Triangulating Homotopy Equivalences

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Abstract

In this thesis we consider the problem of deforming a homotopy equivalence

$$g : M \rightarrow L$$

between compact PL-manifolds $M$ and $L$ into a PL-homeomorphism. If $M$ is simply connected and $\dim M \geq 6$, an obstruction theory is developed for such a deformation from two points of view – one geometric, the other homotopy theoretical.

The obstructions lie in

$$H^i(M; P_1),$$

where

$$P_1 = \begin{cases} 
0 & \text{if odd} \\
\mathbb{Z}_2 & i \equiv 2 \pmod{4} \\
\mathbb{Z} & i \equiv 0 \pmod{4}
\end{cases}$$

$P_1$ arises in the geometric point of view as a cobordism theory and in the second point of view as the homotopy groups of a universal space $F/PL$. 
Introduction

Consider a homotopy equivalence

\[ f : (L, bL) \longrightarrow (M, bm) \]

between two PL n-manifolds. Is \( f \) homotopic (as a map of pairs) to a PL-homeomorphism?

Suppose \( M = M' \times I \) and \( L = L' \times I \) where \( L' \) and \( M' \) are closed and \( f \) is a homotopy between two PL-homeomorphisms \( c_0 \) and \( c_1 \):

\[ c_i : L' \times I \longrightarrow M' \times I \quad i = 0, 1 \]

Can \( f/\pi(0, 1) \) be deformed to a pseudo-isotopy between \( c_0 \) and \( c_1 \)?

The purpose of this thesis is to develop a framework for answering such questions.

The starting point is the observation that these problems are equivalent to a certain "cobordism" problem. We illustrate this for the problem of deforming a homotopy equivalence

\[ f : L \longrightarrow M, \quad bm = 0 \]

into a PL-homeomorphism.

Let \( v \) be the PL-normal disk bundle of \( M \subset int D \) \((D = D^{|D|}, k \) large\) and form \( v' = f^*v \). Let \( b(f) \) be a bundle map \( v' \overset{b(f)}{\longrightarrow} v \) covering \( f \).

Suppose we can build an \((n+k)\)-disk \( D' \) containing \( v' \) in its interior and extend \( b(f) \) to a homotopy equivalence
which carries $D'_1 - v^1$ into $D - v$. Then if $\dim M \geq 6$ and $\partial_1'(M) = 0$ it follows that $f$ is homotopic to a PL-homeomorphism (Theorem 1.16).

The basic idea here is to choose a homotopy $H$ between $F$ and a PL-homeomorphism $C$. Then change $H/D'x(0,1)$ so that $H$ is $t$-regular to $M \subset \text{int } D$ and $H^{-1}(M)$ is an $h$-cobordism between $H^{-1}(M) \cap D'x0 = L$ and $H^{-1}(M) \cap D'x1 = C^{-1}(M)$. Then we can straighten out $(J)$ the $h$-cobordism to obtain the desired homotopy.

In order to carry out these operations we need to work with manifolds with non-empty boundary.

This is essentially the uniqueness theorem in the PL Browder-Novikov theory.
Note that $b(f)$ defines a homotopy equivalence

$$f': bv' \longrightarrow bv$$

and that $Q = D$-int $v$ defines a cobordism between $bv$ and $s = S^{n+k-1}$.

It is easy to see then that the problem of constructing $D'$ and $F$ is equivalent to the problem of constructing a cobordism $Q'$ between $bv'$ and the some $(n+k-1)$-sphere $S'$ and extending $f'$ to a homotopy equivalence

$$F': (Q', bv', S') \longrightarrow (Q, bv, S).$$

$(Q', F')$ is called a cobordism construction for $(Q, bv; f')$.

This indicates how the problem of deforming $f$ into a PL-homeomorphism is reduced to the problem of making a certain cobordism construction.

Thus in section 1 of Part I we consider the general problem of making cobordism constructions,

$$M' \quad W \quad M$$

$$f \downarrow \quad \downarrow f$$

$$L' \quad Q \quad L$$
for some triple \((W,M;f)\). We treat the case where \((W,M)\)
looks like the complement of the normal disk bundle of
some \((M,bM)\) in \((D,bD)\). (Definition 1.4)

Under these conditions we derive an obstruction theory
for the problem of making a cobordism construction for
\((W,M;f)\). The obstructions lie in

\[ H^{i+1}(W,M;P_1) \]

where

\[
P_1 = \begin{cases} 
0 & \text{if } i \text{ is odd} \\
\mathbb{Z}_2 & \text{if } i \equiv 2 \pmod{4} \\
\mathbb{Z} & \text{if } i \equiv 0 \pmod{4}
\end{cases}
\]

(See Theorem 1.12)

Geometrically, \(P_k\) is the set of framed cobordism classes
of framed \(k\)-manifolds \((M,bM)\) in \((D,bD)\) \((D=D^{n+k}, n \text{ large})\)
where \(bM=S^{k-1}(P_k\text{ becomes a group by forming connected sums
along the boundary.})\).

\[
\begin{array}{c}
LxI \\
(fxI)^{-1}(S^k) \\
(H^{-1}_1(S^k))
\end{array} \xrightarrow{(H_t)} \begin{array}{c}
MxI \\
(k+1\text{-handle})
\end{array}
\]
We can illustrate how \( P_k \) enters into the theory by considering the case when \( W = \text{MxI} \cup (k+1\text{-handle}) \). Suppose \( S^k \subset \text{MxI} \) is the core sphere and \( \text{fxI} \) is \( t \)-regular to \( S^k \). Then \( P = (\text{fxI})^{-1}(S^k) \) is an \( \alpha \)-framed submanifold of \( \text{LxI} \). In our case \( P \) has high codimension and can be constructed so that \( \text{P-int} D^k \) is a contractible submanifold of \( \text{LxI} \). Thus \( P \) determines an element in \( P_k \). This element is the precise obstruction to changing \( f \) by a homotopy \( H_t \) so that \( H_t^{-1} \) is a PL-homeomorphism on a neighborhood of \( H^{-1}(S^k) \). (Lemma 1.5)

If \( H_t^{-1} \) has this property we can attach a \((k+1)\)-handle to \( (\text{LxI}) \) (using \( H_t^{-1}/(\text{nghd of } S^k) \)) to form the desired cobordism of \( L \). Then \( f \) extends to a homotopy equivalence of cobordisms using \( H_t \) on \( \text{LxI} \) and the identity on the handle.

This indicates how a cochain with values in \( P_k \) may be defined to measure the possibility extending a given partial construction over the \((k+1)\)-handles of \((W,M)\). (Theorem 1.7)

The fact that an obstruction theory arises from this procedure is due to the fact that there are \( |P_{k+1}| \) different possible extensions of \( H_t \) over the \((k+1)\)-handle on \( \text{LxI} \). (Theorem 1.18)

The main purpose of section 2 of Part I is to prove the following

**Theorem:** Let \( f:(L,bL) \rightarrow (M,bM) \) be a homotopy equivalence. Suppose \( \dim M \geq 6, \mathcal{W}_{1}(M) = 0, \mathcal{W}_{1}(bM) = 0 \). Then \( f \) is homotopic to a PL-homeomorphism if and only if a sequence of obstructions in
vanish, \( i < \dim \mathcal{M} \).

This theorem is proved by relating cobordism constructions to the "normal invariants" of Browder and Novikov (1). This relationship (essentially described above) has other applications to problems where "normal invariants" enter.

It follows immediately from the existence of the obstruction theory and Theorem 1.14 of section 1 that \( f \) is homotopic to a PL-homeomorphism if

1) \( H^{4i+2}(\mathcal{M}, \mathbb{Z}_2) = 0 \quad 4i+2 < n \)

2) \( H^4(\mathcal{M}, \mathbb{Z}) \) is free

3) \( f \) is a correspondence of rational Pontryagin classes. \( (f^*p_1(\mathcal{M}) = p_1(L)) \).

Note that this theorem applies when \( f \) is a homeomorphism because of Novikov’s result (20).

There are other theorems where more is known about \( f \) and many examples showing why conditions 1) and 2) are "necessary". Some of these are reserved for Part II. Others are omitted.

The discussion of the pseudo-isotopy problem is also reserved for Part II.

The existence of the above obstruction theory suggests that the problem of deforming a homotopy equivalence \( f: L \rightarrow \mathcal{M} \) into a PL-homeomorphism should be "classified" by a map
$C_f$ of $M$ into some universal space $X$. We should have $\Pi_f(X) = P_1$ and $C_f \cong pt$ map iiff $f$ is homotopic to a PL-homeomorphism.

This suggestion is correct, and Part II is devoted to developing this point of view.

A PL-structure on $M$ is a pair $(L, g)$ consisting of a PL n-manifold $L$ and a homotopy equivalence $g : (L, bL) \to (M, bM)$. Two PL-structure $(L_0, g_0)$ and $(L_1, g_1)$ are concordant if there exists an h-cobordism $W$ between $L_0$ and $L_1$ and a homotopy equivalence $G : W \to M \times I$ which extends $g_0 \cup g_1$, and preserves appropriate subspaces.

(see Definition 5).

Let $\mathcal{M}_n$ denote the category whose objects are

$$\left\{ \begin{array}{ll}
\text{PL n-manifolds } (M, bM) \text{ such that } \\
\Pi_1(M) = \Pi_1(bM) = 0, \ bM \neq 0 \\
\text{or } n \leq 6 \\
\text{or } n < 6
\end{array} \right\}
$$

whose morphism are

embeddings $N_1 \subset \text{int } N_2$ such that $\Pi_1(N_2 - N_1) = 0$. 

Let $M_n$ denote the category obtained by dropping the condition $\beta M \neq 0$.

In chapter 1 we show that the correspondence

$$M \longrightarrow PL(M)$$

defines a contravariant functor

$$\begin{array}{ccc}
M_n & \longrightarrow & PL
\end{array}$$

from $M_n$ to the category of pointed sets and base point preserving functions. The key technical lemma here is provided by Browder (2). This enables one to change a PL-structure

$$L \longrightarrow M_2$$

on $M_2$ by a concordance so that it "induces" a PL-structure on

$$M_1 \subset \text{int.} M_2.$$ 

Thus we can define an induced map

$$\begin{array}{ccc}
PL(M_2) & \longrightarrow & PL(M_1)
\end{array}$$

We would now like to "classify" elements in $PL(M)$ by defining some natural transformation

$$\begin{array}{ccc}
PL & \longrightarrow & [\ , X]
\end{array}$$

where $X$ is our universal space. This is hard to do on the
level of representatives so we have to employ an intermediate functor $B_k$.

$B_k$ is defined on $\mathcal{C}$ the category of countable, connected, locally finite, simplicial complexes and PL-maps.

To each $K$ in $\mathcal{C}$ we assign the set of equivalence classes of diagrams

$$
\begin{array}{c}
E \xrightarrow{t} D^k \\
p \downarrow \\
K
\end{array}
$$

where $E \xrightarrow{p} K$ is a PL-$k$-disk bundle and $E \xrightarrow{t} D^k$ is a fibre homotopy ($F$-) trivialization. Two diagrams are equivalent if there is a diagram over $K \times I$ which restricts to the appropriate diagram on $K \times bI$. Such a diagram is called an $F(\text{PL})_k$-bundle.

The induced bundle construction makes $B_k$ into a contravariant functor on $\mathcal{C}$.

In chapter 2 we define a natural transformation

$$
\begin{array}{c}
\text{PL} \\
\downarrow \\
B_k
\end{array}
$$

of functors on $\mathcal{M}/n$. ($C$ is actually defined for any $M$.)

Let $a$ in $\text{PL}(M)$ be represented by $(L, bl) \xrightarrow{g} (M, bm)$.

Let $g'$ be a homotopy inverse for $g$ and $g'$ by an embedding.

$$(M, bm) \xrightarrow{i} (L, bl) \times \text{int } D^k, \text{ } k \text{ large}.$$

Choose $i$ and a closed tubular neighborhood

$$
\begin{array}{c}
E \xrightarrow{j} L \times \text{int } D^k
\end{array}
$$
of \( i(M) \) so that

\[
\mathcal{D}(\text{open disk bundle of } E) \supseteq \mathcal{D}_0.
\]

(Lemma 3). Then choose \( r : D^k \rightarrow D^k \) so that

\[
\begin{array}{ccc}
E & \xrightarrow{i} & \mathcal{D}_0^k & \xrightarrow{p_2} & D^k & \xrightarrow{r} & D^k \\
\end{array}
\]

defines an \( F \)-trivialization of \( E \) (Lemma 4). The pair

\((E, rp_2j)\) is called a classifying bundle for the PL-structure

\((L, g)\) and the correspondence

\( (\text{PL-structure}) \rightarrow (\text{classifying bundle}) \)

defines a natural transformation

\[
\begin{array}{ccc}
\text{PL} & \xrightarrow{C} & B_k, \text{ } \text{ } k \text{ large} \\
\end{array}
\]

of functors on \( \mathcal{M}_n \) (Theorem 6).

If \( M \) is embedded in \( bW \) and \( a = (L, g) \) is a PL-structure

on \( M \) we say that \( a \) extends to a PL-structure on \( W \) if there

is a PL-structure \( (Q, G) \) on \( W \) so that

i) \( L \) is embedded in \( bQ \) and \( G/L = g \)

ii) \( (Q, G)/L \) defines a PL-structure on \( M' = bW - \text{int } M \)

\((L' = bQ - \text{int } L)\)

Most extension problems for PL-structures reduce to

one of this type. (e.g. the cobordism construction problem

of part I)

Two extensions of \( a \) are said to be equivalent (rel \( M \))

if the product concordance of \( a \) extends to a concordance

between them. The set of
equivalence classes is denoted by $PL(W, M; a)$. Two $F/PL)_k$-bundles on $W$ which restrict to the same $F/PL)_k$-bundle $v$ on $M$ are said to be equivalent (rel $M$) if the product equivalence on $M$ extends to an equivalence between them. The set of equivalence classes is denoted by $B_k(W, M; v)$.

Note that there are natural projections $PL(W, M; a) \xrightarrow{p} PL(W)$ and $B_k(W, M; v) \xrightarrow{p} B_k(W)$.

Suppose that $bw\text{-int } M \neq 0$, $\Gamma_1(bw\text{-int } M) = \Gamma_1(M) = 0$, and $\dim W \geq 6$. Chapter 3 is denoted to the proof of the.

Theorem 9: Let $(W, M)$ be as above. Let $a = (I, g)$ be a PL-structure on $M$ with $k$-dimensional classifying bundle $\gamma = (E, \rho, t)$, $k$ large. Then there exists a one-to-one correspondence

$$PL(W, M; a) \xrightarrow{C_a} B_k(W, M; v)$$

so that $pC_a = C_p$ where

$$PL(W) \xrightarrow{C} B_k(W)$$

is defined above.

Corollary: If $k \gg n$, $PL \xrightarrow{C} B_k$ is a natural equivalence of contravariant functors on $\mathcal{M}_n$ (not $\mathcal{M}_n$).

Theorem 9 is stated in a relative form so that the pseudo-isotopy problem can be treated more completely.

If $M$ is closed let $W_0 = \text{int } D^n$. (If $bM \neq 0$, let $W_0 = M$.)}
Then $M_o \subset M$ defines a map

$$\text{PL}(M) \xrightarrow{F} \text{PL}(M_o)$$

which is an isomorphism (Lemma 21). Thus

$$\mathcal{M}_n \xrightarrow{\text{PL}} S$$

is determined by $\text{PL}/\mathcal{M}_n$.

In chapter 4 we construct (using Brown's theory\(^4\)) a space $F/\text{PL}$ which classifies stable equivalence classes of $F/\text{PL}_k$-bundles over finite simplicial complexes. Let $(\cdot, F/\text{PL})$ denote the functor which assigns to finite $X$ the set of free homotopy classes of maps $X \to F/\text{PL}$. Then Theorem 9 implies that there is a natural equivalence

$$\text{PL} \xrightarrow{C} (\cdot, F/\text{PL})$$

of contravariant functors on $\mathcal{M}_n$ (Theorem 25).

In fact, it follows from results in chapter 4 that $\text{PL}(W,M;a)$ is in one-to-one correspondence with

$$(W \cup (\text{cone } M), F/\text{PL}).$$

(Theorem 12, 14).

Thus the problem of deforming the homotopy equivalence

$$(L,bl) \xrightarrow{f} (M,bl)$$

to a PL-homeomorphism is classified by a map

$$M_o \xrightarrow{C_f} F/\text{PL}.$$  (Theorem 26)
The problem of deforming a homotopy
\[ \text{between two PL-homeomorphisms to a pseudo-isotopy is classified by a map} \]

\[ \text{susp. } M_0 \xrightarrow{C_f} F/\text{PL} \quad \text{(Chapter 6)} \]

We can use the fact that
\[ \text{is natural to develop an obstruction theory for deforming} \]
\[ (L, bl) \xrightarrow{f} (M, bM) \]

\text{to a PL-homeomorphism "over the skeletons of } M" \text{(Definition 22).}

One skeletal version of this obstruction theory is described in chapter 6. (Compare section I, Part I.)

The obstructions lie in
\[ H^i(M_0, \Pi_i(F/\text{PL})) \]
and are just the obstructions to deforming \( C_f \) to the point map. We prove in chapter \# that \( \Pi_i(F/\text{PL}) = P_i \).

Now there is a map
\[ \text{which is essentially the fibre of } B_{PL} \rightarrow B_F \]

(Theorem 14) It follows from the construction of $C_\gamma$ that the composition

$$
M_0 \xrightarrow{C_\gamma} F/PL \xrightarrow{b} B_{PL}
$$

classifies the normal bundle of $(M_0, bM_0) \xrightarrow{i_0} (L_0, bL_0) \times D^k$,
where $i_0$ is an embedding so that

$$
M_0 \xrightarrow{i_0} L_0 \times D^k \xrightarrow{(f/L_0)^p_1} M_0
$$
is homotopic to the identity (as a map of pairs). Thus by making tangential hypothesis on $f$ we can prove triviality theorems for the obstructions. See the corollaries to Theorem 26.

Using facts about $(X, F/PL)$ we can give an upper bound on the number of PL-homeomorphism classes of manifolds within a given tangential equivalence class. For example for $CP^k$ this upper bound is 4. In the examples in chapter 6 the number is shown to be at least 2.

This and other estimates are described in the Corollaries to Theorem 26.

A provocative fact about the obstructions is described in Theorem 27 of chapter 6. This Theorem relates the homomorphisms determined by the obstructions to a sequence of Bundle Problems.

A Bundle Problem is the problem of showing that a PL-structure on a $k$-disk bundle $E$ over $M$ "comes from" a PL-structure on $M$; i.e. $W \xrightarrow{g} E$ is concordant to $g \overset{b(g)}{\longrightarrow} E$ for some PL-structure $L \overset{f}{\longrightarrow} M$. 
The Bundle Problem per se is discussed in chapter 5. The situation is very nice when \( M \) is in \( M_n \) or when \( M \) in in \( \overline{M}_n \), \( k \geq 3 \), and \( n \) is odd. (Theorem 23, Corollaries 1 and 2). The case \( M \) in \( \overline{M}_n \), \( k \geq 3 \), and \( n \) even is discussed in Theorem 24. Here it is shown that \((W,G)\) in \( PL(E) \) "comes from" \( PL(H) \) ifff the "index" or "Kervaire Invariant" of a certain \( F/PL\) \( K \)-bundle vanishes. A "formula" for the "Kervaire Invariant" of an \( F/PL\) \( K \)-bundle is also described.

The formula for the "Kervaire Invariant" of an \( F/PL\) \( K \)-bundle over \( M \) arises from a homomorphism

\[
\bigcap_{n}^{\text{PL}} (F/PL) \xrightarrow{K} P_n
\]

which is defined geometrically. (Theorems 16, 17.)

This homomorphism is used to study homotopy properties of \( F/PL \). The main point is Theorem 18 which asserts that

\[
\pi_n(F/PL) \to \bigcap_{n}^{\text{PL}} (F/PL) \xrightarrow{K} P_n
\]

is an isomorphism if \( n \neq 4 \). As a corollary we show that for example that the \( k \)-invariants of \( F/PL \) (reduced mod 2) are zero. This fact enables us to compute the \( \mathbb{Z}_2 \)-cohomology algebra of \( F/PL \). (See Corollaries to Theorem 20.)

We remark that there is an analogous smooth theory which attacks the problem of deforming \( f: (L, bL) \to (M, bM) \) into a diffeomorphism. If \( \text{Herm}^2 \neq 0 \), \( f \) may be deformed into a diffeomorphism ifff a sequence of obstructions in

\[
\text{H}^2(M, \text{Herm}_1(F/0))
\]
vanish.

These two theories and that of Hirsch (8) are related by the fibration

\[ PL/0 \to F/0 \to F/PL \]

This smooth theory is omitted for reasons of space.

At this point I would like to acknowledge several debts. I have had numerous useful conversations with Professor Milnor and Professor Steenrod.

Several graduate students have offered much encouragement. I would especially thank George Cooke for asking a particularly appropriate question at the very beginning of this work.

Finally, I express my warmest thanks to my thesis advisor Professor Browder for patiently explaining his recent research, for suggesting the viewpoint of part II and for providing a friendly, informal and inspiring atmosphere for this research.
PART I
The PL Obstruction Theory

Let \( f : L \to M \) be a homotopy equivalence of PL \( n \)-manifolds. Let \((W; M, M')\) be a cobordism between \(M\) and another PL \(n\)-manifold \(M'\):

\[
\begin{array}{c}
M' \\
\uparrow F \quad \uparrow f \\
W \\
\uparrow F \quad \uparrow f \\
M \\
\end{array}
\]

\[
\begin{array}{c}
L' \\
\quad Q \\
\quad L \\
\end{array}
\]

We wish to consider the problem of constructing a cobordism \((Q; L, L')\) such that \(f\) extends to

\[
F : (Q; L, L') \to (W; M, M')
\]

a homotopy equivalence of triples.

If certain hypotheses are satisfied by the pair \((W, M)\), we will show that an obstruction theory for this problem exists. The obstructions lie in

\[
H^{k+1}(W, M ; P_k)
\]

where

\[
P_k = \begin{cases} 
0 & k \text{ odd} \\
\mathbb{Z}_2 & k \equiv 2 \pmod{4} \\
\mathbb{Z} & k \equiv 0 \pmod{4} 
\end{cases}
\]
Section I

First we define $P_k$ geometrically. Let $(M^k, bM^k)$ be an oriented smooth parallelizable $k$-manifold embedded in $(D^{k+r}, bD^{k+r})$ with boundary in the boundary and interior in the interior. Suppose that

1) $r$ is greater than $k$
2) $bM^k$ is PL-homeomorphic to the standard PL $(k-1)$ sphere
3) the normal bundle of $M$ has a given product structure $F$.

The pair $(M, F)$ will be called a framed almost-closed (f.a.c.) $k$-manifold in $D^{k+r}$.

Definition 1.1 Two framed almost-closed $k$-manifolds $(M_0, F_0)$ and $(M_1, F_1)$ are framed cobordant if there is a framed $(k+1)$-manifold $(W, G)$ in $D^{k+r} \times I$ such that

1) $(W, G) \cap (D^{k+r} \times I) = (M_i, F_i)$, $i=0,1$.
2) $W \cap (bD^{k+r} \times I)$ is a $h$-cobordism between $bM_0$ and $bM_1$.

\[\begin{array}{c}
\includegraphics{cobordism.png}
\end{array}\]
This defines an equivalence relation on the set of framed almost-closed \( k \)-manifolds.

**Definition 1.2** Let \( P_k \) denote the set of framed cobordism classes of framed almost-closed \( k \)-manifolds in \( D^{k+r}, r > k \).

Two f.a.c. \( k \)-manifolds \( (M_1, F_1) \) in \( D_1^{k+r} \) and \( (M_2, F_2) \) in \( D_2^{k+r} \) may be added by identifying a framed \( (k-1) \)-cell in \( bM_1 \) to a framed \( (k-1) \)-cell \( bM_2 \). Thus \( D_1^{k+r} \) is attached to \( D_2^{k+r} \) along \((k+r-1)\)-cells in their boundaries. If orientations are respected, an oriented f.a.c. \( k \)-manifold in \( D_3^{k+r} = D_1^{k+r} \cup D_2^{k+r} \) results. This operation, called "connected sum along the boundary," induces an Abelian group structure on \( P_k \).

**Lemma 1.1** With the operation "connected sum along the boundary,"

\[ P_k \text{ is isomorphic to } \begin{cases} 
0 & \text{if } k \text{ is odd} \\
\mathbb{Z}_2 & \text{if } k \equiv 2 \pmod{4} \\
\mathbb{Z} & \text{if } k \equiv 0 \pmod{4}
\end{cases} \]
Lemma 1.2  \( P_k \) is generated by

a) \((S^{k/2} \times S^{k/2} - \text{int} \; D^k)\) with a certain framing if \( k = 2, 6, \text{or} 14 \).

b) the Kervaire manifold obtained by "plumbing together" two copies of the tangent disk bundle of \( S^{k/2} \) with any framing if \( k = 2(\text{mod} \; 4) \) \( \neq 2, 6, \text{or} 14 \).

c) a certain \((k/2 - 1)\)-connected manifold with index \( a_k \) and any framing if \( k = 0(\text{mod} \; 4) \) \((a_4 = 16, \; a_{4r} = 8 \text{ for } r \text{ greater than } 1.)\)

d) any \( k \)-disk and any framing if \( k \) is odd.

Lemma 1.3  If \((M^k, F)\) represents the zero element in \( P_k \), then there is a framed cobordism \((W, G)\) between \((D^k, F)\) and \((M^k, F)\) where \( W \) is the trace of surgeries on \( M \) of dimension \( \leq k/2 \), \( k \neq 3, 4 \) \(( \leq k-1, k=3, 4)\).

These lemmas are proved in (11) except for \( k = 3 \) and \( k = 4 \). For these cases see the appendix to Section I.
Now we will discuss transverse regularity in the PL-category. The theory in this case is analogous to the smooth theory if the pertinent submanifolds have PL normal microbundles (34).

**Definition 1.3** Let \( f: M \to M' \) be a map of PL manifolds. Let \( N' \) be a submanifold of \( M' \) with a normal microbundle. Then \( f \) is \( t \)-regular on \( M \) if \( N = f^{-1}(N') \) is a submanifold of \( M \) with a normal microbundle, and so that \( f \) restricted to a neighborhood of \( N \) is a bundle map of normal microbundles.

**Theorem 1.4** (Mazur, Williamson) Let \( f: M \to M' \) be a map of compact PL-manifolds. Suppose \( N' \) is a closed submanifold of \( M' \) with a normal microbundle. Then \( f \) can be approximated in the \( C^0 \) topology by a mapping \( g: t \)-regular on \( N' \). Furthermore \( N = g^{-1}(N') \) and \( N' \) have PL normal bundles, and \( g \) restricted to a suitable neighborhood of \( N \) is a map of PL normal bundles.

**Proof:** The approximation part is proved by Williamson in (34). By Mazur, the normal microbundle of \( N' \) contains a unique PL bundle \( E' \). Then \( E = g^*E' \) is a PL normal bundle for \( N \). The last statement follows from the \( t \)-regularity of \( g \) and the construction of \( g^*E' \).
There is a relative version of Theorem 1.4 (to the effect that $f$ need not be changed on closed subsets where it is already $t$-regular) and a form for submanifolds with boundary.

Now we relate $t$-regularity and the coefficient groups $P_k$. The lemma will be used to define the obstructions in the cobordism problem.

**Lemma 1.5** Let $f: (M, bM) \rightarrow (M', bM')$ be a map of oriented PL $n$-manifolds with degree 1. Suppose that $M$ and $M'$ are $(k-1)$ connected, that $2k+1 < n$, and $k \geq 1$. Then $f$ determines a homomorphism

$$c_f: \Omega_k(M') \rightarrow P_k$$

with the following two properties.

1) If $f$ is homotopic to $g$ (as a map of pairs), then $c_f = c_g$.

2) Let $S^k_1, \ldots, S^k_r$ be disjoint $k$-spheres in $M'$ representing elements $(S^k_1), \ldots, (S^k_r)$ in $\Omega_k(M')$. Then $f$ is homotopic to $g$ such that $g$ is $t$-regular to each $S^k_i$ and so that each $g^{-1}(S^k_i)$ is PL-homeomorphic to $S^k$ iff $c_f(S^k_1) = \ldots = c_f(S^k_r) = 0$.

**Proof:** $c_f$ is defined as follows: let $x$ be in $\Omega_k(M')$. Embed $S^k$ in $M'$ to represent $x$. Make $f$ $t$-regular to $S^k$ and consider $f^{-1}(S^k)\text{-int } D^k$. 


We can assume that this is a framed submanifold of $M$ which is contractible in $M$. We may assume also by general position that it is contained in an $n$-cell of $M$. Thus $f^{-1}(S^k)$ determines an element $y$ in $P_k$. Let
\[ c_f(x) = y. \]
y is the precise obstruction to changing $f$ by a homotopy so that $f^{-1}(S^k)$ becomes a PL $k$-sphere.

In the appendix to section I we will give a detailed proof that $c_f$ is a well defined homomorphism from $\Pi_k(M')$ to $P_k$ and satisfies 1) and 2).

Now we describe the cobordism problem more precisely and develop the accompanying obstruction theory.

**Definition 1.1** Let $(M^n, bm^n)$ be a (smooth) PL-manifold pair. A (smooth) admissible cobordism of $(M, bm)$ is a (smooth) PL $(n+1)$-manifold pair $(W, bw)$ satisfying:

1) $n \geq 5$
2) $\Pi_1(M) = \Pi_1(W) = 0$
3) $bw = M \cup M'$
   $bn = M \cap M' = bm'$
4) there is an integer $s \leq n$ so that
   \[ H_i(W, M) \xrightarrow{b} H_{i-1}(M) \]
   is an isomorphism for $2i+1 < s$ and
   so that $H_i(W, M) = 0$ for $2i+1 \geq s$.\]
Let $j: \bigcup_{i=1}^r S_i^k \times D^{n-k}_1 \to (\text{int } M) \times 1$ be a (smooth) embedding. Let $A = j(\bigcup_{i=1}^r S_i^k \times D^{n-k}_1)$. A (smooth) $(k+1)$-elementary cobordism of $(M, bM)$ is a (smooth) admissible cobordism of $(M, bM)$ where $W$ is (diffeomorphic) PL-homeomorphic to

$$M \times 1 \cup_A \bigcup_{i=1}^r D^{k+1}_i \times D^{n-k}_1.$$ 

An admissible cobordism of $(M, bM)$ by $W$ will be denoted by $(W; M, M')$.

Note that admissible cobordisms, $(W_1; M_0, M_1)$ and $(W_2; M_1, M_2)$, may be "composed" to obtain a third admissible cobordism, $(W_1+ W_2; M_0, M_2)$ by setting

$$W_1 + W_2 = W_1 \cup_{M_1} W_2.$$ 

The following decomposition lemma discusses smooth admissible cobordisms and implies in particular that they are closed under composition.

![Diagram](image-url)
Lemma 1.6  Let \((W; M, M')\) be a smooth admissable cobordism of \((M, bM)\). Let \(r\) be such that \(H_i(W; M) = 0\) for \(i > r\). Then properties 1), 2), and 3) of Definition 1.5 imply there exist smooth \(k\)-elementary cobordisms
\[
(W_k; M_k, M_{k+1}; M_k) \quad k = 1, 2, \ldots, r
\]
with \(M = M_0\) and \(M' = M_r\) so that
\[
(W; M, M') = (W_1 + W_2 + \cdots + W_r; M_0, M_r).
\]
The decomposition satisfies

a) the inclusion \(i_0: M \to W_1 + \cdots + W_k\)
induces an isomorphism between \(H_i(M)\) and \(H_i(W_1 + \cdots + W_k)\) for \(i > k\),

b) the inclusion \(i_k: M_k \to W_1 + \cdots + W_k\)
induces an isomorphism between \(H_i(M_k)\) and \(H_i(W_1 + \cdots + W_k)\) for \(i < n-k\),

c) property 4) implies \(M_k\) is \((k-1)\) connected when \(r\) is minimal.

**Proof:** Write \(bW = M'' \cup bM \times I \cup M\) by choosing a collar neighborhood of \(bM\) in \(M'\) and setting \(M'' = M' - (bM \times [0, 1]).\) Define \(f: bW \to R\) by
\[
f(x) = \begin{cases} 
\frac{1}{2} & \text{if } x \in M \\
ry + \frac{1}{2} & \text{if } x = (m, y) \in bM \times I \\
r + \frac{1}{2} & \text{if } x \in M''.
\end{cases}
\]
Extend $f$ to a nice Morse function $g$ (i.e., $g$ has finitely many critical points in $\text{int } W$ and $g$ of (critical point of index $i$) equals $i$) with critical points of index $\leq r$. and set

$$W'_k = g^{-1}(k-\frac{1}{2}, k+\frac{1}{2}) \quad k = 1, \ldots, r.$$ Such a $g$ exists by (21).

Now the $W'_k$ may be modified slightly in a neighborhood of $BM \times I$ (using a local product structure) to obtain the desired decomposition

$$(W_1; M, M') = (W_1^+ \ldots + W_k; M, M').$$

Now a) and b) follow from the fact that $W_1^+ \ldots + W_k$ has the homotopy type of $M$ with $(1 \text{ thru } k)$-cells attached or $M_k$ with $(n+1-k \text{ thru } n)$-cells attached.

To prove c) note that the inclusion

$$j: W_1^+ \ldots + W_k \rightarrow W$$

is a homotopy equivalence of the respective $k$-skeletons. However, 2) and 4) imply that $W$ is $(k-1)$-connected for each $k$ such that $2k+1 < s$. Thus $W_1^+ \ldots + W_k$ is
(k-1)-connected if (2k+1) < s. Therefore, by the proof of b) \( M_k \) is (k-1)-connected if (2k+1) < s and \( k < n-k \). If we take \( r \) such that \( 2r+1 < s \) then \( k < r \) implies \( 2k+1 < s \) and \( k < n-k \). This completes the proof of c) and the lemma.

Lemma 1.6 shows that an admissible cobordism of essentially the trace of surgeries on the interior of \( \bar{M} \) which kill the low dimensional homology of \( M \).

