More on Harmonic Functions

1 More on Green's Theorem

In these notes we shall derive some further important facts about harmonic functions, including the Poisson kernel for the unit ball in $\mathbf{R}^n$, where $n \geq 2$. As in Chapter 11, we shall need to use Green's Theorem: if f, g are C^2 in a neighborhood of $\overline{D}$, then

$$\int_D (f\Delta g - g\Delta f) = \int_{\partial D} (f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n})dS, \tag{1}$$

but now for domains D which are more general than annuli, even for certain domains which (unlike the domains in (13.6.21) are not C^1. Fortunately, the cases of Green's Theorem which we shall need are derivable from the annulus case, without a lot of effort.

As usual, if F is a function on $\mathbf{R}^n$ and $p \in \mathbf{R}^n$, $\tau_p F$ is the function defined by $(\tau_p F)(x) = F(x-p)$.

Proposition 1.1 *Green's theorem holds for an annulus centered at any point in $\mathbf{R}^n$. Explicitly, say $p \in \mathbf{R}^n$, say $0 < a < b$, let A_a^b denote the annulus $\{x \in \mathbf{R}^n : a \leq |x| \leq b\}$, let $D = p + A_a^b$, and suppose f, g are C^2 in a neighborhood of $\overline{D}$. Then*

$$\int_D [(\Delta f)g - f\Delta g)] = \int_S ([\frac{\partial \tau_{-p} f}{\partial r}\tau_{-p}g - \tau_{-p}f\frac{\partial \tau_{-p}g}{\partial r}](r\omega)r^{n-1})|_{r=a}^{r=b})d\sigma(\omega). \tag{2}$$

Proof Apply Green's Theorem on an annulus centered at 0 to $\tau_{-p}f$ and $\tau_{-p}g$, and use the translation invariance of Lebesgue measure.

Proposition 1.2 *Green's theorem holds for a ball centered at any point in $\mathbf{R}^n$. Explicitly, say $p \in \mathbf{R}^n$, and say $0 < b$. Suppose f, g are C^2 in a neighborhood of $\overline{B(p,b)}$. Then*

$$\int_{B(p,b)} [(\Delta f)g - f\Delta g)] = \int_S ([\frac{\partial \tau_{-p} f}{\partial r}\tau_{-p}g - \tau_{-p}f\frac{\partial \tau_{-p}g}{\partial r}](b\omega)b^{n-1})d\sigma(\omega). \tag{3}$$

Proof Let $a \to 0^+$ in Proposition 1.1.

Proposition 1.3 *Green's theorem holds for the set-theoretic difference of two balls in $\mathbf{R}^n$, provided the smaller ball is closed and contained inside the other (which is open), even if the balls are* *not concentric. Explicitly, say $p \in \mathbf{R}^n$, say that $\overline{B(p,a)} \subseteq B(0,b)$, and let $D = B(0,b) \setminus \overline{B(p,a)}$. Suppose f, g are C^2 in a neighborhood of $\overline{D}$. Then*

$$\int_D [(\Delta f)g - f\Delta g)] = \int_S ([\frac{\partial f}{\partial r} g - f\frac{\partial g}{\partial r}](b\omega)b^{n-1})d\sigma(\omega) - \int_S ([\frac{\partial \tau_{-p} f}{\partial r}\tau_{-p}g - \tau_{-p}f\frac{\partial \tau_{-p}g}{\partial r}](a\omega)a^{n-1})d\sigma(\omega). \tag{4}$$

Proof Let d denote the distance from p to $\partial B(0,b)$; then $a < d$. We first claim that the result evidently holds in each of the following two cases:
(1) Either f or g vanishes identically outside a ball $B(p,C)$, where $a < C < d$;
(2) Both f and g vanish identically inside a ball $B(p,c)$, where $a < c < d$.
Indeed, in case (1), pick C' with $C < C' < d$. The integral on the left side of (4) may then actually be taken over $p + A_a^{C'}$, so that the result follows from Proposition 1.1 (in this case the first integral on the right side of (4) is zero). In case (2), although f, g were initially defined only on D, they may be extended as C^2 functions on all of $B(0,b)$ by setting them to be zero on $\overline{B(p,a)}$; and then (4) follows, by an application of Green's theorem on the ball $B(0,b)$ (in this case the second integral on the right side of (4) is zero).

