Sample Final Exam

MAT 515, Fall 2019

December 11, 2019

Stony Brook University

Name: (please print)	**ID #**:

	1	2	3	4	5	6	7	8	9	**Total**
	10pts	10pts	10pts	10pts	12pts	12pts	12pts	12pts	12pts	100pts
Grade										

No notes or books.

You must provide explanation, not just the answer (unless otherwise is stated).

Answers without justification will get only partial credit.

Please cross out anything that is not a part of your solution — e.g., some preliminary computations that you didn't need.

Instructor: Dzmitry Dudko

1. *(10 pts)*
Indicate whether each of the statements below is True **(T)** or False **(F)**.
No explanation is required.

(a) There is a triangle ABC such that $AB = 4$, $AC = 6$, and $BC = 9$.
True. (Because $9 < 6 + 4$ – the triangle inequality is satisfied.)

(b) In a triangle ABC, the external angle of A is equal to the sum of the internal angles of B and C.
True.

(c) Suppose $AB > BC$ in a triangle ABC. Then $\angle C > \angle A$.
True.

(d) There is a right triangle ABC such that $\angle A = 60°$ and $\angle B = 50°$.
False. (Because $60° + 50° \neq 90°$.)

(e) Let AB and CD be two chords of a circle with center O. If $AB > CD$, then the distance between O and AB is greater than the distance between O and CD.
False.

(f) The sum of opposite angles in a parallelogram is $180°$.
False.

(g) There is a triangle that has 6 axes of symmetry.
False. (A triangle has at most 3 axes of symmetry.)

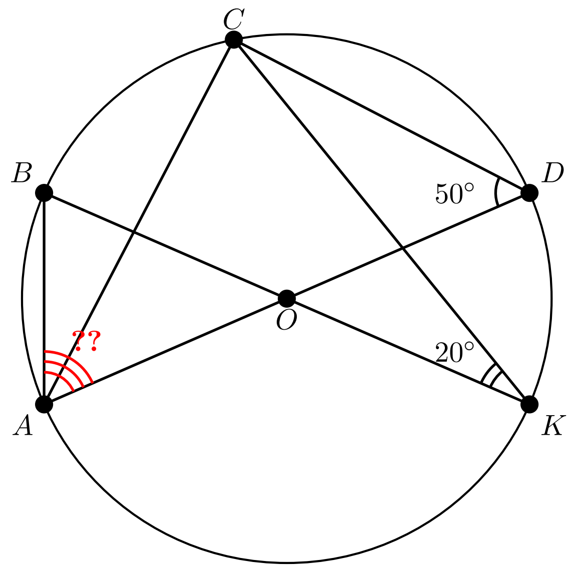
(h) A square is an inscribed and circumscribed quadrilateral.
True.

(i) If $ABCD$ is a rectangle, then the distance between A and B is equal to the distance between AD and BC.
True.

(j) A median always splits a triangle into two similar triangles.
False.

2. *(10 pts)*

On the figure below $\angle BKC = 20^\circ, \angle ADC = 50^\circ$ and BK, AD are diameters. Compute $\angle BAD$.



Solution. We have:

$$\overset{\frown}{BC} = 40^\circ, \quad \overset{\frown}{AB} + \overset{\frown}{BC} = 100^\circ, \quad \overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} = 180^\circ,$$

Therefore, $\overset{\frown}{BC} + \overset{\frown}{CD} = 120^\circ$ and $\angle BAD = 60^\circ$. □

3. *(10 pts)*

Let $ABCD$ be a trapezoid with parallel bases AB and CD. Prove that the internal angle bisectors of the angles adjacent to the lateral side BC are perpendicular to each other.

Solution. Suppose BB_1 is the bisector of $\angle ABC$ and CC_1 is the bisector of $\angle BCD$. Then $\angle B_1BC = \frac{1}{2}\angle ABC$ and $\angle BCC_1 = \frac{1}{2}\angle BCD$. Since $\angle ABC + \angle BCD = 180^\circ$, we have $\angle B_1BC + \angle BCC_1 = 90^\circ$. Therefore, BB_1 and CC_1 are perpendicular.

