}$ (resp. $\mathbf{H}^{(1)}$) be the 2-dimensional space with basis $\{x, y\}$ form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $Q(x) = Q(y) = 0$ (resp. $Q(x) = Q(y) = 1$).

LEMMA. (a) *Let V be a space with form then*

either $\qquad V \cong \mathbf{H}^{(0)} \oplus V'$

or $\qquad V \cong \mathbf{H}^{(1)} \oplus V'$.

$$\text{(b)} \qquad \mathbf{H}^{(0)} \oplus \mathbf{H}^{(0)} \cong \mathbf{H}^{(1)} \oplus \mathbf{H}^{(1)}$$

$$\text{(c)} \qquad \bigoplus_{i=1}^{n} \mathbf{H}_i^{(0)} \not\cong \mathbf{H}^{(1)} \oplus \bigoplus_{i=1}^{n-1} \mathbf{H}_i^{(0)} .$$

Proofs. (a) Let $x \in V$, $x \neq 0$ and choose $y \in V$ with $x \cdot y = 1$ by non-singularity. Let $\mathbf{H}$ be the subspace generated by x and y and V' the orthogonal complement. Then if $Q(x) = Q(y)$ the result is proved, otherwise we have (without loss of generality) $Q(x) = 0$ and $Q(y) = 1$. Then let $y' = x + y$ and then $Q(x) = Q(y') = 0$.

(b) Let $\{x_1, y_1\}\{x_2, y_2\}$ be bases for the $\mathbf{H}^{(0)}$'s and let $a_1 = x_1 + y_1 + x_2$, $b_1 = x_1 + y_1 + y_2$, $a_2 = x_1 + x_2 + y_2$, $b_2 = y_1 + x_2 + y_2$. Then check that $a_i \cdot b_j 1 =$ if $i = j$ and 0 if $i \neq j$, and that $Q(a_i) = Q(b_i) = 1$.

(c) Let V be a space with form and define $n(V)$ to be the number of elements of V on which Q takes the value 0. Then by induction and formula (3) one sees that:

$$n\Big(\bigoplus_{i=1}^{n} \mathbf{H}_i^{(0)}\Big) = 2^{2i-1} + 2^{i-1}$$
$$n\Big(\mathbf{H}^{(1)} \oplus \bigoplus_{i=1}^{n-1} \mathbf{H}_i^{(0)}\Big) = 2^{2i-1} - 2^{i-1}$$

(I first saw this elegant proof in a lecture given by Wall).

Using the lemma we can completely classify spaces with forms since, by part (a), any space $V \cong \bigoplus_j \mathbf{H}_j^{(1)} \oplus \bigoplus_i \mathbf{H}_i^{(0)}$ and by part (b) we can change all but possibly one of the $\mathbf{H}^{(1)}$'s into $\mathbf{H}^{(0)}$'s. Therefore

$$V = \text{either } \mathbf{H}^{(1)} \oplus \bigoplus_i \mathbf{H}_i^{(0)}$$
$$\text{or } \bigoplus_i \mathbf{H}_i^{(0)}$$

and these possibilities are distinct by part (c).

Define the Arf invariant $\mathrm{Arf}(V) \in \mathbf{Z}_2$ to be 1 in the first case and 0 in the second case then we see from part (b) that $\mathrm{Arf}(V)$ can be computed as the number of $\mathbf{H}^{(1)}$ summands in any decomposition of V into $\mathbf{H}^{(0)}$'s and $\mathbf{H}^{(1)}$'s (counted mod 2) and consequently that $\mathrm{Arf}(\ )$ is additive (i.e. $\mathrm{Arf}(V_1 \oplus V_2) = \mathrm{Arf}(V_1) + \mathrm{Arf}(V_2)$).

We also used the following property of the Arf invariant:

LEMMA. *Suppose* $U \subset V$ *is a self-orthogonal subspace* (i.e. $x \cdot y = 0$ *for all* $x, y \in U$) *of dimension* $1/2 \dim(V)$ *and that* $Q(x) = 0$ *for each* $x \in U$. *Then* $\mathrm{Arf}(V) = 0$.

Proof. By induction on dimension. Let $x \in U$, $x \neq 0$ and $y \in V$ with $x \cdot y = 1$. Let $\mathbf{H}$ be the space generated by x and y and V' the orthogonal complement. Then $U' = U \cap V'$ has dimension $1/2 \dim(V')$ and $\mathbf{H} \cong \mathbf{H}^{(0)}$ since

$Q(x) = 0$ (see proof of (a) above). By induction $\mathrm{Arf}(V') = 0$ and hence $\mathrm{Arf}(V) = 0$.

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