To complete the proof, we note that both sides of (4) are bilinear in f, g, so it is enough to write $f = f_1 + f_2$, $g = g_1 + g_2$, where each pair $(f_1, g_1), (f_1, g_2), (f_2, g_1), (f_2, g_2)$ falls into case (1) or (2). But this is easy to accomplish. Select c, C with $a < c < C < d$, then select $\zeta \in C_c^\infty(B(p,C))$ with $\zeta \equiv 1$ in a neighborhood of $\overline{B(p,c)}$. We need only put $f_1 = \zeta f$, $f_2 = (1-\zeta)f$, $g_1 = \zeta g$, $g_2 = (1-\zeta)g$. This completes the proof.

Proposition 1.4 *Green's theorem holds for the set-theoretic difference of a hemisphere and a closed ball in $\mathbf{R}^n$, provided the ball is contained inside the hemisphere. Explicitly, say $b > 0$, and let $B_+(0,b) = \{x \in B(0,b) : x_n > 0\}$. Say $p \in \mathbf{R}^n$, and that $\overline{B(p,a)} \subseteq B_+(0,b)$, and let $D_+ = B_+(0,b) \setminus \overline{B(p,a)}$. Also let $S = \{x \in \mathbf{R}^n : |x| = 1\}$, $S_+ = \{x \in S : x_n > 0\}$, $D' = \{x' \in \mathbf{R}^{n-1} : |x'| \leq b\}$. Suppose f, g are C^2 in a neighborhood of $\overline{D_+}$. Then*

$$\int_{D_+} [(\Delta f)g - f\Delta g)] = I - II - III, \tag{5}$$

where

$$I = \int_{S_+} ([\frac{\partial f}{\partial r} g - f\frac{\partial g}{\partial r}](b\omega)b^{n-1})d\sigma(\omega),$$

$$II = \int_S ([\frac{\partial \tau_{-p} f}{\partial r}\tau_{-p}g - \tau_{-p}f\frac{\partial \tau_{-p}g}{\partial r}](a\omega)a^{n-1})d\sigma(\omega),$$

and

$$III = \int_{D'} ([\frac{\partial f}{\partial x_n} g - f\frac{\partial g}{\partial x_n}])(x',0)dx'.$$

(In III, we are writing points x in $\mathbf{R}^n$ as $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$.)

Proof Choose a nondecreasing function $\phi \in C^\infty(\mathbf{R})$ with $\phi(y) = 0$ for $y < 1$, and $\phi(y) = 1$ for $y > 2$. For $t > 0$, define $\phi_t(y) = \phi(y/t)$. Choose $T > 0$ such that $B(p,a) \subseteq \{x \in D_+ : x_n > 2T\}$.

For $0 < t < T$, define $f_t(x) = \phi_t(x_n)f(x)$, $g_t(x) = \phi_t(x_n)g(x)$; also define $f_t(x), g_t(x)$ to be zero for $x_n < 0$. Let D be as in Proposition 1.3, and note that f_t, g_t are C^2 in a neighborhood of D. We apply Proposition 1.3 to f_t, g_t in place of f, g, to find that

$$\int_{D_+} [(\Delta f_t)g_t - f_t\Delta g_t)] = I_t - II, \tag{6}$$

where

$$I_t = \int_S ([\frac{\partial f_t}{\partial r} g_t - f_t \frac{\partial g_t}{\partial r}](b\omega)b^{n-1})d\sigma(\omega).$$

Note, however, that

$$[\frac{\partial f_t}{\partial r} g_t - f_t \frac{\partial g_t}{\partial r}](x) = \phi_t^2(x_n)([\frac{\partial f}{\partial r} g - f \frac{\partial g}{\partial r}](x),$$

so that, by DCT, $\lim_{t\to 0^+} I_t = I$. On the other hand,

$$\int_{D_+} [(\Delta f)g - f\Delta g)] = \int_{D_+} \phi_t^2(x_n)[(\Delta f)g - f(\Delta g)] + III_t, \tag{7}$$

where

$$III_t = \int_{D_+} 2\phi_t'(x_n)\phi_t(x_n)([\frac{\partial f}{\partial x_n} g - f\frac{\partial g}{\partial x_n}])(x)dx.$$

Clearly, as $t \to 0^+$, the first term in (7) approaches the left side of (5). We therefore need only show that $\lim_{t\to 0^+} III_t = III$, for then by taking the limit of both sides of (6) as $t \to 0$, we will know that

$$\int_{D_+} [(\Delta f)g - f\Delta g)] + III = I - II,$$

as claimed.