□

4. *(10 pts)*
Let $ABCD$ be a circumscribed trapezoid with perimeter 4 (i.e., $AB+BC+CD+DA=4$). What is the length of the midline of $ABCD$?

Solution. Suppose $AD||BC$. Since $ABCD$ is circumscribed, we have $AB+CD=AD+BC$. Hence $AD+BC=2$ because $AB+BC+CD+DA=4$. The midline of $ABCD$ is $\frac{1}{2}(AD+BC)=1$.

□

5. *(12 pts)*

Consider a triangle ABC. Suppose that M is the intersection of the medians of $\triangle ABC$ and N is the intersection of the altitudes of $\triangle ABC$. Show that if $N = M$, then $\triangle ABC$ is equilateral.

Solution. Suppose AA_1, BB_1, CC_1 are the medians of $\triangle ABC$ and AA_2, BB_2, CC_2 are the altitudes of $\triangle ABC$. Since $M = N$, we have

$$AA_1 = AA_2, \quad BB_1 = BB_2, \quad CC_1 = CC_2.$$

Then $\triangle AA_1B = AA_1C$ and $\triangle BB_1A = BB_1C$ by SAS. This shows that $\triangle ABC$ is equilatera. $\square$

6. *(12 pts)*
Four houses A, B, C, D form vertices of a square. The residents would like to dig a well at a point W such that the sum of distances $AW + BW + CW + DW$ from all the houses to the well is the smallest possible. Where should they dig the well?

Answer: W is the intersection of the diagonals. □

Solution. Let O be the intersection of AC and BD. By the triangle inequality, $AO + CO = AC \leq AW + BW$ with the equality if and only if W belongs to AC. Similarly, $BO + DO = BD \leq BW + DW$ with the equality if and only if W belongs to BD. This shows that $AW + BW + CW + DW$ is the smallest possible if and only if W is O. □

7. *(12 pts)*
Construct a trapezoid $ABCD$ with bases $BC < AD$, given AB, BC, CD, DA.

Solution. Let X be a point on AD such that $BX || CD$.

Since $AX = AD - XD = AD - BC$, we can construct AX. We also have $BX = CD$. This allows us to construct $\triangle ABX$.

Let us next construct a parallelogram $XBCD$ so that $ABCD$ is a required trapezoid. On the line AX, we construct AD containing X. Through B we construct a line parallel to AD; on this line we construct BC so that C and D are on the same side of the line AB.

