MAT 515: Geometry for Teachers

Problem Set 3

Stony Brook University Fall 2019
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Problem 1. (4+4 points) A **kite** is a quadrilateral whose four sides can be grouped into two pairs of equal-length sides that are adjacent to each other. Show that:

a) A quadrilateral is kite if it has an axis of symmetry passing through a vertex.

b) Diagonals of a kite are perpendicular.

Solution. **a)** Let us assume that $ABCD$ is a quadrilateral and ℓ is its axis of symmetry passing through A. Since the numbers of vertices on both sides of ℓ are equal, ℓ also passes through C. Thus ℓ is the line AC.

The flip of the plane along the line AC interchanges the segments AB and AD; hence $AB = AD$. Similarly, the flip of the plane along AC interchanges the segments CB and CD; hence $CB = CD$. Finally, the flip of the plane along AC fixes BD. Therefore, BD is orthogonal to AC.

b) Let us assume that $ABCD$ is a kite, $AB = AD$, and $CB = CD$. Then $\triangle ABD$ and $\triangle CBD$ are isosceles triangle.

Suppose D is the middle point of BD. Then AD and CD are medians in $\triangle ABD$ and $\triangle CBD$. Since the triangles are isosceles, AD and CD are altitudes in $\triangle ABD$ and $\triangle CBD$. Therefore, A, D, C are on the same line and AC is perpendicular to BD. □

Problem 2. (1+3 points) A triangle ABC is given. Using a compass, construct a segment $A'B'$ congruent to AB on a given line. Then, using a compass, construct a point C' such that $A'B'C'$ is a triangle congruent to ABC.

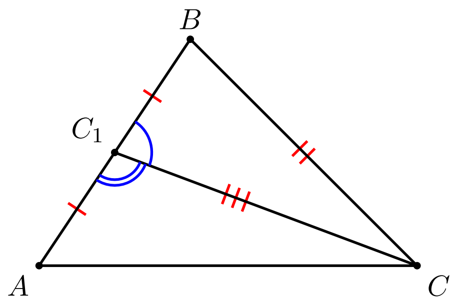
Proof. Suppose ℓ is the given line. Choose a point A' on ℓ. Construct a circle with center at A' and radius AB; denote by B' one of the intersections between the circle and ℓ. Then $A'B' = AB$.

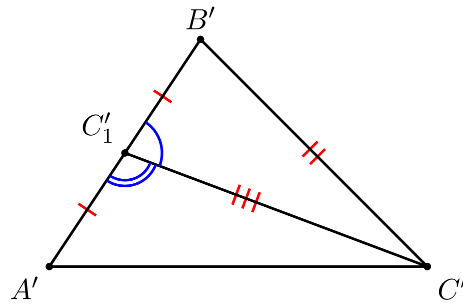
Construct a circle with center at A' and radios AC and a circle with center at B' and radios BC. Denote by C' one of the intersections of these two circles. Then $AC = A'C'$ and $BC = B'C'$.

Triangles ABC and $A'B'C'$ are congruent by SSS-test. □

Problem 3. (5 points)
Prove that if two sides and the median drawn to the first of them in one triangle are respectively congruent to two sides and the median drawn to the first of them in an other triangle, then such triangles are congruent.

Solution. Let us denote the first triangle by ABC and the second triangle by $A'B'C'$. We assume AB, BC, and the median CC_1 are congruent respectively to $A'B'$, $B'C'$, and the median $C'C'_1$.





Since $AB = A'B'$, we have $C_1A = BC_1 = C_1'A' = B'C_1'$. Then $\triangle BCC_1$ and $\triangle B'C'C_1'$ are congruent by SSS-test. Therefore, $\angle BC_1C = \angle B'C_1'C'$. Note that $\angle AC_1C$ is supplementary to $\angle BC_1C$ and $\angle A'C_1'C'$ is supplementary to $\angle B'C_1'C'$. We obtain that $\angle AC_1C = \angle A'C_1'C'$, hence $\triangle ACC_1$ and $\triangle A'C'C_1'$ are congruent by SAS-test. □

Problem 4. (5 points) Prove that in a convex pentagon, if all sides are congruent and all diagonals are congruent, then all interior angles are congruent.

Solution. Consider such pentagon $ABCDE$. Since

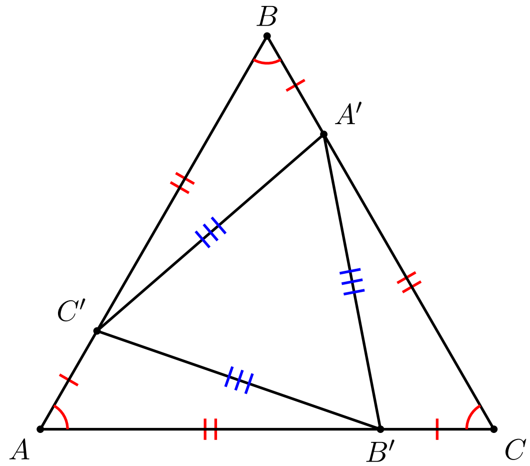
$$AB = BC, \quad BC = CD, \quad \text{and} \quad AC = CD,$$

$\triangle ABC$ and $\triangle BCD$ are congruent by SSS-test. Therefore, $\angle ABC = \angle BCD$.

Similarly, $\angle BCD = \angle CDE = \angle DEA = \angle EAB$. □

Problem 5. (5 points)

On each side of an equilateral triangle ABC, congruent segments AB', BC', and CA' are marked, and points A', B', and C' are connected by lines. Prove that $A'B'C'$ is also equilateral.



Proof. We assume that the point B' is marked on AC, the point C' is marked on AB, and the point A' is marked on BC. Since

$$AB' = BC' = CA' \quad \text{and } AC = BA = CB$$

we also have

$$CB' = AC' = BA'.$$

Since $\angle A = \angle B = \angle C$, the triangles $A'BC'$, $C'AB'$, and $B'CA'$ are congruent by SAS-test. Therefore, $A'C' = C'B' = B'A'$. □