Final Exam
MAT 515, Fall 2019
December 11, 2019
Stony Brook University

Name: (please print)	**ID #**:

	1	2	3	4	5	6	7	8	9	**Total**
	10pts	12pts	11pts	11pts	11pts	11pts	11pts	12pts	11pts	100pts
Grade										

No notes or books.
You must provide explanation, not just the answer (unless otherwise is stated).
Answers without justification will get only partial credit.
Please cross out anything that is not a part of your solution — e.g., some preliminary computations that you didn't need.

Time:

Instructor: Dzmitry Dudko

1. *(10 pts)*
Indicate whether each of the statements below is True **(T)** or False **(F)**.
No explanation is required.

(a) There is a triangle ABC such that $AB = 6$, $AC = 8$, and $BC = 15$.

False

(b) For all A, B, C, D we have $AB + BC + CD \geq AD$.

True

(c) If a triangle has an axis of symmetry, then the triangle is isosceles.

True

(d) The sum of angles of a trapezoid is $360°$.

True

(e) A quadrilateral has 2 diagonals.

True

(f) A pentagon has 3 diagonals.

False

(g) A rhombus is a circumscribed quadrilateral.

True

(h) The three altitudes of a triangle intersect at one point.

True

(i) If $\angle BAC = 110°$, then $\triangle ABC$ is not isosceles.

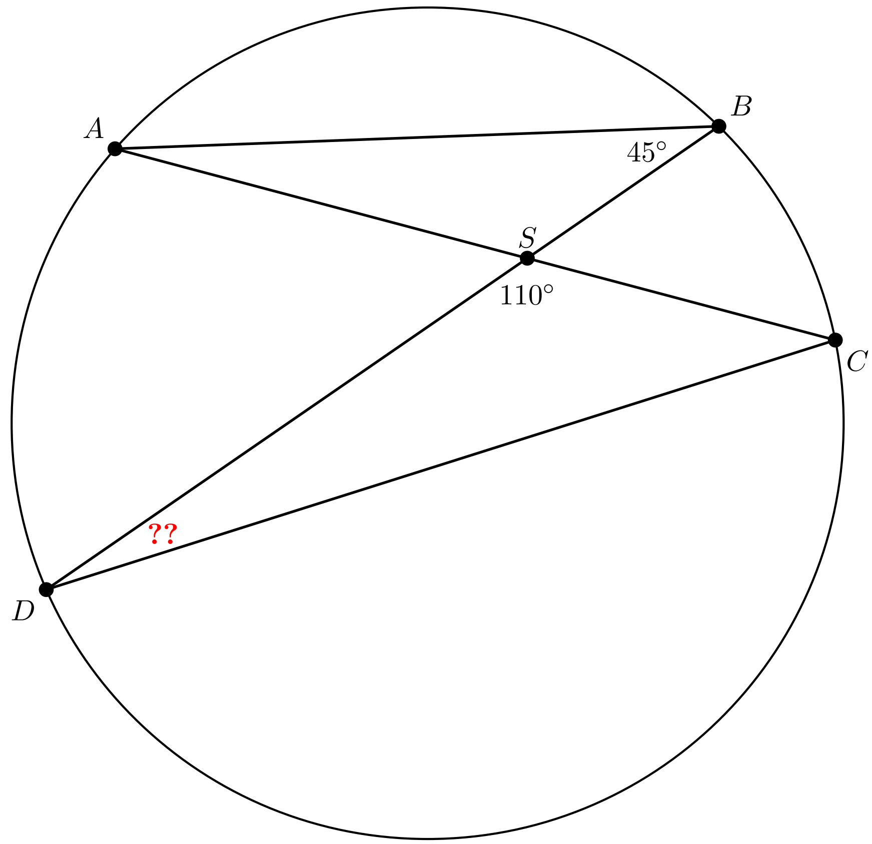
False

(j) If $ABCD$ is a rectangle, then $AC > AB$.

