MAT 544 - Second Midterm – Fall 2006

1. Recall that the distance between two disjoint, nonempty sets $S, T \subseteq \mathbf{R}^n$ is

$$d(S,T) = \inf\{|s-t| : s \in S, t \in T\}.$$

Suppose that $d(S,T) > 0$. Show that $m_e(S \cup T) = m_e(S) + m_e(T)$. (Here m_e denotes outer Lebesgue measure.)

Solution Since m_e is finitely subadditive, we surely have $m_e(S \cup T) \leq m_e(S) + m_e(T)$. We need to show that $m_e(S \cup T) \geq m_e(S) + m_e(T)$. Let V be any open set containing $S \cup T$; it suffices to show that $m(V) \geq m_e(S) + m_e(T)$; the desired result then follows if one takes the inf over all possible V. It suffices to show the following claim: that there are disjoint open sets V_1, V_2 with $S \subseteq V_1$, $T \subseteq V_2$. For then

$$m(V) \geq m((V \cap V_1) \cup (V \cap V_2)) = m(V \cap V_1) + m(V \cap V_2) \geq m_e(S) + m_e(T),$$

as needed. (Here we have used, first, the monotonicity of m; then the disjointness of $V \cap V_1$ and $V \cap V_2$; and then the fact that $S \subseteq V \cap V_1$, $T \subseteq V \cap V_2$.)

To prove the claim, let $\delta = d(S,T) > 0$. Let $V_1 = \{x : d(x,S) < \delta/2\}$, $V_2 = \{x : d(x,T) < \delta/2\}$. (Here, for instance, $d(x,S) = \inf\{|x-s| : s \in S\}$.) It is easy to see that these V_1, V_2 work.

(Note: we did the case where S, T were disjoint compact sets in the book, by the same method.)

2. Suppose $E \subseteq \mathbf{R}$ is Lebesgue measurable, and that $m(E) > 0$. Show that for every ϵ with $0 < \epsilon < 1$, there is an open interval $I \subseteq \mathbf{R}$ with

$$\frac{m(E \cap I)}{m(I)} > 1 - \epsilon.$$

Solution First note that we may assume that $m(E) < \infty$. For otherwise, for $n \in \mathbf{Z}$, we could let $E_n = E \cap (n, n+1)$, and note that $m(E) = \sum_n m(E_n)$. Thus we must have $m(E_n) > 0$ for some n. If we could prove the result for E_n in place of E, then for some I we would have

$$\frac{m(E \cap I)}{m(I)} \geq \frac{m(E_n \cap I)}{m(I)} > 1 - \epsilon$$

and we would be done. In short: by replacing E by some E_n if necessary, we may assume that $m(E) < \infty$.

We argue by contradiction; say the conclusion is false. Then for some ϵ with $0 < \epsilon < 1$, we have that

$$m(E \cap I) \leq (1 - \epsilon)m(I)$$

for every open interval I. Let $U \subseteq \mathbf{R}$ be any open set of finite measure, with $E \subseteq U$. As usual, write U as the disjoint union of open intervals; call the set of those intervals $\mathcal{I}_U$. Then

$$m(E) = m(E \cap U) = \sum_{I \in \mathcal{I}_U} m(E \cap I) \leq (1 - \epsilon) \sum_{I \in \mathcal{I}_U} m(I) \leq (1 - \epsilon)m(U).$$

Taking the inf over all such U we find $m(E) \leq (1 - \epsilon)m(E)$, contradiction.

3. Let c denote the Lebesgue measure of the open unit ball in $\mathbf{R}^n$.
(a) Show that the Lebesgue measure of any open ball of radius $r > 0$ is cr^n. (Only use techniques from chapter 4.)
(b) Show that the Lebesgue measure of any closed ball of radius $r > 0$ is also cr^n. (Again, only use techniques from Chapter 4.)
(c) Suppose that $U \subseteq \mathbf{R}^n$ is open, and that $m(U) < \infty$. Show that for any $\epsilon > 0$, one can select a finite collection of disjoint closed balls $B_1, \ldots, B_N \subseteq U$ such that

$$m(U \setminus \cup_{j=1}^N B_j) < \epsilon.$$

Solution (a) Since Lebesgue measure is invariant under translation, it suffices to prove this for balls centered at the origin. Let B denote the open unit ball.