The Lemma could have been stated in a much sharper form. The pairs

\[(W_1^+ \cdots + W_k, M) \quad k = 1, \ldots, r\]

filter the pair \((W, M)\). This filtration of \((W, M)\) gives us a chain complex whose homology is \( H_\bullet(W, M) \). Furthermore this chain complex has an explicit geometric description in terms of the handlebodies \( W_k \).

Conversely, if we are given any chain complex whose homology is \( H_\bullet(W, M) \), we can write \((W; M, M')\) as a sequence of elementary cobordisms whose geometric (or filtered) complex is chain isomorphic to the given complex.

We will use some of this information in the obstruction theory to follow. The facts used will be discussed in the appendix to Section I.

Note that the filtration of \((W, M)\) by the pairs \((W_1^+ \cdots + W_k, M)\) is, homotopically speaking, just a skeletal decomposition of the C.W. pair \((W, M)\).
Definition 1.5 Let $f : (L, bL) \to (M, bM)$ be a homotopy equivalence of PL manifold pairs. Let $(W; M, M')$ be an admissible cobordism of $(M, bM)$. A cobordism construction for the triple $(W, M; f : L \to H)$ is an admissible cobordism $(Q; L, L')$ of $(L, bL)$ and a homotopy equivalence of pairs $F : (Q, bQ) \to (W, bW)$ such that $F/\sim$ induces $f : (L, bL) \to (M, bM)$.

and

$F/\sim'$ induces $f' : (L', bL') \to (M', bM')$, where $f'$ is a homotopy equivalence of pairs.

To be brief we shall write $(Q, F)$ is a construction for $(W, M; f)$.

Definition 1.6 Suppose that $(W; M, M')$ is a smooth admissible cobordism of $(M, bM)$. Let $(W_k; M_k, M_{k-1})^k$, $k=1, \ldots, r$, define a skeletal decomposition of $(W; M, M')$, i.e. $(W; M, M') = (W_1 + \cdots + W_r; M_0, M_r)$.

Let $f : (L, bL) \to (M, bM)$ be a homotopy equivalence. A cobordism construction for $(W, M; f)$ over the k-skeleton of $(W, M)$ is a cobordism construction for $(W_1 + \cdots + W_k, M; f)$.

Let $(Q_k, F_k)$ be such a construction. We say that $(Q_k, F_k)$ extends to a construction over the $(k+1)$-skeleton of $(W, M)$ if there is a construction for $(W_{k+1}, M_k; g)$, where $g = F_k/(bQ_k - \text{int } L)$. 
If \((R,G)\) is a construction for \((W_k, M_{k-1}; g)\), then by setting \(Q_{k+1} = R \cup_{\partial Q_k} Q_k\) and \(F_{k+1} = F_k \cup_g G\) we get a construction for \((W, M; f)\) over the \((k+1)\)-skeleton of \((W, M)\) which naturally extends \((Q_k, F_k)\).

Note that the partial constructions of Definition 1.6 are defined with respect to a specific skeletal decomposition of \((W; M, M')\). We fix now a decomposition with minimal \(r\) and discuss the stepwise extension of a construction for \((W, M; f)\) over the skeletons of \((W, M)\).

Let \((Q_{k-1}, F_{k-1})\) be a construction for \((W, M; f)\) over the \((k-1)\)-skeleton of \((W, M)\). Let \(L_{k-1} = \partial Q_{k-1} - \text{int} L\). Then according to Definition 1.5 \(F_{k-1}\) induces \(g: (L_{k-1}, \partial L_{k-1}) \to (M_{k-1}, \partial M_{k-1})\) a homotopy equivalence of PL manifold pairs. By Lemma 1.6 c) \(M_{k-1}\) is \((k-2)\)-connected since the fixed decomposition has minimal \(r\). Thus Lemma 1.5 applies to \(g\).

Definition 1.7 (The obstruction cochain) Let \(c_k = c_k(Q_{k-1}, F_{k-1})\) in \(\text{Hom} (H_k(W_k, M_{k-1}), F_{k-1})\) be defined by the composition

\[
H_k(W_k, M_{k-1}) \xrightarrow{b} H_{k-1}(M_{k-1}) \xrightarrow{h} \tilde{H}_{k-1}(M_{k-1}) \xrightarrow{c_g} F_{k-1},
\]

where \(c_g\) is defined by Lemma 1.5, and \(h\) is the Hurewicz Isomorphism.
Theorem 1.7  Let $(Q_{k-1}, F_{k-1})$ be a cobordism construction for $(W, M; f: L \to M)$ over the $(k-1)$-skeleton of $(W, M)$. Then $(Q_{k-1}, F_{k-1})$ extends to a construction over the $k$-skeleton of $(W, M)$ iffi

$$c_k(Q_{k-1}, F_{k-1}) = 0.$$  

**Proof:** Let $L_{k-1} = bQ_{k-1}$ and set $g = F_{k-1}/L_{k-1}$. We want to show that there is a construction for $(W_{k-1}, M_{k-1}; g: L_{k-1} \to M_{k-1})$ iffi $c_k(Q_{k-1}, F_{k-1}) = 0$. Now $c_k(Q_{k-1}, F_{k-1})$ is zero iffi $c_g = 0$. In fact, in the composition

$$H_k(W_{k-1}, M_{k-1}) \xrightarrow{b} H_{k-1}(M_{k-1}) \xrightarrow{h} \varpi_{k-1}(M_{k-1}) \xrightarrow{c_g} P_{k-1}$$

$b$ and $h$ are isomorphisms.

Suppose first that $c_g = 0$. Write

$$W_k = M_{k-1} \times I \cup A \left( \bigcup_{i=1}^{r} D_i^k \times D_i^{n+1-k} \right),$$

where $A = j(\bigcup_{i=1}^{r} S_i^{k-1} \times D_i^{n+1-k})$ in int($M_{k-1} \times 1$).

Let $N_{k-1} = j(\bigcup_{i=1}^{r} S_i^{k-1} \times p_i)$ where $p_i$ is a point in $D_i^{n+1-k}$. By Lemma 1.5, $c_g = 0$ implies there exists

$$H: (L_{k-1}, bL_{k-1}) \times I \to (M_{k-1}, bM_{k-1})$$

so that $H/(L_{k-1}, bL_{k-1}) \times 0 = g$ and

so that $\overline{g} = H/(L_{k-1}, bL_{k-1}) \times 1$ is
t-regular to $N_{k-1}$ with $g^{-1}(N_{k-1}) = R_{k-1}$

PL-homeomorphic to $N_{k-1}$. Since $R_{k-1}$ is a disjoint union of $(k-1)$-spheres, $\bar{g}/R_{k-1} : R_{k-1} \to N_{k-1}$ is homotopic to a PL-homeomorphism. Thus by using the PL-covering homotopy theorem (19) and the homotopy extension theorem we can assume that $\bar{g}/R_{k-1}$ is a PL-homeomorphism.

Now $R_{k-1}$ has a product neighborhood

$$p : \left( \bigcup_{i=1}^{r} S_{i}^{k-1} \right) \times D_{n+k-1} \to (\text{int } L_{k-1}) \times I$$

so that $\bar{g} \cdot p = j$. Let $B = p(\bigcup_{i=1}^{r} S_{i}^{k-1} \times D_{n+k-1})$ and form $R = L_{k-1} \times I \cup B (\bigcup_{i=1}^{r} D_{i}^{k} \times D_{i}^{n+k-1})$. Define

$G : (R, bR) \to (W_{k}, bW_{k})$ by $G = \Pi \times I \cup g \times I$ where

$$i : \left( \bigcup_{i=1}^{r} D_{i}^{k} \times D_{n+k-1} \right) \to \left( \bigcup_{i=1}^{r} D_{i}^{k} \times D_{n+k-1} \right)$$

is the obvious equivalence.

We claim that $(R, G)$ is a cobordism construction for $(W_{k}, N_{k-1}; g)$. Let $L_{k} = bR \to \bar{g}$. Then $G$ induces

$$g' : (L_{k}, bL_{k}) \to (W_{k}, bW_{k}).$$

We need to show that $G$ and $g'$ are homotopy equivalences of pairs.
We proceed in steps:

1) \( g'/b L_k \) is a h.e. by construction.

2) It follows from a standard lemma in homotopy theory that

\[
G: R \rightarrow W_k
\]

is a h.e. (See Lemma H in the appendix to section I.)

3) \( G: (R, bR) \rightarrow (W_k, bW_k) \) has degree one.

This follows from the commutative diagram 3).

4) \( G^*: H^*(W_k, M_{k-1}) \rightarrow H^*(R, L_{k-1}) \) is an isomorphism. This follows from diagram 2), diagram 4) and the five lemma.

5) \( G_*: H_*(W_k, M_k) \rightarrow H_*(R, L_k) \) is an isomorphism. This follows from 3), 4) and diagram 5).

6) \( g'_*: H_*(L_k) \rightarrow H_*(M_k) \) is an isomorphism. This follows from 5) and the 5-lemma by an argument like that of 4).

7) Therefore \( g' \) is a h.e. when \( L_k \) and \( M_k \) are simply connected by the Whitehead Theorem.

8) \( (G/b R)_*: H_*(b R) \rightarrow H_*(b W_k) \) is an isomorphism. This follows from 6) and diagram 8).

9) Therefore \( G/b R \) is a h.e. when \( b R \) is simply connected.

This takes care of all cases except possibly when \( k = 1 \) or \( k = 2 \). For these cases see the Appendix to section I.
Thus \( c_k(Q_{k-1}, F_{k-1}) = 0 \) implies that there is a construction for \((W_k, M_{k-1}; g)\).

3) \[
H_{n+1}(R, bR) \xrightarrow{\cong} H_{n+1}(R, bR \cup L_{k-1} \times I) \xrightarrow{\cong} H_{n+1}^{\text{exc}}(R_{k-1} \times D, b(R_{k-1} \times D)) \xrightarrow{i_*} H_{n+1}(W_k, bW_k) \xrightarrow{\cong} H_{n+1}(W_k, bW_k \cup M_{k-1} \times I) \xrightarrow{\cong} H_{n+1}^{\text{exc}}(N_{k-1} \times D, b(N_{k-1} \times D))
\]

4) \[
\xymatrix{ H^1(W_k) & H^1(W_k, M_{k-1}) & H^1(L_{k-1}) \ar[l] \ar[r] \ar[d] \ar[l] & H^1(R, L_{k-1}) & H^1(L_{k-1}) \ar[l] \ar[r] \ar[d] \ar[l] } 
\]

5) \[
H^*(W_k, M_{k-1}) \xrightarrow{G^*} H^*(R, L_{k-1}) \xrightarrow{\cong} \cap \mu_{W_k} \xrightarrow{\cong} \cap \mu_R \\
H_k(W_k, M_{k-1}) \xrightarrow{G^*} H_k(R, L_{k-1})
\]

8) \[
\xymatrix{ H_1(L_k) \otimes H_1(L_{k-1}) \ar[r] & H_1(bR) \ar[r] & H_{i-1}(bL_k) \ar[d] \ar[r] & H_1(bM_k) \ar[r] & H_1(bM_{k-1}) \ar[d] \ar[r] & H_{i-1}(bM_k) \ar[d] \ar[r] & } 
\]

Now suppose that \((R, G)\) is a construction for \((W_k, M_{k-1}; g)\). We want to show that \( c_g \) is the zero homomorphism. Represent \( x \) in \( T_{k-1} M_{k-1} \) by an embedded \( S^{k-1} \). Suppose that
G restricted to a collar neighborhood of bR is equal to \((G/bR) \times I\), where we assume that G/bR is \(t\)-regular to \(S^{k-1}\). Then the class determined by \(g^{-1}(S^{k-1}) = (G/bR)^{-1}(S^{k-1})\) in \(P_{k-1}\) is the same as the class determined by
\[
G^{-1}(S^{k-1} \times 1/2).
\]

This means that
\[
\begin{array}{ccc}
\Pi_{k-1}(N_{k-1}) & \xrightarrow{c_E} & P_{k-1} \\
i_* & \downarrow & \\
\Pi_{k-1}(W_k) & \xrightarrow{c_G} & P_{k-1}
\end{array}
\]
is commutative. But \(\Pi_{k-1}(W_k) = 0\).

Therefore,
\[
c_k(Q_{k-1}, F_{k-1}) = c_E = 0.
\]
Recall that the filtering pairs \((W_1^*\ldots+W_k^*, M)\)

\[ k = 1, \ldots, r \]

give us a chain complex whose homology is \(H_k(W, M)\). The \(k\)th chain group is \(H_k(W_1^*\ldots+W_k^*, W_{k+1}^*\ldots+W_{k+2}^*)\), and the boundary map \(\partial : C_k \to C_{k-1}\) is the boundary map of the triple

\[(W_1^*\ldots+W_k^*, W_1^*\ldots+W_{k-1}^*, W_1^*\ldots+W_{k-2}^*)\]  

Using excision we identify \(H_k(W_k^*, M_{k-1})\) and \(C_k\). Thus \(c_k(W_k^*, M_{k-1})\) determines a cochain in the cochain complex for \(H^*(W, M)\).

**Theorem 1.8**  
1) \(c_k \in (P_{k-1}^*, Q_{k-1}^*)\) is a cocycle. Let \((c_k)\) be the cohomology class of \(c_k\) in \(H^k(W, M; P_{k-1})\).

2) \((c_k)\) determines the zero homomorphism in \(\text{Hom}(H_k(W, M), P_{k-1})\) if and only if \(c_k\) vanishes on the image of the composition

\[
0 \to H_{k-1}(M) \xrightarrow{i_k} H_{k-1}(W_1^*\ldots+W_{k-1}^*) \xrightarrow{\partial} H_{k-1}(M) \xrightarrow{i_{k-1}} \cdots \xrightarrow{\partial} H_{k-1}(M) \xrightarrow{i_1} H_{k-1}(M). 
\]

**Proof:**  
1) We want to show that \(c_k\) vanishes on the image of

\[
\partial : H_{k+1}(M^*, M_{k+1}) \to H_k(W_{k+1}, M_{k+1}) \quad \text{and} \quad \partial : C_{k+1}(W, M) \to C_k(W, M)
\]

But this follows from commutative diagram i) since by Lemma 1.6 b) \(i_{k-1}\) is an isomorphism.

To prove ii) note that \((c_k)\) determines the zero homomorphism in \(\text{Hom}(H_k(W, M), P_{k-1})\) if and only if \(c_k\) vanishes
on the kernel of
\[ \mathcal{D} : H_k(W, M) \to H_{k-1}(W_{k-1}, M_{k-2}) \]
Thus ii) follows from commutative diagram ii) since
1) image \( j = \) kernel \( k \) and
2) \( i_\circ \) is an isomorphism by Lemma 1.6 a).
Now we want to show that cohomology class determined by \( c_k(Q_{k-1}, R_{k-1}) \) depends precisely on the construction over the \((k-2)\) skeleton of \((W, M)\).

We need two preliminary definitions and lemmas.

**Definition 1.8** Write \( W_k \) as

\[
M_{k-1} \times I \cup A \left( \bigcup_{i=1}^{r} D_i^k \times D_i^{n+1-k} \right).
\]

Then the \( \overline{D}_i = D_i^k \cup (bD_i^k \times I) \) \( i = 1, \ldots, r \) with orientations generate \( H_k(W_k; M_{k-1}) \). Let \( DW_k \) denote \( W_k \cup M_{k-1} \cdot W_k \). Define the **doubling homomorphism**

\[
d: H_k(W_k; M_{k-1}) \to H_k(DW_k)
\]

on generators by

\[
d(\overline{D}_i^k) = (i_1 \cdot \overline{D}_i^k - i_2 \cdot \overline{D}_i^k)
\]

where \((x)\) denotes the homology class of a chain \( x \) and \( i_1 \) and \( i_2 \) are the two inclusions of \( W_k \) in \( DW_k \).

**Lemma 1.9** The following diagram is commutative.

\[
\begin{array}{ccc}
H_k(W_k) & \xrightarrow{j} & H_k(W_k; M_{k-1}) \\
\downarrow i_1 - i_2 & & \downarrow d \\
H_k(DW_k) & \xrightarrow{\text{}} & H_k(DW_k)
\end{array}
\]

**Proof:** Let \( x \) be a cycle representing \((x)\) in \( H_k(W_k) \). We may choose \( x \) so that

\[
x = a_1 \cdot \overline{D}_i^k + \cdots + a_r \cdot \overline{D}_i^k + y
\]

where \( y \) is a chain on \( M_{k-1} \).
Then \( dj(x) = d(a_1(\bar{D}^k_i) \ldots a_r(\bar{D}^k_r)) \)
and \( (i_1 - i_2)(x) = a_1(i_1\bar{D}^k_i - i_2\bar{D}^k_i) \ldots a_r(i_1\bar{D}^k_r - i_2\bar{D}^k_r) \).
Therefore \( dj(x) = (i_1 - i_2)(x) \). Q.E.D.

Let \((Q_{k-1}; P_{k-1})\) be a construction for \((W,M; f)\) over the \((k-1)\) skeleton of \((W,M)\). Suppose that \((R,G)\) and \((R',G')\) are two constructions for \((W_k, M_{k-1}; g)\)
\((g = P_{k-1}/I_{k-1} = bQ_{k-1} - \text{int } L)\) defining extensions of \((Q_{k-1}; P_{k-1})\) over the \(k\)-skeleton of \((W,M)\). Then by using Van Kampen's Theorem, the Mayer Vietoris sequence, and the Hurewicz Theorem we see that

\[
R \cup I_{k-1} R' \quad \text{and} \quad W_k \cup M_{k-1} W_k = DW_k
\]

are \((k-1)\)-connected and that

\[
G \cup G': (R \cup I_{k-1} R', b(R \cup I_{k-1} R)) \to (DW_k, bDW_k)
\]

has degree one. Thus Lemma 1.5 applies to \( G \cup G' \).
Let \( c_{G \cup G'} \) denote the homomorphism given to us by Lemma 1.5.

\[
c_{G \cup G'} : \Pi_k(DW_k) \to P_k.
\]
Definition 1.9  (The difference cochain).
If $(R,G)$ and $(R',G')$ define extensions of $(Q_{k-1},F_{k-1})$ over the $k$-skeleton of $(W,M)$, let $d(G,G')$ denote the composition

$$H_k(W_k,M_{k-1}) \xrightarrow{d} H_k(DW_k) \xrightarrow{\sim} T_k(DW_k) \xrightarrow{c_{G,G'}} P_k.$$

$d(G,G')$ is called the difference cochain associated to the two extensions $(R,G)$ and $(R',G')$.

Lemma 1.10  If $(R,G),(R',G')$ and $(R'',G'')$ define three extensions of $(Q_{k-1},F_{k-1})$ over the $k$-skeleton of $(W,M)$, then

$$d(G,G'') = d(G,G') + d(G',G'').$$

Proof: Let $W_k$ and $D_i^k$ be as in Definition 1.8.
Let $g = F_{k-1}/L_{k-1}$. By Theorem 1.7, $c_g = 0$. Therefore we may change $g$ by a homotopy so that it has the nice form of $\tilde{g}$ in the proof of Theorem 1.7. We may suppose by the homotopy extension theorem that $g$, $g'$, and $G''$ extend this new $g$.

Now we may approximate $G$ by a map which is $t$-regular to $(D_i^k, bD_i^k)$ in $(W_k,M_{k-1})$. Then $G^{-1}(D_i^k)$ is a framed submanifold of $R$ whose boundary is PL-homeomorphic to $S^{k-1}$, and which is contractible in $R$. Let $G(D_i^k)$ denote the element in $P_k$ determined by $G^{-1}(D_i^k)$. Make similar definitions for $G'$ and $G''$. 
Now \( (G \cup G')^{-1}(i_1D^k_i \cup i_2D^k_i) \) is equal to \( i_1G^{-1}(D^k_i) \cup i_2G'^{-1}(D^k_i) \). Thus it is clear that

\[
d(G,G') (D^k_i) = G(D^k_i) + G'(-D^k_i)
\]

\[
= G(D^k_i) - G'(D^k_i)
\]

The lemma follows immediately.

**Theorem 1.11** Let \((R,G)\) and \((R',G')\) define extensions of \((Q_{k-1}^l,F_{k-1}^l)\) over the \(k\)-skeleton of \((W,M)\). Denote the extensions by \((Q_k^l,F_k^l)\) and \((Q_k'^l,F_k'^l)\). Then

\[
c_{k+1}(Q_k^l,F_k^l) - c_{k+1}(Q_k'^l,F_k'^l) = \delta d(G,G')
\]

Conversely, suppose that \((R,G)\) defines an extension \((Q_k^l,F_k^l)\) of \((Q_{k-1}^l,F_{k-1}^l)\) and that \(u\) in \(\text{Hom}(W_k,M_{k-1};P_k)\) is given. Then there exists \((R',G')\) so that

\[
u = d(G,G')
\]

Thus we may change \((Q_k^l,F_k^l)\) on the \(k\)-skeleton of \((W,M)\) so that \(c_{k+1}(Q_k^l,F_k^l)\) varies by an arbitrary coboundary.
Proof: For the first part we consider the diagram

![Diagram]

where \( g = G/L_k = Q_k\) - int \( L \) and

\[ g' = G'/L_k = Q_k' - \text{int } L \]

Commutativity around region I is just Lemma 1.9.

Commutativity around region II is easily verified on representatives. Commutativity around region III is just the naturality property of \( c_g \) demonstrated in the proof of Theorem 1.7.

Therefore the diagram is commutative and this proves the first part of Theorem 1.11.

For the second part suppose that \((R, G)\) is given and that \( g = G/L_{k-1} \) has the form of \( \tilde{g} \) in Theorem 1.7. Let \((R_0, G_0)\) denote the construction given in the proof of Theorem 1.7 with \( H = g \times I \).
If $x=(x_1, \ldots, x_r)$ denotes a sequence of elements in $\mathbb{P}_k$, let $(G_x, \mathbb{R}_x)$ denote the construction of Theorem 1.7 with $H=g \times 1$ and $i$ replaced by $i_x$ where

$$i_x^{-1}(\bar{D}_i^k \times p_i) \text{ represents } x_i,$$

and $i_x$ is a h.e. of pairs. (For a proof that $i_x$ exists see Lemma F in the appendix.) Then by the formula derived in the proof of Lemma 1.10,

$$d(G_x, G_o)(\bar{D}_i^k) = G_x(\bar{D}_i^k) - G_o(\bar{D}_i^k)$$

$$= x_i - 0$$

$$= x_i.$$

If $(R, G)$ and $u$ in $\text{Hom}(H_k(W_k, \mathbb{R}_{k-1}), \mathbb{P}_k)$ are given, let $x = (G(\bar{D}_i^k) - u(\bar{D}_i^k), \ldots, G(\bar{D}_r^k) - u(\bar{D}_r^k))$ and set $(R'_x, G'_x) = (R_x, G_x)$. Then

$$d(G, G')(\bar{D}_i^k) = (d(G, G_o) + d(G_o, G_x))(\bar{D}_i^k)$$

$$= G(\bar{D}_i^k) - (G(\bar{D}_i^k) - u(\bar{D}_i^k))$$

$$= u(\bar{D}_i^k).$$

So $d(G, G') = u$. This completes the proof of Theorem 1.11.
We summarize the above results in the following theorem.

**Theorem 1.12** Let \((W;M,M')\) be a smooth admissible cobordism of \((W,bM)\) with the skeletal decomposition

\[
(W; M,M') = (W_1 + \ldots + W_r; M,M')
\]

where \(2r+1 < n\). Let \(f: (L,bL) \rightarrow (M,bM)\) be a homotopy equivalence of PL-manifold pairs. Then there exists an obstruction theory for the problem of making a stepwise cobordism construction for \((W,M; f)\) over the skeletons of \((W,M)\). The obstructions \(c_{k+1}\) lie in

\[
H^{k+1}(W,M; P_k)
\]

where

\[
P_k = \begin{cases} 
0 & \text{k odd} \\
\mathbb{Z}_2 & k \equiv 2 \text{(mod 4)} \\
\mathbb{Z} & k \equiv 0 \text{(mod 4)}
\end{cases}
\]

**Corollary:** Let \((W,M; f)\) be as above and suppose that

1) \(H^{k+1}(W,M; \mathbb{Z}) = 0\)

2) \(H^{k-1}(W,M; \mathbb{Z}_2) = 0\)

Then there exists a cobordism construction for \((W,M; f)\).
From the statement of Theorem 1.12 it appears that the obstruction theory depends on the particular skeletal decomposition of \((W; M)\). This is not the case, however. In fact one can use Smale's theory of handle decompositions to pass from one decomposition to another and thereby relate the corresponding obstruction theories.

We will not do this because there is another interpretation of the obstruction theory in terms of an extension problem in homotopy theory (see Part II). From this point of view the independence of the theory from particular skeletal decompositions follows from a lemma in homotopy theory.

One can also ask if the obstructions described in Theorem 1.12 are actually representable as obstructions to making particular cobordism constructions. There are many examples. (See Chapter 6.) For instance the following lemma is easy to prove.

**Lemma 1.13** Let \( p \) be an element of \( P_k \). Then there exists a smooth admissible cobordism \((W; M, M')\), a h.e. \( f: (L, bL) \rightarrow (M, bM) \), a construction \((Q_k, F_k)\) for \((W, M; f)\) over the \( k \)-skeleton of \((W, M)\), and an element \( x \) of \( H_{k+1}(W, M) \), so that

\[
c_{k+1}(Q_k, F_k) \cdot x = p.
\]
Proof: Let $i_p: D^k_1 \times D^n_1 \to D^k \times D^n$ for $n > k$ be a homotopy equivalence of PL-manifold pairs so that $i_p^{-1}(D^k \times *)$ represents $p$. Then there is a product neighborhood

$$j: S^{k-1} \times D^n \to b(D^k_1 \times D^n_1)$$

so that $i_p \cdot j$ is just the natural inclusion of $S^{k-1} \times D^n$ in $b(D^k \times D^n)$ so that $S^{k-1} \to bD^k \times *$.

Define $L = D^k_1 \times D^n_1 \cup \bigcup_j D^k_1 \times D^n_1$ by identifying $b(D^k_1 \times *) \times D^n$ with $j(S^{k-1} \times D^n)$. Let

$$M = D^k \times D^n \cup_{id} D^k_1 \times D^n = S^{k} \times D^n,$$

and $W = D^{k+1} \times D^n$. Define $f: L \to M$ by $f = i_p \cup j_1$.

Then $(W,M,f)$ has a construction over the $k$-skeleton, (namely $(L \times I, f \times I)$), $H_{k+1}^*(W,M) = \mathbb{Z}$, and $c_{k+1} \cdot (\text{gen}) = p$. Q.E.D.

Now we relate the obstructions in $H_{k+1}^*(W,M; P_{4r})$ to $(L,bL)$.

Let $p_1(N,Q)$ denote the $i$th rational Pontryagin class of the PL-manifold $N$. Let $E: H^*(X; A) \to \text{Hom}(H_*(X), A)$ be the evaluation map.
Theorem 1.14. Let \((Q_k, F_k)\) be a construction for \((W, M; f)\) over the \(k\)-skeleton of \((W, M)\). Let 
\[ c_{k+1} = c_{k+1}(Q_k, F_k). \]
Then
\[ a) \text{ if } k > 4r, \text{ then } p_r(L, Q) = 0 \]
\[ b) \text{ if } k = 4r, \text{ then } p_r(L, Q) = 0 \]
iffi

\[ E \cdot (c_{k+1}) = 0. \]

Proof: Let \((Q_{i-1}, F_{i-1})\) be a construction for \((W, M; f)\) over the \((i-1)\)-skeleton of \((W, M)\), where \(i = 4r\). Let 
\[ L_{i-1} = bQ_{i-1} - \text{int } L \]
\[ g = F_{i-1} / L_{i-1}. \]

Consider the diagram

\[ \begin{array}{ccccccc}
H_1(M_1, Q) & \overset{i_0}{\longrightarrow} & H_1(W_1, W_{i-1}, Q) & \overset{\cong}{\longrightarrow} & H_1(M_{i-1}, Q) & \overset{\sim}{\longrightarrow} & M_1(M_{i-1}) \otimes Q \\
\cong f & \overset{\cong}{\longrightarrow} & F_{i-1} & \overset{g}{\cong} & II & \overset{\text{index}}{\downarrow} & \\
H_1(L, Q) & \overset{i_0}{\longrightarrow} & H_1(Q_{i-1}, Q) & \overset{\cong}{\longrightarrow} & H_1(L_{i-1}, Q) & \overset{L_{r}(L_{i-1})}{\longrightarrow} & Q \\
& \overset{L_{r}(L)}{\longrightarrow} & & & & & \\
\end{array} \]

where \(L_r(X)\) denotes the \(r\)th Hirzebruch polynomial of any PL-manifold \(X\).
We show first that the commutativity of this diagram implies Theorem 1.14.

a) Suppose that $k' < k$ and that $p_\lambda(L,Q) = \ldots = p_{k-1}(L,Q) = 0$. Let $i \leq k'$ and choose $(Q_{i-1}, F_{i-1})$ so that $c_i(Q_{i-1}, F_{i-1}) = 0$. Then $c_g = 0$, and the commutativity of the diagram implies $L_r(L) = 0$. Thus $p_r(L,Q) = 0$.

b) By Theorem 1.8 b), $E(c_{k+1}) = 0$ if $i \leq k'$ and $c_g = 0$. By the diagram $c_e = 0$ if $i \leq k'$. Thus $p_r(L,Q) = 0$ if $L_r(L) = 0$.

Now the commutativity around region II is verified immediately on representatives. The commutativity around I is established as follows:

For each $k: (L, bL) \to (S^{n-i}, p)$ there exists $K$ and $k'$ so that

\[
\begin{array}{ccc}
(L, bL) & \xrightarrow{k} & (S^{n-i}, p) \\
\downarrow & & \downarrow \\
(Q_{i-1}, bL) & \xrightarrow{K} & (S^{n-i}, p) \\
\downarrow & & \downarrow \\
(L_{i-1}, bL_{i-1}) & \xrightarrow{k'} & \\
\end{array}
\]

is commutative. ($\vartheta^i_j(Q_{i-1}, L; \Pi^i_{j-1}(S^{n-i})) = 0$ for $j > n-i$.) Suppose that $k, k', K$ are $t$-regular to $y$ in $S^{n-i} - p$. Then $K^{-1}(y)$ is a cobordism between $k^{-1}(y)$ and $k'^{-1}(y)$. Thus if $u$ generates $H^{n-i}(S^{n-i}, p; Q)$
and \( LDx \) denotes the Lefschetz Dual of \( x \), then

\[
L_i(L)(LDk^*u) = \text{index } k^{-1}(y) \\
= \text{index } k'^{-1}(y) \\
= L_i(L_{i-1})(LDk'^*u) \\
= L_i(L_{i-1})(i^{-1}_{i-1}LDk^*u).
\]

Therefore \( I \) is commutative for elements in \( H_i(L,\mathbb{Q}) \) which are of the form \( LDk^*u \) where

\[
k:(L,bL) \to (S^{n-1},p).
\]

It follows from a theorem of Serre that these generate \( H_i(L,\mathbb{Q}) \).

This completes the proof of Theorem 1.14.
Appendix to Section 1

In this appendix we clarify a few points raised in Section 1.

1) Proof of Lemma's 1, 2, and 3 for \( k = 3, 4 \).

**Proof:** \( k = 3; P_3 = 0 \) by (27). The proof of Lemma 3 for \( k = 3 \) is the same as that for \( k = 4 \). Lemma 2 is clear.

\( k = 4 \): There is a homomorphism \( \mathbb{P}_4 \rightarrow \mathbb{Z} \).

By (14) this is onto \( 16(\mathbb{Z}) \).

Suppose \( \text{Index} (\mathbb{L}^4) = 0 \). Since \( bM^4 = S^3 \), \( M^4 \in \text{cone bM^4} \).

is a PL-manifold which is smoothable. By (28) \( M^4 \) is cobordant to \( S^4 \) by \( W_1 \) where \( \pi_1 (W_1) = 0 \) and \( \widehat{\pi}_i (W_1) = 0 \) for \( i \neq 2 \). Let \( W = W_1 \) (tubular neighborhood of arc connecting \( S^4 \) and \( M^4 \)) and extend the embedding of \( M^4 \) to an embedding of \( (W; M^4, bM^4, D^4, bD^4, bD^4 \times I) \) in \((D^4 + k, bD^4 + k) \times 0, (D^4 + k, bD^4 + k) \times I) \). Since \( H^4 (W, M^4; \pi_1 - 1 (S^0)) = 0 \), the framing of \( M^4 \) extends over \( W \). Thus

\[ \mathbb{P}_4 \rightarrow \mathbb{Z} \]

is a monomorphism.

Using a nice Morse function on \( W \) whose critical points are on the interior of \( W \) we can write \( W = W_0 \cup \bigcup_{i=1}^{n} D^4_i \), where \( W_0 \) is the trace of framed surgeries beginning on \( M^4 \) whose spheres have dimension \( \leq 3 \). Now \( bW_0 - h^4 = L = D^4 \cup \bigcup_{i=1}^{n} S^4_i \).
So we can make $L$ connected by framed surgery to get a four disk. Thus Lemma 3 is verified for $k = 4$. The proof for $k = 3$ is identical.

2) **Proof of Lemma 1.5**

**Proof:** Let $x$ be in $\mathcal{T}_k(M')$. Choose an $S^k \subseteq \text{int } M'$ to represent $x$. $S^k$ has a normal microbundle by (6). Make $f$ $t$-regular to $S^k$, assume $f^{-1}(S^k)$ is connected and let $T = f^{-1}(S^k) \cap D^k$. ($f^{-1}(S^k)$ is non-void because $f$ has degree one.) $T$ is a contractible subcomplex of $M$, so by general position we may assume that $T$ is contained in the interior of a closed $n$-cell $D^n$ of $M$. Choose an embedding of $S^{k-1} \times I$ in $D^n - T$ so that $S^{k-1} \times 0$ goes into $bT$ and $S^{k-1} \times I$ lies in $bD^n$. Let $T^* = T \cup \text{image}(S^{k-1} \times I)$.

Now $f$ induces a null-homotopic map $T \to S^k \subseteq M'$. We can change $f$ by a homotopy $f_t$ so $f_t$ is $t$-regular to $S^k$, $f_t^{-1}(S^k) = f^{-1}(S^k)$, and $f_1(T) = p$ in $S^k$. By the $t$-regularity of $f_1$, there is an embedding $F: T \times D^{n-k} \to D^n \subseteq M$ extending the inclusion $T \subseteq M$ so that $f_1 F = ip_2$ where $i: D^k \to M'$ is a normal disk at $p$. Using (8) we can change $F$ (and $f_1$) by an ambient isotopy of $D^n$ (fixed on $bD^n$) and put a smooth structure on $T$ so that $F$ becomes a smooth embedding.

