MAT 515: Geometry for Teachers
Problem Set 9

Stony Brook University
Dzmitry Dudko

Fall 2019

Problem 1. (3+3 points)

(a) Prove that the midpoints of the sides of a rectangle are vertices of a rhombus.

(b) Prove that the midpoints of the sides of a rhombus are vertices of a rectangle.

Solution. **(a)** Let $ABCD$ be a rectangle, and let K, L, M, N be the midpoints of AB, BC, CD, DA respectively. Since $ABCD$ is a rectangle, we have $AC = BD$. By the midline theorem:

- $KL = \frac{1}{2}AC$ (for $\triangle ABC$);
- $LM = \frac{1}{2}BD$ (for $\triangle BCD$);
- $MN = \frac{1}{2}CA$ (for $\triangle CDA$);
- $NK = \frac{1}{2}DB$ (for $\triangle DAB$).

Therefore, $KL = LM = MN = NK$; i.e. $KLMN$ is a rhombus.

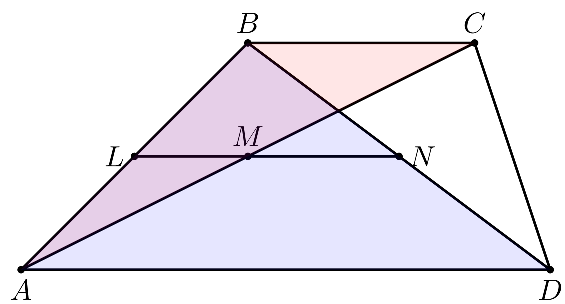
(b) Let $ABCD$ be a rhombus, and let K, L, M, N be the midpoints of AB, BC, CD, DA respectively. By the midline theorem:

- $KL||AC$ (for $\triangle ABC$);
- $LM||BD$ (for $\triangle BCD$);
- $MN||CA$ (for $\triangle CDA$);
- $NK||DB$ (for $\triangle DAB$).

Therefore, $\angle KLM$, $\angle LMN$, $\angle MNK$, $\angle NKL$ are equal to the angle between AC and BD which is a right angle because $ABCD$ is a rhombus. This implies that $KLMN$ is a rectangle. $\square$

Problem 2. (6 points)
Let $ABCD$ be a trapezoid where $BC < AD$ are its bases (i.e., BC and AD are parallel sides). Denote by M and N the midpoints of the diagonals AC and BD. Prove that MN is congruent to $\frac{1}{2}(AD - BC)$.
Hind: consider $\triangle ABD$, $\triangle ABC$ and use the midline theorem.

Solution. Let LM and LN be the midlines of $\triangle ABC$ and $\triangle ABD$ respectively.



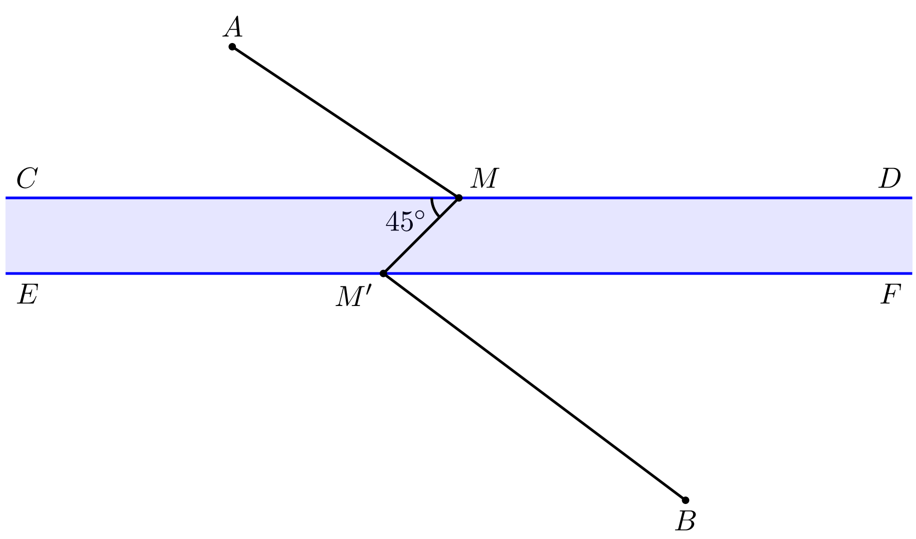
By the midline theorem:

- $LM = \frac{1}{2}BC$ and $LM||BC$ (for $\triangle ABC$);
- $LN = \frac{1}{2}AD$ and $LN||AD$ (for $\triangle ABD$).

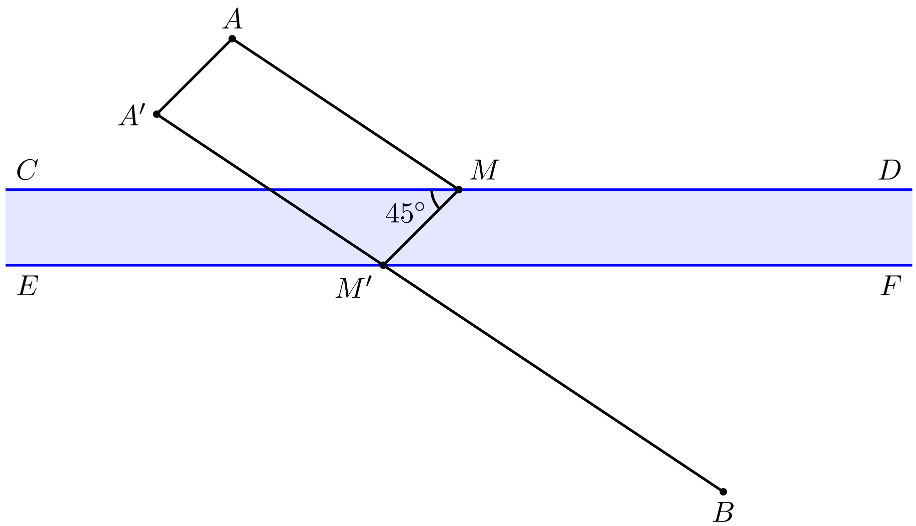
From $BC||AD$ we obtain that $LM||LN$. Therefore, the lines LM and LN coincide. We have $NM = LN - LM = \frac{1}{2}(AD - BC)$.

□

Problem 3. (6 points)
Two towns A and B are situated on opposite sides of a river whose banks CD and EF are parallel straight lines. At which point should one build a slant bridge MM' across the river, where M is on the line CD, such that $\angle CMM' = 45°$ and such that $AM + MM' + M'B$ is the shortest possible path between A and B? Describe how to construct M or M' and explain your answer.



Solution. Note that all such $45°$-slant bridges across the river are parallel and equal. Let AA' be the unique line segment parallel and congruent to every $45°$-slant bridge such that A' is closer to CD than A. In other words, if MM' is a $45°$-slant bridge, then $AA'M'M$ is a parallelogram:



Since $AA' = MM'$ and $AM = A'M'$, the sum $AM + MM' + M'B$ is minimal if and only if $AA' + A'M' + M'B$ is minimal. The latter sum is minimal if and only if $A'M' + M'B$ is minimal. This implies that M' is the intersection of $A'B$ and EF.

Answer: M' is the intersection of $A'B$ and EF.

□