INTRODUCTION:

We consider the problem of deforming a homotopy equivalence \( f: (L, \partial L) \to (M, \partial M) \) between smooth manifolds into a diffeomorphism. There is a theory for this problem analogous to that in (6).

We describe this theory—concentrating on points of difference between the two theories and on points omitted in (6).

THE OBSTRUCTION THEORY:

**Definition 1:** If \( n > 5 \), let \( \mathcal{H}_n \) denote the category whose objects are smooth compact \( n \)-manifolds \( M \) such that \( \pi_1 M = \pi_1 \partial M = 0 \) and whose morphisms are embeddings \( \overline{M}_1^n \subset \text{interior } M_2^n \) such that \( \pi_1 (M_2^n - M_1^n) = 0 \). If \( n \leq 5 \) let \( \mathcal{H}_n = \emptyset \).

**Definition 2:** If \( M \in \mathcal{H}_n \), a \( k \)-skeleton of \( M \) is an embedding \( M_k \subset M \) in \( \mathcal{H}_n \) such that \( \pi_1 (M, M_k) = 0 \) for \( i < k \). A homotopy equivalence \( f: (L, \partial L) \to (M, \partial M) \) is homotopic to a diffeomorphism over the \( k \)-skeleton of \( L \) if there is a \( k \)-skeleton \( L_k \subset L \) and a map \( g: (L, \partial L) \to (M, \partial M) \) such that
1) \( f \) is homotopic to \( g \) as maps of pairs

2) \( g/\mathcal{L}_k : \mathcal{L}_k \to \mathcal{M} \) is an embedding

3) \( g(\mathcal{L} - \mathcal{L}_k) \subset \mathcal{M} - g(\mathcal{L}_k) \).

**Definition 3:** Let \( A_i \) denote the Abelian group of (almost) framed cobordism classes of almost parallelizable manifolds \( \mathcal{M}_i \subset \mathbb{R}^{i+k}, k \gg i \).

**Theorem 1:** Let \( f : (\mathcal{L}, \partial \mathcal{L}) \to (\mathcal{M}, \partial \mathcal{M}) \) be a homotopy equivalence between manifolds in \( \mathcal{M}_n \). Let \( \mathcal{L}_{k-1} \subset \mathcal{L}_k \subset \mathcal{L}_{k+1} \subset \mathcal{L} \) be skeletons of \( \mathcal{L} \). Suppose that \( f/\mathcal{L}_k \) is an embedding, \( f(\mathcal{L} - \mathcal{L}_k) \subset \mathcal{M} - f(\mathcal{L}_k) \), and \( k + 1 < n \). Then there is a homomorphism

\[
\mathcal{H}_{k+1}(\mathcal{L}_{k+1}, \mathcal{L}_k) \xrightarrow{C_{k+1}} A_{k+1}
\]

with the following properties:

1) \( C_{k+1} = 0 \) iff \( f \) is homotopic to a diffeomorphism over the \((k+1)\)-skeleton of \( \mathcal{M} \) by a homotopy which is fixed on \( \mathcal{L}_k \) and keeps \( \mathcal{L} - \mathcal{L}_k \) in \( \mathcal{M} - f(\mathcal{L}_k) \).

2) Under the identification of \( \mathcal{H}_{k+1}(\mathcal{L}_{k+1}, \mathcal{L}_k) \) with the \((k+1)\)-chain group for \( \mathcal{H}_*(\mathcal{L}) \), \( C_{k+1} \) becomes a cocycle. Let \( \theta_{k+1} \) denote the cohomology class of \( C_{k+1} \).

3) \( \theta_{k+1} = 0 \) in \( H^{k+1}(\mathcal{L}; A_{k+1}) \) iff \( f \) is homotopic
to a diffeomorphism over the \((k+1)\)-skeleton of \(L\) by a homotopy which is fixed on \(L_{k-1}\) and keeps \(L - L_{k-1}\) in \(M - f(L_{k-1})\).