Thus we may assume that $T^*$ is a smooth f.a.c. submanifold of $D^n$. Define

$$c_k(x) = \text{ (framed cobordism class of } T^*) \text{ in } P_k.$$
Now we show that $c_k(x)$ is well-defined. Suppose $f_o: M \to M'$ is $t$-regular to $S^k \subset M'$ and $f_o^{-1}(S^k)$ is connected. Suppose $f_t$ is a homotopy between $f_o$ and $f_1$. Then we can change $f_t$ for $t$ in $(0,1)$ so that $f_t \times 1: M \times 1 \to M'$ is $t$-regular to $S^k$ and so that $(f_t \times 1)^{-1}(S^k)$ is connected. (The connected part follows from the techniques described below.) Then if we have made constructions to produce $T^*_1$ and $T^*_o$ from $f_1^{-1}(S^k)$ and $f_o^{-1}(S^k)$ as above these can be extended to $(f_t \times 1)^{-1}(S^k)$ to produce a framed cobordism between $T^*_1$ and $T^*_o$ in $D^n \times 1$.

Thus $c_k(x)$ is unchanged if we alter $f_1$ by a homotopy. Now any two choices of $S^k \subset M'$ to represent $x$, the $t$-regular approximation to $f$, the homotopy making $f^{-1}(S^k)$ connected, $D^k \subset f^{-1}(S^k)$, the isotopy of $D^n$, etc. are related by homotopies of $f$. Thus $c_k(x)$ is well-defined for each $x$ in $\tilde{\pi}_k(M')$.

That $c_k(x)$ is a homomorphism is verified by an easy geometric argument with representatives.

Now we show that $c_k(x) = 0$ if $f_1$ we can change $f$ by a homotopy $f_t$ so that $f_1$ is $t$-regular to $S^k$, and $f_1^{-1}(S^k) = S^k$.

The "only if" follows from property 1) of Lemma 1.5 and the construction of $c_k$.

Now suppose that $c_k(x) = 0$. Then by Lemma 1.2 there is a framed cobordism $(\nu, G)$ of $(T^*_1, F)$ in $D^n \times 1$ so that
i) $W \cap D^n x 1$ is a k-disk

ii) $W$ is trace of surgeries on r-spheres, $r \leq k-1$, and

iii) $W$ is the product cobordism on a neighborhood of $bT^*$

We may suppose that the neighborhood in iii) contains $T^* - T$; so $(f^{-1}(S^k) - T) x I \cup W$ defines a cobordism of $f^{-1}(S^k)$, $W$ in $M x I$, and $(W \cap M x 1)$ is a PL-homeomorphic to $S^k$.

Define $f_*$ in $(M - \text{int } D^n) x I$ by $f x t$, on $W$ by $f(W) = p$, and on $N$, a product neighborhood of $W$ by $i_2 p^* g^{-1}$.

Now we want to extend $f_*$ over $D^n x I - N$ so that $f^{-1}_*(S^k) = W \cap M x 1$. Using a nice Morse function on $W$ we can write $W$ as the composition of traces of single surgeries and reduce the extension problem to this case. We may have to change $D^n$ (and thus the embedding of $W$ in $M x I$) to make the extension.

So let $W$ be the trace of a framed surgery on some $S^r \subset \text{int } T$, $r \leq k-1$.

1) The embedding of $S^r \subset T$ extends to an embedding of $D^{r+1}$ in $M x 0$ so that

i) $D^{r+1}$ intersects $f^{-1}(S^k)$ transversely along $S^r$

ii) $f$ may be changed by a homotopy which is constant on a neighborhood of $f^{-1}(S^k)$ so that a neighborhood of $D^{r+1}$ is mapped to the normal disk at $p$. 
Proof: i) follows from general position since $M$ is $k-1$ connected. There is no obstruction to deforming $f$ as required by ii if $r+1 \leq k-1$ since $M'$ is $(k-1)$-connected.

If $r=k-1$, the obstruction to doing this may be identified with an element of $\overline{\pi}_k(M' - S^k)$ which is isomorphic to $\overline{\pi}_k(M')$ under the inclusion $M' - S^k \subset M'$ ($k$ is small wrt $n$).

Now $f_*: \overline{\pi}_k(M) \rightarrow \overline{\pi}_k(M')$ is onto since $f$ has degree one. Thus we may alter $D^k = D^{r+1}$ by adding some $S^k$ to the interior of $D^k$ (connected sum) to make the obstruction zero.

Now let $D^n_1$ be an $n$-cell which intersects $f^{-1}(S^k)$ in a product neighborhood of $S^k$ and which contains $D^{r+1}$ in its interior. Then we may choose our embedding of $W$ in $M \times I$ so that $W$ is the product cobordism with the product framing outside $D^n_1 \times I$.

Let $N = L \times D^{n-k}_1$ be a product neighborhood of $L = W \cap (D^n_1 \times I)$ and define a partial $f_t = g$ as above on the complement of $(D^n_1 \times I) - N$ where $D^n_1$ plays the role of $D^n$. Let

$$X = (D^n_1 \times I) - (L \times \text{int } D^{n-k}_1), A = bX - \text{int}(X \cap M \times I),$$

and $D = D^{n-k}$ be the normal disk at $p$. We may assume that $g$ maps $A$ to $bD$. 

![Diagram](attachment:image.png)
The obstructions to extending
\[ A \xrightarrow{g} bD \]
over \( X \) lie in
\[ H^{i+1}(X, A; \pi_i(bD)) \]

Now \( L \) is just a \( k+1 \) disk so using excision, the exact sequence, and Mayer-Vietoris we see that
\[
H^{i+1}(X, A) = H^{i+1}(D^n_{\|}xI, N \cup bD^n_{\|}xI \cup D^n_{\|}x0)
\]
\[ = H^i(N \cup (bD^n_{\|}xI \cup D^n_{\|}x0)) \]
\[ = H^i(L \cup s^r(n+1)-\text{disk}) \]
\[ = H^{i-1}(s^r) \]

which is zero if \( i \neq 1, r+1 \). \( \pi_i(bD) = 0 \) if \( i < n-k-1 \).

But \( r+1 \leq k < n-k-1 \). So the desired extension can be made.

We proceed in this way to construct the homotopy required by Lemma 1.5.

We can carry out these steps independently for a number of \( S^k_i \)'s so this completes the proof.
3) Theorem 1.7 for \( k=1,2 \).

Proof: There is an easy direct proof that \( G \) is a homotopy equivalence. Or one can appeal to Lemma II of Part II, Chapter 2.

4) Lemma (Homotopy theory) Let

\[ f: X \rightarrow Y \]

be a homotopy equivalence of CW complexes. Let \( a: S^{n-1} \rightarrow X \). Then if \( X_1 = X \cup_a D^n, Y_1 = Y \cup_f a \cdot D^n \), and \( f_1 = f \cup_a \text{id} \), then

\[ f_1: X_1 \rightarrow Y_1 \]

is a homotopy equivalence.

Proof: This also follows from Lemma II Chapter 2, Part II.

5) Smale Theory: No Smale Theory other than Lemma 1.6 was needed to develop the obstruction theory. However if one uses explicit minimal handlebody decompositions for \((W, M)\) then attractive geometric proofs of the theorems can be given.

6) Lemma F: Let \((P, bP)\) in \((D, bD)\) \((D = D^{n+k})\) represent an element of \( P_n, k+2 > n \). Then there exists a homotopy
equivalence

\[(D, bD) \rightarrow (D^m \times 0, b(D^n \times 0))\]

which is t-regular to \(D^n \times 0\) and such that

\[t_p^{-1}(D^n \times 0) = P \subset D\]

**Proof:** Assume there is a small disk \(D_1^{n+k} \subset D\) so that 
\(P \cap D_1^{n+k} = (D^n \times 0) \cap D_1^{n+k} = n\)-disk. Define \(t_p\) to be the identity on \(D_1^{n+k}\). Let \(S_1 = bD_1^{n+k}\), \(f_1 = t_p/S_1\). Then 
\(D - D_1^{n+k} = S_1 \times I\) and \(P \cap (S_1 \times I)\) defines a framed cobordism of 
\(f_1^{-1}(b(n\text{-disk}))\). Now we proceed as in Lemma 1.5 to construct 
a homotopy \(H: S_1 \times I \rightarrow S_1 \times I\) so that \(H\) is t-regular to 
\((D^n \times 0) \cap S_1 \times I\) and \(H^{-1}(D^n \times 0) \cap S_1 \times I = P \cap S_1 \times I\). Then \(H \cup t_p\) is the 
desired map.

The condition on \(k\) is needed to make the obstruction 
groups vanish. We can actually construct \(t_p\) for highly 
connected generators of \(P_n\) when \(k \sim \frac{n}{2}\), by the same 
argument. By a different (and much more complicated) 
argument we can construct \(t_p\) for these generators when 
\(k \geq 3\).
Application to the Study of Normal Invariants

Let \( M \) be a closed connected PL \( n \)-manifold with PL normal \( k \)-disk bundle \( \nu = \nu(M) \), \( k \gg n \). Let \( T(\nu) = \nu \cup (\text{cone on } b\nu) \) be the Thom complex of \( \nu \). Then an important invariant of \( M \) is any homotopy element \( c_M \) in \( \mathbb{T}^{n+k}_{n+k}(T(\nu)) \) which can be obtained by collapsing \( \mathbb{S}^{n+k} \) onto \( T(\nu) \) where \( \nu \subset \mathbb{S}^{n+k} \). \( c_M \) is a very strong invariant of \( M \). In fact, if \( M \) is simply connected and \( n \geq 6 \), then \( c_M \) determines the PL-homeomorphism type of \( M \).

We make this last statement precise in Theorem 1.15.

**Definition 1.10** A homotopy equivalence \( g : L \to M \) preserves normal invariants if \( g \) is covered by a bundle map

\[
\begin{align*}
v(L) & \xrightarrow{b(g)} v(M) \\
\mathbb{S}^{n+k} & \xrightarrow{c_L} T(v(L)) \quad \xrightarrow{T(b(g))} T(v(M)) \\
& \xrightarrow{c_M} T(b(g))
\end{align*}
\]

is homotopy commutative, where

\[ T(b(g) = b(g) \cup (\text{cone on } b(g)/b\nu(L)). \]
Theorem 1.15  If $M$ is simply connected and $n \geq 6$, then $g : L \to M$ preserves normal invariants iff $g$ is homotopic to a PL-homeomorphism.

Proof:  This is essentially the uniqueness theorem in the PL Browder-Novikov theory. We indicate the proof below.

In this section we will try to construct PL-homeomorphisms from homotopy equivalences, $g : L \to M$. Thus by Theorem 1.15 this problem is equivalent (when $M$ is simply connected and $n \geq 6$) to the problem of showing that $g$ preserves normal invariants.

We first show how the latter problem is related to a certain cobordism construction. Since Theorem 1.15 can be stated for manifolds with boundary we consider that case.

So let $g : (L, bL) \to (M, bM)$ be a homotopy equivalence and assume that $(v(M), v(bM))$ is embedded in $(D^{n+k}, bD^{n+k})$. Then the collapsing map

$$(D^{n+k}, bD^{n+k}) \xrightarrow{c_M} (Tv(M), Tv(bM))$$

defines a normal invariant for $(M, bM)$. Let $W = D^{n+k}$ (open disk bundle of $v(M)$) and $S = $ sphere bundle of $v(M)$. If $R = g^* S = $ sphere bundle of $g^*(v(M))$, then we have a homotopy equivalence

$$f : (R, bR) \to (S, bS),$$
given by the bundle map $b(g)$ covering $g$.

Thus we can consider the problem of making a cobordism construction for $(W,S;f)$.

\textbf{Theorem 1.16} \ Let $g:(I^n,bL) \rightarrow (M^n,bM)$ be a homotopy equivalence and let $v(M)$ and $b(g)$ be as above. Consider the propositions

i) $g$ is homotopic to a PL-homeomorphism

ii) there is a cobordism construction for $(W,S;f) = (D^{n+k}_{\text{interior}}\cdot v(M),\text{sphere bundle } v(M); b(g)/\text{sphere bundle } g^*v(M))$

iii) $g$ preserves normal invariants.

Then $i) \implies ii) \implies iii)$, and $iii) \implies i)$ if $n > 6$ and $\pi_{1}(M) = \pi_{1}(bM) = 0$.

\textbf{Proof:} \ i) \implies ii): \ Suppose $g$ is homotopic to a PL-homeomorphism. Then by the covering homotopy theorem so is $b(g)$. Let

$$H: (R,bR) \times I \longrightarrow (S,bS)$$
be a homotopy between \( b(g) \) and a PL-homeomorphism \( c \). Then

\[
(Q,F) = (W \cup c^* R x I, \text{id} \cup c^* H)
\]

is a cobordism construction for \((W,S;F)\).

\[
\text{ii) } \Rightarrow \text{ iii): If } (Q,F) \text{ is a construction for } (W,S;F) \text{ consider}
\]

\[
F \cup b(g) : Q \cup b^* v(M) \longrightarrow W \cup v(M).
\]

It follows from Lemma H of Chapter 2 Part II that

\( F \cup b(g) \) is a homotopy equivalence of manifolds with boundary. Now there are natural collapsing maps so that

\[
\begin{array}{ccc}
(W \cup v(M), b(W \cup v(M))) & \rightarrow & (T v(M), T v(bM)) \\
\uparrow & & \uparrow \\
F \cup b(g) & \rightarrow & T(b(g)) \\
\downarrow c_L & & \downarrow c_L^M \\
(Q \cup g^* v(M), b(Q \cup g^* v(M))) & \rightarrow & (T g^* v(M), T g^* v(bM))
\end{array}
\]

is commutative. \( W \cup v(M) \) is just \( D^{n+k} \) so we may choose a homotopy inverse \( c \) for \( F \cup b(g) \) which is a PL-homeomorphism. Then \((g^* v(M)) = v(L)\) is a normal bundle for \( L \), \( c_L = c_L^M c \) is a normal invariant for \( L \), and

\[
\begin{array}{ccc}
(D^{n+k}, bD^{n+k}) & \rightarrow & (T v(M), T v(bM)) \\
\downarrow c_L & & \downarrow T(b(g)) \\
& & \downarrow c_L \\
(T v(L), T v(bL))
\end{array}
\]

is homotopy commutative. Thus \( g \) preserves normal invariants.
iii) \( \Rightarrow i \) if \( n \geq 6 \) and \( \overline{\pi}_1(M) = \overline{\pi}_1(bM) = 0 \):

We first consider the case \( bM \neq 0 \). Suppose there is a homotopy

\[
H: (D^n, bD) \times I \to T(v(M), v(bM)) \quad D = D^{n+k}
\]

between \( c_M \) and \( T(b(g)) \cdot c_L \).

Note that \( H \) is \( t \)-regular to

\[
(M, bM) \quad \xrightarrow{0\text{-section}} \quad (Tv(M), Tv(bM))
\]

on \( D^{n+k} \times 0 \) and \( D^{n+k} \times 1 \). Thus we can change \( H \) slightly in \( D^{n+k} \times (0,1) \) so that \( H \) is \( t \)-regular to \( (M, bM) \) on all of \( D^{n+k} \times I \).

Now we apply the technique described by Browder and Hirsch in (1') to change \( H \) in \( D^{n+k} \times (0,1) \) so that \( H^{-1}(M) \) defines an \( h \)-cobordism between \( M' = H^{-1}(M) \cap D^{n+k} \times 0 \) and \( L' = H^{-1}(M) \cap D^{n+k} \times 1 \). Here we use \( bM \neq 0 \) when \( n \) is odd.

Let \( J: L \times I \to H^{-1}(M) \) be a \( PL \)-homeomorphism which carries \( L \times i \) onto \( H^{-1}(M) \cap D^{n+k} \times i \), \( i = 0, 1 \). Let \( c_i = J/L \times i, \quad i = 0, 1 \).

Then \( H \cdot J \) defines a homotopy between

\[
(c_M/M)c_0 \quad \text{and} \quad (T(b(g))/0\text{-section})(c_L/L)c_1
\]

or

\[
(c_M/M)c_0 \quad \text{and} \quad g(c_L/L)c_1
\]
So \( g \) is homotopic to a PL-homeomorphism.

If \( M \) is closed let \( M_0 = M - \text{int} D^n \) and \( L_0 = L - \text{int} D^n \).
Change \( g \) by a homotopy so that \( g_0 = g/L_0 \) induces a homotopy equivalence

\[
g_0: (L_0, bL_0) \rightarrow (M_0, bM_0)
\]

and \( g/D^n \) is a PL-homeomorphism. (see Lemma 21)

Then if \( g \) preserves normal invariants we get a homotopy

\[
H: S^{n+k} \times I \rightarrow T(v(M))
\]

between \( c_M \) and \( T(b(g))c_L \) which is \( t \)-regular to \( M \rightarrow T(v(M)) \). Then we can remove a tubular neighborhood in \( S^{n+k} \times I \) of an arc in \( H^{-1}(M) \) connecting \( L \) and \( M \) in \( bH^{-1}(M) \) to show that \( g_0 \) preserves normal invariants. Then \( g_0 \) is homotopic to a PL-homeomorphism and thus \( g \) is. (Lemma 21).

Now the obstruction theory of Section I may be applied to the construction problem for \((W,S; f)\). We need to satisfy the conditions of Definition 1.4.

1) First write \( W \) as the composition of admissible cobordisms \( W_L \cup W' \) where \( W_L = S \times I \cup (2\text{-handles}) \) and \( S' = bW_L - \text{int} S \) and \( W' \) are each simply connected. We can do this using standard surgery techniques in \( D^{n+k} \) since

\[
\overline{\nu}_1(W) = \overline{\nu}_1(D^{n+k} - \text{n complex}) = 0 \text{ and } \overline{\nu}_1(S) \text{ is finitely generated. Since } P_1 = 0 \text{ we can make a construction } (Q_1, F_1) \text{ for } (W_L, S; f) \text{. Then if } f' = F_1/\partial S', \text{ we have reduced the problem to making a construction for } (W', H'; f') \text{ where } \overline{\nu}_1(W') = \overline{\nu}_1(S') = 0.\]
2) \((W', S')\) is a smoothable admissible cobordism.
   a) \(W'\) is an \(n+k\)-PL submanifold of \(D^{n+k}\).
Thus \(W'\) can be smoothed by (8).

   b) Using excision and the exact sequence of the pair \((D^{n+k}, M^*)\) where \(M^* = \nu(M) \cup \Sigma_1 = \nu(M) \cup (2 \text{ handles})\) we see that

\[
\begin{align*}
H^{i+1}(W', S') &= H^{i+1}(D^{n+k}, M^*) \\
&= H^i(M^*) \\
&= H^i(M \cup 2 \text{ handles}) \\
&= 0 \text{ for } i > n.
\end{align*}
\]

Furthermore since \(W' = D^{n+k}\) is \((n\text{-complex})\), \(W'\) is \((k-1)\)-connected.
Thus if we take \(k\) large the conditions of Definition 1.4 are satisfied. This proves

**Theorem 1.17** Let \(g: (L, bL) \to (M, bM)\) be a homotopy equivalence of connected PL \(n\)-manifolds, \(n \geq 1\). Then the cobordism obstruction theory may be applied to the problem of showing that \(g\) preserves normal invariants. The obstructions may be considered to lie in

\[
H^i(M^*, \Sigma_1), \quad 0 < i < n
\]

where \(M^*\) is any simply connected finite complex obtained from \(M\) by attaching 2-disks.

We state Theorem 1.17 for non-simply connected manifolds because there is some hope of proving an analog of \(\text{iii) } \Rightarrow \text{i)}\)
of Theorem 1.16 for certain non-simply connected manifolds (see 30).

We will apply Theorem 1.17 to construct PL-homeomorphisms from homotopy equivalences. Other applications may be made to "construct normal invariants" needed for certain embedding and isotopy theorems (see 2,3).

Corollary 1: Let \( g:(L,bL) \rightarrow (M,bM) \) be a homotopy equivalence of PL \( n \)-manifolds and suppose that \( n \geq 6 \), \( \pi_1(M) = 0 \), and \( \pi_1(bM) = 0 \). Then there exists an obstruction theory for the problem of deforming \( g \) into a PL-homeomorphism. The obstructions lie in

\[ H^i(M, \mathbb{Z}), \quad 0 < i < n. \]

Corollary 2: Let \( g:(L,bL) \rightarrow (M,bM) \) be a homotopy equivalence, \( M \) as above, and suppose that

\[ H^{4i+2}(M, \mathbb{Z}_2) = 0, \quad 4i+2 < n. \]

\[ H^{4i}(M, \mathbb{Z}) = 0, \quad 0 < 4i < n. \]

Then \( g \) is homotopic to a PL-homeomorphism.

Corollary 2 is almost best possible if one only demands that \( g \) be a homotopy equivalence. See Examples, Chapter 6 Part II. There are counterexamples to Corollary 2 for most values of \( k \) and \( d \) when \( M \) has the homotopy type of

\[ S^{2k} \quad \text{or} \quad S^{2k-1} \cup_{deg \cdot d} e^{2k}. \]
Corollary 3: Let \( g : (L, bL) \to (M, bM) \) be as above and let \( p_i(X) \) denote the \( i^{th} \) rational Pontryagin class of the PL-manifold \( X \). Then if

1) \( H^{4i+2}(M, \mathbb{Z}_2) = 0 \) \( , \) \( 4i+2 < n \)
2) \( H^{4i}(M, \mathbb{Z}) \) is free
3) \( g^*(p_i(M)) = p_i(L) \)

then \( g \) is homotopic to a PL-homeomorphism.

Proof: By Corollary 1 the only possible non-zero obstruction to deforming \( g \) into a PL-homeomorphism lie in \( H^{4i}(M, \mathbb{Z}) \).

These are related by the isomorphisms in b) above to those in making a cobordism construction for \((W, S; \hat{f})\). Condition 3) implies that the rational Pontryagin classes of the tangent bundle of \( g^*v(M) \) are zero. Thus Theorem 1.14 applies to show that the above obstructions are zero.

Corollary 4: If \( h : L \to M \) is a homeomorphism, 
\( H^{4i+2}(M, \mathbb{Z}_2) = 0 \) \( , \) \( 4i+2 < n' \), and \( H^{4i}(M, \mathbb{Z}) \) is free, then \( h \) is homotopic to a PL-homeomorphism. (\( \dim M \geq 6 \), \( \Pi_1(M) \to \Pi_1(bM) = 0 \).)

Proof: According to Novikov (20) Corollary 2 applies to \( h \).
The condition $H^i(M)$ is free in Corollary 4 may be weakened somewhat. See Corollary 6 to Theorem 26 in Part II. It may be weakened even more if $g$ is known to be a PL-tangential equivalence. This is still not enough, however, to conclude even that $M$ and $L$ are PL-homeomorphic. For example when $M$ is complex projective $4$-space this is not true. The crucial point here is the fact that

$$H^6(\mathbb{C}P^4,\mathbb{Z}_2) \neq 0.$$ 

In general the obstructions in $H^{4i+2}(M,\mathbb{Z}_2)$ are hard to handle, and little is known.

Triviality theorems for these obstructions (in dimensions such that $\theta_{4i+1}(\mathcal{F}_M) = \mathbb{Z}_2$) can be related by geometric arguments to hoped for properties of the Hurewicz homomorphism for PL/O or $B_{PL}$, but this is not too helpful because these problems are quite hard.

Also there is a Bockstein operation relating the obstructions in the above theory to those in an analogous smooth obstruction theory. This relationship would be useful when $\theta_{4i+1}(\mathcal{F}_M)$ is a non-trivial direct summand of $\theta_{4i+1}$.

Some of these relationships can be readily seen from the viewpoint of Part II.

Now we give a brief skeletal description of the obstruction theory. Suppose that $g$ is homotopic to a PL-homeomorphism over the $k$-skeleton of $M$ in the sense of
Definition 22. Let $M_i$ denote a thickened $i$-skeleton of $M$, with $M_i \subset \text{int } M_{i+1}$.

Now using the fact that $g/M_k$ is a PL-homeomorphism we can make a partial cobordism construction for $(W,S;f)$ over the $k$-skeleton of $(W,S)$.

By making the argument in Theorem 1.16 relative we can prove that this construction can be extended over the $(k+1)$ skeleton of $(W,S)$ iff $g$ can be changed by a homotopy in the complement of $M_k$ so that it becomes a PL-homeomorphism over the $(k+1)$-skeleton of $M$.

In this way we show that the obstruction theory of Section I corresponds precisely to an obstruction theory for the problem of deforming $g$ to a PL-homeomorphism "over the skeletons of $M$." (Such an obstruction theory is derived in a different manner in Part II.)

One can see the geometric significance of the obstructions by viewing the cobordism construction for $(W,S;f)$ as a process for building a disk, $D^{n+k}$ around $g^*(v(M))$ and extending the bundle map covering $g$ to a homotopy equivalence of $(D^{n+k}, bd^{n+k})$ which carries the complement of $g^*(v(M))$ to the complement of $v(M)$.

This point of view provides the geometric motivation for proving lemmas about the obstructions and provides a good model for working with the obstruction theory.
Another geometric point of view may be obtained by working directly in the manifold. Recall that $g$ is a PL-homeomorphism over the $k$-skeleton of $M$, and $M_i$ denotes a thickened $i$-skeleton of $M$.

![Diagram](image)

We can think of forming $M_{k+1}$ from $M_k$ by attaching $(k+1)$-handles to $bM_k$. Now change $g$ in the complement of $M_k$ so that it is $t$-regular to the framed core disks of the handles and then look at $g^{-1}$ of these. We get a sequence of framed submanifolds of $L$, which intersect $bM_k$ in a sequence of $k$-spheres.

These framed submanifolds may be constructed so that they are contractible in $L$ and thus determine elements in $P_{k+1}$ (by using $L \times D^n$ if necessary).

A careful look at the construction for $(V, S; f)$ arising from $g/\Pi_k$ shows that these submanifolds correspond in $P_{k+1}$ to the values of the cobordism obstruction cochain on a basis for $H_{k+1}(\Pi_{k+1}M_k)$. 
This computation of the obstruction is useful when \( g \) is known to be a tangential equivalence. In fact, the essence of Corollary 7 Theorem 26 can be seen this way.
Piecewise Linear Structures on Poincaré Spaces

A connected CW pair \((X,A)\) is called a relative Poincaré space of dimension \((n+1)\), if \(H_{n+1}(X,A) = \mathbb{Z}\), \(\partial : H_{n+1}(X,A) \to H_n(A)\) is an isomorphism and the vertical maps in the diagram

\[
\begin{array}{cccccc}
\ldots & \to & H^k(X,A) & \to & H^k(X) & \to & H^k(A) & \to & \ldots \\
& & \downarrow \wedge g & & \downarrow \rho g & & \downarrow \wedge \partial g & \\
\ldots & \to & H_{n+1-k}(X) & \to & H_{n+1-k}(X,A) & \to & H_{n-k}(A) & \to & \ldots
\end{array}
\]

are isomorphisms.

If we suppose that \(A\) and \(X\) are simply connected, then Spivak (33) has shown that \((X,A)\) has a unique stable normal bundle in the category of homotopy objects. That is, there is a unique (up to fibre homotopy equivalence) spherical fibre space over \((X,A)\) whose (relative) Thom space has spherical fundamental homology class.

Let \(f : X \to B_T\) be the classifying map for this spherical fibre space.

\[
\begin{array}{c}
\text{\overline{f}} \\
\downarrow \\
X \\
\downarrow f \\
B_T
\end{array} \quad \begin{array}{c}
\rightarrow B_{PL} \\
\rightarrow B_{PL} \end{array}
\]

The problem of factoring \(f\) through the classifying space \(B_{PL}\) is essentially the problem of finding a
PL-manifold with boundary \((X, bM)\) which is homotopically equivalent to \((X, A)\). In fact, if \(n \geq 5\) and \((X, A)\) is simply connected \(A \neq 0\), Browder and Hirsch (Compare (26) and (1)) have shown that if \(\tilde{f}\) is a lifting of \(f\), then there exists a PL-manifold pair \((\tilde{M}, b\tilde{M})\) and a map \(g: (\tilde{M}, b\tilde{M}) \to (X, A)\) so that \(g\) is a homotopy equivalence and \(\tilde{f}g\) classifies the stable PL normal bundle of \(\tilde{M}\).

In the next three sections we will classify the pairs \((M, g)\) by the different liftings \(\tilde{f}\).

We assume then that at least one lifting exists. Thus by the theorem of Browder and Hirsch we may replace \((X, A)\) by a PL manifold with boundary \((M, bM)\).

We will work in the category \(C\) whose objects are pairs \((K, L)\) consisting of a locally finite simplicial complexes \(K\) and a subcomplex \(L\) and whose morphisms are PL-maps

\[
f: (K, L) \to (K', L').
\]

\(f: K \to K'\) is PL if there exists a rectilinear subdivision of \(\tilde{K}\), \(\tilde{K} \to K\) so that \(\tilde{K} \to \tilde{K}' \to K'\) maps each simplex of \(K\) linearly into a simplex of \(L\).

By a homotopy of \(f\) we mean a homotopy of \(f\) in the category, that is the subspace is mapped to the subspace during the homotopy. Thus

\[
f: (K, L) \to (K', L')
\]

is a homotopy equivalence if there exists
\[ g: (K', L') \to (K, L) \]

so that \( fg \) is homotopic to the identity of \((K', L')\)
(keeping \( L' \) in \( L' \)) and \( gf \) is homotopic to the identity
of \((K, L)\) (keeping \( L \) in \( L \)).

**Remark** Let \( f: (K, L) \to (K', L') \) be such that
\( f: K \to K' \) and \( f/L: L \to L' \) are homotopy equivalences.
Then \( f \) is a homotopy equivalence in \( C \).

**Proof:** We prove this in the category of CW-complexes.
The result then follows by simplicial approximation.

By taking the mapping cylinder of \( f \) we may suppose
that \( f \) is an inclusion so that \( L' \cap K = L \).

**Step 1** \( K \) is a strong deformation retract of \( L' \cup K \).
This follows from the fact that \( L \subset L' \) is a homotopy
equivalence and \( L' \cap K = L \).

**Step 2** \( \pi_i(K', L' \cup K) = 0 \) for all \( i \). This
follows from the exact sequence for the triple \( K \subset L' \cup K \subset K' \).
Thus \( L' \cup K \) is a strong deformation retract of \( K' \).

**Step 3** Compose the deformation of step 2 with
that of step 1 to get a strong deformation retraction
of \((K', L')\) onto \((K, L)\) (preserving subspaces).

Let \( \mathcal{M} \subset C \) be the subcategory of compact PL-manifold
pairs \((N, \partial N)\) with or without boundary and PL-maps.
We will define what we mean by a PL-structure on \((N, \partial N)\)
and then compute these on certain subcategories \( \mathcal{M}_n \); \( n \geq 5 \).
Definition 1. A piecewise linear homotopy structure on \((M, bM)\) is a PL-manifold pair \((L, bL)\), and a homotopy equivalence \(g: (L, bL) \to (M, bM)\). To be brief we call the pair \((L, g)\) a PL-structure on \(M\).

Definition 2. We say that the PL-manifold \(M^n\) is properly embedded in the PL-manifold \(W^{n+k}\), \(k = 0, 1\) if

1) for \(k = 0\), \(M\) is PL-embedded in the interior of \(W\) or \(M\) is PL embedded in \(W\) so that \(bM \cap bW\) is an \((n-1)\) dimensional submanifold of \(bM\).

2) for \(k = 1\), \(M\) is PL-embedded in \(bW\).

Note that if \(M\) is properly embedded in \(W\), \(W\)-int \(M\) is a PL-manifold with boundary in case 1) and \(bW\)-int \(M\) is a PL-manifold with boundary in case 2).

Definition 3. Let \(a = (Q, g)\) be a PL-structure on \(W\) and suppose that \(M\) is properly embedded in \(W\). We say that \(a\) induces a PL-structure on \(M\) if there is a PL-manifold \(L\) properly embedded in \(Q\) so that

\[
\begin{array}{ccc}
L & \xrightarrow{g/L} & M \\
\downarrow & & \downarrow \\
Q & \xrightarrow{g} & W \\
\downarrow & & \downarrow \\
Q\text{-int} L & \xrightarrow{g/Q\text{-int } L} & W\text{-int } M
\end{array}
\]

is commutative and
(L, g/L) is a PL-structure on M
(Q-int L, g/Q-int L) is a PL-structure on W-int M.
in case 1).
(bQ-int L, g/bQ-int L) is a PL-structure on bw-int M
in case 2).

We write a/M for the induced PL-structure on M.

We will show later that in most cases any structure
on W may be changed slightly so that it induces a
PL-structure on a properly embedded submanifold M.

**Definition 4** Let a be a PL-structure on M
which is properly embedded in W. A PL-structure
b on W is said to extend a if b induces a PL-structure
on M and b/M = a.

**Remark** We note here that if M^n is properly embedded
in W^{n+1} and a = (L, g) is a PL-structure on M, then
the problem of extending a to a PL-structure on all of
W is precisely the cobordism problem discussed in section 1
for admissible pairs (W, M).

**Definition 5** (Concordance) Let a_i = (M_i, \phi_i)
i = 0, 1 be two PL-structures on (M, bM). Then a_0 and a_1
induce a PL-structure on (M x 0) \cup (M x 1) which is properly
embedded in M x I. We say that a_0 is concordant to a_1
if this PL-structure extends to a PL-structure on all of
M x I.
Concordance defines an equivalence relation on the set of PL-structures on $M$. Let $\text{PL}(M)$ denote the set of equivalence classes. Let $0$ in $\text{PL}(M)$ denote the concordance class of $(M, \text{id})$. (*) See Lemma H in Chapter 2.

**Definition 6** a) (The category $\mathcal{M}_n$) For $n \geq 6$ let $\mathcal{M}_n$ denote the category

1) whose objects are PL $n$-manifolds $(M, bM)$ such that $bM \neq 0$, $\pi_i(M) = 0$ and $\pi_i(bM) = 0$.

2) whose morphisms are embeddings $i : M_1 \to \text{int } M_2$ so that $\pi_i(M_2 - M_1) = 0$.