True

2. *(6+6 pts)*

(a) Points A, B, C, D are on a circle, AC and BD intersect at S as it shown on the figure below. Assume that $\angle DSC = 110°$ and $\angle DBA = 45°$. Find $\angle BDC$.

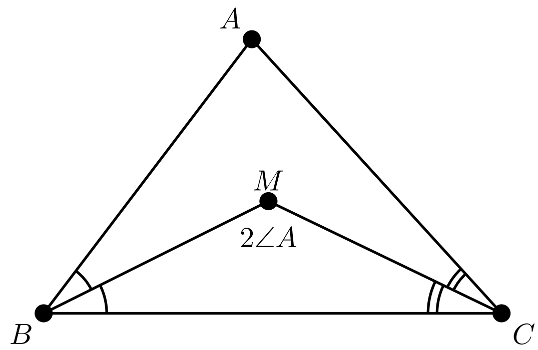


Solution. We have: $\angle DCA = \frac{1}{2}\overset{\frown}{DA} = \angle DBA = 45°$; hence

$$\angle BDC = 180° - \angle DSC - \angle SCD = 180° - 110° - 45° = 25°.$$

□

(b) Consider a triangle ABC and assume that the bisectors of $\angle B$ and $\angle C$ intersect at M. Suppose $\angle BMC = 2\angle A$. Compute $\angle A$.



Solution. We have: $\angle A = 180^\circ - \angle B - \angle C$ and

$$2\angle A = \angle BMC = 180^\circ - \frac{1}{2}\angle B - \frac{1}{2}\angle C = 90^\circ + \frac{1}{2}(180^\circ - \angle B - \angle C) = 90^\circ + \frac{1}{2}\angle A.$$

Therefore, $\frac{3}{2}\angle A = 90^\circ$ and $\angle A = 60^\circ$.

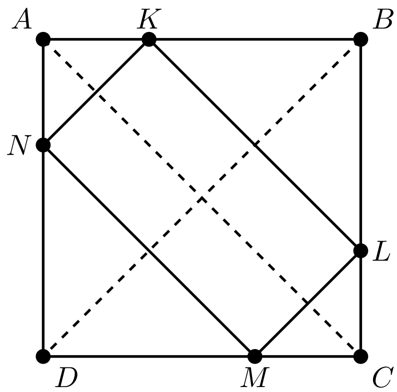
□

3. *(5+6 pts)*

Let $ABCD$ be a square, and let K, L, M, N be points on the sides AB, BC, CD, DA respectively such that $KL||AC||MN$ and $KN||BD||LM$. Prove that

(a) $KLMN$ is a rectangle; and

(b) $KL + LM + MN + NK = AC + BD$.



Solution. **(a)** $KLMN$ is a parallelogram because $KL||MN$ and $KN||LM$. The angle between KN and KL is equal to the angle between AC and BD which is $90°$. This shows that $KLMN$ is a rectangle.

(b) Since $AC = BD$, it is sufficient to prove that $KL + LM + MN + NK = 2BD$.

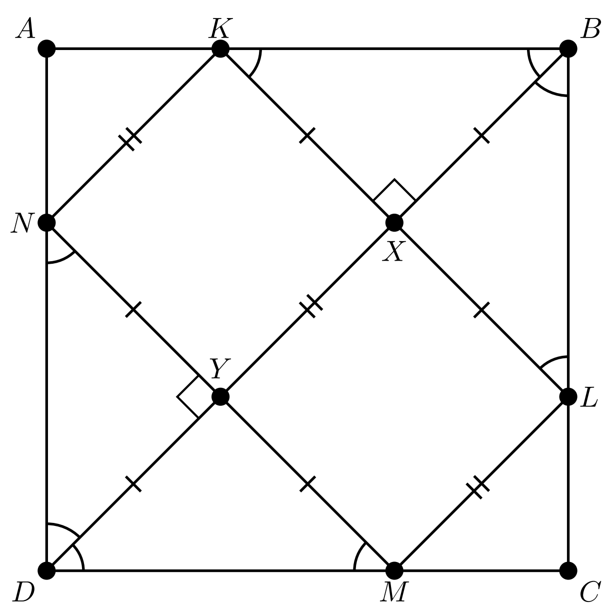
Let us denote by X and Y the intersections of BD with KL and MN respectively. Observe that

$$\angle XKB = \angle KBX = \angle XLB = \angle LBX = \angle NDY = \angle YND = \angle YDM = \angle DMY = 45°.$$