**Corollary:** If \(f : (L, \partial L) \to (M, \partial M)\) is a homotopy equivalence with \(L\) and \(M\) in \(\mathcal{U}_n\) and \(\partial L \neq 0\), then \(f\) is homotopic to a diffeomorphism iff a sequence of obstructions in
\[
H^i(L; A_i) \quad 0 < i < \dim M
\]
vanish.

**Remark:** If a \((k+1)\)-skeleton of \(M\) is obtained by attaching \((k+1)\)-handles to \(\partial g(L_k)\),
\[
M_{k+1} = g(L_k) \cup \bigcup_i D_i^{k+1} \times D_i^{n-k-1},
\]
then \(C_{k+1} : H_{k+1}(L_{k+1}, L_k) \to A_{k+1}\) may be defined by the framed submanifolds
\[
\widetilde{g}^{-1}(D_i^{k+1})
\]
where \(\widetilde{g}\) is a suitable \((t\text{-regular to} \cup_i D_i^{k+1} \times 0)\) approximation to \(g\) such that \(\widetilde{g}/L_k = g\).

\[
\begin{array}{ccc}
\text{diffeomorphism} & \overset{g}{\longrightarrow} & \text{homotopy equivalence} \\
\text{FIGURE}
\end{array}
\]
Remark: If $\theta_i$ denotes the group of differentiable structures on $S^i$ and $P_i$, $i = 1, 2, 3, \ldots$ denotes the sequence of Abelian groups, $0, Z_2, 0, Z, 0, Z_2, 0, Z, \ldots$, there is an exact sequence
\[ \cdots \rightarrow P_{i+1} \xrightarrow{\partial} \theta_i \rightarrow A_i \xrightarrow{1} P_i \xrightarrow{\partial} \theta_{i-1} \rightarrow \cdots \]
See (10).

Using this sequence, (3), and (9) we compute $A_i$ for $i \leq 19$ as follows:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_i$</td>
<td>0</td>
<td>$Z_2$</td>
<td>0</td>
<td>$Z_2$</td>
<td>0</td>
<td>$Z \oplus Z_2$</td>
<td>$(Z_2)^2$</td>
<td>$Z_6$</td>
<td>0</td>
<td>$Z$</td>
<td>$Z_3$</td>
<td>$(Z_2)^2$</td>
<td>$15$</td>
<td>$16$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_i$</td>
<td>$Z_2$</td>
<td>$Z \oplus Z_2$</td>
<td>$(Z_2)^3$</td>
<td>$Z_8 \oplus Z_2$</td>
<td>$Z_2$</td>
</tr>
</tbody>
</table>

**HOMOTOPY INTERPRETATION:**

**Definition 1:** A smooth structure on $M$ in $\mathcal{N}_n$ is a pair $(L, g)$ where $L$ is in $\mathcal{N}_n$ and $g: (L, \partial L) \rightarrow (M, \partial M)$ is a homotopy equivalence.

**Definition 2:** A smooth structure $g: (L, \partial L) \rightarrow (M, \partial M)$ restricts to a smooth structure on $M' \subset M$ if $M' \subset M$ and $L' = g^{-1}(M') \subset L$ are morphisms of $\mathcal{N}_n$ and $(L', g/L')$ is a smooth structure on $M'$.

**Definition 3:** $(\tilde{\mathcal{M}}(M))$ Two smooth structures, $(L, g)$ and $(L', g')$ on $M$, are equivalent (or concordant)
if there is a diffeomorphism \( d : L \to L' \) so that \( g' \cdot d \) is homotopic to \( g \). Denote the set of equivalence classes by \( \mathcal{J}(M) \).

**Remark:** The preferred element in \( \mathcal{J}(M) \) is the concordance class of \((M, \text{identity})\). \((L, g)\) is concordant to \((M, \text{identity})\) iff \( g \) is homotopic to a diffeomorphism.