For $n < 6$ let $\mathcal{M}_n$ be the void category.

b) (The category $\overline{\mathcal{M}}_n$) For $n \geq 6$ let $\overline{\mathcal{M}}_n$ be the enlarged category defined by conditions 1) and 2) without the assumption that $bM \neq 0$. Let $\overline{\mathcal{M}}_5$ be the category of closed simply connected PL 5-manifolds and PL-homeomorphisms. Let $\overline{\mathcal{M}}_n$ be the void category for $n < 5$.

An application of Van Kampen's shows that $\mathcal{M}_n$ and $\overline{\mathcal{M}}_n$ are categories.

We will compute $\text{PL}(M)$ for $M$ in $\mathcal{M}_n$ (and then $\overline{\mathcal{M}}_n$). Our applications to the problem of deforming a homotopy equivalence into a PL-homeomorphism and the study of pseudo-isotopy of PL-homeomorphisms will make use of the PL-h cobordism theorem for $\overline{\mathcal{M}}_n$. 
Theorem (h-cobordism) If $M$ is in $\mathcal{N}_n^c$, then the relative h-cobordism theorem is true for $(M, bM)$.

(The relative h-cobordism theorem asserts that if $(W, W')$ is an h-cobordism of $(M, bM)$ then $W'$ is equivalent to the product cobordism $bM \times I$ and any such equivalence extends to an equivalence of $W$ with the product cobordism $M \times I$.)

If $M$ is smoothable this theorem follows from Smale (21). In the general PL-case it is indicated in work of Mazur (13) and Zeeman (32).

We will use this theorem in the applications to straighten out concordances.

Lemma 1 Let $g: (L, bL) \to (M, bM)$ be a homotopy equivalence. Then if $g$ is homotopic to a PL-homeomorphism, $(L, g)$ is concordant to $(M, \text{id})$.

Conversely, if $(L, g)$ is concordant to $(M, \text{id})$ and $M$ belongs to $\mathcal{N}_n^c$ then $f$ is homotopic to a PL-homeomorphism.

Proof: Suppose there exists $H: (L, bL) \times I \to (M, bM)$ so that $H(x, 0) = f(x)$ and $H(x, 1)$ is a PL-homeomorphism $c$.

\[
\begin{array}{ccc}
L \times I & \xrightarrow{H \times I} & M \times I \\
\downarrow & & \downarrow \\
M & \xrightarrow{\text{identity}} & M \times I \\
\end{array}
\]
Then \((W, G) = (L \times I \cup c \cdot 1, H \times I \cup c^{-1} \cdot \text{identity})\) gives a concordance between \((L, f)\) and \((M, \text{id})\). Thus if \(f\) is homotopic to a PL-homeomorphism \((L, f)\) is concordant to zero.

Now suppose \((W, G)\) is a concordance between \((L, f)\) and \((M, \text{id})\).

Now \(W\) defines an h-cobordism between \((L, bL)\) and \((M, bM)\). By the h-cobordism theorem for \(M\) we may choose a PL-homeomorphism

\[
H: L \times I \to W
\]

so that \(H/L \times 1\) is the identity map of \(L\) and \(c = H/L \times 0\) is onto \(M\). Then \(G \cdot H\) is a homotopy between \(f\) and the PL-homeomorphism \(c\) . Q.E.D.

Let \(\text{PL}\) be the function which assigns to each \(M\) in \(\mathcal{M}_n\) the set \(\text{PL}(M)\). We will now show how a morphism of \(\mathcal{M}_n\)

\[
i: M_1 \to M_2
\]

induces a map

\[
i^*: \text{PL}(M_2) \to \text{PL}(M_1)
\].
Theorem (Browder) Let $i: M_1 \rightarrow M_2$ be a morphism of $\mathcal{H}^n_\mathbb{R}$ and suppose $g: (L_2, bL_2) \rightarrow (M_2, bM_2)$ defines a PL-structure on $M_2$. Then we can change $g$ by a homotopy so that $g$ induces a PL-structure on $M_1 \subset M_2$. The induced PL-structure is unique up to a concordance which is embedded in $L_2 \times I$.

Proof: This is a restatement in our terminology of Browder's relative codimension one embedding theorem (2). He has a further hypothesis on $H_2$ which has since been removed by J. B. Wagoner. (25).

Therefore given $i: M_1 \rightarrow M_2$ in $\mathcal{H}^n_\mathbb{R}$ we may define

$$i^*: \text{PL}(M_2) \rightarrow \text{PL}(M_1)$$

using the Browder Codimension One Theorem. This is well-defined on concordance classes by the uniqueness part of the Browder Theorem. (The construction of the concordance only uses the fact that $L_2 \times I$ is an $h$-cobordism.)

Corollary 1: If we let $S$ be the category of pointed sets and base point preserving functions then the assignment

$$M \rightarrow \text{PL}(M)$$

$$i \rightarrow i^*$$

makes

$$\mathcal{H}^n_\mathbb{R} \xrightarrow{\text{PL}} S$$

into a contravariant functor.
Proof: Let $0$ be the preferred element of $\text{PL}(M)$. On representatives we verify that

$$i^*(0) = 0.$$ 

To show naturality let $M_0 \rightarrow M_1$ and $M_1 \rightarrow M_2$ be morphisms of $\mathcal{M}_n$ and let $(L_2, g)$ define a PL-structure on $M_2$. Change $g$ to $g'$ by a homotopy so that $g'/L_1 = g'^{-1}(M_1)$ induces a PL-structure on $M_1$. Change $f = g'/L_1$ by a homotopy so that $f/L_0 = f^{-1}(M_0)$ induces a PL-structure on $M_0$. Extend the homotopy of $f$ to a homotopy of $g'$, then $g'$ induces a PL-structure on $M_0$. Then the naturality is verified on representatives.

We remark that the Browder theorem actually implies that

$$i^* : \text{PL}(M_2) \rightarrow \text{PL}(M_1)$$

may be defined for proper embeddings

$$i : M_1 \rightarrow M_2$$

such that $L = bM_2 \cap bM_1 \neq 0$ if the further conditions

$$\pi_1(bM_2-L) = \pi_1(bM_1-L) = 0$$

and $L$ is in $\mathcal{M}_{n-1}$ are satisfied.

If $i_1 : M_0 \rightarrow M_1$ and $i_2 : M_1 \rightarrow M_2$ are two such embeddings, then one can show just as in
the corollary that
\[(i_2 i_1)^* = i_1^* i_2^*\]
in case \(BM_0 \cap BM_1 \cap BM_2 = 0\).

In the next few sections we will show that \(PL/\mathcal{M}\)
is naturally equivalent to a functor which is defined on
a category of topological spaces and continuous maps
(containing \(\mathcal{M}_n\)). We then apply Brown's theory (4) to
represent \(PL\) as
\[
\left[\mathcal{C}, X\right]
\]
for some space \(X\). With this approach we can get a
fairly good hold on homotopy properties of \(X\).

Another approach might be the following. Define
\(\mathcal{M}_n \rightarrow \mathcal{M}_{n+1}\) by \(M \rightarrow M \times I \subset M \times 3I, i \rightarrow i \times I\).
Let \(\mathcal{F} = \text{dir lim } \mathcal{M}_n\), and replace embeddings by isotopy
classes of embeddings. Let \(\mathcal{F}\) be the category of
finite simply connected complexes and continuous maps
and define a covariant functor
\[
\mathcal{F} \xrightarrow{\text{RN}} \mathcal{M}
\]
by mapping objects to regular neighborhoods in the interior of
\(D^n \subset D^n \times I \subset \ldots\) and maps to isotopy classes of embeddings
which are covered by ambient isotopies of \(D^n \subset D^n \times I \subset \ldots\).

One can show that \(PL\) is well-defined on \(\mathcal{M}\) (see
Theorem 23). Therefore we have a contravariant functor \(H\)
\[
\mathcal{F} \xrightarrow{\text{RN}} \mathcal{M} \xrightarrow{\text{PL}} B \xrightarrow{H} B
\]
from $\mathcal{S}$ to $S$, which we can try to represent as $[\mathcal{S}, Y]$ using Brown's theory.

The homotopy groups of $Y$ can be computed directly using surgery, Theorem 23 implies that $PL$ is determined by its behaviour on image (HN), thus we have obtained quite a bit of information about $PL$.

One difficulty is the fact that $\mathcal{S}$ contains only simply connected finite complexes. So there may be a problem applying Brown's theory.

This can probably be overcome. A more serious drawback is the fact that this definition of $Y$ is somewhat remote, and a closer study of its homotopy properties and its relation to $B_{PL}$ seems harder.

This approach can be used to study the concordance classes of smoothings of a PL-manifold.
The Classifying Bundle of a PL-Structure

Now we consider a bundle theory which will be used to study the functor

\[ \nu_n : \text{PL} \to \mathcal{S}. \]

**Definition 7** Let \( X \) be a locally finite simplicial complex. An \( (F/\text{PL})_n \)-bundle over \( X \) is a pair \((E,t)\) consisting of a PL \( n \)-disk bundle \( E \to X \) and an \( F \)-trivialization \( t \). That is, we have

\[
\begin{array}{ccc}
E & \xrightarrow{t} & D^n \\
\downarrow & & \downarrow \\
X & \xrightarrow{} & D^n
\end{array}
\]

where \( t \) induces a homotopy equivalence of each fibre pair with \((D^n, bD^n)\).

**Definition 8** Two \((F/\text{PL})_n\)-bundles \((E_i, t_i)\) \(i=0,1\) are said to be equivalent if there is an \((F/\text{PL})_n\)-bundle \((E, T)\) over \(X \times I\) so that

\[
\begin{array}{ccc}
E_0 & \xrightarrow{b_0} & E_i \\
\downarrow & & \downarrow \\
X & \xrightarrow{x_i} & X \times I
\end{array}
\]

is commutative where \( b_0 \) and \( b_1 \) are PL bundle maps.

Let \( B_n(X) \) denote the set of equivalence classes of \((F/\text{PL})_n\)-bundles over \(X\).
Definition 9 (The Induced Bundle)
If \( f: X \to Y \) is a piecewise linear map, define \( B_n(f): B_n(Y) \to B_n(X) \) on representatives by

\[
\begin{array}{ccc}
  f^*E & \xrightarrow{\tilde{f}} & E & \xrightarrow{t} & D^n \\
  \downarrow & & \downarrow & & \\
  X & \xrightarrow{f} & Y
\end{array}
\]

that is,

\[
(E, t) \mapsto (f^*E, t \tilde{f}).
\]

It's easy to show that \( B_n(f) \) is well defined on equivalence classes.

Definition 10 (The Whitney Sum) If \( x_1 \) in \( B_n(X) \) is represented by \((E_1, t_1)\) and \( x_2 \) in \( B_m(X) \) is represented by \((E_2, t_2)\) define the Whitney sum \( x_1 \oplus x_2 \) in \( B_{n+m}(X) \) on representatives by

\[
\begin{array}{ccc}
  E_1 \oplus E_2 & \xrightarrow{\alpha} & E_1 \times E_2 & \xrightarrow{t_1 \times t_2} & D^n \times D^m \\
  \downarrow & & \downarrow & & \\
  X & \xrightarrow{\text{diag.}} & X \times X
\end{array}
\]

where \( E_1 \oplus E_2 = (\text{diag})^* E_1 \times E_2 \) and \( t_1 \oplus t_2 \) is the indicated composition.

Let 0 in \( B_n(X) \) be represented by \((X \times D^n, p_2)\).
Let \( \mathcal{S} \) be the category of countable connected locally finite simplicial complexes and PL-maps and \( S \) the category of pointed sets and base point preserving functions.
The association
\[ X \rightarrow B_n(X) \]
\[ f \rightarrow (B_n(Y) \rightarrow B_n(X)) \]
defines a contravariant functor from \( \mathcal{C} \) to \( S \).

\[ \mathcal{C} \rightarrow S \]

\( B_n \) satisfies the following properties:

1) \( B_n(\text{pt}) = 0 \).

2) \( B_n(S^n) \) is countable. \( B_n(S^1) = \{0\} \), \( n > 2 \).

3) If \( X = \bigcup X_i \), \( X_i \) increasing subcomplexes of \( X \), and if there exists \( x_i \) in \( B_n(X_i) \) so that \( x_i/X_{i-1} = x_{i-1} \), then there exists \( x \) in \( B_n(X) \) so that \( x/X_i = x_i \).

4) If \( X = X_1 \cup X_2 \) with \( X_1 \cap X_2 = A \), and \( x_i \) in \( B_n(X_i) \), \( x_2 \) in \( B_n(X_2) \) satisfy

\[ x_1/A = x_2/A \]

then there exists \( x \) in \( B_n(X) \) so that \( x/X_i = x_i \), \( i = 1, 2 \).

5) \( B_n(f) \) only depends on the homotopy class of \( f \).

**Proof:** The first statement is clear. Now for the properties.

1) \( B_n(\text{pt}) = 0 \) because any homotopy equivalence of \( (D^n, \partial D^n) \) is homotopic to a PL-homeomorphism.
2) The statement about $B_n(S^k)$ follows from Lemma 1.5 and Lemma 19.

$B_n(S^k)$ is countable because

a) $B = \prod_{k-1}$ (PL automorphisms of $D^n$)
is countable, ($D^n$ is a finite simplicial complex).

b) $T = \text{the set of homotopy classes of \text{F-trivializations of the trivial n-disk bundle over}}$

$S^k$ is countable.

c) A subset of $B \times T$ maps onto $B_n(S^k)$.

3) Consider $X' = (X_1 \times I) \cup X_2 \times I \cup \ldots$

where $X_1 \times 1$ is attached to $X_1 \subset X_{i+1} \times 0$. If we

choose representatives $v_i$ for the $x_i$ and apply the definition of equivalence we see that there is an $(F/PL)_n$-bundle

$V$ over $X'$ so that $V/X_1 \times 0 = v_i$. Now there is a

homotopy equivalence $X' \rightrightarrows X$ so that

\[
\begin{array}{c}
X' \xrightarrow{p} X \\
\downarrow \quad \quad \quad \quad \downarrow c \\
X_1 \quad \quad \quad \quad X_i
\end{array}
\]

is commutative. So 3) follows from 5). (Let $x$ in $B_n(X)$ be such that $p^*x = (v)$.)

4) Consider $X' = X_1 \cup A \times I \cup X_2$ where

$A \times 1$ is identified to $A$ in $X_{i+1}$. Now proceed as in 3).

5) If $f$ is homotopic to $g$ by $H$ then

$H^*(V)$ gives an equivalence between $f^*v$ and $g^*v$. 
Let \( a \) belong to \( PL(M) \). We want to "classify" \( a \) by an element in \( B_k(M) \), \( k \) large.

**Definition 11** Let \( f: (M,bM) \rightarrow (L,bL) \) a map of PL \( n \)-manifold pairs. A \( k \)-tubular neighborhood of \( f \) is a pair \( (E,i) \) consisting of a PL \( k \)-disk bundle over \( M \) \( (E \xrightarrow{p} M) \) and a PL embedding of \( (E,E/bM) \) in \( (L,bL) \times \text{int } D^k \) so that

\[
\begin{array}{c}
(E,E/bM) \xrightarrow{i} (L,bL) \times \text{int } D^k \\
\downarrow p \quad \downarrow p \\
(M,bM) \xrightarrow{f} (L,bL)
\end{array}
\]

is homotopy commutative.

In what follows \( \bar{f} \) denotes a bundle map covering \( f \), and \( f' \) denotes a homotopy inverse for \( f \) if \( f \) happens to be a homotopy equivalence.

**Lemma 3** Let \( g:(L,bL) \rightarrow (M,bM) \) be a homotopy equivalence. Then \( g \) has a \( k \)-tubular neighborhood \( (E,i) \), if \( k \geq n + 3 \). Furthermore, \( i \) can be chosen so that

\[
(M,bM) \times 0 \subset i(E',E'/bM)
\]

where \( E' \) denotes the associated open disk bundle.

**Proof:** Approximate \( g \) by an embedding \( i: (L,bL) \rightarrow (M,bM) \times \text{int } D^k \). By the stable PL-tubular neighborhood theorem (6) we can find a bundle \( E \xrightarrow{p} L \)
and an embedding $i$ extending $\mathcal{I}$ (p/0-section). Then $(E,i)$ is a $k$-tubular neighborhood of $g$.

Let $j:(M,bM)\to (E',E'/bM)$ be an embedding approximating $\mathcal{I}g'$. Then $j$ and the inclusion

$$(M,bM) \xrightarrow{x_0} (M,bM) \times \text{int } D^k$$

are homotopic and therefore isotopic since $k \geq n+3$. By (31) we may choose an ambient isotopy $I$ of the identity of $(M,bM) \times D^k$ relating $x_0$ and $j$. Then $(E, I(x,1)^{-1} i)$ has the desired property.

**Lemma 4** Let $g:(L^n,bL)\to (M^n,bM)$ be a homotopy equivalence. Let $(E,i)$ be a $k$-tubular neighborhood of $g'$ which has the additional property of Lemma 3. Then if $L$ is compact, there is a $k$-disk $0 \subset D^k_1 \subset \text{int } D^k$ so that

$$\text{fibre} : E \xrightarrow{i} M \times D^k \xrightarrow{D^k} D^k$$

induces a homotopy equivalence $t_x$

$$(p^{-1}(x), b(p^{-1}(x)) \xrightarrow{t_x} (D^k, D^k - D^k_1))$$

for each $x$ in $L$.

**Proof:** Using the compactness of $L$ it is easy to see that $D^k_1$ may be chosen (independent of $x$) so that $t_x$ is well-defined for each $x$ in $M$. 

Now each fibre disk represents the Lefshetz Dual of a generator of \( H^n(B,B) = H^n(L,bl) \). Now \( i(L,bl) \) is homologous to \((M, bM) \times 0 \) so it is clear that the composition

\[
\text{fibre} \xrightarrow{i} M \times D^k \xrightarrow{p_2} D^k
\]

induces the desired relative homology isomorphism in dimension \( k \). The lemma follows.

From Lemma 4 we see that a "good" \( k \)-tubular neighborhood of \( g: (L,bl) \rightarrow (M, bM) \) (with the additional property of Lemma 3) yields an \( F/PL)_k \) bundle, namely

\[
(E, rp_2, l)
\]

where

\[
r \text{ is a h. e. } (D^k, D^k - D^k_\perp) \rightarrow (D^k, bD^k). \]

**Definition 12** Let \( (L, g) \) be a PL-structure on \( M \) and \( g' \) a homotopy inverse for \( g \). A **classifying bundle** for \( (L, g) \) is an \( F/PL)_k \) bundle \( (E, t) \) where

1) \( t \) is given by the composition

\[
E \xrightarrow{j} L \times D^k \xrightarrow{r} D^k
\]

where \( (E, j) \) is a "good" tubular nghd of \( M \xrightarrow{g'} L \) and,

ii) \( r \) is an \( F \)-trivialization of \( L \times D^k \rightarrow L \) such that \( r^{-1}(0) = L \times 0 \) and \( r \) agrees with \( L \times D^k \rightarrow D^k \) on a neighborhood of \( L \times 0 \).
Theorem 6: Suppose \( k > n + 3 \) and \((L, g)\) is a PL-structure on \( M^n \). Then the equivalence class in \( B_k(M) \) of any \( k \)-dimensional classifying bundle for \((L, g)\) depends only on the concordance class of \((L, g)\).

The correspondence

\[ \text{PL(structure)} \rightarrow \text{(classifying bundle)} \]

defines a natural transformation of contravariant functors on \( M^n \).

Proof: The first statement follows immediately from Lemma \(^4\) below applied to \((M \times I, M \times 0 \cup M \times 1)\).

If \( M_1 \rightarrowtail M_2 \) is a morphism of \( \overline{M}_n \), and \((L_2, g_2)\) is a PL-structure on \( M_2 \) which restricts to a PL-structure \((L_1, g_1)\) on \( M_1 \), then any classifying bundle for \((L_1, g_1)\) restricts to a classifying bundle for \((\text{BL}_1, g_1/\text{BL}_1)\).

Using Lemma \(^4\) we can extend the latter classifying bundle over \( M_2 - \text{int} M_1 \) so that it is classifying for \((L_2 - \text{int} L_1, e_2/ L_2 - \text{int} L_1)\).

The conditions of Lemma \(^4\) imply that these fit together to give a classifying bundle for \((L_2, e_2)\). (See first part of proof of Theorem 9 for more remarks on this.)

The naturality is then verified for these representatives.
Lemma 31. Let $\mathbb{R}^n$ be embedded in $\mathbb{R}^{n+1}$. Suppose $(L,g)$ defines a PL-structure on $L$ which extends to a PL-structure $(Q,G)$ on $W$. Let $(E_1;i_1)$ be a "good" $k$-tubular neighborhood of $g$, $k > n + 3$. Then there exists a "good" $k$-tubular neighborhood of $G,(E,j)$, and a bundle map $E \xrightarrow{b} B$ covering $L \subset Q$ so that

$$
\begin{array}{ccc}
E/(Q;L,bL,L',bL') & \xrightarrow{j} & (W;M,bM,M',bM') \\
\downarrow p & & \downarrow p_1 \\
(Q;L,bL,L',bL') & \xrightarrow{G} & (W;M,bM,M',bM')
\end{array}
$$

is homotopy commutative and

$$j/(E/L) \circ b = i_1.$$

$L' = bQ\text{-int } L, M' = bw\text{-int } M$

Furthermore, we may assume that there is a homotopy between $p_1j$ and $Gp$ which extends (using $b$) a given homotopy between $p_1i_1$ and $Gp$.

Proof: Let $h$ be a given homotopy between $p_1i_1$ and $Gp$. Recall that $G$ defines a map of $5$-tuples

$$
\begin{array}{ccc}
(Q;L,bL,L',bL') & \xrightarrow{G} & (W;M,bM,M',bM')
\end{array}
$$

which is a homotopy equivalence on each factor.
l) There exists \((E,j_1)\), a tubular neighborhood of \(G\), so that

\[
\begin{align*}
E/(Q;L,bL,L',bL') & \xrightarrow{j_1} (W;M,bM,M',bM') \times \text{int } D^k \\
\downarrow p & \downarrow p_1 \\
(Q;L,bL,L',bL') & \xrightarrow{G} (W;M,bM,M',bM')
\end{align*}
\]

is homotopy commutative by homotopy extending \(h\), and

ii) \((E/L, j_1/(E/L))\) defines a "good" tubular neighborhood of \(g = G/L\),

\[
g: (L,bL) \xrightarrow{} (N,bM)
\]

Proof of l):

1) Extend \(h \times p_1/0\)-section to a homotopy \(H'_1\) of \((x0)G\). Change \(H'_1\) by a homotopy which is
fixed on the 0-section of \(E_1\) to an embedding. Let
\(H_1\) be the sum of these two homotopies.

Apply the PL-tubular neighborhood theorem to find
a PL-bundle \(E\) and an embedding

\[
j_1: E/(Q;L,bL,L',bL') \xrightarrow{} (W;M,bM,M',bM') \times \text{int } D^k
\]

so that \(j_1/0\)-section is the embedding

\[
H_1/(0\text{-section}): (Q;L,bL,L',bL') \xrightarrow{} (W;M,bM,M',bM') \times \text{int } D^k.
\]
There is no obstruction to finding a homotopy between \( p_1 j_1 \) and \( Gp \) which extends \( h \).

Now we deform \( j_1 \) so that \( ii) \) is satisfied.

Now \( i_1 \) and \( j_1 / E/(L,bL) \) define two tubular neighborhoods of \( i_1(0\text{-section}) = j_1(0\text{-section}) \).

So by the PL-tubular neighborhood theorem there is an isotopy \( I_t \) of the identity of \( (M',bM) \times \text{int } D^k \) which is constant at infinity and so that \( I_t \) carries \( j_1(E/(L,bL)) \) onto \( i_1(E_1/(L,bL)) \). Thus

\[
(E/L, I_t j_1)
\]
defines a "good" tubular neighborhood of \( g \).

Now we apply the IEP (Isotopy Extension Theorem) successively to extend \( I_t \) to an isotopy \( K_t \) of the identity of \( (W,M,bM,M',bM') \times \text{int } D^k \). Then \( (E,K_t j_1) \) is the desired tubular neighborhood.

2) a) There is an isotopy \( K_t \) of

\[
(M',bM')^0 \rightarrow (M',bM') \times \text{int } D^k
\]

so that

i) \( K_t /bM' \) constant isotopy

ii) \( K_t ((M',bM')^0) \subset j_1(E'(L',bL')) \)

(where \( E' \) is the associated open disk bundle of \( E \)).
b) $K_t'$ defines an isotopy of
\[ bW \xrightarrow{x_0} bW \times \text{int} \ D^k \]
\((bW = M \cup M')\) which is the constant isotopy on \( M \).
There exists an extension of \( K_t' \) to \( K_t \), an isotopy of
\[ (W, bW) \xrightarrow{x_0} (W, bW) \times \text{int} \ D^k \]
c) There exists an isotopy \( I_t \) of
\[ (W, bW) \xrightarrow{K_{t'}} (W, bW) \times \text{int} \ D^k \]
so that
1) \( I_t/bW = \) constant isotopy
2) \( I_t (W, bW) \subset j_1(E', E'/bQ) \).

Proof of 2):

a) First we produce a homotopy \( H_t' \) of
\[ (M', bM') \xrightarrow{x_0} (M, bM) \times \text{int} \ D^k \]
with the desired properties.

Now \( bM' \times 0 = bM \times 0 \) which is contained in
\( j_1(E'/bL) \subset j_1(E'/bL') \), by 1b). Therefore we may consider the problem of deforming
\[ M' \times 0 \text{ into } j_1(E'/L) \]
while keeping \( bM' \times 0 \) fixed. This problem gives rise to
a sequence of obstructions in
\[ H^i((M^1, bM^1) \times 0; \ W_i(M^1 \times \text{int } D^k, j_1(E'/L))) \].

However the coefficient groups all vanish by 1a).

Therefore \( H^i_t \) exists.

To get \( K^i_t \), we "approximate" \( H^i_t \) using general position.

Proof of b): Apply the IEP.

Proof of c): \( K^1(bW) \subset j_1(E'/bQ) \) by construction so we may proceed just as in the proof of 2a).

3) There is an isotopy \( J_t \) of the identity of \((W;M,bM,M^1,bM^1) \times D^k \) so that
   a) \( J_t/(M,bM) \times D^k = \text{constant isotopy} \)
   b) \( J_1((W;M,bM,M^1,bM^1) \times 0) \subset j_1(E'/(Q;L,bL,L',bU)) \).

Proof of 3):

Step 1) Apply the IEP to get an isotopy \( M^i_t \) of the identity of \( M^1 \times D^k \) which is fixed on the boundary and covers \( K^i_t \).

Step 2) \( M^i_t \) extends to an isotopy \( bW^i_t \) of the identity of \( b(W \times D^k) - \text{int } (M^1 \times D^k) \). \( bW^i_t \) may be extended to an isotopy \( W^i_t \) of the identity of \( W \times D^k \) using the IEP.

Step 3) Apply 2c) to \( K_1 = W^i/\{W \times 0\} \). We get an isotopy of \( W^i/\{W \times 0\} \) which carries \( W \times 0 \) into \( j_1(E') \). We apply the IEP to extend this to an isotopy \( W_t \) of
\( W \times D^{k} \) which is fixed on the boundary.

Then

\[
J_t = \begin{cases} 
  W'_{2t} & 0 \leq t \leq \frac{1}{2} \\
  W_{2t-1} \cdot W'_{1} & \frac{1}{2} \leq t \end{cases}
\]

is the desired isotopy.

Now 3) means that we can take a tubular neighborhood \((E, j_1)\) of

\[
(Q; L, bL, L', bL') \xrightarrow{G} (W; M, bM, M', bM')
\]

which restricts to a "good" tubular neighborhood of

\[
(I, bL) \xrightarrow{G} (M, bM)
\]

and change it by an isotopy (of 5-tuples) which is fixed on \( L \) so that it becomes "good" everywhere (Isotop \( j_1 \) by \( J_t^{-1} \cdot j_1 \)).

We apply the PL-stable tubular neighborhood theorem to produce a bundle map \( E_1 \xrightarrow{b} E/L \) covering the identity and an isotopy \( L_t \) between

\[
(E_1, E_1/bL) \xrightarrow{i_1} (M, bM) \times \text{int } D^{k}
\]

and

\[
(E_1, E_1/bL) \xrightarrow{b} (E/L, E/bL) \xrightarrow{j_1} (M, bM) \times \text{int } D^{k}.
\]

Now

\[
L_t \times I: E_1 \times I \rightarrow (M, bM) \times I \times \text{int } D^{k}
\]

defines a tubular neighborhood of
\[ (L_x I; L_x 0, L_x 1, b L_x (0, 1)) \xrightarrow{x} (M_x I; M_x 0, M_x 1, b M_x (0, 1)) \]

which is "good" when restricted to \( L_x 0 \cup L_x 1 \), \( (y = p_1(L_x x I)) \).
We apply 3) to this situation to isotopy \( L_t x I \) to a
PL- embedding

\[ H: E_{x I} \rightarrow (M, b M) \times I \times \text{int } D^k \]

so that

\[ H/E_{x I} x s = i_{s+1} \quad s = 0, 1 \]

and

\[ H(E_{x I} / (L_x I; L_x 0, L_x 1, b L_x I)) \subseteq (M_x I; M_x 0, M_x 1, b M_x I) \times 0. \]

Now let us suppose that over some product neighborhood \( N \) of \( (L, b L) \) in \( (Q, b Q) \) \( E \) is identified in the product bundle \( (E / (L, b L)) x I \) so that \( j_{x I} \) on this subset is just \( (j_{x I} / (E / (L, b L))) x I: E / N \rightarrow N' x \text{int } D^k : N' = (M, b M) x I \) is a collar neighborhood of \( (M, b M) \) in \( (W, b W) \). Suppose also that \( I_t \) is fixed on \( N' x \text{int } D^k \).

So define \( E \xrightarrow{j} W' \times \text{int } D^k \) by

\[ j(x) = \begin{cases} i_{-1} - j_1(x) & x \text{ in } E / Q - N \\ H(b^{-1} x I) & x \text{ in } E / N \end{cases} \]

Then

a) \( j(E' / (Q; L, b L, L', b L')) \supset (W; M, b M, M', b M') x 0 \)

b) \( j / (E / L) \cdot b = H(x, 0)(b^{-1} x 0) b = i_l \)
c) $j_{p_{L}}$ and $G_{p}$ are homotopic (as maps of 5-tuples) by a homotopy extending $bh_{L}b^{-1}$. These are clear from the construction. This completes the proof of Lemma 3'.

**Lemma 4'** Let $M$ be embedded in $bW^{n+1}$. Let $(L,g)$ be a PL-structure on $M$ which extends to a PL-structure $(Q,G)$ on $W$. Let $(E_{1},r_{1},l_{1})$ be a $k$-dimensional classifying bundle for $(L,g)$, $k>n+3$. Then $(Q,G)$ has a classifying bundle $(E,r_{j})$ so that

1) $(E,r_{j})$ restricts to a classifying bundle for $(L',G/L')$

2) $j/(E/M)\cdot b = l_{1}$, $r/LxD^{k} = r_{1}$, where $b: E_{1} \to E$ is a bundle map covering $M \subset W$.

3) There is a homotopy making

$$E/(W;M,bM,M',bM') \stackrel{j}{\longrightarrow} (Q;L,bl_{L},L',bl_{L}') \times \text{int } D^{k}$$

$$\downarrow p \quad \quad \quad \quad \quad \downarrow p_{L}$$

$$(W;M,bM,M',bM') \longrightarrow (Q;L,bl_{L},L',bl_{L}')$$

commutative which extends (using $b$) a given homotopy making

$$E_{1}/(M,bM) \stackrel{i_{1}}{\longrightarrow} (L,bl_{L}) \times \text{int } D^{k}$$

$$\downarrow q \quad \quad \quad \quad \quad \downarrow p_{L}$$

$$(M,bM) \stackrel{g_{1}}{\longrightarrow} (L,bl_{L})$$

commutative.
Proof: By Lemma H below there is homotopy inverse for G,

\[ G': (W; M, bM, M', bM') \rightarrow (Q; L, bL, L', bL') \]

which we may assume restricts to a given inverse \( g' \) for \( g = G/L \).

Now \( (E_1, i_1) \) is a "good" tubular neighborhood of \( g' \). So by Lemma 3' there is a "good" tubular neighborhood \((E, j)\) of \( G' \) and a bundle map \( E_1 \rightarrow E \) so that 3) and the first part of two are satisfied. Also \((E, j)\) restricts to a "good" tubular neighborhood of \( G'/M' \). Thus if we can find a F-trivialization

\[ r: Q \times D^k \rightarrow D^k \]

such that \( r^{-1}(0) = Q \times 0 \), \( r \) agrees with \( p_2 \) on a neighborhood of \( Q \times 0 \), \( r/L \times D^k = r_1 \), and so that \( rj \) defines an F-trivialization of \( E \), then the proof of Lemma 4' will be completed.

It follows from Definition 12 that there is a small concentric disk \( D^k_1 \) about the origin in \( D^k \) so that \( r_1/L \times D^k_1 = p_2 \), and \( r_1^{-1}(D^k_1) = L \times D^k_1 \). Suppose the radius of \( D^k_1 \) is 2a. Let

\[ r_1^*: D^k \rightarrow D^k \]

be defined by \( d \rightarrow a(|d|)d \), where
\[ a(|d|) = \begin{cases} 1 & |d| \leq a \\ \frac{1-2a}{2a} - \frac{a-1}{2a} & a \leq |d| \leq 2a \\ \frac{1}{|d|} & |d| \geq 2a \end{cases} \]

Then \( r_a : L \times D^k \to D^k \) defined by \( r_a(x,d) = r^1(d) \) is an \( F \)-trivialization of \( L \times D^k \to L \) which satisfies all the appropriate properties and is easily extendable to
\[ Q \times D^k \to D^k. \]

Define \( H : L \times D^k \times I \to D^k \) by
\[
H(x,d,t) = ((1-t)+t a(|d|))d, \quad |d| \leq 2a
\]
\[
= \left( \frac{r^1(x,d)}{|r^1(x,d)|} \right) (1-t+\frac{t}{|r^1(x,d)|}) \frac{r^1(x,(1-t+\frac{2at}{d})d)}{|r^1(x,(1-t+\frac{2at}{d})d)|}, \quad |d| \geq 2a
\]

Then if \(|d| = 2a\),
\[
H(x,d,t) = (1-t + \frac{t}{|r^1(x,d)|}) \quad r^1(x,d)
\]
\[
1-t+\frac{ta}{|d|}. \quad \text{So}
\]
\[ H(x,d,t) \] defines a homotopy (thru \( F \)-trivializations) between \( r_a \) and \( r^1 \). Furthermore,
\[
|H(x,d,t)| \geq |H(x,d,0)| = |r^1(x,d)|.
\]
Now choose a so small that \( D^k_1 \subset (D^k - r_1 i_1 \text{ (sphere bundle of } E_1)) \). Then \( H_t \cdot i_1 \) defines a homotopy between \( r_1 i_1 \) and \( r_a i_1 \). Let \( r_a \) also denote

\[
Q \times D^k \xrightarrow{p_2} D^k \xrightarrow{r_1^i} D^k.
\]

Then define

\[
r: Q \times D^k \to D^k
\]

by

\[
r = \begin{cases} 
H_t & \text{on collar neighborhood of } (L, bL) \text{ in } (Q, bQ) \\
r_a & \text{complement of collar neighborhood}
\end{cases}
\]

Then its easy to check that \( r \) has the desired properties.