We have

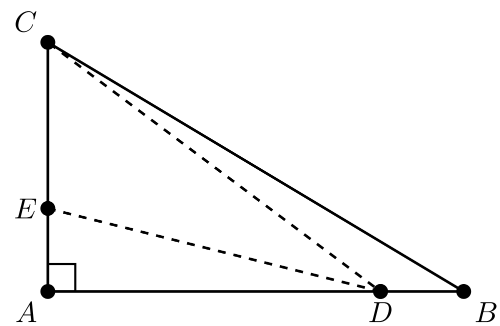
- $KN = XY = LM$ because $KNYX$ and $LMYX$ are parallelograms;
- $KX = XB = LX$ and $NY = YD = YM$ because $\triangle KXB, \triangle LXB, \triangle NYD, \triangle MYD$ are isosceles.

Therefore, $KL + LM + MN + NK = 2BD$.



□

4. *(5+6 pts)*
Let ABC be a right triangle with $\angle A = 90°$. Suppose D is a point on AB strictly between A and B.
(a) Prove that $DC < BC$.
Suppose also that E is a point on AC strictly between A and C.
(b) Prove that $DE < BC$.



Solution. **(a)** Consider $\triangle CDB$. Since $\angle CDB > 90° > \angle ABC$, we have $DC < BC$. (In a triangle, one side is longer than another side if and only if the angle opposite the first side is larger than the angle opposite the second side.)

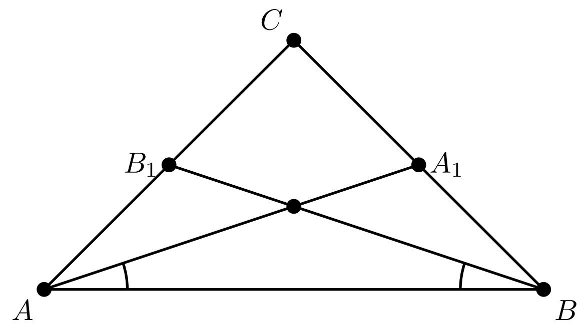
(b) Consider $\triangle CDE$. Since $\angle CED > 90° > \angle ECD$, we have $ED < DC$. By (a), $ED < DC < CB$. □

5. *(6+5 pts)*

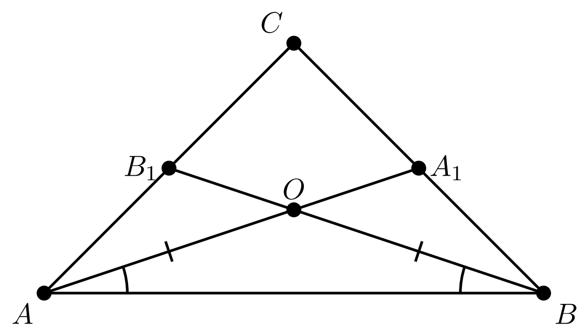
Let AA_1 and BB_1 be medians of $\triangle ABC$. Suppose that $\angle A_1AB = \angle B_1BA$. Prove that

(a) $AA_1 = BB_1$; and

(b) $AC = BC$.



Solution. **(a)** Let O be the intersection of AA_1 and BB_1. Then OAB is isosceles and we have $AO = OB$. Since $AO = \frac{2}{3}AA_1$ and $BO = \frac{2}{3}BB_1$, we obtain $AA_1 = BB_1$.