\( \mathcal{J}(M) \) may be regarded as the set of homotopy equivalence classes of smooth manifold structures on an underlying CW pair for \((M, \partial M)\).

\( \mathcal{J} \) is a functor:

**Theorem (Browder)** Let \( M' \subset M \) be a morphism in \( \mathcal{M}_n \) and let \((L, g)\) be a smooth structure on \( M \). Then \( g \) is homotopic to \( \overline{g} : (L, \partial L) \to (M, \partial M) \) so that \((L, \overline{g})\) restricts to a smooth structure \((L', g')\) on \( M' \). The concordance class of \((L', g')\) depends only on the concordance class of \((L, g)\).

**Proof:** This is the codimension one embedding theorem of Browder (1) without the \( \mathcal{A}_2 \) hypotheses. This form follows from properties of \( \Theta^*_M \) in Theorem 2.

**Corollary:** The assignment \( M \to \mathcal{J}(M) \) extends to a contravariant functor from \( \mathcal{M}_n \) to the category of based sets.
A PRODUCT OPERATION IN $\mathcal{S}(M)$:

We use the restriction homomorphism to define a binary operation in $\mathcal{S}(M)$. Let $\tau \to M$ denote the tangent $n$-disk bundle of $M$. Then there is an embedding $i: \tau \to M \times M$ representing $\tau$ as a tubular neighbourhood of the diagonal in $M \times M$.

**Definition:** Suppose either $\partial M \neq 0$ or $\dim M \neq 4i$.

Define a product

$$\mathcal{S}(M) \times \mathcal{S}(M) \to \mathcal{S}(M)$$

by the composition

$$\mathcal{S}(M) \times \mathcal{S}(M) \xrightarrow{e_{\mathcal{X}}(M \times M)} \mathcal{S}(\tau) \xrightarrow{i^*} \mathcal{S}(M).$$

$e$ is defined on representatives by $(L, g) \times (L', g')$.

$i^*$ is the map corresponding to the inclusion $i: \tau \subseteq M \times M$. $\pi^*$ is defined on representatives by $(L, g) \xrightarrow{\pi^*} (\pi^* L, \pi^* g)$.

$\pi^*$ is an injection which contains the image of $i^* e$ if $\partial M \neq 0$ or $\dim M \neq 4i$. (These facts are proved in (8) where $\mathcal{S}(M)$ and $\mathcal{S}(bundle over M)$ are compared.)

THE CLASSIFYING SPACE FOR $\mathcal{S}(M)$.

Let $F/0 \xrightarrow{\mathcal{F}} B_0$ be the fibre of the homomorphism

$$B_0 \xrightarrow{\mathcal{J}} B_F,$$

which maps equivalence classes of stable vector bundles to equivalence classes of stable spherical fibre spaces.
Theorem 2:  a) For each $M$ in $\mathcal{M}_n$ there is a homomorphism

$$\mathcal{M}(M) \xrightarrow{\theta_M} [M, F/0].$$

b) The collection $\{\theta_M\}$ comprise a natural transformation of functors on $\mathcal{M}_n$,

$$\mathcal{M} \xrightarrow{\theta} [\ , F/0].$$

c) $\theta_M$ is an isomorphism if $\partial M \neq 0$.

Remark: In a) replace the word "homomorphism" by "function" if $\partial M = 0$ and $\dim M = 4i$ since $\mu$ is not defined in this case. A product operation is always defined in the piecewise linear analogue of $\mathcal{M}(M)$, $PL(M)$, because $PL(M) = PL(M_0)$.

If $\partial M = \emptyset$, $\theta$ need not be surjective nor injective. We can describe the situation however.