**Lemma H**: Let

\[
(X; A_1, \ldots, A_n) \xrightarrow{f} (Y; B_1, \ldots, B_n)
\]

be a mapping of \((n+1)\)-tuples. Suppose that \( X \) and \( Y \) are complexes, \( A_i \) and \( B_i \) are subcomplexes, and \( X = \bigcup A_i, Y = \bigcup B_i \).
If \( S \) is a non-void subset of \( \{1, \ldots, n\} \) let \( A_S = \bigcap_{i \in S} A_i \), \( B_S = \bigcap_{i \in S} B_i \). Suppose \( f \) induces a homotopy equivalence

\[
f_S: A_S \to B_S \quad \text{for each non-void } S.
\]

Then \( f \) is a homotopy equivalence of \((n+1)\)-tuples.
Proof: Suppose first that \( n = 2 \). We want to show that

\[
    f: X \longrightarrow Y
\]

is a homotopy equivalence. By forming the mapping cylinder of \( f \) we can suppose that \( f \) is an inclusion so that

\[
    B_3 \cap X = A_3.
\]

Using the exact sequence of an appropriate triple (see Remark chapter 1) we can show that

\[
    \overline{H}_*(B_1, A_1 \cup B_1 \cap B_2) = 0 \quad i = 1, 2.
\]

Thus \( X \cup B_1 \cap B_2 \) is a strong deformation retract of \( Y \), and thus so is \( X \).

By induction we obtain that \( f: X \longrightarrow Y \) is a homotopy equivalence for any \( n \).

Now we can finish the proof by a similar argument with the mapping cylinder. (See 17)
C Is a Natural Equivalence

Let $M^n$ be properly embedded in $W^{n+k}$, $k = 0, 1$. Let $(L, g)$ be a PL-structure on $M^n$. We want to consider the problem of extending this PL-structure over $W$.

Suppose $k = 0$ and $M^n$ is embedded in the interior of $W$. Then $bM^n$ is properly embedded in $b(W - \text{int } M)$ and $(bL, g/bL)$ defines a PL-structure on $bM^n$. The problem of extending $(L, g)$ over $W$ is equivalent to the problem of extending $(bL, g/bL)$ over $W - \text{int } M$.

Suppose $k = 0$ and $bM^n \cap bW = M \neq 0$. If $N$ and $bM - \text{int } N$ belong to $\mathcal{M}'_{n-1}$, we can apply the Browder Theorem to change $g$ by homotopy so that $(L, g)$ induces a PL-structure say $(L', g')$ on $bM - \text{int } L$. The problem of extending $(L', g')$ over $W - \text{int } M$ is equivalent to the problem of extending the concordance class of $(L, g)$ over $W$.

Thus it suffices in most instances to consider the case where $M$ is embedded in $bW$.

Definition 13 (An admissable pair) Let $M$ be void or let $N$ be a PL $n$-manifold which is embedded in $bW^{n+1}$. Then $(W, M)$ is called an admissable pair if $n \geq 5$, $bW - M \neq 0$ if $n$ is odd, and $\tau_{1}(W) = \tau_{1}(bW - M) = 0$. 
Now we give an equivalent formulation of the extension problem for admissable pairs.

**Theorem 7** Let \( W^n \) be embedded in \( BW^{n+1} \). Let \( a = (L, g) \) be a PL-structure on \( W^m \) with \( k \)-dimensional classifying bundle \( v = (B_1, t_1) \). Then if \( a \) extends to a PL-structure on \( W \), \( v \otimes \mathcal{O}^r \) extends to a \( F/PL_{k+r} \)-bundle over \( W \) for some \( r \). Conversely, if \( v \otimes \mathcal{O}^r \) extends and \( (W, M) \) is admissable, then \( a = (L, g) \) extends to a PL-structure on \( W \).

**Proof:** Suppose that \( (L, g) \) extends to a PL-structure \( (Q, G) \) on \( W \). Then \( v \otimes \mathcal{O}^r \) is a classifying bundle for \( (L, g) \) and if \( r \) is large Lemma 4 applies to give the desired extension.

Now the converse. Suppose \( (B_1, t_1) \otimes \mathcal{O}^r \) extends over \( W \). Then we claim that there is an \( F/PL_s \)-bundle \( (E, T) \) over \( W \) so that

1) \( t = T/(E/M) \) is \( t \)-regular to \( 0 \) in \( D^s \)

2) \( t^{-1}(0) \xrightarrow{\text{incl}} E/M \xrightarrow{p} M \) defines a PL-structure on \( M \)

3) there is a PL-homeomorphism \( c: L \to t^{-1}(0) \)

so that \( p(\text{incl})c \) is homotopic to \( g \), (\( p \) is the projection map of \( E \)).
Proof: Since \((E_1, t_1)\) is a classifying bundle for \((L, g)\), \((E_1, t_1) = (E_1, r_1)\) where \((E_1, r_1)\) defines a "good" tubular neighborhood of \(g' : (M, bM) \to (L, bL)\). If we change notation and replace \(E_1\) by \(E_1 \otimes (\text{trivial } D^r\text{ bundle})\), \(i_1\) by \(i_1 \times \text{id } D^r\), and \(r\) by \(r \times \text{id } D^r\), then

\[(E_1, t_1) \otimes D^r = (E_1, r_i)\] where \((E_1, r_1)\) defines a "good" tubular neighborhood of \(g'\) and \(r = p_1\) in a neighborhood of \(L \times 0\).

By hypothesis there is bundle \((E, T)\) over \(W\) and a bundle map \((E_1, r_1) \xrightarrow{b} (E, T)\) covering the inclusion \(M \subset W\). (That is, \(E_1 \xrightarrow{b} E/M\) is a bundle map covering the identity and \(r_1 = T/(E/M) \cdot b\).)

Let \(t = T/(E/M)\). Now \(b\) is a PL-homeomorphism, so \(t = r_1 b^{-1}\). Thus it is clear that \(t\) is \(t\)-regular to \(0\) in \(D^S\). Since \(i_1(E_1) \supset L \times 0 \subset L \times \text{int } D^S\) we may define

\[c = b_i^{-1}(x_0)\]

Then

\[p(\text{incl.}) \cdot c = p(\text{incl.}) b_i^{-1}(x_0)\]

\[= q_i^{-1}(x_0)\] (where \(q\) is the projection map of \(E_1\))

which is homotopic to \(g\) (as a map of pairs) since \((E_1, i_1)\) is a tubular neighborhood of \(g'\).
Thus it is clear that Theorem 7 follows from Lemma 8.

**Lemma 8**  Let \((v^{n+1}, \mathbb{R}^n)\) be an admissible pair and let \((v, T)\) be an \(F/PL)\)
bundle over \(W\) with projection map \(p\), \(k > n + 3\). If \(M \neq 0\), suppose that \(t = T/(v/h)\) is \(t\)-regular to \(0\) in \(D^k\) and

\[
t^{-1}(0) \subset v/h \xrightarrow{p} M
\]
defines a PL-structure \((L, g)\) on \(M\). Then there exists

\[
H: v \times I \rightarrow D^k
\]
so that \(H(x, 0) = T(x); H(x, t)\) is an \(F\)-trivialization of \(v; h(m, t) = T(m)\) for \(m \in v/h, t \in I; H\) is \(t\)-regular to \(0\) in \(D^k\); and if \(H(x, 1) = T_1: v \rightarrow D^k\), then

\[
T_1^{-1}(0) \subset E \xrightarrow{p} W
\]
defines a PL-structure on \(W\) extending \((L, g)\).

**Proof:** Suppose that \(T\) is \(t\)-regular to \(0\) in \(D^k\) on all of \(v\). Then

\[
T^{-1}(0) = (Q!, b^Q!) \subset v/(W, bW) \xrightarrow{p} (W, bW)
\]
is a first approximation to a PL-structure on \(W\) which extends the PL-structure on \(M\) defined by \((L, g)\).

By this we mean that \(L\) is embedded in \(b^Q!\) and \(G^I\) induces \(g, g^I, bG^I\) so that
is commutative. However we do not know that $g'$, $bG'$, and $G'$ are homotopy equivalences.

First we examine the algebra of the situation. Let $L' = bQ'\text{-int } L$ and $M' = bw\text{-int } M$. Then we have

$$\begin{array}{ccc}
L' & \xrightarrow{g'} & M' \\
\wedge & bG' & \wedge \\
bQ' & \xrightarrow{G'} & bw \\
U & G' & U \\
Q' & \rightarrow & W
\end{array}$$

It follows from Lefschetz Duality (LD) in $v/(w, bw)$ that there exists $/\beta_\ast$ in $H_{n+1}(Q', bQ')$ so that $G_\ast /\beta_\ast = /\mu_{W'}$, a generator of $H_{n+1}(W, bw)$. Thus we have the diagram

$$\begin{array}{cccccccc}
& O & \xrightarrow{k} & O & \xrightarrow{j} & O & \xrightarrow{b} & \\
\rightarrow & A_\ast & \xrightarrow{i} & B_\ast & \xrightarrow{i} & C_\ast & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
(D) & H_\ast(L') & \xrightarrow{\alpha} & H_\ast(Q') & \xrightarrow{\beta} & H_\ast(Q', L') & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
& H_\ast(M') & \xrightarrow{\gamma} & H_\ast(W, M') & \rightarrow & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& O & \xrightarrow{0} & O & \xrightarrow{0} & O & \rightarrow
\end{array}$$
where \( A_* = (\ker g')_* \), \( B_* = (\ker G')_* \), \( C_* = (\ker G_{o'})_* \) and \( i \) denotes the various inclusions. \( \alpha, \beta, \) and \( \gamma \) are defined by

\[
\begin{array}{c}
\downarrow \gamma_{M'}^{L'} \\
\alpha: H_*(M') \leftarrow_{\cong} H^*(M', b'M') \rightarrow H^*(L', bL') \rightarrow H_*(L') \\
\downarrow \gamma_{W}' \\
\beta: H_*(W) \leftarrow_{\cong} H^*(W, bW') \rightarrow H^*(Q', bQ') \rightarrow H_*(Q') \\
\downarrow \gamma_{W,M'} \\
\gamma: H_*(W, M') \leftarrow_{\cong} H^*(W, M) \rightarrow H^*(Q', L) \rightarrow H_*(Q', L') \\
\end{array}
\]

where the vertical maps come from the exact sequence for a pair or a triple. (e.g. \( H^*(M', bM') \cong H^*(bW, M') \).

From all this and the fact that

\[
g: (L, bL) \rightarrow (M, bM)
\]

is a homotopy equivalence we derive the following

**Lemma A**

1) 

\[
\ldots \rightarrow A_k \rightarrow B_k \rightarrow C_k \rightarrow A_{k-1} \rightarrow \ldots
\]

is exact.

2) 

\[
\begin{align*}
H_*(L') &= A_k \oplus H_*(M') , & g_{\alpha}^I \beta &= 1 \\
H_*(Q') &= B_k \oplus H_*(W) , & G_{\beta}^I \gamma &= 1 \\
H_*(Q', L') &= C_k \oplus H_*(W, M') , & G_{\gamma}^I \gamma &= 1
\end{align*}
\]
3) $A_\ast$ satisfies Poincaré Duality wrt dimension $n$ (i.e. $A_k = A^{n-k} = \text{Hom}(A_{n-k},Z) \oplus \text{Ext}(A_{n-k-1},Z)$)

$B_\ast$ and $C_\ast$ satisfy Lefschetz Duality wrt dimension $n+1$ (i.e. $B_{k+1} = C^{n-k}$, $C^{k+1} = C_{n-k}$)

**Proof of Lemma A:** First we show that $g_\ast |_1 = 1$, $G_\ast |_1 = 1$, and $G_\ast |_n = 1$.

Let $x$ be in $H_k(M')$, then

$x = \mu_{M'} \cap u$ for some $u$ in $H^{n-k}(M', bm')$.

But

$g_\ast |_1 (\mu_{M'} \cap (g_\ast |_1 u)) = g_\ast |_1 \mu_{M'} \cap u$

$= \mu_{M'} \cap u$

$= x$.

Therefore $g_\ast |_1 = 1$. Similarly $G_\ast |_1 = 1$ and $G_\ast |_n = 1$.

This proves 2) and shows that $g_\ast |_1$, $G_\ast |_1$, and $G_\ast |_n$ are onto as indicated in (D).

Now 1) follows from 2) and the commutativity of (D) and (5) by a diagram chase.

To prove 3) we have that

$H_1(L') = A_1 \oplus H_1(M')$.

We also have
The vertical isomorphisms come from the fact that
\[ g/\text{bl}: \text{bl} \rightarrow bM \]
is a homotopy equivalence, \( bM' = bM' \) and \( \text{bl} = \text{bl}' \). \( \alpha_0 \) is defined (like \( \alpha \) was) so that \( g_0^1 \alpha_0^1 = 1 \). Thus \( \ker g_0^1 \subset \ker g_0^1 \), and \( \ker g_0^1 \subset \) is a direct summand of \( H_i(L', \text{bl}' \rangle \). Therefore \( H_i(L', \text{bl}') = A_i \oplus H_i(M', bM') \).

We use this to show that
\[
A_i \oplus H_i(M') = H_i(L') \\
= H^{n-1}(L', \text{bl}') \\
= \text{Hom}(H_{n-1}(L', \text{bl}'), Z) \oplus \text{Ext}(H_{n-1}(L', \text{bl}'), Z) \\
= \text{Hom}(A_{n-1} \oplus H_{n-1}(M', bM')) \\
\quad \oplus \text{Ext}(A_{n-1} \oplus H_{n-1}(M', bM'), Z) \\
= \text{Hom}(A_{n-1}, Z) \oplus \text{Ext}(A_{n-1}, Z) \oplus H^{n-1}(M', bM') \\
= A^{n-1} \oplus H_i(M')
\]

Since everything is finitely generated this means that
\[ A_i = A^{n-1} \].
To prove the second statement of 3) we use

\[ \cdots \rightarrow H_k(bQ', L') \rightarrow H_k(Q', L') \rightarrow H_k(Q', bQ') \rightarrow H_k(bW, L') \rightarrow H_k(W, L') \rightarrow H_k(W, bW) \rightarrow \cdots \]

define \( \beta_0 \) so that \( G_\star \beta_0 = 1 \) and conclude that

\[ H_k(Q', bQ') = H_k(W, bW) \oplus C_k. \] (By excision \( G_\star \) is an isomorphism since \( g_\star \) is.)

Therefore \( B_k \circ H_k(W) = H_k(Q') = H^{n+1-k}(Q', bQ') \)

\[ = \text{Hom}(H_{n+1-k}(Q', bQ'), Z) \ominus \text{Ext}(H_{n-k}(Q'; bQ'), Z) \]

\[ = \text{Hom}(C_{n-k+1} \oplus H^{n+1-k}(W, bW), Z) \]

\[ = \text{Ext}(C_{n-k} \oplus H^{n-k}(W, bW), Z) \]

\[ = C^{n-k+1} \oplus H^{n+1-k}(W, bW) \]

\[ = C^{n-k+1} \oplus H_k(W). \]

This completes the proof of Lemma 1.

Lemma 2: Suppose that \( S^r \subset \text{int } L' \) \( (S^r \subset \text{int } Q') \) represents an element of \( \mathcal{X} \) \( (\mathcal{Y}) \). Then a neighborhood of \( S^r \subset \text{int } L' \) \( (S^r \subset \text{int } Q') \) is smoohtable.
Proof: Consider the case $S^r \subset \text{int } L'$. $S^r \subset v/\text{int } M'$ is contractible. So by general position we may suppose that a neighborhood of $S^r \subset v/\text{int } M'$ is contained in a top dimensional cell of $v/\text{int } M'$. Then

$$t_{L'_1}/S^r \oplus v_{L'_1}/S^r = \text{normal microbundle} \oplus t_{S^r}$$

of $S^r \subset \text{cell}$

where $t_{L'_1} = \text{tangent microbundle of } L'$

$v_{L'_1} = \text{normal microbundle of } L' \text{ in } v/\text{int } M'$.

By the transverse regularity of $T$, $v_{L'_1}$ is trivial. Thus $t_{L'_1}/S^r$ is stably trivial. Similarly $t_{Q'_1}/S^r$ is stably trivial.

Therefore we may smooth a neighborhood of $S^r \subset \text{int } L'$ or $S^r \subset \text{int } Q'$. Q.E.D.

Suppose $(Q,bQ)$ is a PL-framed submanifold of $v/(W,bW)$ which contains $L'$ in its boundary. We say that $R \subset v \times I$ is an elementary framed cobordism on the interior of $(Q,L')$, where $L' = bQ \cap \text{int } L$, if

1) there is a closed $(n+k)$-cell $U$ in the interior of $v$ so that

a) $(U,bU) \cap Q = (S^r \times D^s \cup (S^r \times D^s))$

where $r + s = n + 1$

b) $(U' \times I) \cap R = (U' \times I) \cap Q \times I$

where $U' = v \cap \text{int } U$

c) $(U \times I, b(U \times I)) \cap R = (D^{n+2}, bD^{n+2})$,

where $D^{n+2}$ is the $(n+2)$-disk.
2) there is an extension of \(((Q \cap U^1) x I) \cup (\text{framing of } Q x 0)\) to a framing of \((R,bR)\) in \(v/(W,bW)\).

We say that \(R \cap v x I\) is an elementary framed cobordism on the boundary of \((Q^1,L^1)\) if there is a closed \(n+k\)-cell \(U\) in \(v\) so that

i) \(U \cap v/bQ = U \cap v/\text{int} M = U^1\), an \((n+k-1)\)
cell in \(L^1\).

ii) a), b), c) and 2) hold for \(U\).

iii) \((U_1,bU_1) \cap bQ = (S^r x D^{s-1}, b(S^r x D^{s-1}))\) and \((U_1 x I, b(U_1 x I)) \cap bR = (D^{r+1}, bD^{n+1})\).

Let \(Q_1 = R \cap v x I\). Then it is clear that

\[(Q_1, bQ_1) \subset v/(W,bW) \xrightarrow{p} (W,bW)\]

has degree one, and

\[(L_1, bL_1') \subset v/(M', bM') \xrightarrow{p} (M', bM')\]

has degree one. Thus Lemma A applies to \(Q_1 \subset v x I\).

An e.f. (elementary framed) cobordism on the interior or the boundary of \((Q,L^1)\) can arise by doing framed surgery on the interior of \(Q\) or by doing framed surgery on \(bQ\) and then thickening.
**Lemma T:** Let \( T: v \to D^k \) be an \( F \)-trivialization of \( v \) such that \( T \) is \( t \)-regular to 0 in \( D^k \) and \( T^{-1}(0) = Q \).

Suppose \((Q, bQ)\) is framed in \( v/(W, bW)\) by \( T \). Let \( R \subset v \times I \) be an elementary framed cobordism on the interior or on the boundary of \((Q, L^i)\). Then there is an \( F \)-trivialization of \( v \times I \)

\[
H: v \times I \to D^k
\]

so that \( H \) is \( t \)-regular to 0 in \( D^k \), \( H^{-1}(0) = R \), and \( H = T \times I \) on \( L \times I \).

**Proof:** Consider the case when \( R \) is an e.f. cobordism on the interior of \((Q, L^i)\). We define

\[
H: v \times I \to D^k
\]

as follows

1) define \( H \) on \( U \times I \) by \( T \times I \)

2) map \( D^{n+2} = R \cap (U \times I) \) to 0 and map a product neighborhood \( N \) of \( D^{n+2} \) using the framing of \( R \).

3) if \( N = D^{n+2} \times D^k \), we may assume that

\((U \times 0 \cup bU \times I)-(D^{n+2} \times \text{int} D^k)\) is mapped to \( bD^k \).

Extend \( H \) over \( U \times I \) using obstruction theory to map \( U \times I-(D^{n+2} \times \text{int} D^k) \) to \( bD^k \). (See Lemma 1.5 in the appendix to section I for details.)

Thus \( H \) can be constructed to satisfy the geometric conditions. That \( H \) is an \( F \)-trivialization follows
from these conditions and the nature of $R$ using Lefshetz Duality as in Lemma 4.

To prove Lemma 1 when $R$ is an e.f. cobordism on the boundary of $(Q, L')$ we apply the above process to define $H$ on $v/bW$ and then we are essentially reduced to the interior case.

Let $(P, bP) \in (D^{n+1+k}, bd^{n+1+k})$ represent an element of $P_{n+1}$, $n$ odd. Let $Q_{\partial}^P$ denote the oriented connected sum of $Q$ and $P$ along their boundaries.

1) Assume that $D^n_l$ in $bP = b(D^{n+1} \times D^k)$ coincides with $D^n_l = D^n_x0 \cup bD^n_xI$ where $D^{n+1}_l = D^n_xI$, and that the framing of $D^n_l$ is standard.

2) Assume that $W$ and $Q$ coincide on some collar neighborhood $D^n_xI$ of $D^n_xI$ in $\text{int} L'$, that $v/D^{n+1}$ is identified with $D^{n+1} \times D^k = D^{n+1+k}$, and that the framing of $Q \cap D^{n+1+k}$ is the standard framing of $D^{n+1}$ in $D^{n+1+k}$.

If we let $Q' = (Q - \text{int} D^{n+1}) \cup P$, then $Q' = Q_{\partial}^P$ and $(Q', bQ')$ is naturally framed in $v/(W, bW)$. 
Lemma P: Let \((Q, F)\) be a framed submanifold \(v/(W, bw)\) and \((P, G)\) a framed submanifold of \((D^{n+1+k}, bD^{n+1+k})\) representing an element of \(P_{n+1}^d\), \(n \) odd. Let \(P_{n+1}^d\) denote the natural framing of \(Q_{n+1}^d\) in \(v\), where \(Q\) and \(P\) are connected in the interior of \(I^1\). Then \((Q, P)\) and \((Q_{n+1}^d, P_{n+1}^d)\) are related by a sequence of e.f. cobordisms in \(v \times I\).

Proof: Assume \(Q\) has the form of 2) above. Define a cobordism \(R\) between \(Q\) and \(Q_{n+1}^d\) in \(v \times I\) by

\[
R = \begin{cases} 
Q \times I & \text{in } (v-\text{int}D^{n+1+k}) \times I \\
C & \text{in } D^{n+1+k} \times I 
\end{cases}
\]

where \(C\) is a submanifold of \(D^{n+1+k} \times I\) which is constructed as follows:

a) Write \(D^{n+1+k} = D^n \times D^k\) and assume \(P\) is embedded in \(D^{n+1+k} \times I\) as above.

b) Then \(-P\) is naturally embedded (by interchanging coordinates) in \(D^n \times (D^n \times I)\) so that \(P \cap (-P) = bP \cap b(-P) = bP - \text{int } D^n_1\).

c) \(P \cup (-P) \subset (D^n \times D^k \times I) \cup D^n \times (D^n \times I) \subset b(D^{n+1} \times I)\) is just \(P_{n+1}^d\) \((-P)\) which is framed cobordant to the \((n+1)\)-disk in \(D^{n+1} \times I\). Let \(C\) be such a framed cobordism so that \(bC = C \cap b(D^{n+1} \times I) = (P_{n+1}^d \cap (-P)) \cup (D^n_1 \times 0) \times I \cup (D^{n+1}_1)\).
Assume that the framing of \( C \) is standard on \( D^{n+1} \cup D_1^n \times 0 \).

d) Then the product framing of \( Q \times I \cap (v \times \text{int}D^{n+1}) \times I \) extends using the framing of \( C \) to a framing \( H \) of \( R \) in \( v \times I \). Furthermore \( (R, H) \cap (v \times 0) = (Q, F) \) and \( (R, H) \cap (v \times I) = (Q, P, H) \). 

e) We factor \( R \) into e.f. cobordism on the boundary and interior using general position and a Morse function \( f \) on \( C \) such that

\[
 f(x) = \begin{cases} 
 0 & x \text{ in } D^{n+1} \\
 y & x = (d, y) \text{ in } D_1^n \times I \\
 1 & x \text{ in } P 
\end{cases}
\]

and \( f \) has distinct critical values. The critical points of \( f \) on the interior of \( C \) yield interior e.f. cobordisms, and those in \((-P)\) yield e.f. cobordisms on the boundary.

Now we use Lemmas A, S, and P to construct a sequence \( (R_i, F_i), i=0, \ldots, r \), of e.f. cobordisms in \( v \times [i, i+1] \) so that \( (R_i, F_i) \cap v \times (i+1) = (R_{i+1}, F_{i+1}) \cap v \times (i+1) \), \( (R_0, F_0) = (Q', F') \times I \) and if \( Q_r = R_r \cap v \times (r+1) \) and \( H_r = bQ_r - \text{int}L \times (r+1) \),
\[(Q_r, L_r) \subset v/(W, M') \longrightarrow (W, M')\]

is a homotopy equivalence. This construction (with trivial modifications) is done by Browder and Hirsch (1) in their proof of Wall's Theorem (26) in the PL-case. Lemmas A, S, and P provide the main ingredients of the construction.

Then we use Lemma T \( r \)-times to build a homotopy of \( T \) with the required properties. If \( T_r \) denotes the end of the homotopy then

\[(Q_r; L'_r, bL'_r, Lx(r+1)) \subset v/(W; M', bM', M) \overset{p}{\longrightarrow} (W; M', bM', M)\]

induces a homotopy equivalence on each factor. It follows from Lemma \( H \) of Chapter 2 that

\[bQ_r \subset v/bW \overset{p}{\longrightarrow} bW\]

is a homotopy equivalence. Thus

\[Q_r = T_r^{-1}(0) \subset v \times (r+1) \overset{p}{\longrightarrow} W\]

defines a PL-structure on \( W \) extending \( (L, g) \). This completes the proof of Lemma 8.
Let $M$ be properly embedded in $W$. We want to
c onsider the different possible extensions of a given
$PL$-structure on $M$ to $PL$-structures on $W$. We will classify
the extensions for admissible pairs such that $W-M \neq 0$.

Definition 14. Let $M$ be properly embedded in $W$
and let $a = (b, g)$ be a $PL$-structure on $M$. Let $b_i = (Q_i, G_i)$
define extensions of $a$ to $PL$-structures on $W$, $i = 0, 1$.
(So there are embeddings $e_i: L \subseteq Q_i$ so that $G_i e_i = g$.)
Let $b_0 \cup axI \cup b_1$ denote the $PL$-structure $^1$ on
$Wx0 \cup WxI \cup Wx1 \subseteq WxI$ defined by

\[(Q_0 \cup e_0 IXI \cup e_1 0 \cup G_0 \cup e_0 GXI \cup e_1 G_1)\]

We say that $b_0$ and $b_1$ are concordant relative to $M$ if
$b_0 \cup axI \cup b_1$ extends to a $PL$-structure on all of $WxI$.

This defines an equivalence relation $^1$ on the set
of extensions of $a$. Let $PL(W, M; a)$ denote the set of
equivalence classes.

Definition 15. Let $(X, A)$ belong to $C$, and let
$v = (E, t)$ be an $F/PL)_k$ bundle over $A$. Let $v_i = (E_i, T_i)$
define extensions of $(E, t)$ to $(F/PL)_k$ bundles over $X$,
(i.e. there are $PL$-bundle maps $b_i: E \to E_i$ covering the

---

$^1$ See Lemma H of previous chapter.
inclusion \( A \subset X \) so that \( t = T_i b_i \), \( i = 0, 1 \). Then

\[
v_0 \cup v \times I \cup v_1 = (E_0 \cup b_0 \times I \cup b_1 \times I, T_0 \cup b_0 \times I \cup b_1 \times I)
\]
defines on \( F/PL_k \) bundle over \( X \times 0 \cup X \times I \cup X \times 1 \subset X \times I \). We say that \((E_0, T_0)\) and \((E_1, T_1)\) are equivalent relative to \( M \) if \( v_0 \cup v \times I \cup v_1 \) extends to an \( F/PL_k \)-bundle over \( X \times I \).

This defines an equivalence relation on the set of extensions of \( v \) over \( X \). Let \( B(X; A, v) \) denote the set of equivalence classes.

We remark that there are maps

\[
B(X; A, v) \xrightarrow{P} B(X) \quad P
\]

\[
P; (W; M, a) \xrightarrow{P} \text{PL}(W)
\]
defined on representatives by considering an extension of \( v \) (or \( a \) as a bundle over \( X \) (or a PL-structure on \( W \)).

If \( F: (X, A) \to (Y, B) \) is a PL map, then the induced bundle operation induces \( F^*: B(Y, B; v) \to B(X, A; f^*v) \) where \( f = F/A \), and

\[
\begin{array}{ccc}
B(Y, B; v) & \xrightarrow{F^*} & B(X, A; f^*v) \\
\downarrow P & & \downarrow P \\
B(Y) & \xrightarrow{f^*} & B(X)
\end{array}
\]
is commutative.

If \( I: (Q, L) \subset (W, M) \) is an embedding so that the Browder theorem applies to \( L \subset M \) and \( Q \subset W \) in a relative manner then \( I^*: \text{PL}(W, M; a) \to \text{PL}(Q, L; f^*a) \) may be
defined, where $i = I/L$ and $i^\#a$ is a specific restricted PL-structure on $M$. An analogous diagram results.

**Theorem 2.** Let $(W^m, M^n)$ be a PL-manifold pair.

Let $a = (L, g)$ be a PL-structure on $M$ with $k$-dimensional classifying bundle $v = (E, t), k > n+3$. Then there exists a correspondence

$$
\begin{array}{ccc}
\text{PL}(W, M; a) & \xrightarrow{C_a} & \text{B}_k(W, M; v) \\
\text{PL}(W) & \xrightarrow{C} & \text{B}(W)
\end{array}
$$

so that

$$
\begin{array}{ccc}
\text{PL}(M^1) & \xrightarrow{C} & \text{B}_k(M^1) \\
\downarrow{p} & & \uparrow{p} \\
\text{PL}(w, M; a) & \xrightarrow{C_a} & \text{B}(W, M; v) \\
\downarrow{p} & & \downarrow{p} \\
\text{PL}(W) & \xrightarrow{C} & \text{B}(W)
\end{array}
$$

is commutative, and

- $C_a$ is injective if $(W^*, M^*)$ is admissible
- $C_a$ is onto if $(W, M)$ is admissible

$((W^*, M^*) = (W \times I, W \times O U M \times I U W \times I)).$

**Proof:** Let $b = (Q, G)$ define an extension of $a = (L, g)$ over $W$. Let $(E, ri)$ be the classifying bundle for $(L, g)$, $g$ a homotopy inverse for $g$, and $H$ a homotopy between $g^p$ and $p_1i$.

Then there exists a classifying bundle $v_b$ for $(Q, G)$ which extends $(E, ri)$ in the sense of 2) and 3) of Lemma 4.
and which has an associated homotopy which extends h as in 3) of Lemma 4'. Define $C_a = C_a(L, g, E, r, i, g', h)$ on representative by

$$b \mapsto v_b$$

Note then that the desired commutativity holds if $C_a$ is well-defined.

**Claim:** $C_a$ is well-defined and injective.

**Proof:** If $b_o$ and $b_1$ denote two extensions of a over $M$ with classifying bundles $v_o$ and $v_1$ extending $v$, then $v_o \cup v x I \cup v_1$ is a classifying bundle for $b_o \cup a x I \cup b_1$.

To see this, write $b_s = (Q_s, G_s)$, $v_s = (E_s, r_s g_s)$, $v = (E, r_i)$, $s = 0, 1$. Let $j_s$ denote the embedding of $L$ in $Q_s$ and $b_s$ the embedding (as a subbundle) of $v$ in $v_s$, $s = 0, 1$. Let $g'$ and $G'_s$ denote homotopy inverses for $g$ and $G_s$, $s = 0, 1$. By Lemma H we may choose $G'_s$ so that $(G'_s / j_s L) \cdot j_s = g'$.

Then $i^* = i_0 \cup b_0 i x I \cup b_1 i$ defines an embedding of $E^* = E_0 \cup b_0 E x I \cup b_1 E$ in $q^* x \text{int} \ D^k = (Q_0 \cup j_0 L x I \cup j_1 L \cup Q_1) x \text{int} \ D^k$.

$E^*$ is a bundle over $M^*$ with projection $p^* = p_0 \cup p x I \cup p_1$.

If we let $G^* = G'_0 \cup G'_1 x I \cup G'_1$, and apply Lemma 4' 3)
is homotopy commutative. (Let \( h \) be a fixed homotopy between \( p_1 i \) and \( g' p \), \( H_S \) the extended homotopies between \( p_1 j_S \) and \( G' p_S \). Then \( E^* = H_o \cup h \times I \cup H_1 \) is a homotopy between \( p_1 i^* \) and \( G^* i^* \)).

Also \( i^*(E^*,i^*;b^*) \supset (Q^*,bQ^*) \times 0 \), so \( (E^*,i^*) \) defines a "good" tubular neighborhood of \( G^* \).

Thus if \( r^* = r_o \cup r \times I \cup r_1 \), then \( v_o \cup v \times I \cup v_1 \) \((E^*,r^*;i^*)\) a classifying bundle for \( b_o \cup a \times I \cup b_1 \).

Thus by Theorem 7 \( C_a \) is well-defined for all pairs \((W,M)\), and it is injective when \((W^*,M^*)\) is admissable.