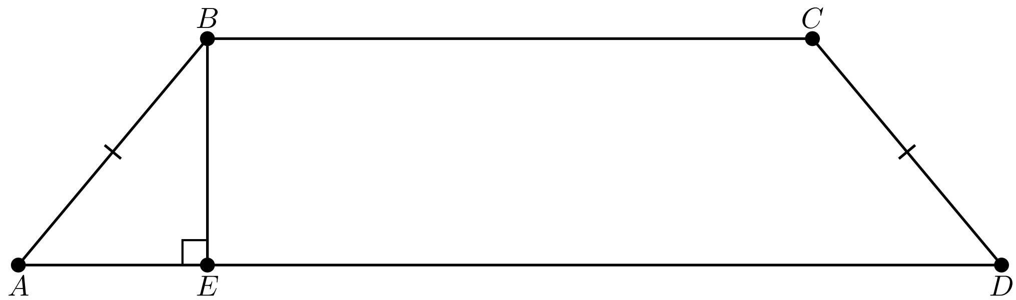


(b) Using (a), $\triangle ABB_1 = \triangle BAA_1$ be SAS. Therefore, $AB_1 = BA_1$; this implies $AC = BC$.

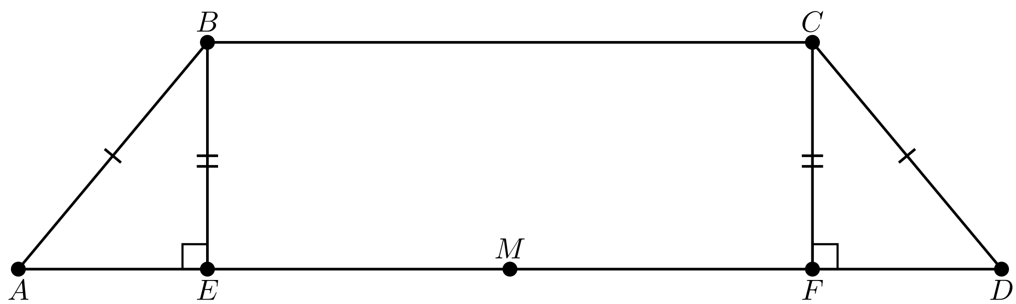
□

6. *(11 pts)*

Let $ABCD$ be an isosceles trapezoid, where $BC < AD$ are its bases and $AB = CD$ are its lateral sides. Let BE be the perpendicular dropped from B onto AD. Construct $ABCD$ given segments congruent to AD, BC, and BE.

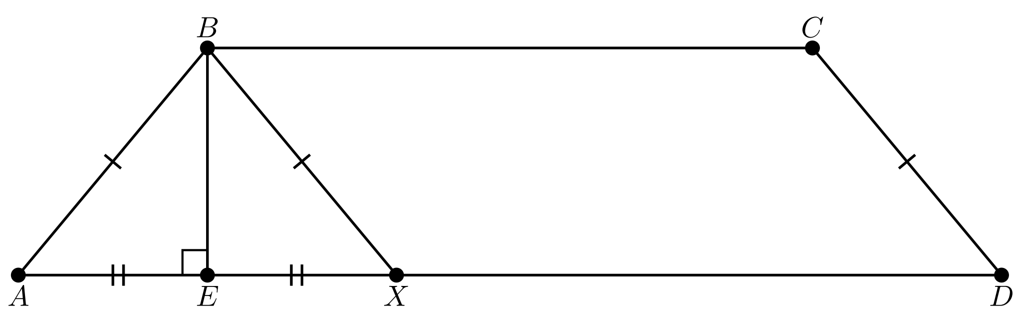


Solution 1. Construct AD. Find the midpoint M of AD. On AD construct EF congruent to BC such that M is the midpoint of EF. Construct BE perpendicular to AD (the length of BE is given) and construct CF perpendicular to AD such that CF is congruent to EB and such that B, C are on the same side of the line AD. Then $BEFC$ is a rectangle and $ABCD$ is a require trapezoid: AD, BC, and BE are congruent to the required segments.