Definition: Define an action (connected sum)

$$\theta_n \times \mathcal{M}(M) \xrightarrow{\#} \mathcal{M}(M)$$
on representatives by

$$(\sigma, (L, g)) \mapsto (\sigma \# L, \text{pt map} \# g) \equiv \sigma \# (L, g)$$

Definition: Define functions

$$[M^n, F/0] \xrightarrow{K} P_n$$
on representatives $M^n \xrightarrow{f} F/0$ by

$$K(M^n, f) = \begin{cases} 0 & \text{n odd} \\ ([W(M) \cup f^*(U)] [M]) & n \equiv 2 \pmod{4} \\ ([L(M) \cup f^*(\alpha)] [M]) & n \equiv 0 \pmod{4} \end{cases}$$
Here \( U \in H^{1*+2}_{n}(F/0, Z_2) \) is defined in (8), \( W(M) \) is the total Stiefel Whitney class of \( M \), and \([M]_2\) is the generator of \( H_n(M, Z_2) \). 

\( \chi = 1/8j^*(L - 1) \) in \( H^{1*}(F/0; Q) \), where \( j:F/0 \to B_0 \) and \( L \) is the Hirzebruch \( \chi \)-genus in \( H^{1*}(B_0, Z_2) \), \( L(M) \) is the total Hirzebruch class, and \([M]\) is a generator of \( H_n(M; Q) \).

**Remark:** If \( n \equiv 2 \pmod{4} \), the expression for \( K(M^n, f) \) represents a formula for the Kervaire Invariant of an \( F/0 \)-bundle. See (8). In this case \( K \) is a homomorphism.

If \( n \equiv 0 \pmod{4} \), then \( K(M^n, f) \) is actually an integer (3). \( K \) is not a homomorphism in this case (it is \( \text{mod} \ 2 \), however).

**Theorem 3:** Let \( M^n \) belong to \( \mathcal{M}_n \) and consider
\[ \Theta^M: \mathcal{F}(M) \to \mathcal{M}, \ F/0. \]

\( a) \) \( \Theta^M(L, g) = \Theta^M(L', g') \) iff there is a \( \sigma \) in \( \mathcal{F}(M) \) such that \( (L, g) \# \sigma = (L', g') \).

\( b) \) If \( \alpha \in \mathcal{M}, \ F/0 \), then \( \alpha = \Theta^M(L, g) \) iff \( \alpha K = 0 \) in \( P_n \).

**Corollary:** The sequence
\[ \Theta^M(\partial \pi) \#(M, 1\text{id}) \to \mathcal{F}(M) \to \mathcal{M}, \ F/0 \to P_n \]

is exact.

**Properties of \( \Theta^M \):**

Let \( g:(L, \partial L) \to (M, \partial M) \) be a homotopy equivalence and denote \( \Theta^M(L, g) \) by \( \Theta g: M + F/0 \). Note \( \Theta g = \text{pt. map} \) iff \( g \) is homotopic to a diffeomorphism (mod \( \Theta^M \)).
In (6) we described a map \( \zeta g : M \rightarrow F/PL \) which is homotopic to zero iff \( g \) is homotopic to a PL-homeomorphism. (\( M_0 = M \) if \( \partial M \neq \emptyset \) and \( M_0 = M - \text{pt} \) if \( \partial M = \emptyset \).)

If \( g \) is a PL-homeomorphism, then \( g : L \rightarrow M \) defines a smoothing of the underlying PL-manifold of \( M \). This smoothing is classified by a map \( a_g : M \rightarrow PL/0 \). (2) and (4).

Consider the fibration

\[
\begin{array}{ccc}
PL/0 & \xrightarrow{i} & F/0 \\
| & | & \downarrow j \\
M_0 & \xrightarrow{\zeta g} & F/PL \\
\end{array}
\]

**Theorem 4:**

a) Let \( k \) denote the inclusion \( M_0 \subseteq M \). Then

\[
\begin{array}{ccc}
M & \xrightarrow{\partial g} & F/0 \\
\uparrow k & & \downarrow j \\
M_0 & \xrightarrow{\zeta g} & F/PL \\
\end{array}
\]

is homotopy commutative.

b) If \( g \) is a PL-homeomorphism, then \( a_g \) is defined and

\[
\begin{array}{ccc}
& & PL/0 \\
& \xrightarrow{i} & \\
M & \xrightarrow{\partial g} & F/0 \\
\end{array}
\]

is homotopy commutative.