**Claim:** \( C_a \) is onto.

**Proof:** Let \( v = (E_o, T) \) represent \( x \) in \( B(W,M;v) \). We will use Lemma 8 to construct a PL-structure \( b = (Q, \sigma) \) on \( W \) extending \( a = (L, \sigma) \) where \( Q \) is embedded in \( E_o \). Then we carefully deform \( E_o \) over itself until it has the form of some \( v_b \). The process is complicated by the subspace \( M \).
Let \( b : E \to E_0 \) be a bundle map covering the inclusion \( M \subseteq W \) so that 
\[ \pi(T/(E/M)) b = t. \]

If \( t = r_i \) we make the following assumptions about 
\( (L, g, E, r, i, g', h) \)

A) \( g : L \to M \) is given by the composition
\[
L \xrightarrow{x_0} L \times \text{int } D^k \xleftarrow{i} E \xrightarrow{p} M.
\]

B) there is an embedding
\[
L \times D^k \xrightarrow{e} E
\]
so that
i) \( e_i \) is the identity on a neighborhood of the 0-section of \( E \)

ii) \( r_i e = p_2 \) on some neighborhood of \( L \times 0 \) in \( L \times D^k \).

C) \( g' : M \to L \) is given by the composition
\[
M \xrightarrow{\text{0-section}} E \xrightarrow{i} L \times \text{int } D^k \xrightarrow{p_1} L
\]

D) \( h \) is given by the composition
\[
E \xrightarrow{H} E \xrightarrow{i} L \times \text{int } D^k \xrightarrow{p_1} L,
\]
where \( H \) is a deformation retraction of \( E \) onto its zero section.

The proof that \( C_a \) is onto proceeds in steps.
1) There is a PL (n+1)-manifold $Q$, an $F$-trivialization $T_o : E_o \to D^k$ of $E_o$; an embedding

$$
\begin{array}{c}
\text{L} \\
\text{Q} \\
\text{E}_o
\end{array}
\xrightarrow{k}
\begin{array}{c}
\text{D}^k \\
\text{bQ} \\
\text{E}_o
\end{array},
$$

and an embedding

$$
Q \times D^k \xrightarrow{e_o} E_o
$$

so that

i) $e_o(Q \times 0) = T_o^{-1}(0)$

ii) the composition

$$
\begin{array}{c}
Q \\
Q \times D^k \\
E_o \\
W
\end{array}
\xrightarrow{x_0}
\xrightarrow{e_o}
\xrightarrow{p_0}

defines a PL-structure on $W$ extending

$$(kL, g_k^{-1}).$$

iii) $T_oe_o = p_2$ and $be = e_o(k \times \text{id.})^1$

iv) $(E_o, T_o)$ is equivalent in $B(W, M; v)$ to $(E_o, T)$.

**Proof:** We apply Lemma 8 and change $T$ to $T_o$ by a homotopy which is fixed on $E_o/M$ so that $T_o$ is $t$-regular to 0 in $D^k$ and

$$
\begin{array}{c}
\text{Q} \\
\text{E}_o \\
\text{W}
\end{array}
\xrightarrow{p_o}
\xrightarrow{T_o^{-1}(0)}
\xrightarrow{G}

defines a PL-structure on $W$ which extends the PL-structure defined by

(1) on a neighborhood of $Q \times 0$. 

\[(tb^{-1})^{-1}(0) \subset E_o/M \rightarrow M.\]

Let \(k: L \subset bQ\) be defined by the composition

\[
L \xrightarrow{x0} L \times \text{int}\ D^k \xrightarrow{i} E \xrightarrow{b^{-1}} (E_o/M \cap bQ)
\]

Then from A) we get that

\[G/kL = gk^{-1}.\]

Now iv) is clear from our construction, so only iii) is left to prove.

Let \(e_o\) be defined on \(kL \times D^k\) by \(be(k^{-1}x \text{id})\). Then on \(kL \times D^k\) we have that

\[T_o e_o = T_o be(k^{-1}x \text{id})\]

\[= rie(k^{-1}x \text{id})\]

\[= p_2(k^{-1}x \text{id}), \text{by B)}\]

\[= p_2, \text{on some neighborhood of } L \times 0 \text{ in } L \times D^k.\]

Because of the \(t\)-regularity of \(T_o\) we can extend the embedding of \(Q \subset E_o\) to an embedding

\[Q \times D^k \xrightarrow{e_o^i} E_o\]

so that \(T_o e'_o = p_2\). Thus our partial \(e_o\) and \(e'_o/kL \times D^k\) represents \(kL \times D^k\) as tubular neighborhoods of \(e'_o kL\).
The PL-tubular neighborhood theorem and the equation

\[ T_0(\text{partial } e_0) = T_0(e_0'/kL \times D^k) = p_2 \]

imply they are related by an ambient isotopy of the identity of \( E_0/(M,bM) \) which is fixed on \( e_0'kL \).

We can use this isotopy to alter \( T_0 \) and \( e_0' \) on a collar neighborhood of \( E_0/(M,bM) \) in \( E_0/(W,bW) \) and \( (kL,bkL) \times D^k \) in \( (Q,bQ) \times D^k \) so that iii) is satisfied.

2) Let \( E_0^e \) denote the "enlarged" bundle whose space is \( E_0 \cup (\text{sphere bundle}) \times I \) and whose maps (the projection and the \( F \)-trivialization) are induced by

\[
\begin{align*}
\text{id} \cup p_1 = r \\
E_0^e & \longrightarrow E_0
\end{align*}
\]

Then there exists an isotopy \( M_t \) of the identity of \( E_0^e/(M,bM) \) so that

i) \( M_t/(E_0/(M,bM)) \) is the composition \( s \),

\[
E_0/(M,bM) \rightarrow E/(M,bM) \rightarrow (L,bL) \times D^k \rightarrow E_0/(M,bM) \subseteq E_0^e/(M,bM)
\]

ii) \( M_t(E_0^e/(M,bM)) \supseteq e_0((L,bL)x0), \ t \ in \ I \).

**Proof:** \( s \) and the inclusion \( E_0/M \subseteq E_0^e/M \) each represent \( E_0/(M,bM) \) as a tubular neighborhood of the zero section of \( E_0^e/(M,bM) \). Since \( s \) may be written as

(inclusion) \( b \circ i \circ b^{-1} \),

these embeddings agree on a neighborhood of the zero-section of \( E_0^e/(M,bM) \). Thus it follows from the PL-tubular
neighborhood theorem that there is an ambient isotopy $M_t'$ of the identity of $E_0^e/(\bar{H}, bH)$ relating them. We may alter $M_t'$ so that ii) also holds using the technique of 3) in the proof of Lemma 3'.

3) There is an isotopy $W_t$ of the identity of $E_0^e/(W; M', bM', M', bM')$ so that
   i) $W_t/(E_0^e/(H, bH)) = M_t'$
   ii) $W_t(E_0^e/(W; M, bM, M', bM')) \subset e_0((Q; L, bL, L', bL') \times \text{int } D^k)$
   iii) $e_0((Q; L, bL, L', bL') \times \{0\}) \subset W_t(E_0^e/(W; M, bM, M', bM'))$
   for each $t$ in $I$

Proof: First we construct $W_t$ so that i) and ii) are satisfied. This construction proceeds in steps —

a) Extend $M_t$ to an isotopy $W_t'$ of the identity of $E_0^e/(W; M, bM, M', bM')$.

b) Deform (first by a homotopy, then by an isotopy) $W_t'(W; M, bM, M', bM')$ over $E_0^e/(W; M, bM, M', bM')$ so that it is contained in $e_0((Q; L, bL, L', bL') \times \text{int } D^k)$. Keep $(M, bM)$ fixed.

c) Cover the isotopy by an ambient isotopy to get $W_t$.

We get $W_t$ to satisfy iii) by doing a), b), and c) again in $E_0 \times I$. See proof of Lemma 3', 2) and 3'), for details.
4) Let \( E_1 = W_1(E_0) \). Let \( q_1 : E_1 \to W \) and \( j : E_1 \to Q \times D^k \) be defined by \( p_0 W_1^{-1} \) and \( W_1(E_0) \subset E_0 \subset Q \times D^k \) respectively. Let \( b_1 = M_1 \cdot b \). Then there exists \( r_1 : Q \times D^k \to D^k \) so that

i) \( r_1 = p_2 \) on a neighborhood of \( Q \times 0 \), \( r_1^{-1}(0) \),

ii) \( E_1 \xrightarrow{q_1} W \) is a PL-bundle and \( E_1 \xrightarrow{r_1 j} D^k \) is an \( F \)-trivialization,

iii) \( b_1 \) is a bundle map covering \( k \), in fact \( r_1(k \times \text{id}) = r \) and \( j b_1 = (k \times \text{id}) i \),

iv) \((E_1, r_1 j)\) represents \( C_a(Q, G) \).

**Proof:** Let \( 0 < D^k \subset D^k \) be such that

\[ 2D^k \subset D^k - T_0(j_1(\text{sphere bundle of } E_1)) \]

and \( r | L \times D^k_1 \) is the identity. Let \( r_1' = r' p_2 \) where \( r' : (D^k, D^k - 2D^k) \to (D^k, bD^k) \) be a homotopy equivalence which is the identity on \( D^k_1 \). Then \( r_1' j \) is an \( F \)-trivialization of \( E_1 \). We can now alter \( r_1' \) on a collar neighborhood of \( k(L, bL) \) in \((Q, bQ)\) so that \( r_1(k \times \text{id}) = r \). Thus i), ii), and iii) are satisfied by \( r_1 \).

It is clear that \( E_1 \xrightarrow{q_1} W \) is a PL-\( k \)-disk bundle.
Now
\[ j_{b_1} = e_o^{-1}(\text{inclusion}) \quad b_1 \quad b \]
\[ = e_o^{-1}(\text{inclusion}) \quad e_o(k \times \text{id})i, \quad \text{by 2} \]
\[ = (k \times \text{id})i. \]

So only iv) needs to be proved.

Recall that \( b_1 E \downarrow \) is a deformation retraction of \( b_1 E \) onto its zero section. The obstructions to extending \( b_1 E \downarrow \) to a deformation retraction of \( E_1 \) onto its zero section vanish since \( \prod (E_1, bE \cup \emptyset \text{ section } E_1) = 0 \).

So let \( R_1 \) be a deformation retraction extending \( b_1 E \downarrow \).

Then if we let \( G' = p_1 j(0\text{-section}), \) \( p_1 j R_1 \) yields a homotopy between \( G' q_1 \) and \( p_1 j \). Thus \((E_1, j)\) defines a "good" tubular neighborhood of

\[ G' : (W; M, bM, M', bM') \rightarrow (Q; kL, kbL, L', bL') \]

which extends the "good" tubular neighborhood

\( (b_1 E, (k \times \text{id})i \quad b^{-1}_1) \), of \( kg' : (M, bM) \rightarrow (kL, kbL) \). in fact,

\[ G' / (M, bM) = p_1 j(0\text{-section}) / (M, bM) \]
\[ = p_1 j b_1 (0\text{-section}) \]
\[ = p_1 (k \times \text{id})i \quad (0\text{-section}) \]
\[ = kp_1 i \quad (0\text{-section}) \]
\[ = kg', \]
\[ \frac{j}{b}E = (k \times \text{id})i \cdot b^{-1} \]

and

\[ p_{\perp} \cdot j \cdot h_{\perp} / b_{\perp}E = p_{\perp} \cdot j \cdot b_{\perp} \cdot h_{\perp} \cdot b_{\perp}^{-1} \]

\[ = k(p_{\perp} \cdot i \cdot h) \cdot b_{\perp}^{-1} \]

which is the prescribed homotopy between \( k(p_{\perp} \cdot i) \cdot b_{\perp}^{-1} \)
and \( k(g' \cdot p) \cdot b_{\perp}^{-1} \).

Thus \((E_{1}, r_{\perp} \cdot j)\) satisfies all the properties of
Lemma 4; and so represents \( C_{a}(Q, G) \).

5) \((E_{1}, r_{\perp} \cdot j)\) is equivalent in \( B(W, M; v) \) to \((E_{0}, T)\).

Proof: Consider the diagram
All regions are commutative except \(*\) because of 2),3) and 4).

Let \((B, T_B')\) denote the pair

\[
(E_0 \times I \cup \mathcal{W}_1 E_1, \quad p_1(TxI)(rxI)((W_c \cdot \text{incl})xI))
\]

Then \(B\) is a bundle over \(WxI\) which contains the bundle \(E_0 \cup_b \mathcal{W}_1 E_1 = E^*\) so that \(T_B'\) restricted to \(E^*\) gives the \(F\)-trivialization \(T \cup_b txI \cup_b T_1\).

This follows from the diagram.

Now \(*\) is commutative for a neighborhood of \(QxO\) in \(QxD^k\). So 3 iii) implies \(T_B'\) induces a homotopy equivalence of each fibre pair with \((D^k, D^k - 0)\). Thus we can modify \(T_B'\) slightly on the complement of \(E^*\) so that it becomes an \(F\)-trivialization \(T_B\). Then \((B; T_B)\) provides the desired equivalence.

This proves that

\[C_a(L, g; E, r, i; g', h)\]

is onto. Now we show that this result is independent of our choices of \((L, g)\) and \((E, r, i)\). This follows from

6) i) We can change a given \((L, g)\) and \((E, r, i)\)
by equivalences so that \(A\) and \(B\) are satisfied
ii) Suppose \((L, g), (E, r, i), g'\) and \(h\) satisfy

\(A, B, C,\) and \(D.\) Suppose \((L, g)\) and \((E, r, i)\) are respectively equivalent to \((L, g_1)\) and \((E, r_1 i_1)\). Let \(C_a^*\) denote a map defined as above using Lemma 4. (for some choice of \(g_1\) and \(h_1\)). Let \(C_a\) denote

\[C_a (L, g; E, r, i; g', h).\]

Then there is a commutative diagram

\[\begin{array}{ccc} C_a & \xrightarrow{e} & B_k(W, M; (E, r, i)) \\ (D) \downarrow e & & \downarrow e^* \\ PL(W, M; (L, g)) & \xrightarrow{C_a^*} & B_k(W, M; (E, r_1 i_1)) \end{array}\]

where \(e\) and \(e^*\) are bijective maps.

Proof: i) Let \(e: L \times D^k \to E\) be defined by the composition

\[L \times D^k \xrightarrow{id \times r} L \times \frac{1}{n}(D^k) \xrightarrow{i^{-1}} L \times D^k \to E\]

where \(n\) is chosen so that \(i^{-1}\) is defined and \(r: D^k \to \frac{1}{n} D^k\) is the identity on some neighborhood of \(0.\)

Then B) is satisfied.

We can change \(g\) by a homotopy so that A) is satisfied.

ii) Let \((R, f)\) be a concordance between \((L, g)\) and \((L, g_1)\). Define
$PL(W, M; (L, g)) \xrightarrow{e} PL(W, M; (L_1, g_1))$

on representatives by

$$(Q, G) \xrightarrow{\cdot} (Q \cup_{L_1} R, c^{-1}(G \cup f))$$

where

$$c^{-1} \quad W \cup M \xrightarrow{\cdot} W$$

is a PL-homeomorphism which restricts to $M \xrightarrow{PL} M$ on $M \cup L$ and the identity on the complement of a collar neighborhood of $M$ in $W$.

If $(S, T)$ is an equivalence between $(E, ri)$ and $(E_1, r_1 i_1)$ define

$$B_k(W, M; (E, ri)) \xrightarrow{e^*} B_k(W, M; (E_1, r_1 i_1))$$

on representatives by

$$(E_0, T_0) \xrightarrow{c^*} (E_0 \cup M, \overline{c}(T_0 \cup T))$$

where $\overline{c}$ is a bundle map covering $c$.

If we choose, $(S, T) = (S, r_2 i_2)$ to be a classifying bundle for $(R, f)$ which extends $(S, ri) \cup (E_1, r_1 i_1)$ with respect to $g' \cup g_1$ and $h_1 \cup h_1$ using Lemma 1', then an easy but tedious argument shows that (D) is commutative.

This completes the proof of Theorem 9.

Corollary: $C$ is a natural equivalence of contravariant functors on $\mathcal{M}_n$. 
The Classifying Space \( F/PL \)

Theorem 9 implies that
\[
\text{PL} \xrightarrow{C} B_k
\]
is a natural equivalence of functors on \( \mathcal{M}_n \), \( k \gg n \).
We now study the functor \( B_k \).

**Definition 16** Let \( X \) be a space. Then \((\ _\ ,X)\) denotes the contravariant functor on \( C \) which assigns to \( K \) in \( C \) the set of *free homotopy classes of maps* from \( K \) to \( X \) and to \( f:K \to L \) the induced map
\[
(L,X) \xrightarrow{f_*} (K,X)
\]
defined on representatives by \( g \to gf \).

If \( X \) is in \( C \) and \( u \) is in \( B_k(X) \), let \( u \) denote the natural transformation
\[
(\ _\ ,X) \xrightarrow{u} B_k
\]
defined on representatives by \( (K \to X) \xrightarrow{\to} (f^*v) \),
where \( v \) is an \( F/PL)_k \)-bundle representing \( u \).

**Theorem 10** Let \( \mathscr{G} \) be the category of countable connected locally finite simplicial complexes and PL-maps. Then if \( k \gg 3 \), there is a space \( F/PL)_k \) in \( \mathscr{G} \) and an element \( u_k \) in \( B_k(F/PL)_k \) so that
\[
(\ _\ ,F/PL)_k) \xrightarrow{u_k} B_k
\]
is a natural equivalence of contravariant functors on \( \mathscr{G} \).
Proof: One can carry through Brown's construction a la Milnor (19) to construct \( F/\text{PL}_k \) using the properties of \( B_k \) described in Lemma 2. We can forget base points because \( B_k(S^1) = \{0\} \).

Lemma 11. Let \( v \) be an \( F/\text{PL}_k \)-bundle over \( D^n \) and let \( v_k \) represent the universal bundle \( v_k \). Suppose \( f: bD^n \rightarrow F/\text{PL}_k \) is such that \( f^*v_k \) is embedded (as a subbundle) in \( v \). Then there exists an extension \( g \) of \( f \) over \( D^n \) so that the bundle

\[
v \cup (f^*v_k \times I) \cup g^*v_k
\]

over \( D^n \cup bD^n \times I \cup D^n \) extends over \( D^n \times I \).

Proof: Let \( A \) be a subcomplex of \( X \). Let \( u \) be an \( F/\text{PL}_k \) bundle over \( A \) and \( f: A \rightarrow X \) be such that \( f^*v_k \) is equivalent in \( B_n(A) \) to \( u \). Then Theorem 10 implies that \( u \) extends over \( X \) if and only if \( f \) extends over \( X \).

In our case \( f^*v_k \) extends over \( D^n \) so \( f \) has at least one extension \( g \). Then \( v_g = v \cup (f^*v_k \times I) \cup g^*v_k \) extends over \( D^n \times I \) if and only if a certain element \( C_g \) in \( \pi_n(F/\text{PL}_k) \) vanishes (\( C_g \) is the classifying map for \( v_g \)).

If \( C_g \neq 0 \), we merely alter \( g \) in the interior of \( D^n \) by adding \(-C_g \). If \( g' \) is the new map, then \( C_g' = 0 \) and \( v_{g'} \) extends over \( D^n \times I \).
Theorem 12. Let $X$ belong to $\mathcal{C}$ and let $A$ be a subcomplex of $X$. Let $v_k$ represent the universal bundle $u_k$ over $F/PL)_k$, and let $f: A \to (F/PL)_k$ be given. Let $(X, A; f)$ denote the homotopy classes (by homotopies fixed on $A$) of extensions of $f$ over $X$. Then the induced bundle construction defines a one-to-one correspondence

$$ (X, A; f) \rightarrow \ast \rightarrow B_k(X, A; f^*v_k) $$

Proof: It's easy to see that $\ast$ is well defined and injective.

Let $v$ belong to $B_k(X, A; f^*v_k)$. Let $X_r = A \cup (r$-skeleton of $X$). Suppose inductively that there is an extension, $g_r$, of $f$ over $X_r$ so that $g_r^*v_k$ is equivalent to $v/X_r$ in $B_k(X_r, A; f^*v_k)$. This means that

$$ v/X_r \cup f^*v_k \times I \cup g_r^*v_k $$

extends over $X_r \times I$. $g_{r+1}$ extending $g_r$ is constructed with this property by applying Lemma 11 to each $r$ cell of $X_{r+1} - X_r$. If we let $g = \bigcup g_r$, then $g$ extends $f$ and $\ast(g) = v$. Thus $\ast$ is onto.
Now we relate \( \mathcal{F}/\mathcal{PL}_n \) to certain other universal spaces.

**Definition 17** Let \( \mathcal{F}(n) \) denote the set of homotopy equivalences of \( S \) of degree +1 endowed with the compact open topology.

Let \( \mathcal{B}_{\mathcal{PL}}n \) denote the universal space for PL \( n \)-disk bundles (19).

Let \( \mathcal{B}_{\mathcal{F}}(n) \) denote the universal space for \((n-1)\)-spherical fibre spaces (22).

**Theorem 13** There exist maps \( \mathcal{F}(n) \xrightarrow{0_n} \mathcal{F}/\mathcal{PL}_n \)
\( \mathcal{F}/\mathcal{PL}_n \xrightarrow{b_n} \mathcal{B}_{\mathcal{PL}}n \), and \( \mathcal{B}_{\mathcal{PL}}n \xrightarrow{J_n} \mathcal{B}_{\mathcal{F}}(n) \) so that

\[
\begin{align*}
(\ , \mathcal{F}(n)) & \xrightarrow{0_n} (\ , \mathcal{F}/\mathcal{PL}_n) \\
& \xrightarrow{b_n} (\ , \mathcal{B}_{\mathcal{PL}}n) \\
& \xrightarrow{J_n} (\ , \mathcal{B}_{\mathcal{F}}(n))
\end{align*}
\]

is an exact sequence of functors on \( \mathcal{C} \).

**Proof:** Let \( \nu_n = (E_n, t_n) \), \( p_n \), and \( f_n \) denote universal bundles over \( \mathcal{F}/\mathcal{PL}_n \), \( \mathcal{B}_{\mathcal{PL}}n \), and \( \mathcal{B}_{\mathcal{F}}(n) \), respectively. Let \( b_n \) and \( J_n \) be defined by \( J_n \) \( f_n \) is fibre homotopically equivalent to the sphere bundle of \( p_n \), and \( b_n \) \( p_n \) is PL-bundle equivalent to \( E_n \).

Now it follows from (22) that \( \mathcal{F}(n) \) is homotopically equivalent to an element of \( \mathcal{C} \). Define \( \mathcal{F}(n) \times D^n \xrightarrow{t} D^n \) by \( (f, d) \xmapsto (\text{cone on } f)(d) \). Then \( (\mathcal{F}(n) \times D^n, t) \) defines an \( \mathcal{F}/\mathcal{PL}_n \)-bundle over \( \mathcal{F}(n) \) which is classified by \( 0_n : \mathcal{F}(n) \rightarrow \mathcal{F}/\mathcal{PL}_n \).
**Exactness at $B_{PL}^n$:** Let $X$ be in $\mathcal{F}$ and $f$ be in $(X, B_{PL}^n)$. Then $J_n(f^*p_n) = 0$ if and only if $f^*p_n$ admits an $F$-trivialization $t$. $(f^*p_n, t)$ is classified by $g: X = F/PL_n$ and $(b_n g)^*p_n = g^*b_n^*p_n = g^*E_n = f^*p_n$. So $b_ng \sim f$. Thus $J_n f = 0$ if and only if $f \sim b_ng$.

**Exactness at $F/PL_n$:** Let $f$ belong to $(X, F/PL_n)$. Then $b_nf = 0$ if and only if $f^*E_n$ is the trivial bundle. Thus $b_nf = 0$ if and only if $f^*v_n$ is equivalent to $(X \times D^n, t_o)$ for some trivialization $t_o$. Assume $t_o(x, d) = (\text{cone on } g(x)(d))$ where $g: X \to F(n)$. Then

$$(0_{ng})^*(E_n, t_n) = g^*(F(n) \times D^n, t)$$

$$= (X \times D^n, t(g \times \text{id}))$$

$$= (X \times D^n, t_o)$$

since $f(g(x), d) = (\text{cone on } g(x))(\delta)$. Therefore $0_{ng}$ is homotopic to $f$. The exactness follows from the

**Lemma:** Let $(X \times D^n, t)$ be an $F/PL_n$-bundle. Let $g: X \to F(n)$ be defined by

$$(g(x))(y) = t(x, y), \quad y \text{ in } bD^n.$$ 

Then $(X \times D^n, t)$ is equivalent to $(X \times D^n, \text{cone on } g)$.
Proof: A homotopy between $t$ and the cone on $g$ is given by

$$H(x,d,s) = \begin{cases} 
|d| t(x, \frac{d}{|d|}) & \text{if } |d| \geq 1-s \text{ or } s = 1 \text{ and } |d| \neq 0 \\
(l-s)t(x, \frac{d}{1-s}) & \text{if } |d| < 1-s, s < 1 \\
0 & \text{if } s = 1, d = 0.
\end{cases}$$

We also have maps so that

$$F(n) \xrightarrow{0_n} F/PL_n \xrightarrow{b_n} B_{PL} n \xrightarrow{J_n} BF(n)$$

is homotopy commutative. These are defined respectively by suspension, Whitney sum with the trivial one-dimensional bundle, and Whitney join with the trivial 0-dimensional bundle.

Definition 18: By forming mapping cylinders we can suppose that the vertical maps are inclusions. Let

$$F = \bigcup F(n), \quad F/PL = \bigcup F/PL_n, \quad B_{PL} = \bigcup B_{PL} n, \quad \text{and} \quad B_F = \bigcup B_{F(n)}.$$
Then by successively applying the homotopy extension theorem we can make the diagram actually commutative. Let \( O = \bigcup O_n \), \( b = \bigcup b_n \), and \( J = \bigcup J_n \).

We get a sequence

\[
\begin{array}{c}
F & \xrightarrow{0} & F/PL & \xrightarrow{b} & B_{PL} & \xrightarrow{J} & B_F \\
\end{array}
\]

**Theorem 14:** Let \( X \) be a finite complex. Then

a) \( F/PL \) is the classifying space for stable equivalences classes of \( F/PL)_k \)-bundles over \( X \).

b) If \( f : X \rightarrow F/PL \) is the classifying map for the class of \( (E,t) \), then \( bf \) is the classifying map for the stable equivalence class of \( E \).

c) The sequence

\[
(X, F) \xrightarrow{O_*} (X, F/PL) \xrightarrow{b_*} (X, B_{PL}) \xrightarrow{J_*} (X, B_F)
\]

is exact.

d) The Whitney sum operation induces a \( \mathbb{H} \)-space structure on \( F/PL \).

e) This \( \mathbb{H} \)-space structure on \( F/PL \) makes \( (X, F/PL) \) into an Abelian group.

**Proof:** If \( A = \bigcup A_n \), then any map of a finite complex into \( A \) is homotopic to a map into \( A_n \) for some \( n \).

Thus a) follows from Theorem 10, b) follows from the definition of \( b \), and c) follows from Theorem 13.
i) $b$ and $J$ are homomorphisms (where the operations in $(X, B_{PL})$ and $(X, B_F)$ are induced by Whitney sum)

ii) $(X, B_{PL})$ is an Abelian group, (see (15) and (12))

iii) $(X, F)$ is finite. (The homotopy groups of $F$ are finite (22).)

Now inverses are easily constructed using c). This proves e).
The Homotopy Properties of F/PL

**Lemma 15** (Stability) \( \pi_i(F/PL) \xrightarrow{i_n} \pi_{i+1}(F/PL) \) is an isomorphism if \( i < n - 6 \).

**Proof:** It's easy to check that

\[
\begin{array}{c}
\xymatrix{ C \ar[r] & B_n(S^1 \times D^3) \ar[d] & \pi_n(F/PL) \ar[d] \ar[l] \ar[r] & C \ar[l] \ar[d] & B_{n+1}(S^1 \times D^3) \ar[r] & \pi_{n+1}(F/PL) \ar[l] \ar[d] \ar[r] & C \ar[l] }
\end{array}
\]

is commutative, (here we identify \((S^1 \times D^3, X)\) with \( \pi_i(X) \) by \( p_1 : S^1 \times D^3 \rightarrow S^1 \)). Now \( u_n \) and \( u_{n+1} \) are isomorphisms and \( C \) is an isomorphism if \( n - 3 > i + 3 \) by Theorem 9.

**Definition 19** Let \( M_1 \) and \( M_2 \) be two oriented closed connected PL n-manifolds. Let \( f_i : M_i \rightarrow F/PL \) and \( D^n_i \subset M_i \) be such that \( f_i(D^n_i) = p \) in \( F/PL, i = 1, 2 \). Let \((M_1, f_1) \# (M_2, f_2)\) denote the pair \((M_1 \# M_2, f_1 \# f_2)\), where \( M_1 \# M_2 \) denotes the oriented connected sum of \( M_1 \) and \( M_2 \) using \( D_1^n \) and \( D_2^n \) and \( f_1 \# f_2 : M_1 \# M_2 \rightarrow F/PL \) is defined by \( f_1 \cup f_2 \).
Now $F/PL$ is simply connected so we actually get a well-defined map

$$(M_1, F/PL) \times (M_2, F/PL) \xrightarrow{\#} (M_1 \# M_2, F/PL)$$

by choosing representatives which satisfy $f_i(D^n) = p$.

If $M_1 = M_2 = S^i$, then $\#$ is just the group operation in $\pi_i(F/PL)$.

**Theorem 16** Let $M_i^n$ be simply connected closed oriented PL $n$-manifolds $i = 0, 1$. Let $f_i$ in $(M_i^n, F/PL)$ classify the $F/PL$-$k$-bundles $(E_i, t_i)$, $i = 0, 1$; $k \gg n$. Then there exists an element

$$n(M_i, f_i) \text{ in } \begin{cases} 0 & \text{if } n \text{ is odd} \\ Z_2 & \text{if } n \equiv 2 \pmod{4} \\ Z & \text{if } n \equiv 0 \pmod{4} \end{cases}$$

so that

i) if $n \geq 4$, $n(M_0^n, f_0) + n(M_1^n, f_1) = n((M_0^n, f_0) \#: (M_1^n, f_1))$

ii) if $n \geq 5$, $n(M_1^n, f_1) = 0$ if $t_i$ "splits", i.e. $t_i$ is homotopic (through $F$-trivializations) to $t_i'$, where $t_i'$ is $t$-regular to $0$ in $D^k$ and the composition

$$t_i'^{-1}(0) \subset E_i \longrightarrow M_i$$

is a homotopy equivalence.
Proof: Let \( M_1 \) be an oriented \( n \)-manifold and suppose \( M_1 \times D^k \) is embedded in \( E \) so that \((M_1 \times 0)\) is homologous to \((M)\). Then for \( n \leq 4 \), let

\[
n(M,f) = \frac{1}{8} (\text{index } M', \text{index } M).
\]

It follows from (16) that \( n(M,f) \) is an integer.

The additivity follows from an easy geometric argument using the additivity of the index.

Now we can apply the technique of Lemma 8 and change \( t_i \) by a homotopy through \( P\)-trivializations so that

\[
f_i^{-1}(0) \subset E_i \xrightarrow{P_i} M_i
\]

\( f_i \)

is \( \left\lfloor \frac{n-1}{2} \right\rfloor \) -connected. If \( n = 4k \), then \( n(M_1,f_i) \) is the precise obstruction to making \( f_i \) \( \frac{n}{2} \) -connected (and thus a homotopy equivalence) when \( n \geq 5 \), (see 1).

If \( n \) is odd and \( n \geq 5 \) there is no obstruction.

If \( n = 4k+2 \), there is a \( \mathbb{Z}_2 \) obstruction (the Kervaire invariant of the kernel of

\[
H_{2k+1}(f_i^{-1}(0)) \to H_{2k+1}(M).
\]

whose vanishing implies \( f_i \) can be made \( \frac{n}{2} \) -connected, \( n \geq 5 \). (see 1).
An easy application of the proof of Lemma 8 where 
\((W,M)\) there corresponds to \((M \times I, M \times 0)\) here shows 
the converse. A similar application shows that the 
obstruction only depends on the equivalence class of 
\((E,t)\).

Thus we can define \(n(M,f)\) so that ii) holds 
using this obstruction. i) will also hold, in fact 
this is used in the proof of Lemma 8 (see 1).

**Theorem 17** Let \(W^{n+1}\) be a simply connected 
PL-manifold whose boundary components \(M_1, M_2, \ldots, M_s\) 
are simply connected, \(n \geq 4\). Let \(M_1\) be oriented by 
the boundary of an orientation of \((W, \partial W)\). If 
\(F: W^{n+1} \to F/PL\), let \(f_i = F/M_i\), \(i = 1, \ldots, s\). Then 
\[
n(M_1, f_1) + n(M_2, f_2) + \ldots + n(M_s, f_s) = 0 .
\]

**Proof:** A homology calculation takes care of the 
case \(n = 4\) (indeed \(n = 4k\)). We proceed to the case 
\(n > 4\).

It follows immediately from the definition of \(n(M,f)\) 
that \(n(-M,f) = -n(M,f)\), where \(-M\) denotes \(M\) with 
the opposite orientation.

We use this to reduce Theorem 17 to the case when 
all except possibly one of the \(n(M_i, f_i)\) are zero.
Choose an arc $\lambda$ in int $W$ connecting a point in $M_1$ to a point in $M_2$. If $M$ is an oriented $n$-manifold let $(W; M_1, M_2) \neq M$ denote the manifold obtained from $W$-open tubular neighborhood of $\lambda$ and $(M$-open disk)x I by the obvious identification of their mutual $(S^{n-1} \times I)$'s. To be more precise do this in an orientation preserving manner so that the oriented components of $b((W; M_1, M_2) \neq M)$ are $M_1 \neq M$, $M_2 \neq -M$, $M_3$, ..., $M_s$.

If $f: M \to F/PL$ is given, and $F: W \to F/PL$ and $f_0: M \times I \to F/PL$ map the $(S^{n-1} \times I)$'s to $p$ in $F/PL$, there is an obvious extension over $(W; M_1, M_2) \neq M$, say $(F; f_1, f_2) \neq f$. In general, we can change $f_0$ and $F$ so that this condition is satisfied and $(F; f_1, f_2) \neq f$ is well defined up to homotopy (since $F/PL$ is simply connected).