□

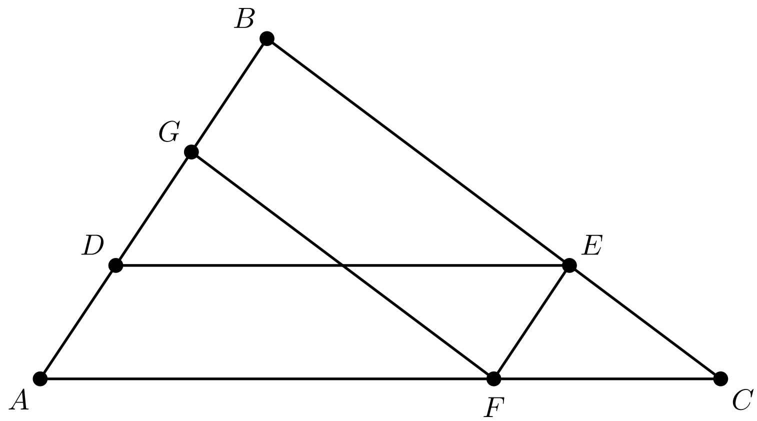
Solution 2 (sketch). Let X be a point on AD such that $BX||CD$. Then $BX = CD$ (because $BXDC$ is a parallelogram) and BE is a median and an altitude of $\triangle ABX$. We first construct a required $\triangle ABX$, then we construct a parallelogram $BXDC$ – compare with Problem 7 from the Sample Final.



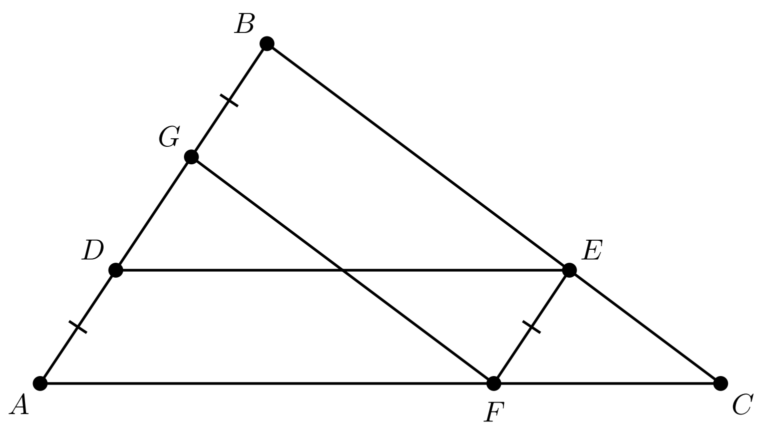
□

7. *(11 pts)*

Consider $\triangle ABC$. Suppose D is a point on AB such that $AD = \frac{1}{3}AB$. Let E, F, G be points on BC, CA, AB respectively such that $DE||AC$, $EF||BA$, $FG||CB$. Prove that $AD = DG = GB$.



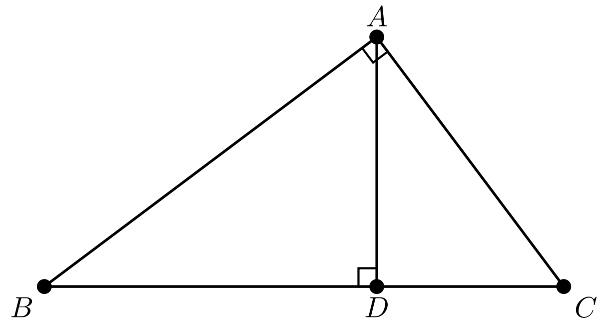
Solution. Note that $ADEF$ and $GBEF$ are parallelograms (opposite sides are parallel). Therefore, $AD = FE = GB = \frac{1}{3}AB$. This implies that $AD = DG = GB$.