**Corollary:**

a) \( g \) is homotopic to a PL-homeomorphism iff \( \partial g \) lifts to \( PL/0 \).
b) If $\partial M \neq 0$, the smoothings of $M$ corresponding (under $a$) to $\ker ([M, PL/O] \xrightarrow{i^*} [M, F/O])$ are determined by manifolds diffeomorphic to $M$. In fact the smoothing map $g : L \to M$ is homotopic to a diffeomorphism.

**Theorem 5:** Consider the diagram

\[
\begin{array}{ccc}
F & \to & Top/O \\
\downarrow & & \downarrow \\
M & \xrightarrow{\theta g} & F/O \\
\end{array}
\]

a) $\theta g$ lifts to $F$ iff $g$ is a tangential equivalence.

b) $\theta g$ lifts to $Top/O$ if $g$ is a homeomorphism.

**Remark:** a) explains the existence of the obstructions (in $H^i(M, \pi_1 F)$) defined by Novikov in (5). The vanishing of these was a sufficient (but not necessary) condition to deform a tangential equivalence to a diffeomorphism (mod $\theta_n \partial \pi$). In effect, Novikov has chosen an arbitrary lifting of $\theta g$ to $F$ to define his obstructions.

We relate $a$ and $\theta$ in a simple example.

\[
\begin{array}{ccc}
& c g & \to \text{PL/O} \\
\downarrow & \downarrow 1 & \\
M & \xrightarrow{-\theta g} & F/O \\
\end{array}
\]

**Example:** Let $c : S \to S^n$ be a smoothing of $S^n$ $n \geq 5$. Then $c \times \text{identity} = s$
s: \sigma \times D^k \to S^n \times D^k

defines a smoothing of \( S^n \times D^k \). This smoothing is
classified by \( as: S^n \times D^k \to PL/O \), where as is given by
the composition

\[ S^n \times D^k \xrightarrow{p_1} S^n \xrightarrow{\sigma} PL/O. \]

(We identify \( \theta_n \) and \( \pi_{nPL/O} \).)

The "smooth structure" in \( (S^n \times D^k) \) determined
by \( s \) is classified (\( k \geq 3 \)) (according to Theorem 4 part b)
by the composition \( \theta s \),

\[ S^n \times D^k \xrightarrow{as} PL/O \xrightarrow{i} F/O \]

Thus we obtain

1. \( s \) is (PL) weakly isotopic to a diffeomorphism

iff \( as = 0 \), i.e. iff \( \sigma = 0 \) in \( \theta_n \). (2) and (4).

2. \( s \) is homotopic to a diffeomorphism

iff \( \theta s = ias = 0 \), i.e. iff \( \sigma \) is in the subgroup

\( \theta_n \delta \pi \subseteq \theta_n \). (Since \( \ker (\pi_{nPL/O} \xrightarrow{i^*} \pi_{nF/O}) = \theta_n \delta \pi \).)

3. If \( k > n \), then \( \sigma \times D^k \) and \( S^n \times D^k \) are diffeomorphic.

So if \( \theta_n \delta \pi \neq \theta_n \), there is a PL-homeomorphism

\( s: S^n \times D^k \to S^n \times D^k \) which is not homotopic to a
diffeomorphism. For example this happens if \( n = 8 \).

4. If \( \theta_n = \theta_n \delta \pi \), all smoothings of \( S^n \times D^k \) are
determined by diffeomorphic manifolds (\( k \geq 3 \)). For
example if $n = 7$ or $n = 11$ this holds. If $n = 15$, there are no more than two (diffeomorphism classes of) manifolds PL-homeomorphic to $S^n \times D^k$. (There are respectively 28, 992, and 16, 256 smoothings of $S^n \times D^k$ for $n = 7$, 11, and 15.)

BIBLIOGRAPHY