Now replace $(W, F)$ by $(W', F') = ((W; M_1, M_2) \neq (-M_1), (F; f_1', f_2') \neq f_1)$. Then if $W'$ has components $M'_1, M'_2, M'_3, ..., M'_s$, and $f'_1 = F'/M'_1$, we have:

\[ n(M'_1, f'_1) = 0 \]
\[ n(M'_2, f'_2) = n(M_1, f_1) + n(M_2, f_2) \]
\[ n(M'_i, f'_i) = n(M_1, f_1) \text{ for } 2 < i \leq s. \]

We continue in this manner, and we get a pair $(W^*, F^*)$ so that

i) $W^*$ is simply connected
ii) \( bW^* \) consists of simply connected components
\[ M_1^*, M_2^*, \ldots, M_s^* \]

iii) if \( f_i^* = F^*/M_i^* \), then
\[
n(M_i^*, f_i^*) = \begin{cases} 
0 & i \neq s \\
n(M_1, f_1) + \ldots + n(M_s, f_s) & i = s
\end{cases}
\]

Let \( F^* \) determine the bundle \((E^*, t^*)\). Let \((E, t) = (E^*, t^*) / bW^* - M_s\). Apply Theorem 16 to each component of \( bW^* - M_s \) and "split" \( t \). Use a collar neighborhood of \( bW^* - M_s \) in \( W \), to change \( t^* \) on the complement of \( E^*/M_s \) so that \( t^*/E \) is "split". Now let \((W^*, M_s, bW^* - M_s, t^*)\) correspond to \((W, M, M_1, T)\) of Lemma 8 to see that \( t^* / (E^*/M_s) \) can be "split."

Thus \[ n(M_s^*, f_s^*) = n(M_1, f_1) + \ldots + n(M_s, f_s) = 0. \]

Let \( \prod_{n}^{PL} (F/PL) \) denote the \( n \)th oriented PL-bordism group of \( F/PL \). Since \( \prod_{n} (F/PL) = 0 \), each bordism class in \( \prod_{n}^{PL} (F/PL) \) contains a pair \((M, f)\) where \( M \) is simply connected, \( n \geq 1 \). Furthermore any cobordism between two such simply connected pairs may be replaced by a simply connected cobordism. (Any oriented \( n \)-manifold is oriented cobordant (by one-dimensional surgery) to a simply connected manifold, \( n \geq 1 \).)

Thus Theorem 17 has the
Corollary: The correspondence \((M,f) \rightarrow n(M,f)\)
defines a homomorphism

\[
\bigcap_n \mathbb{P}L (F/PL) \xrightarrow{K} \mathbb{P}_n
\]

for \(n \geq 4\), where

\[
P_n = \begin{cases} 
0 & n \text{ odd} \\
\mathbb{Z}_2 & n \equiv 2 \pmod{4} \\
\mathbb{Z} & n \equiv 0 \pmod{4}
\end{cases}
\]

We remark that \(K\) has a natural definition for
\(n < 4\). This is unique for \(n = 1, 3\). For \(n = 2\) define
\(K(M^2, f)\) to be 0 if \(t\) splits and 1 otherwise
\((f \sim (E, t))\). The additivity of \(K\) for this case follows
exactly as in the case \(n = 6\) or \(n = 14\). The fact that
\(K\) is well-defined on cobordism classes follows from
Lemma 19 and the fact that \(\mathbb{T}_2(F/PL) = \bigcap_n \mathbb{P}L (F/PL) = \mathbb{Z}_2\);

Then \(K\) has the property that if \(n \neq 3, 14\), \(K(M^n, f) = 0\)
iff \(t\) "splits" where \(f \sim (E, t)\).

We relate \(K\) and \(\mathbb{T}_n(F/PL)\) by

Theorem 18 Let \(\mathbb{T}_n(F/PL) \xrightarrow{h^1} \bigcap_n \mathbb{P}L (F/PL)\) denote
the "\(\bigcap_n \mathbb{P}L\) Hurewicz Homomorphism." Then if \(n \neq 4\), the
composition

\[
\mathbb{T}_n(F/PL) \xrightarrow{h^1} \bigcap_n \mathbb{P}L (F/PL) \xrightarrow{K} \mathbb{P}_n
\]

is an isomorphism.
If \( n = 4 \), \( \mathcal{K}^i \) is a monomorphism onto the even integers.

**Proof:** We make use of the following

**Lemma 19:** Let \( (E,\iota) \) denote an \( \text{F/PL}_k \)-bundle over closed \( M^i \), \( k \gg i \). Then \( \iota \) splits so that \( \iota^{-1}(0) \) is PL-homeomorphic to \( M^i \). If \( (E,\iota) \) is equivalent to the trivial \( \text{F/PL}_k \)-bundle, \( (M^i \times D^k, p_2) \).

**Proof:** Extend the embedding of \( M \times 0 = \iota^{-1}(0) \subseteq E \times 0 \) to an embedding \( \iota \) of \( (M \times I, M \times (0,1)) \) in \( (E' \times I, E' \times (0,1)) \) so that \( M \times 1 \) goes onto the zero section of \( E \). Now \( M \times I \) has a trivial normal bundle in \( E' \) (because \( M \times 0 \) does) so by the PL tubular neighborhood theorem \( E \) is the trivial bundle, \( M \times D^k \xrightarrow{\text{PL}} M \).

It follows from Lemma 3', or by an easy direct argument, that there is an embedding \( \epsilon \) of \( M \times I \times D^k \) in \( E' \) so that

1) \( e/M \times I \times 0 = i \)
2) \( e/M \times I \times D^k \) is a PL-homeomorphism \( \gamma \) onto \( E \times 1 \)
3) \( t(e/(M \times C \times D^k)) = p_2 \).
Then define \( T: E \times I \rightarrow D^k \) on the image of \( e \cup E \times 0 \) using \( p_2 \) and \( t \) and on the complement of image of \( e \cup E \times 0 \) using obstruction theory so that \( T \) is an \( F \)-trivialization. Then \((E \times I \cup C \times D^k, T \cup C \circ p_2)\) gives an equivalence between \((E, t)\) and \((M \times D^k, p_2)\).

The converse follows from the PL covering htpy theorem.

Now we prove Theorem 18.

Let \((L_n, bL_n)\) be a framed, almost closed, almost parallelizable \( \left[ \frac{n-1}{2} \right] \) - connected \( n \)-manifold in \((D^{n+k}, bD^{n+k})\) which generates \( F \) (Definition 1.2). Let \( \overline{L}_n \) denote \( L_n \cup \text{(cone) } bL_n \). Let \( \overline{bL}_n \)

\[
\begin{align*}
\overline{h}_n: \ (D^n, bD^n) \times D^k & \rightarrow (D^n, bD^n) \times D^k \\
\end{align*}
\]

be the map given by \( \overline{F} \) so that \( \overline{h}_n \) is \( t \)-regular to \( D^n \times 0 \) and \( \overline{h}_n^{-1} (D^n, bD^n) \times 0 = (L_n, bL_n) \subset (D^n, bD^n) \times D^k \).

We may assume that \( \overline{h}_n/ (\text{nghd of } bL_n \text{ in } bD^n \times D^k) \)

is a PL-homeomorphism \( c \) which is the identity on \( S^{n-1} \times 0 = bL_n \). Let

\[
\begin{align*}
E & = (D^n, bD^n) \times D^k \cup (D^n, bD^n) \times D^k \\
c(bL_n \times D^k) \\
\end{align*}
\]

and

\[
t = p_2 (h_n \cup c \text{id}).
\]
An easy application of the PL-tubular neighborhood theorem shows \( E \) is the total space of a PL \( k \)-disk bundle over \( S^n \). Then \( (E,t) \) is an \( F/\text{PL}_k \) bundle over \( S^n \), \( t \) is \( t \)-regular to 0 in \( \mathcal{D}_k \), and \( t^{-1}(0) = \mathcal{L}_n \).

Suppose \( (E,t) \) is classified by \( f: S^n \to F/\text{PL} \).

Then if \( n = 4i \),

\[
\nu(S^n,f) = \pm \left( \text{index } L_n \right)
\]

\[
= \begin{cases} 
\pm 1 & \text{if } i > 1 \\
\pm 2 & \text{if } i = 1.
\end{cases}
\]

If \( n = 4i + 2, \ n \geq 6 \), then

\[
\nu(S^n,f) = 1, \ \text{by (1)}.
\]

Thus \( K_h^1 \) is onto if \( n \geq 5 \) and onto \( 2P_n \) if \( n = 4 \).

Suppose \( f: S^n \to F/\text{PL} \) classifies \( (E,t) \) and

\( K_h^1(S^n,f) = \nu(S^n,f) = 0 \). If \( n = 4 \), we may change \( t \) by a homotopy so that \( t^{-1}(0) \) is PL-homeomorphic to \( S^k \).

If \( n \geq 5 \), Theorem 16 and the Generalized Poincare Conjecture imply that \( t \) can be split so that \( t^{-1}(0) \) is PL-homeomorphic to \( S^n \). Then Lemma 19 implies \( f \) is homotopic to zero. Thus \( K_h^1 \) is a monomorphism, \( n \geq 4 \).

---

1 See Lemma 1.5
Lemma 1.5 and Lemma 19 imply $\bar{T}_1(F/PL) = T_3(F/PL) = 0$. $T_2(F/PL)$ can be computed in a number of ways. One way is to use the exact sequence of Theorem 14 to show that $T_2(F/PL)$ has 2 elements. Lemma 19 implies $Kh'$ is an isomorphism. This completes the proof.

**Corollary:** $T_n(F/PL) = P_n$.

Now we apply Theorem 18 to study the $k$-invariants and $Z_2$-cohomology of $F/PL$.

Let $X_i$ denote the space which is homotopically equivalent to $F/PL$ over the $i$-skeleton and such that $\bar{T}_k(X_i) = 0$ for $k > i$. Let $x_{i+1}$ in $H^{i+2}(X_i; T_{i+1}(F/PL))$ be any $k$-invariant (or characteristic class) associated to the fibration

$$K(i+1, T_{i+1}(F/PL)) \rightarrow X_{i+1} \rightarrow X_i$$

in a Postnikov tower for $F/PL$, (see 23).

The $k$-invariants are closely related to the Hurewicz homomorphism.

**Theorem 20** Let $h$ denote the Hurewicz homomorphism, $h: T_*(F/PL) \rightarrow H_*(F/PL)$, and $h_2$ the (mod 2) Hurewicz homomorphism $h_2: T_*(F/PL) \rightarrow H_*(F/PL; Z_2)$. Then $h$ is a monomorphism, and in all positive even dimensions $h_2$ is non-trivial.
Proof: We first study the mod 2 Hurewicz homomorphism

\[ \tilde{\Omega}_n(F/PL) \xrightarrow{h_2} H_n(F/PL, \mathbb{Z}_2). \]

Let \( \Omega_n(F/PL) \) and \( \gamma_n(F/PL) \) denote the oriented and unoriented (smooth) bordism groups of \( F/PL \), respectively. Then \( h_2 \) may be factored as follows:

Now \( h' \) has a left inverse if \( n \neq 4 \) by Theorem 18. Therefore \( h'' \) has a left inverse, \( n \neq 4 \). The kernel of \( r' \) consists of elements divisible by 2, so this means that \( r'h'' \) is a monomorphism. If \( f:S^n \to F/PL \), represents a generator of \( \tilde{\Omega}_n(F/PL) \), then we have shown that \( (S^n, f) \) is non-trivial in \( \gamma_n(F/PL) \).

Thus there exists a polynomial \( w \) in the Stiefel-Whitney classes of \( S^n \) and an element \( u \) in \( H^*(F/PL, \mathbb{Z}_2) \) so that \( ((f^*u) \cup w) \neq 0 \) in \( H^n(S^n, \mathbb{Z}_2) \). This can only happen when \( w \) equals 1 and \( u \) is in \( H^n(F/PL, \mathbb{Z}_2) \).

Thus \( h_2 \) is non-trivial. For \( n = 4 \) see Corollary 5.

This shows that \( h \) is a monomorphism (onto a direct summand) in dimensions \( 

\)

To see that \( h \) is a monomorphism in dimensions 4i we can look at the Pontryagin numbers of \( (S^{4i}, f) \).
For the facts about bordism theory used here see (5).

**Corollary 1** The \( n \)th \( k \)-invariant of \( F/PL \) is zero if \( n \neq k \).

**Proof:** We make use of Theorem 20.

There is an exact sequence

\[
\begin{align*}
    H^i(X_i, \overline{\nu}_i) &\xrightarrow{i^*} H^i(K(i, \overline{\nu}_i), \overline{\nu}_i) \xrightarrow{t} H^{i+1}(X_{i-1}, \overline{\nu}_i) \\
\end{align*}
\]

where \( i \) is the inclusion map, \( t \) is the transgression map, and \( \overline{\nu}_i = \overline{\nu}_i(F/PL) \), (see 7). If \( u \) denotes the fundamental class of \( K(i, \overline{\nu}_i) \), then \( u \xrightarrow{t}(k\text{-invariant}) \).

Thus we need to look at image \( i^* \). Consider

\[
\begin{align*}
    \overline{\nu}_i(X_i) &\xleftarrow{i} \overline{\nu}_i(K_i) \\
h &\downarrow r & h &\downarrow h \\
H_i(X_i) &\xleftarrow{r^*} H_i(K_i) \\
\end{align*}
\]

where \( r \) is defined using the homotopy equivalence with \( X \) over the \( i \)-skeleton and the result about \( h_2 \) in dimensions \( \neq 4k \). If \( (x) \) in \( H^i(X_i, \overline{\nu}_i) \) determines the homomorphism \( r: H_i(X_i) \rightarrow \overline{\nu}_i(X_i) \), then the
homomorphism $H^i(X, T) \to H^i(K, T)$ determined by $i^*(r)$ may be identified with $i^{-1} r_i$. So $r_i h$ is essentially $r_i h$. (See proof of Corollary 2 for details.) If $i \neq k$, $h r$ is the identity so $(r)$ goes to the fundamental class of $K(i, T)$ since its associated homomorphism is $h^{-1}$. The Corollary follows from the exactness of $(S)$.

**Corollary 2** The $k^{th}$ $k$-invariant of $F/PL$ is a torsion element, and its reduction mod $2$ is zero.

**Proof:** There is a class in $H^k(X, T)$ which restricts to some non-zero multiple of the fundamental class of $K(k, T)$ where the latter space is the fibre of

$$X_{k+1} \longrightarrow X_{k+1-2},$$

in the Postnikov tower $F/PL$. This follows from the fact that

$$h: H_k(F/PL) \to H_k(F/PL)$$

is a monomorphism. The first statement now follows from the exact sequence of Serre,

$$0 \to H_{k+1}(X_{k+1}, T) \to H_k(k(k, T), T) \to H_{k+1}(X_{k+1}, T) \to H_{k+1}(X_{k+2}, T).$$
For the second statement we proceed in the same manner using the mod 2 Serre sequence.

**Addendum:** Let

\[ S_k = \frac{B_k j_k D_k}{\text{ord} (e_{4k-1}(\mathcal{E} \mathcal{W}))} \]

where \( B_k = (2k-1)! (k+1, 2) \), the Bott index

\[ j_k = \text{index (ker(} \pi_{4k-1}(O) \to \pi_{4k-1}(F))) \]

\[ D_k = \text{index of the image of the composition} \]

\[ H^{4k}(B_{PL}, \mathbb{Z}) \xrightarrow{1} H^{4k}(B_0, \mathbb{Z}) \xrightarrow{\text{coeff } P} \mathbb{Z} \]

Then the order of the \( k \)-th \( k \)-invariant of \( F/PL \) divides \( S_k \). \( (S_1 = 48, S_2 = 360) \)

**Proof:** One shows that there is a class \( u \) in \( H^{4k}(B_{PL}, \mathbb{Z}) \) such that \( u \) pulls back to \( S_k(\text{gen}) \) under the composition

\[ S^{4k}_{\text{gen. of } \pi_k(F/PL)} \xrightarrow{b} F/PL \xrightarrow{b} B_{PL} \]

In fact choose \( u \) so \( i^*u \) has coefficient \( D_k \) on \( P_k \). Then the diagram
\[ H^{4k}(\mathcal{S}^{4k}) \xrightarrow{\text{gen ker. J}^*} H^{4k}(B_0) \]
\[ \xrightarrow{\text{ord } \theta_{4k-1}(\mathcal{S}^n)} \]
\[ H^{4k}(\mathcal{S}^{4k})(\text{b-gen } \Pi_{4k}(F/PL)) \xrightarrow{i^*} H^{4k}(B_{PL}) \]
\[ S_k \xleftarrow{\text{u}} \]
is commutative. So
\[ (\text{ord } \theta_{4k-1}(\mathcal{S}^n)) S_k = D_k B_k^i S_k. \]

Note that
\[ S_k = \frac{B_k D_k \text{ ord } \Pi_{4k-1}(F)}{\text{ord } \Pi_{4k-1}(PL/0)}. \]

**Corollary 3:** Let \( X \) be a finite complex, and \( d = \left[ \frac{\dim X}{4} \right] \). Let \( P_* \subset (X,F/PL) \) be the subset consisting of maps \( f: N \to F/PL \) such that \( f^* \otimes Q = 0 \) in all positive dimensions. Then \( P_* \) is finite. In fact,
\[ \text{ord } P_* \leq \text{ord} \left( \bigoplus_{i=0}^d (H^{4i+2}(X,Z_2) \otimes \text{Torsion } H^{4i}(X,Z_2)) \right) \]

**Proof:** See proof of Corollary 4.

**Corollary 4:** Let \( X \) be a finite complex and let \( T_* \) denote the kernel of
\[ (X,F/PL) \xrightarrow{b_*} (X,B_{PL}) \].
Then $T_* \subset P_*$, and there exists a sequence of integers $S_1, S_2, \ldots$ such that

$$\text{ord } T_* \leq \text{ord} \left( \bigoplus_{i=0}^{d} H^{i+2}(X, \mathbb{Z}_2), \bigoplus (\text{Torsion } H^{i+1}(X, \mathbb{Z}) \otimes \mathbb{Z}_{S_i}) \right).$$

$S_1$ is described in the Addendum to Corollary 2.

**Proof:** It follows from a spectral sequence argument that

$$H^\ast(B_{PL}, \mathbb{Q}) \xrightarrow{b^\ast} H^\ast(F/PL, \mathbb{Q})$$

is an isomorphism. Thus $T_* \subset P_*$. This also follows from below.

To get the estimate on the orders of $T_*$ and $P_*$, we look at the Postnikov tower of $F/PL$,
Now $X_{i+1} \xrightarrow{\pi_{i+1}} X_i$ is a principal fibration so the homotopy classes of liftings $\tilde{\pi}$ of $\pi$ are in one to one correspondence with $(X, K_{i+1})$. If $\tilde{\pi}$ is one lifting then any other lifting is homotopic to the composition

$$\tilde{\pi} \cdot x g : X \xrightarrow{\tilde{\pi} \cdot x} X_{i+1} x K_{i+1} \xrightarrow{\mu} X_{i+1}$$

where $g : X \to K_{i+1}$ and $\mu$ is the action.

If $\tilde{\pi}^* u$ belongs to $H^{i+1}(X, Z)$, $\tilde{\pi}^* g = \mu(\tilde{\pi} \cdot x g)$, and $i : K_{i+1} \to X_{i+1}$ is the inclusion of a fibre, then it is easy to check that

$$\text{(e)} \quad (\tilde{\pi}^* g)^* u = (\tilde{\pi}^* + (ig)^*) u$$

Now choose $v_{i+1}$ in $H^i(\mathbb{R}P^l, Z)$ so that

$$(b^* v_{i+1}) \wedge (h(\text{gen } W_{i+1}(F/PL))) = S_i.$$

Let $w_{i+1}$ denote the image of $(b^* v_{i+1})$ in $X_{i+1}$. Then $i^*(w_{i+1}) = S_i$ (fundamental class of $K_{i+1}$).

Suppose by induction that the estimates of Corollary 3 and Corollary 4 are valid for maps of $X$ into $X_k$ where the sum is taken over cohomology dimensions $\leq k$.

To get an upper bound on the number of liftings of these to $X_{i+1}$ when $k+1 = i+2$ we merely multiply these estimates by $\text{ord}(X, K_{i+1}) = \text{ord } H^{i+2}(X, Z_2)$.
If \( k+1 = 4i \), \( \hat{f}: X \rightarrow X_{k+1} \) is in \( P_* \) for \( X_{k+1} \) and covers \( f: X \rightarrow X_k \) then \( \hat{f} \circ g \) covers \( f \) and is in \( P_* \) only if

\[(\hat{f} \circ g)^* u = \hat{f}^* u + (ig)^* u \]

has finite order in \( H^i(X, Z) \) for all \( u \) in \( H^i(X_{k+1}, Z) \).

If we take \( u = w_{i+1} \) then we see that we need only multiply \( \hat{f} \) by the subset of \( (X, X(Z, 4i)) \) corresponding to some subset of \( \text{Torsion}(H^i(X, Z)) \) to construct all elements of \( P_* \) for \( X_{k+1} \) which project to \( \hat{f} \). This completes the inductive step for Corollary 3.

We prove Corollary 4 in the same way. The inductive step uses the fact that \( \hat{f} \) is in \( T_* \) of \( X_{k+1} \) only if

\[\hat{f}^* w_{i+1} = 0.\]

So if \( (\hat{f} \circ g)^* w_{i+1} = 0 \) then \( g^*(i^* w_{i+1}) = S_i(g^* i) = 0. \)

Thus we only need multiply \( \hat{f} \) by certain maps \( g: X \rightarrow X_{i+1} \) (corresponding to elements in \( H^i(X, Z) \) of order \( S_i \)) to get the other liftings of \( f \) which are in \( T_* \) of \( X_{k+1} \).

For \( k \) large,

\[
P_*(X, X_k) \xrightarrow{P_k} P_*(X, F/PL)
\]

is an isomorphism, where \( p_k: F/PL \rightarrow X_k \) is the projection.

Similarly for \( T_* \). This completes the proof.
**Corollary 5:** Let $\prod$ denote the infinite product,

$$K(Z_2; 2) \times K(Z_2; 4) \times \ldots \times K(Z_2; 4^{i-2}) \times K(Z, 4^1) \times \ldots$$

Then $H^*(F/PL; Z_2)$ and $H^*(\prod; Z_2)$ are isomorphic as algebras.

**Proof:** We first fill the gap remaining in the proof of Theorem 20, namely that the reduction of $k_4$ to $Z_2$ coefficients is zero.

Let $k$ denote the reduction of $k_4$ to $Z_2$ coefficients. Then $k$ belongs to $H^5(K(Z_2; 2); Z_2)$. If $u$ denotes the fundamental class of $K(Z_2; 2)$ then $k$ must equal

$$a(u \, S^1_q u) + b(S^2_q \, S^1_q u)$$

where $a$ and $b$ are in $Z_2$. If $m^*$ denotes a co-multiplication in $H^*(K(Z_2; 2); Z_2)$ which is compatible with that coming from the $H$-space structure in $F/PL$ then we must have

$$m^*(k) = k \otimes 1 + 1 \otimes k,$$

i.e., $k$ must be primitive. Also we must have $S^1_q (k) = 0$ since $k$ is the reduction (mod 2) of an integral class.

Now

$$m^*(u) = u \otimes 1 + 1 \otimes u$$

so

$$m^*(S^2_q \, S^1_q u) = S^2_q \, S^1_q u \otimes 1 + 1 \otimes S^2_q \, S^1_q u$$

(1)

$$m^*(u \, S^1_q u) = (u \otimes 1 + 1 \otimes u)(S^1_q u \otimes 1 + 1 \otimes S^1_q u)$$

$$= (u \, S^1_q u \otimes 1 + 1 \otimes u \, S^1_q u) + (u \otimes S^1_q u + S^1_q u \otimes u).$$
Also
\[ s^1_q (s^1_q u) = (s^1_q u)^2 \] (2)
\[ s^1_q (s^2_q s^1_q u) = s^3_q (s^1_q u) = (s^1_q u)^2. \]

The first set of equations implies \((a, b) = (0, 1)\) or \((a, b) = (0, 0)\). The second set implies \((a, b) = (1, 1)\) or \((a, b) = (0, 0)\). Thus \((a, b) = (0, 0)\) and \(k = 0\). (I am indebted to J. Milnor for this argument.)

Now consider a fibration in the Postnikov tower for \(F/PL\),
\[
\xrightarrow{K_{i+1}}
\]
\[ X_{i+1} \xrightarrow{K_{i+1}} X_i. \]

Since there exists \(u \in H^{i+1}(X_{i+1}; \mathbb{Z}_2)\) which restricts to the fundamental class of \(K_{i+1}\) (reduced mod 2) the restriction map takes
\[ H^*(X_{i+1}; \mathbb{Z}_2) \text{ onto } H^*(K_{i+1}; \mathbb{Z}_2). \]
Thus the Serre spectral sequence (with \(\mathbb{Z}_2\) coefficients) collapses and \(E_2 = E_{\infty}\).

Then it follows that \(H^*(X_{i+1}; \mathbb{Z}_2)\) is isomorphic to \(H^*(K_{i+1}; \mathbb{Z}_2) \otimes H^*(X_1; \mathbb{Z}_2)\)
as an algebra if \(H^*(X_1; \mathbb{Z}_2)\) is a polynomial algebra (since \(H^*(K_{i+1}; \mathbb{Z}_2)\) is a polynomial algebra.)

The Corollary follows by induction.
The PL-Structures on a Bundle over \( M \)

Let \( E \rightarrow M \) be a PL \( k \)-disk bundle over the PL \( n \)-manifold \( M \). Then \( E \) is a PL\((n+k)\)-manifold and PL\((E)\) is defined. We would like to consider the relationship between PL\((E)\) and PL\((M)\).

**Definition 20** Define

\[
\text{PL}(M) \xrightarrow{q^*} \text{PL}(E)
\]

on representatives by

\[
(L, g) \mapsto (g^*E, b(g)) ,
\]

where \( b(g) \) is a bundle map covering \( g \).

**Definition 21** If \( M^k \) is closed, let \( M_0 \) denote \( \text{int} \, D^k \). If \( M^k \) is in \( \mathbb{R}^k \), let

\[
\text{PL}(M) \xrightarrow{r} \text{PL}(M_0)
\]

denote the map induced \( M_0 \subset M \). Let

\[
\text{PL}(M_0) \xrightarrow{c} \text{PL}(M)
\]

denote the map defined on representatives by

\[
(L_0, g) \mapsto (L_0 \cup \text{cone on } bL_0), g \cup \text{cone on } g/bL_0).
\]
**Lemma 21** Let $M^n$ belong to $\mathcal{M}_n$ and let $r$ and $c$ be the maps defined in Definition 21. Then $rc$ equals the identity of $\mathrm{PL}(M_0)$ and $cr$ equals the identity of $\mathrm{PL}(M)$.

**Proof:** The first statement follows immediately on the level of representatives.

The second statement follows from the following:

Let $f: M \to L$ be a homotopy equivalence of closed simply connected $\mathrm{PL}$-$n$-manifolds. Then $f$ is homotopic to $f_1: M \to L$ where $f_1/D^n \subset M$ is a PL-homeomorphism and $f_1(M-D^n) \subset M-f_1(D^n)$.

This follows from an easy argument using $t$-regularity and framed cobordism of finite discrete sets.

**Lemma 22** Let

$$
\begin{array}{ccc}
(M,F/\mathrm{PL}) & \xrightarrow{q^*} & (E,F/\mathrm{PL}) \\
\downarrow \mathrm{PL}(M) & & \downarrow \mathrm{PL}(E) \\
(M,F/\mathrm{PL}) & \xrightarrow{c} & (E,F/\mathrm{PL})
\end{array}
$$

be defined by $f \mapsto f_q$. Then

$$
\begin{array}{ccc}
(M,F/\mathrm{PL}) & \xrightarrow{q^*} & (E,F/\mathrm{PL}) \\
\downarrow \mathrm{PL}(M) & & \downarrow \mathrm{PL}(E) \\
(M,F/\mathrm{PL}) & \xrightarrow{c} & (E,F/\mathrm{PL})
\end{array}
$$

is commutative.
Proof: Let \((L,g)\) be a PL-structure on \(M\) with classifying bundle \((v, r_j)\). \((v, j)\) is a good tubular neighborhood of
\[
g': M \rightarrow L,
\]
a homotopy inverse for \(g\). Now in the diagram
\[
\begin{array}{ccc}
V & \xrightarrow{j} & L \times \text{int } D^r \\
\downarrow p & & \downarrow p_1 \\
M & \xrightarrow{g'} & L
\end{array}
\]
each map is a homotopy equivalence. Thus we may lift this diagram using \(\begin{array}{c} E \\ \downarrow \\ M \end{array}\) to obtain
\[
\begin{array}{ccc}
p^* E & \xrightarrow{b(j)} & g^* E \times \text{int } D^r \\
\downarrow b(p) & & \downarrow b(p_1) \\
E & \xrightarrow{b(g')} & g^* E
\end{array}
\]
where \(b(f)\) denotes a bundle map covering \(f\). Now \(b(p)\) is a bundle projection since \(p\) was, and \(b(j)\) is an embedding (since \(j\) was). Now the open disk bundle \(p^* E \rightarrow E\) contains \(g^* E \times 0\) since \(j\) was contained in \(L \times 0\). Thus \((p^* E, b(j))\) is a good tubular neighborhood of \(b(g')\).
For suitable \( r_1: D^k \rightarrow D^k \) then \((p^*E, r_1 p_2 b(j))\) is a classifying bundle for \((g^*E, b(g)) \sim C(q^*(L, g))\).

So we want to show that it is equivalent to

\[(q^*v, r_1 p_2 b(g)) = q^*(C(L, g)).\]

We know that

\[
\begin{array}{c}
p^*(E) \xrightarrow{b(j)} g^*E \times D^k \xrightarrow{p_2} D^k \\
p^*(q) \downarrow \quad g^*(q) \times \text{id.} \downarrow \\
v \quad \quad j \downarrow \quad \quad \quad L \times D^k \xrightarrow{p_2}
\end{array}
\]

is commutative. Also, there is a bundle map I so that

\[
\begin{array}{c}
p^*(q) \xrightarrow{p^*E} E \\
v \downarrow \quad I \downarrow \\
b(q) \quad q^*v \xrightarrow{q^*(p)} E
\end{array}
\]

is commutative. (Let I be the identity.) Therefore

\[
C(q^*(L, g)) = (p^*E, r_1 p_2 b(j))
\]

\[
= (p^*E, r_1 p_2 j (q))
\]

\[
= (q^*v, r_1 p_2 j b(q))
\]

\[
= q^*(C(L, g))
\]

since the choice of \( r \) doesn't affect the equivalence class.
Theorem 23 Let $M$ be in $\mathcal{M}_n$ and suppose $E \xrightarrow{q} M$ be a $k$-disk bundle over $M$. Define

$$q^* : \text{PL}(M) \longrightarrow \text{PL}(E)$$

on representatives by

$$(g : L \longrightarrow M) \longrightarrow (b(g) : g^* E \longrightarrow E).$$

Then $q^*$ is bijective.

Proof: We first show that $(E, bE)$ is a simply-connected manifold pair. Now

$$bE = E/bM \cup_{E_s/bM} E_s$$

where $E_s$ is the associated sphere bundle. From the homotopy sequence of the fibration we have

$$\pi_1(S^{k-1}) \longrightarrow \pi_1(E_s) \longrightarrow \pi_1(M) = 0$$

and

$$\pi_1(S^{k-1}) \longrightarrow \pi_1(E_s/bM) \longrightarrow \pi_1(bM) = 0.$$

So for $k \geq 2$,

$$\pi_1(E_s) = \pi_1(E_s/bM) = 0.$$

For $k = 2$,

$$\pi_1(E_s) \text{ and } \pi_1(E_s/bM)$$

are generated by the fibre sphere which goes to zero in $bE$. For $k = 1$,

$bE = b(M \times I) = \text{double of } M$. So in all cases $\pi_1(bE) = 0$ by
Van Kampen's Theorem. (\( \tau_{1}(E) \) and \( \tau_{1}(E/\partial M) \) are clearly zero.)

By Theorem 9, \( PL(E) \xrightarrow{c} (E,F/PL) \) and \( PL(M) \xrightarrow{c} (M,F/PL) \) are bijective. \( (M,F/PL) \xrightarrow{q^{*}} (E,F/PL) \) is bijective. So the theorem follows from Lemma 22.

Theorem 23 is germane to a current problem in differential and PL topology.

**The Bundle Problem:** Let \( M \) be a PL \( n \)-manifold, \( E \xrightarrow{p} M \) a PL \( k \)-disk bundle over \( M \), and \( (W,bW) \xrightarrow{G} (E,bE) \) a homotopy equivalence. The Bundle Problem for \( (E,M;G) \) is the following: find a PL \( n \)-manifold \( L \), a homotopy equivalence \( g:(L,bL) \rightarrow (M,bM) \), a bundle projection \( W \xrightarrow{q} L \); and change \( G \) by a homotopy to \( b \) so that \( b \) is a bundle map covering \( g \),

\[
\begin{align*}
W & \xrightarrow{b} E \\
\downarrow q & \quad \quad \quad \downarrow p \\
L & \xrightarrow{g} M
\end{align*}
\]
We note that if the h-cobordism theorem applies to \((W, bW)\) then the Bundle Problem for \((E, M; G)\) is essentially the problem of changing \(G\) by a homotopy so that it is t-regular to the 0-section of \(E\) and

\[(g^{-1}(M), bG^{-1}(M)) \xrightarrow{G} E/(M, bM)\]
is a homotopy equivalence.

**Corollary 1**  Let \(N\) be in \(\mathcal{M}_n\) and \(E\) be a \(k\)-disk bundle over \(M\). Let

\[(W, bW) \xrightarrow{G} (E, bE)\]

be a homotopy equivalence. Then there is a solution to the Bundle Problem for \((N, M; G)\).