□

8. *(2+6+4 pts)*

Let ABC be a right triangle, where $\angle A = 90°$, and let AD be the altitude dropped from the vertex A.



(a) Prove that $\triangle ABD$ and $\triangle CAD$ are similar.
(b) Prove that $AD^2 = BD \cdot DC$.

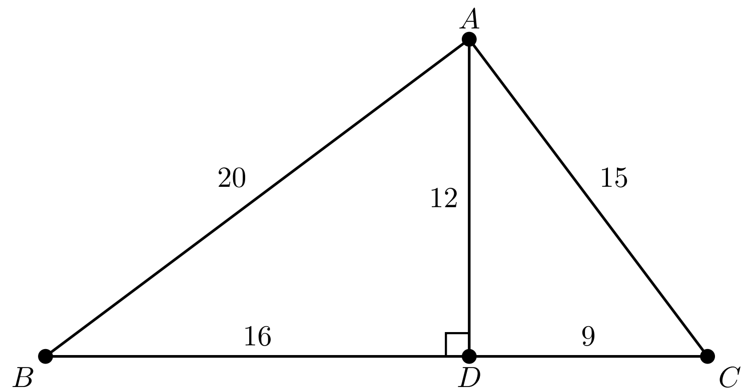
Solution. **(a)** Since $\angle ABD = 90° - \angle DCA = \angle DAC$, the triangle $\triangle ABD$ and $\triangle CAD$ are similar by AA.

(b) Since $\triangle ABD$ and $\triangle CAD$ are similar, we have $\frac{BD}{DA} = \frac{DA}{DC}$. Therefore, $DA^2 = BD \cdot DC$. □

(c) Compute AD, AB, and AC if $BD = 16$ and $DC = 9$.

Solution. Using (b), $AD = \sqrt{16 \cdot 9} = 12$. By the Pythagorean theorem:

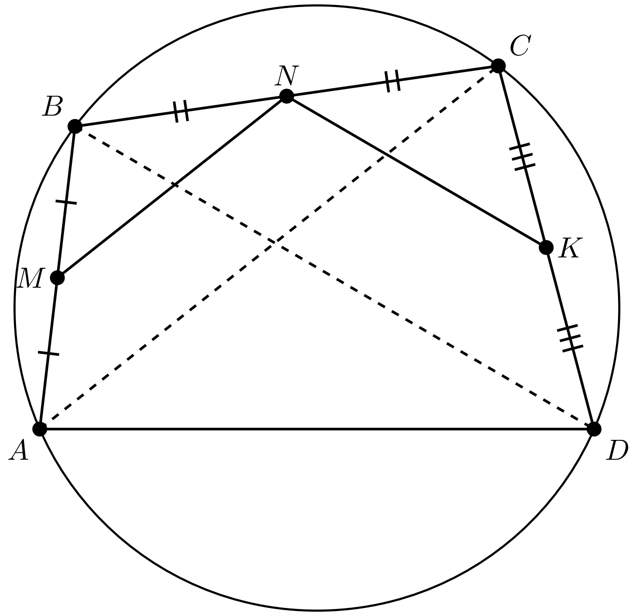
$$AB = \sqrt{BD^2 + DA^2} = 20,$$
$$AC = \sqrt{CD^2 + DA^2} = 15.$$



□

9. *(11 pts)*

Suppose $ABCD$ is a convex inscribed quadrilateral. Let M, N, and K be the midpoints of AB, BC, and CD respectively. Prove that $\angle BMN = \angle NKC$.



Solution. We have:

- $\angle BMN = \angle BAC$ because MN is the midline of $\triangle BAC$,
- $\angle BAC = \frac{1}{2}\overset{\frown}{BC} = \angle BDC$,
- $\angle BDC = \angle NKC$ because NK is the midline of $\triangle BDC$.

$\square$