If $S \subseteq \mathbf{R}^n$, and $r > 0$, define $rS = \{rx : x \in S\}$. It suffices to show that, if $U \subseteq \mathbf{R}^n$ is open, then $m(rU) = r^n m(U)$ (for then we could let $U = B$). Of course, if I is a rectangle, then $m(rI) = r^n m(I)$. Now rU is the disjoint union of the cubes rI (as I ranges over $\mathcal{D}_U$). Thus

$$m(rU) = \sum_{I \in \mathcal{D}_U} m(rI) = \sum_{I \in \mathcal{D}_U} r^n m(I) = r^n m(U),$$

as desired.
(b) Again we may assume that the ball is centered at the origin; call its measure t. For any positive integer k, this ball contains $(r - 1/k)B$ and is contained in $(r + 1/k)B$, so $c(r - 1/k)^n \leq t \leq c(r + 1/k)^n$ for any k. Passing to the limit as $k \to \infty$, we find that $t = cr^n$.
(c) If $U \subseteq \mathbf{R}^n$ is open, let us say that S is a *subcluster* of U if, for some finite collection of disjoint closed balls $B_1, \ldots, B_N \subseteq U$, we have $S = \cup_{j=1}^N B_j$. It suffices to show the following claim: there exists $0 < r < 1$ such that every open $U \subseteq \mathbf{R}^n$, with finite measure, has a subcluster S_1 with

$$m(U \setminus S_1) \leq rm(U).$$

For we could then iterate: $U \setminus S_1$ has a subcluster S_2 with

$$m(U \setminus (S_1 \cup S_2)) = m((U \setminus S_1) \setminus S_2) \leq rm(U \setminus S_1) \leq r^2 m(U),$$

and surely $S_1 \cup S_2$ is a subcluster of U, since S_1 and S_2 are disjoint. Proceeding recursively we obtain disjoint subclusters $S_1, S_2, \ldots$ of U, such that for any N, $m(U \setminus \cup_{k=1}^N S_k) \leq r^N m(U)$. This will be less than ϵ if N is sufficiently large. (In fact, we would find that $m(U \setminus \cup_{k=1}^\infty S_k) = 0$, so that there is a countable collection of disjoint closed balls $B_1, B_2 \ldots \subseteq U$, with $m(U \setminus \cup_{j=1}^\infty B_j) = 0$.)

To show the claim, we need only show that there exists $0 < s < 1$, such that for every open $U \subseteq \mathbf{R}^n$, with finite measure, there is a finite collection of disjoint closed balls $B_1, \ldots, B_N \subseteq U$ with

$$\sum_{j=1}^N m(B_j) \geq sm(U).$$

For then we could set $r = 1 - s$. But this is easy to see: first use the dyadic cube decomposition of U to select a finite collection of disjoint closed cubes $I_1, \ldots, I_N \subseteq U$ with $\sum_{j=1}^N m(I_j) > \frac{1}{2} m(U)$. For $1 \leq j \leq N$, let B_j be the closed ball with the same center as I_j and whose radius is half the side length of I_j, so $B_j \subseteq I_j$. By (b),

$$\sum_{j=1}^N m(B_j) = \frac{c}{2^n} \sum_{j=1}^N m(I_j) \geq \frac{c}{2^{n+1}} m(U).$$

This is what we want, with $s = \frac{c}{2^{n+1}}$.

The method of solution of this problem (removing at least a fixed fraction of U at each stage) is akin to the construction of Cantor sets.

4. Show that there are open neighborhoods U, V of 0 in $\mathbf{R}^2$ and a C^1 function $g : U \to V$, with a C^1 inverse mapping V to U, with the following property:

Whenever $F : V \to \mathbf{R}$ is C^1, then for all $(x, y) \in U$,

$$\frac{\partial}{\partial x}(F \circ g)(x, y) = (5x^4 + y + 1)(F_1 \circ g)(x, y) + 2xye^{(x^2+1)y}(F_2 \circ g)(x, y) \text{ and}$$

$$\frac{\partial}{\partial y}(F \circ g)(x, y) = x(F_1 \circ g)(x, y) + (x^2 + 1)e^{(x^2+1)y}(F_2 \circ g)(x, y).$$

Here $F_1(s, t) = \frac{\partial}{\partial s} F(s, t)$, $F_2(s, t) = \frac{\partial}{\partial t} F(s, t)$, for all $(s, t) \in V$.

Solution Let $g(x, y) = (x^5 + xy + x, e^{(x^2+1)y} - 1)$. Then $g(0, 0) = (0, 0)$, and the Jacobian of g at $(0, 0)$ is the identity matrix. Thus the existence of U, V and the C^1 inverse of g are guaranteed by the inverse function theorem. The formulas for differentiating $F \circ g$ are immediate from the chain rule.