**Proof:** By Theorem 23 \((W, G)\) is concordant to \((g^*E, b(g))\) for some \(g: (L, bL) \to (M, bM)\). We apply the h-cobordism theorem to straighten out this concordance, i.e. if \((Q, J)\) is the concordance choose a PL-homeomorphism

\[H': W \times I \to Q\]

so that

\[H'/W \times 0 = \text{identity of } W\]  and  \[H'/W \times 1\]  is onto \(g^*E\).
If we let
\[ q = g^*(p) \star H/W \times 1 \]
\[ b = b(g) \star H/W \times 1 \]
\[ H = JH^1 \]
then \( q \) and \( b \) are as desired and \( H \) is a homotopy between \( G \) and \( b \).

Corollary 1 is true in such generality because \( bH \neq 0 \). Now we consider the Bundle Problem for closed \( M \).
Let \( M_0 = M - \text{int} D^m \) and \( E_0 = E/M_0 \). We can apply the Browder Theorem to get a map
\[ \begin{array}{c}
\text{PL}(E) \\
\downarrow \text{c}^* \\
\text{PL}(H)
\end{array} \quad \xrightarrow{b(r)} \quad \begin{array}{c}
\text{PL}(E_0) \\
\downarrow \text{c}^* \\
\text{PL}(M_0)
\end{array} \]
if \( k \geq 3 \), and \( M \) belongs to \( M_n \). It's easy to check that

\[ \begin{array}{c}
\text{PL}(E) \\
\downarrow \text{c}^* \\
\text{PL}(H)
\end{array} \quad \xrightarrow{b(r)} \quad \begin{array}{c}
\text{PL}(E_0) \\
\downarrow \text{c}^* \\
\text{PL}(M_0)
\end{array} \]
is commutative. Thus we obtain

**Corollary 2** If \( M \) belong to \( M_n \), \( n \) odd and \( k \geq 3 \), then the Bundle Problem for \( (M,E;G) \) has a solution.
Proof: Consider the commutative diagram

\[
\begin{array}{c}
\text{PL}(E) \quad \text{C} \rightarrow \quad (E,F/\text{PL}) \\
\downarrow \quad \quad \quad \quad \downarrow \\
\quad b(r) \quad \quad \quad \quad \quad \quad \quad \quad \quad r^* \\
\downarrow \\
\text{PL}(E_0) \quad \text{C} \rightarrow \quad (E_0,F/\text{PL})
\end{array}
\]

where \( r^* \) is induced by \( E_0 \subset E \).

Since the Hurewicz homomorphism is a monomorphism for \( F/\text{PL} \), \( r^* \) is onto. Since \( \pi_n(F/\text{PL}) = 0 \) when \( n \) is odd \( r^* \) is injective. Thus \( b(r) \) is bijective by Theorem 9.

\( r \) and \( q_0^* \) are bijective by Lemma 21 and Theorem 23. Thus \( q^* \) is bijective. The proof is now completed just as in Corollary 1.

The case when \( n \) is even is harder to deal with. We must have additional hypothesis on \( G \) in order that the Bundle Problem for \((E,M;G)\) be solvable.

Let \( f: M \rightarrow F/\text{PL} \) and \( u \) belong to \( H^4(F/\text{PL},A) \) where \( A = Q \) or \( Z_2 \). Let \( p \) denote a monomial in the rational Pontryagin classes of \( M \) and \( w \) a monomial in the Stiefel-Whitney classes of \( M \). Then \( (p \cup f^* u)(M) \) is called the Pontryagin number \((u,p)\) of \((M,f)\) (when \( A = Q \)) and \( (w \cup f^* u)(M) \) is called the Stiefel-Whitney number \((u,w)\) of \((M,f)\) (when \( A = Z_2 \)).
Let $E \rightarrow M$ be a $k$-disk bundle over $M$ in $\mathcal{M}_n$ and 
\[
(W,bW) \xrightarrow{G} (E,bE)
\]
a homotopy equivalence, $k \geq 3$. Let $G' : (E,bE) \rightarrow (W,bW)$ 
be a homotopy inverse for $G$. Let $t(X)$ denote the tangent 
bundle of $X$, where $X$ is a PL-manifold.

**Theorem 2**: Let $M$ be in $\mathcal{M}_n$ and $k \geq 3$. Then 
there exists 
\[
f : M \rightarrow F/PL
\]
so that 
\[
\begin{array}{ccc}
P & \xrightarrow{f} & F/PL \\
E & \xrightarrow{p} & M & \xrightarrow{f} & F/PL & \rightarrow B/PL
\end{array}
\]
classifies $(G')^*(t(W)) \oplus t(E)$ and so that the 
Bundle Problem for $(E,M;G)$ is solvable when 

1) $n = 4i$ if and only if a certain linear combination 
of Pontryagin numbers of $(M,f)$ vanishes; 

2) $n = 4i+2$ and $M$ is smoothable if and only if a certain 
linear combination of Stiefel-Whitney numbers of $(M,f)$ 
vanishes.

**Proof**: First let $f : M \rightarrow F/PL$ correspond to the 
$F(PL)_{h}$-bundle, $(E_0,t)$, $k \gg n$, and consider the sign-
ificance of $n(M,f)$. Then 
\[
p_0 \times t : E_0 \rightarrow M \times D^k
\]
determines a PL-structure on $M \times D^k$. 

Now \( n(M, f) \) is defined so that \( n(M, f) = 0 \) if it may be changed by a homotopy \( H \) so that it "splits". Then \( p_o \times H \) is a homotopy between \( p_o \times t \) and \( b \) say, where \( b \) is \( t \)-regular to \( K \times 0 \) and

\[
\begin{array}{c}
L = b^{-1}(c) \subset E_o \xrightarrow{b} H \times D^k \\
\end{array}
\]

is a homotopy equivalence. Now it is easy using the \( h \)-cobordism theorem to alter \( b \) slightly so that \((I, g, b)\) is a solution to the Bundle Problem for \((K \times D^k, M; p_o \times t)\).

Let \(-E\) denote a high dimensional inverse for \( E \xrightarrow{p} M \). Then \((p^*(-E), p^*(p))\) is a bundle over \( E \) and \((p^*(-E) = E \oplus -E, p \cdot p^*(p))\) may be identified with the trivial bundle \((K \times D^k, p_1)\) over \( M \).

Now \( b(G) \) and \( q = p \cdot p^*(p) \) are homotopy equivalences so we can find an embedding \( i : N \rightarrow \text{int}(pG)^*(-E) \) so that

\[
qb(G) i \simeq \text{id} M \quad \text{and} \quad iq b(G) \simeq \text{id}(pG)^*(-E).
\]
Using the $h$-cobordism theorem we may identify 
$(pG)^*(-E)$ with the normal bundle of $i(M)$ in $G^*(-E)$
(which exists if $(pG)^*(-E)$ has large enough dimension.)

Thus $(pG)^*(-E)$ is a PL-bundle over $M$ and 
$t = p_2b(id)b(G)$ is an $F$-trivialization. Let 
$r:(pG)^*(-E) \rightarrow M$ be the projection of this bundle. Then

\[ r \times t = r \times p_2(b(id)) \cdot b(G) \]
\[ \simeq qb(G) \times p_2(b(id)) b(G) \]
\[ = p_1b(id)b(G) \times p_2(b(id)) b(G) \]
\[ = b(id) b(G), \]

where the homotopy is a homotopy of maps of pairs, 

\[ (pG)^*(-E), bpg^*(-E)) \rightarrow \pi \times (D^k, bd^k). \]

If we let $f:M \rightarrow F/PL$ classify $(pG)^*(-E), t)$,
then the above analysis shows that $n(M, f) = 0$ if and
only if we can solve the bundle problem for $(M, (pG)^*(-E); b(G))$. 
Let $(L, g)$ be such a solution.

Now consider the commutative diagram

```
PL(M)  \rightarrow^p \rightarrow PL(E)  \rightarrow^q \rightarrow PL(p^*(-E)) \rightarrow \rightarrow (p^*(p))^* \rightarrow \rightarrow
```

\[ (p^*(p))^* \]
Since $k \geq 3$ $(p^*(p))^*$ is bijective. Now

$$q^*(L, g) = ((pG)^*(-E), b(G)) = (p^*(p))^*(W, G).$$

Thus $p^*(L, g)$ is concordant to $(W, G)$.

Let $v_i(M)$ = normal bundle of $M \xrightarrow{i}(pG)^*(-E)$
and $t(x)$ the tangent bundle of $X$. Then

$$t(M) \oplus v_i(M) = i^*(t((pG)^*(-E)))$$

$$= i^*((q_b(G))^*(-E) \oplus (G^*(p^*(p))^*t(W)))$$

$$= (q_b(G)i)^*(-E) \oplus (G'(O\text{-section})^*t(W))$$

$$= (-E) \oplus (G' \cdot O\text{-section})^*t(W).$$

So

$$p^*v_i(M) = (-p^*t(M)) \oplus p^*(-E) \oplus G^*t(W)$$

$$= G^*(t(W)) \oplus -(t(E)).$$

This shows that $(M, f)$ has almost all of the desired properties. The theorem follows since $n(M, f)$ is given by the homomorphism

$$\bigoplus_{n}^{\text{PL}} (F/\text{PL}) \xrightarrow{K} P_n.$$ If $n = 0$, $P_n = Z$ and elements in $\bigoplus_{n}^{\text{PL}} (F/\text{PL}) \otimes Q$ are determined by their Pontryagin numbers. Thus $K$ may be
expressed as a rational linear combination of Pontryagin numbers. If $n = 4i + 2$, $P_n = \mathbb{Z}_2$ and $K/\Sigma_n^-(F/PL)$ may be factored through $\gamma_n(F/PL)$ since

a) kernel $(\Sigma_n(F/PL) \to \gamma_n(F/PL))$ is divisible by 2

b) $\gamma_n(F/PL) = \text{direct sum of } \mathbb{Z}_2$'s

Then the factor homomorphism

$$\gamma_n(F/PL) \to \mathbb{Z}_2$$

may be expressed as a linear combination of Stiefel-Whitney numbers.

**Addendum:** The form of the linear combinations appearing in Theorem 24 depends only on the dimension of $M$.

We remark that there is a similar solution when $n = 4i + 2$ and $K$ is merely a PL-manifold.

In case $n = 4i$, the classes of $H^*(F/PL, \mathbb{Q})$ occurring in i) come from $H^*(B_{PL}, \mathbb{Q})$ so tangential information about $G$ is sufficient (and necessary) to solve the bundle problem.

It is quite an interesting and difficult problem to determine the classes of $H^*(F/PL, \mathbb{Z}_2)$ occurring in ii) and their relationship to $H^*(B_{PL}, \mathbb{Z}_2)$.

This has an important bearing on the obstruction theory described in Theorem 26.
Applications to PL-Homeomorphisms

Now we may apply the results of the previous sections to study the manifolds within a given homotopy equivalence class.

Theorem 25: Let \( \mathscr{M} \) belong to \( \mathcal{H}_n \). Then there is a one to one correspondence

\[ \text{PL}(M) \xrightarrow{C} (M,F/PL) \]

between the set of concordance classes of PL-structures on \( M \) and the set of free homotopy classes of maps of \( M \) into \( F/PL \).

If \( M \) is in \( \mathcal{H}_n \) and \( bM = 0 \), then \( \text{PL}(M) \) is in one to one correspondence with \( (M_0,F/PL) \) where \( M_0 = M - \text{int} \ D^n \). Such a correspondence is given by the composition

\[ \text{PL}(M) \xrightarrow{r} \text{PL}(M_0) \xrightarrow{C} (M_0,F/PL), \]

where \( r \) is induced by \( M_0 \subseteq M \).

\( C \) is natural with respect to inclusions \( M_1 \subseteq \text{int} M_2 \) in \( \mathcal{H}_n \) and \( C(M,\text{id}) = \text{pt map} \). In general if \( (L,g) \) is a PL-structure on \( M \) and \( g' \) is a homotopy inverse for \( g \) then

1) \( C(L,g) = 0 \) if \( g \) is homotopic to a PL-homeomorphism
ii) $b^*(C(L,g))$ classifies

\[ t(M) \otimes -(g')^*t(L) \quad \text{if } bM \neq 0 \]
\[ (t(M) \otimes -(g')^*t(L))/\mathcal{O} \quad \text{if } bM = 0 \]

where $t(M)$ is the tangent bundle of $M$ and $F/\mathcal{O} \xrightarrow{b} B_{PL}$ is the fibre of $B_{PL} \xrightarrow{J} B_F$.

**Proof:** This follows from Theorems 9, 10, 14 and Lemmas 1 and 21.

**Definition 22:** By a k-skeleton of $M$ we mean a subcomplex $L \xrightarrow{i} M$ where $L$ has the homotopy type of a k-dimensional complex and $i$ is k-connected i.e. the relative groups $\pi_i(M,L)$ are zero for $1 \leq k$.

We say that the k-skeleton $L \xrightarrow{i} M$ is a thickened k-skeleton of $M$ if $L \xrightarrow{i} M$ is a morphism of $\vec{M}_n$.

We say that $f:(L,bl) \xrightarrow{f} (M,bh)$ is homotopic to a PL-homeomorphism over the k-skeleton of $M$ if for any thickened k-skeleton $L_k \xrightarrow{f} L$ there is a homotopy equivalence $f:(L,bl) \xrightarrow{f} (M,bh)$ such that

1) $f$ is homotopic to $g$
2) $f/L_k$ is a PL-embedding.
3) if $L_k^1 = L$-int $L_k$ and $M$-int $f(L_k)$ then $f$ induces a homotopy equivalence $(L_k^1,bl_k^1) \xrightarrow{f} (M_k^1,bh_k^1)$.
Theorem 26: Let $\mathcal{M}$ be in $\mathcal{M}_n$ and let $\mathcal{M}_o$ denote $\mathcal{M}$ if $b\mathcal{M} \neq 0$ and $\mathcal{M}$-int $\mathcal{D}^n$ if $b\mathcal{M} = 0$. Let $g: (L, bL) \to (M, bM)$ be a homotopy equivalence.

Then there is a map

$$\mathcal{M}_o \xrightarrow{C_g} \mathcal{F}/\mathcal{PL}$$

with the following two properties:

i) $\mathcal{M}_o \xrightarrow{C_g} \mathcal{F}/\mathcal{PL} \xrightarrow{b} \mathcal{B}_{\mathcal{PL}}$ classifies

$$(t(M) \ominus (g')^*t(L))/\mathcal{M}_o$$

ii) $C_g/(k$-skeleton of $M)$ is homotopic to the point map if and only if $g$ is homotopic to a PL-homeomorphism over the $k$-skeleton of $M$.

Proof: Let $C_g = C(L, g)$ and suppose $M_k \simeq M$ is a thickened $k$-skeleton of $M$. We can change $g$ by a homotopy so that it induces a PL-structure $(L_k, g_k)$ on $M_k$. Since $C$ is a natural transformation, $C(L_k, g_k) = Cg/M_k$. Since $C$ is injective, $g_k: (L_k, bL_k) \to (M_k, bM_k)$ is homotopic to a PL-homeomorphism if and only if $C_g/M_k$ is homotopic to zero. Now $C_g/(k$-skeleton of $M_o)$ is homotopic to zero if and only if $C_g/M_k$ is homotopic to zero (by obstruction theory) and $g_k$ is homotopic to a PL-homeomorphism if and only if $g$ satisfies 1), 2), and 3) of Definition 22 above by the homotopy extension theorem. This completes the proof.
Corollary: If $\tilde{M}$ belongs to $\tilde{H}_n$ there is an obstruction theory for the problem of deforming a homotopy equivalence $(L, bL) \sim_{C} (M, bM)$ into a PL-homeomorphism. The obstructions lie in

$$H^{2}(\tilde{M}_0, \mathbb{Z}_2), H^{4}(\tilde{M}_0, \mathbb{Z}), \ldots, H^{4}(\tilde{M}_0, \Omega_1(F/PL)), \ldots.$$ 

Suppose $M_1 \subset \text{int} \ M_2 \subset \text{int} \ M_3 \subset \ldots \subset M_n = M$

form an increasing sequence of thickened skeletons of $M$ and $g$ is homotopic to $f$, a PL-homeomorphism over $M_k$. Then there is a cohomology class $O_{k+1}$ in $H^{k+1}(\tilde{M}, \Omega_{k+1}(F/PL))$ which is the precise obstruction to changing $f$ by a homotopy on the complement of $M_{k-1}$ so that it becomes a PL-homeomorphism over the $k+1$-skeleton $M_{k+1}$.

The obstruction class $O_{k+1}$ may be computed as follows. Let $M_{k+1}^! = M_{k+1} \cap \text{int} \ M_k$ and $L_{k+1}^! = L_{k+1} \cap \text{int} \ L_k$.

Assume $f/L \cap \text{int} \ L_k$ induces a PL-structure on $M_{k+1}^!$
\( f/\partial L_k \) is a PL-homeomorphism so let

\[
M_\ast = (M_k + 1 \cup b M_k \cdot M_{k+1}) \times D^r
\]

\[
L_\ast = (M_{k+1} \cup_{f-1} L_{k+1}) \times D^r
\]

and define

\[
f_\ast: L_\ast \longrightarrow M_\ast
\]

by (identity \( \cup f \)) \( x \) identity \( D^r \), \( r \) large.

Now \( H_{k+1}(M_{k+1}, bM_k) \) corresponds to the \( k \)th chain group of \( H_\ast(M_0) \) and is naturally embedded in \( H_{k+1}(M_\ast) \).

Each class \( u \) in \( H_{k+1}(M_{k+1}, bM_k) \) may be represented by a relative \( (k+1) \) disk in \( (M_{k+1}, bM_k) \times D^r \) and thus corresponds (by doubling) to an \( S^{k+1} \times D^s \) in \( M_\ast \), where \( k+1+s = n+r \).

Now change \( f_\ast \) by a homotopy so that it induces a PL-structure on \( S^{k+1} \times D^s \) and let

\[
(E, t) = (f_\ast^{-1}(S^{k+1} \times D^s), f_\ast/f_\ast^{-1}(S^{k+1} \times D^s)).
\]

Then \( (E, p_2 \cdot t) \) may be identified to an \( F/PL \) \( r \)-bundle over \( S^{k+1} \) and thus corresponds to some element \( C_{k+1}(u) \) in \( \mathcal{M}_{k+1}(F/PL) \). The cochain \( C_{k+1}(u) \)

a) is a cocycle

b) vanishes iff if \( f \) can be changed by a homotopy (which is constant on \( M_k \)) to a PL-homeomorphism over \( M_{k+1} \).
c) is cohomologous to zero iff $f$ can be changed by a homotopy (which constant on $M_{k-1}$) to a PL-homeomorphism over $M_{k+1}$.

This may be proved using the cobordism obstruction theory of Part I.

Now we relate the obstructions to deforming $g:L \to M$ into a PL-homeomorphism to a sequence of Bundle Problems.

**Lemma** (Stability) $(L, bL) \xrightarrow{f} (N, bN)$ is homotopic to a PL-homeomorphism over the $k$-skeleton of $M$ iff $g \times \text{id} : (L, bL) \times D^r \to (N, bN) \times D^r$ is homotopic to a PL-homeomorphism over the $k$-skeleton of $M \times D^{k}$.

**Proof:** This follows from Lemma 22 and Theorem 26.

The lemma shows that it suffices to consider the obstructions in dimensions which are small with respect to that of the ambient manifold.

So assume that $i \ll n$ and consider $x$ in $H_{i+1}(M_0; \mathbb{Z})$ or $H_{i+2}(M_0; \mathbb{Z}_2)$. Using (5) we may represent some odd multiple of $x$ by an embedded submanifold $M_i \subset \text{int } M$. We may assume that $M_i$ is simply connected (since $M$ is), smoothable, and that $M_i$ has a normal disk bundle $E$. 
(We may cross $K$ with $D^k$ to construct $E$ if necessary.)

Now change
\[ g : (L, bL) \longrightarrow (M, bM) \]

by a homotopy so that it induces a PL-structure on $E$
and let $(W, G) = (L, g)/E$. Then $(W, G)$ defines a Bundle Problem for $E$. $(E, M^i, G)$ is called a Bundle Problem associated to $x$ by $g$.

**Definition 23:** We say that $x$ in $\tilde{H}_{n+i}(M, Z)$ or $\tilde{H}_{n+i+2}(M, Z_2)$ "splits" the homotopy equivalence $g : (L, bL) \longrightarrow (M, bM)$ if some Bundle Problem $(E, M^i, G)$ associated to $x$ by $g$ has a solution.

**Theorem 27:** Let $K$ be in $\pi_n$ and $g : (L, bL) \longrightarrow (M, bM)$ be a homotopy equivalence. Let $C^g : M_0 \rightarrow F/PL$ represent $C(L, g)$ and suppose $C^g$ is homotopic to the point map over the $k$-skeleton of $M_0$, $k$ odd, $k < n$. Let $O_{k+1}$ in $H^{k+1}(M_0, \pi_{k+1}(F/PL))$ be the obstruction to extending the homotopy of $C^g$ with the point map over the $k+1$ skeleton of $M_0$. Then if $x$ is in $H^{k+1}(M_0, \pi_{k+1}(F/PL))$, $O_{k+1}(x) = 0$ if $x$ "splits" $g$. 
Proof: Let $L \subset \text{int } M$ represent some odd multiple of $x$ in $H^k(M; \pi_k(F/PL))$. Suppose $E$ is a normal disk bundle of $L \subset M$ and $(W, G) = (L, g)/E$. Let $C_g = C(W, G)$. By the naturality of $C$, $C_g/(E/L\cdot pt) = C_g/(E/L\cdot pt) = 0$ by hypothesis. The obstruction to deforming $C_g = C_g/E$ to a point then it just $O_{k+1}(\text{gen } H_{k+1}(E))$ by the naturality of the obstruction. Thus $G$ is homotopic to a PL-homeomorphism if $O_{k+1}(x) = 0$. Thus if $O_{k+1}(x) = 0$ it is clear that $x$ "splits" $g$. In fact, we have shown that in this case any Bundle Problem associated to $x$ is solvable.

Now suppose that $x$ "splits" $g$. Let

$$
\begin{array}{ccc}
W & \xrightarrow{G} & E \\
\downarrow & & \downarrow \\
N & \xrightarrow{f} & L'
\end{array}
$$

be the solution of some Bundle Problem associated to $x$. Now $G$ is homotopic to a PL-homeomorphism over the $k$-skeleton of $E$ so it follows from Lemma 22 and Theorem 26 that $f$ is homotopic to a PL-homeomorphism over the $k$-skeleton of $L$. But then Lemma 21 implies $f$ is homotopic to a PL-homeomorphism. Thus the same is true of $G$, so $O_{k+1}(x) = 0$.

The rationale behind Theorem 27 is the following. It is possible to "compute" the homomorphisms determined by obstructions to deforming $g: L \rightarrow H$ to a PL-homeomorphism "before" we begin the deformation. We merely
cross \( M \) with a high-dimensional disk, represent odd multiples of all the pertinent homology classes by embedded submanifolds and look at the associated Bundle Problems.

We state three Corollaries which follow from the existence of the obstruction theory. Let \( M \) belong to \( \mathcal{M}_n \) and let \( f : (L, bL) \to (M, bM) \) be a homotopy equivalence.

**Corollary 1** Suppose
\[
\begin{align*}
H^{i+2}_i(M, \mathbb{Z}_2) &= 0 \\
H^i(M, \mathbb{Z}) &= 0 \quad i > 0.
\end{align*}
\]

Then \( f \) is homotopic to a PL-homeomorphism.

**Corollary 2** Suppose \( f^*(p_1(M,\mathbb{Q})) = p_1(L,\mathbb{Q}) \) for
\[
i \leq \left\lfloor \frac{\dim M}{4} \right\rfloor,
\]
and
\[
\begin{align*}
H^{i+2}_i(M, \mathbb{Z}_2) &= 0 \\
H^i(M, \mathbb{Z}) &\text{ is free}
\end{align*}
\]
then \( f \) is homotopic to a PL-homeomorphism. (Note that Corollary 2 applies if \( f \) is a homeomorphism.)

**Corollary 3** Suppose \( f \) is a stable PL-tangential equivalence and that
\[
H^{i+2}_i(M, \mathbb{Z}_2) = 0 \quad \text{and}
\]
(Torsion $H^i(M, Z) \otimes \mathbb{Z}_p = 0$, where $S_i$ is defined in Corollary 2 of Theorem 20. Then $f$ is homotopic to a PL-homeomorphism.

Proof: These results follow from Theorem 25 and the Corollaries to Theorem 20.

Corollaries 1 and 2 are positive results towards the Hurewicz Conjecture and the Hauptvermutung. There are many examples showing why the conditions of Corollary 1 are necessary. (See below).

Definition 23: Let $(M)$ denote the set of PL-homeomorphism classes of manifolds which are homotopically equivalent to $M$. Let $L$ represent an element of $(M)$ and let $g:L \rightarrow M$ be a homotopy equivalence. Let $P_\ast(M)$, $H(M)$, and $T_\ast(M)$ be the subsets of $(M)$ defined respectively by those $L$ such that $g$ can be chosen to be

- $P_\ast(M)$: a correspondence of rational Pontryagin classes
- $H(M)$: a homeomorphism
- $T_\ast(M)$: a stable PL-tangential equivalence.

Remark: It is clear that $T_\ast(M) \subset P_\ast(M)$. It follows from Novikov (20) that $H(M) \subset P_\ast(M)$. 
Recall that $P_\ast(X,F/PL)$ denotes the subset of $(X,F/PL)$ consisting of those maps which induce trivial cohomology homomorphisms with $\mathbb{Q}$-coefficients in positive dimensions and $T_\ast(X,F/PL)$ denotes the kernel of

$$(X,F/PL) \xrightarrow{b_\ast} (X,B_{PL}).$$

Let $M$ belong to $\overline{\mathcal{M}}_n$ and let $M_0$ denote $\text{int } D^n$ if $bM = 0$ and $M_0 = M$ if $bM \neq 0$. Then as corollaries to Theorem 25 we obtain

Corollary \textsuperscript{4} There is a natural projection of $(M_0,F/PL)$ onto $(M)$. $P_\ast(M_0,F/PL)$ is carried onto $P_\ast(M)$ and the image of $T_\ast(M_0,F/PL)$ contains $T_\ast(M)$.

**Proof:** The first statement follows from the fact that there is a natural projection of $PL(M)$ onto $(M)$. The map $(L,g) \rightarrow (L)$ is well-defined because of the $h$-cobordism theorem. So

$$(M_0,F/PL) \xrightarrow{G^{-1}} PL(M) \xrightarrow{\pi} (M)$$

is the desired projection.

The second statement follows from b) of Theorem 25. The fact that $P_\ast(M_0,F/PL)$ goes "into" $P_\ast(M)$ is clear when $bM \neq 0$ or $\dim M \neq 4i$. In the excluded case it is still true because of the Hirzebruch Index Formula.
It is not so clear that $T_\mathbb{K}(N_0,F/PL)$ is carried into $T_\mathbb{K}(N)$ if $bM = 0$ and $n = 5k+2$.

**Corollary 5** $P_\mathbb{K}(N)$ is finite. In fact if $d = \lceil \frac{\dim \mathbb{K}}{k} \rceil$, 

$$\text{card } P_\mathbb{K}(N) \leq \text{ord}(\bigoplus_{i=0}^{d} (H^{4i+2}(\mathbb{K}_0,\mathbb{Z}_2) \otimes \text{Torsion } H^{4i}(\mathbb{K}_0,\mathbb{Z}))).$$

**Corollary 6** $H(N)$ is finite if $H^{4i}(\mathbb{K}_0,\mathbb{Q}) = 0$ for $i > 0$. $H(N)$ is finite. In fact if $H_1 = (E_1/D_1)S_1$ where $E_1$ is the index of $H^{4i}(B_{\text{Top}}) \to H^{4i}(B_0) \otimes P_1 \mathbb{Z}$.

then

$$\text{card } H(N) \leq \text{ord}(\bigoplus_{i=0}^{d} (H^{4i+2}(\mathbb{K}_0,\mathbb{Z}_2) \otimes (\text{Torsion } H^{4i}(\mathbb{K}_0,\mathbb{Z}) \otimes H_1))).$$

**Corollary 7**

$$\text{card } T_\mathbb{K}(N) \leq \text{ord}(\bigoplus_{i=0}^{d} (H^{4i+2}(\mathbb{K}_0,\mathbb{Z}_2) \otimes (\text{Torsion } H^{4i}(\mathbb{K}_0) \otimes Z_{S_1}))).$$

**Proof:** 5) and 7) follow from Corollaries 3) and 4) of Theorem 20. 6) follows by a completely analogous argument.

**Examples:**

1) We give some examples which illustrate the conditions in Corollary 1.

These examples arise from the maps

$$s^{2k} \xrightarrow{\text{gen}} \mathbb{K}^{2k} \xrightarrow{f} F/PL$$
and
\[ S^{2k-1} \cup_{e^{2k}} d \cong S^{2k} \xrightarrow{\text{gen}} \mathbb{P}^{2k}_F \]

In the first case if we let \( M = S^k \times D^r \), \( r \) large, and \((L,g)\) in \( \text{PL}(M) \) correspond to \( f \), then for \( k \geq 2 \) and \( 0,1,2(\text{mod } 4) \) \( L \) is not even homeomorphic to \( M \). This follows from the fact that \( \mathbb{P}^* \rightarrow \mathbb{P}^*\text{B}_{40}^r \) is a monomorphism (24).

In the second case if we let \( M \) (regular neighborhood of \( S^{2k-1} \cup_{d} e^{2k} \) in \( S^r \)), \( r \) large, and \((L,g)\) correspond to \( f_d \) then for certain \( d \), \( M \) and \( L \) are not tangentially equivalent.

If \( k = 2i \), then \( L \) is not a \( \mathbb{W} \)-manifold if \( d \) does not divide \( S^i \).

For any \( k \) \( L \) is not smoothable if \( d \) does not divide the generator of \( \mathbb{E}_{2k-1}(\mathbb{W}) \) in \( \mathbb{E}_{2k-1} \) (e.g. if \( k = 8m + 2 \), \( m > 0 \), and \( d \) is some large power of 2 then \( L \) is not smoothable.)

The first set of examples illustrates the Hom conditions in Corollary 1; the second set the Ext conditions.

The phenomenon of \( L \) above not being smoothable can be explained by a Bockstein operation relating the \( F/\text{PL} \) obstructions to \( F/\text{O} \) obstructions in an analogous smooth theory.
2) We construct a closed smooth $S$-manifold $M$ such that

i) there is homotopy equivalence $g: M \to \mathbb{C}P^k$ such that $g^* t(\mathbb{C}P^k) = t(M)$, where $t(X)$ denotes the smooth stable tangent bundle of $X$

ii) $M$ is not PL-homeomorphic to $\mathbb{C}P^k$.

Consider the map

$$
\mathbb{C}P^k - \text{pt} \xrightarrow{h} \mathbb{C}P^3 \xrightarrow{\text{deg} 1} S^6 \xrightarrow{\text{gen}} \prod_6 \to F/\text{PL},
$$

where $h$ is some homotopy equivalence. Let $(M_0, g_0)$ correspond to $f$ in $\text{PL}(\mathbb{C}P^k_0)$. Let $(M, g) = \text{cone on } (M_0, g_0)$ in $\text{PL}(\mathbb{C}P^k)$.

i) Now $g_0$ is a stable PL tangential equivalence because $\pi_6(F/\text{PL}) \xrightarrow{b} \pi_6(\mathbb{E}_{\text{PL}})$ is the zero map.

Thus $g$ is a stable PL tangential equivalence because any stable fibre homotopically trivial PL bundle over $S^6$ with zero $p_2$ is trivial.

Thus we can smooth $M$ so that i) holds.

ii) $(M_0, g_0)$ is not concordant to zero because $f^* (\mathbb{Z}_2 \text{coeff}) = 0$ in dimension 6, by Theorem 20.

Thus $(M, g)$ is not concordant to zero.

Now $\mathbb{C}P^k$ has two homotopy classes of homotopy equivalences and each of these is represented by a
PL-homeomorphism. Thus \( M \) is not PL-homeomorphic to \( \mathbb{CP}^4 \).

The same argument works for \( \mathbb{CP}^6 \) using

\[
\mathbb{CP}^7 \longrightarrow S^{14} \longrightarrow F/\text{PL}.
\]

Now we briefly describe how Theorem 9 may be used to study the pseudo-isotopy classes of PL-homeomorphisms within a given homotopy class. It suffices to consider the homotopy class of the identity.

Let \((c, H)\) be a pair consisting of a PL-homeomorphism \(M \overset{c}{\longrightarrow} M\) and a homotopy

\[
(M, \partial M) \times I \overset{H}{\longrightarrow} (M, \partial M) \times I
\]

between \(c\) and the identity. The pair \((c, H)\) determines a PL-structure on \(M \times I\), \((M \times I, \cup^c M, H \cup \text{id})\) which restricts to \(0 = (M \times \{1\}, \text{id})\) on \(M \times \{1\}\). Thus \((c, H)\) determines an element in \(\text{PL}(M \times (I, \partial I); 0)\).

![Diagram](image-url)
If $M$ is in $\overline{M}_0$ we can apply the relative h-collapsing theorem to show that if $(c_1, H_1)$ and $(c_2, H_2)$ determine the same element in $\text{PL}(h^x(I, bI), 0)$ then $c_1$ and $c_2$ are pseudo-isotopic. Furthermore this pseudo-isotopy extends to a homotopy $H'$ between $H_1$ and $H_2$. The converse is true if $H'$ preserves appropriate subspaces.

If $M$ is in $\overline{M}_0$, Theorem 9 implies $\text{PL}(h^x(I, bI); 0)$ is in one-to-one correspondence with $B_k(h^x(I, bI); v)$.

By Theorems 12 and 14 $B_k(h^x(I, bI); v)$ is in one-to-one correspondence with $(\text{susp}, M, F/\text{PL})$.

Thus we can classify pairs $(c, H)$ for $M$ in $\overline{M}_0$ by elements of

$$(\text{susp } M_0, F/\text{PL})$$

\[ F/\text{PL} \leq B_{\text{PL}} \rightarrow B_{\text{PL}} \]

1) $F/\text{PL} = K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6) \times K(\mathbb{Z}, 8) \times \cdots$

mod class of odd torsion groups.

2) $F/\text{PL} \rightarrow B_0$

mod class of 2-torsion groups

\[ (1 + l_1 + l_2^+ + \cdots) \times (1 + l_1 + l_2^+ + \cdots) \]

$\Rightarrow \lambda = \beta_0$
Bibliography


3. ________, Manifolds with $\pi_1 = \mathbb{Z}$, (to appear).