□

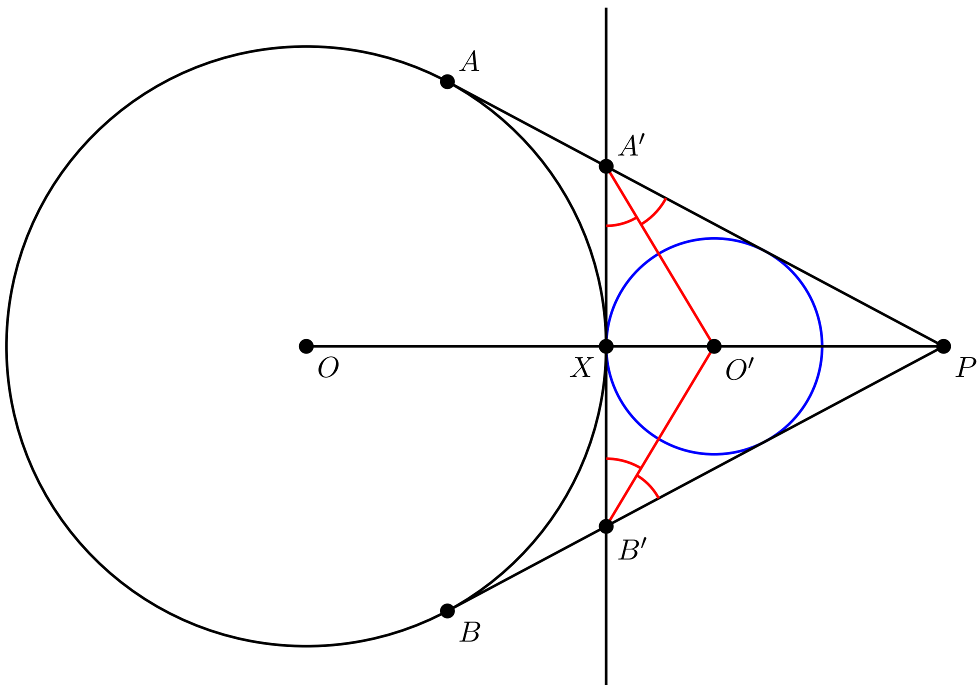
8. *(12 pts)*

Let PA and PB be two tangents from point P to a given circle such that points A and B are the points of tangency. Construct a circle tangent to the given circle and to both lines PA and PB.

Solution. Construct the center O of the given circle. Construct the segment OP; let X be the intersection of OP with the given circle. Construct the line passing through X and perpendicular to OP; this line intersects PA and PB; we denote by A' and B' the intersection points.

Note that $\triangle A'PB'$ is isosceles ($A'P = B'P$) because PX is a bisector and an altitude. The inscribed circle s of $\triangle A'PB'$ is a required circle because s is tangent to $A'B'$ at X.

Construct the bisectors of $\angle B'A'P$ and $\angle A'B'P$; mark their intersection point by O'. Then O' is the center of s while XO' is its radius. This allows us to construct s.



Remark. There is another circle s_2 tangent to the given circle and PA, PB. The center of s_2 is also on the line OP but on the left of O (assuming that P is on the right of O). The circle s_2 can be constructed in a similar way. □

9. *(6+6 pts)*

Consider a triangle ABC and let D be a point on the side AC. Suppose $AB = 20$, $AD = 16$, $BD = 12$, $DC = 9$.

(a) Prove that $\triangle ABD$ is right.

(b) Compute BC.

Solution. Let us prove that if a triangle LMN satisfies $LM^2 + MN^2 = LN^2$, then $\angle LMN$ is right. Consider a right triangle $L'M'N'$ with $L'M' = LM$ and $M'N' = MN$. By the Pythagorean theorem,

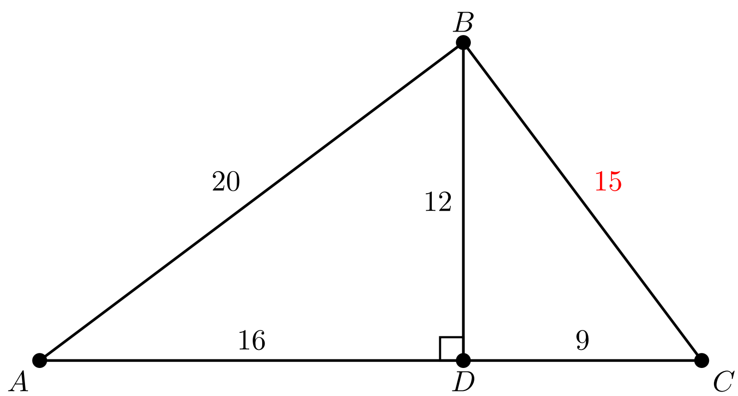
$$L'N'^2 = L'M'^2 + M'N'^2 = LM^2 + MN^2 = LN^2.$$

By the SSS-test, the triangles LMN and $L'M'N'$ are congruent; i.e., $\triangle LMN$ is right and LN is its hypotenuse

(a) Since $20^2 = 12^2 + 16^2$, the triangle BDA is right and $\angle D = 90°$.

(b) By (a), $\triangle CDB$ is also right. By the Pythagorean theorem:

$$BC = \sqrt{BD^2 + DC^2} = 15.$$



□