For $y \in \mathbf{R}$, let

$$\psi_t(y) = 2\phi_t'(y)\phi_t(y) = [\phi_t^2]'(y) = \frac{1}{t}(\phi^2)'(\frac{y}{t}).$$

It follows easily that $\{\psi_t\}$ is a standard approximate identity on $\mathbf{R}$. Note also that supp $\psi_t \subseteq [t, 2t]$.

Define $h = \frac{\partial f}{\partial x_n} g - f\frac{\partial g}{\partial x_n}$; then h is continuous in a neighborhood of $\overline{D_+}$. We need to show that

$$\lim_{t\to 0^+} \int_{D_+} \psi_t(x_n)h(x)dx = \int_{D'} h(x', 0)dx'. \tag{8}$$

For $x' \in D'$, let $I_{x'}$ denote the real interval $[0, \sqrt{b^2 - |x'|^2}]$; then $\int_{D_+} \psi_t(x_n)h(x)dx = \int_{D'} \int_{I_{x'}} \psi_t(x_n)h(x', x_n)dx_n dx'$. In the inner integral we may instead integrate over $J_{x',t} = I_{x'} \cap [t, 2t]$. Altogether,

$$\int_{D_+} \psi_t(x_n)h(x)dx = \int_{D'} H_t(x')dx', \tag{9}$$

where $H_t(x') = \int_{J_{x',t}} \psi_t(x_n)h(x', x_n)dx_n$. For all sufficiently small $t > 0$, $J_{x',t} = [t, 2t]$. Thus, since $\{\psi_t\}$ is a standard approximate identity with supp$\psi_t \subseteq [t, 2t]$, we have that $\lim_{t\to 0^+} H_t(x') = h(x', 0)$. On the other hand, if M is the maximum of h on D_+, we have $|H_t(x')| \leq M \int_t^{2t} \psi_t(x_n)dx_n = M$. Thus, (8) follows from (9) and DCT, as desired.

2 Converse of the Mean Value Theorem

We defined u to be harmonic on an open subset U of $\mathbf{R}^n$ if it is twice differentiable and if $\Delta u = 0$ on U. Until further notice we change this definition: u is harmonic on U if $u \in C^2(U)$ and if $\Delta u = 0$ on U. (We will eventually see that the definitions are equivalent, but is we want to apply Green's theorem to u, we had better know that it is C^2.)

Theorem 2.1 *Suppose $D \subseteq \mathbf{R}^n$ is open, and that $u : D \to \mathbf{C}$ is continuous. Then u is harmonic on D if and only if u satisfies the mean-value property: for every $x \in D$ and every $r > 0$ such that $\overline{B(x,r)} \subseteq D$, we have*

$$u(x) = \frac{1}{\omega_n} \int_S u(x + r\omega) d\sigma(\omega).$$

Proof The forward direction was shown in Exercise 11.4.4: after a translation, we may assume that $x = 0$. Choose $R > 0$ with $\overline{B(0,R)} \subseteq D$, and say $0 < r < R$. Applying Green's theorem on $B(x,r)$ to the functions u and 1, we find that

$$\int_{B(0,r)} \Delta u = \int_S \frac{\partial u}{\partial r}(r\omega) r^{n-1} d\sigma(\omega). \tag{10}$$

Since the left side of (10) is zero and we can just take the factor of r^{n-1} out of the integral on the right side, we see that

$$\frac{d}{dr} \int_S u(r\omega) d\sigma(\omega) = \int_S \frac{\partial u}{\partial r}(r\omega) d\sigma(\omega) = 0. \tag{11}$$

Therefore $\int_S u(r\omega)d\sigma(\omega)$ is constant for $0 < r < R$, and consequently equals its limit as $r \to 0^+$, which is $\omega_n u(0)$, as needed.

For the converse, we may assume u is real-valued (otherwise consider its real and imaginary parts). If we knew u were C^2 on D (and satisfies the mean-value property there), we could simply reverse the steps of the forward direction. That is, suppose to obtain a contradiction that $\Delta u(x) \neq 0$ for some $x \in D$. By means of a translation we may assume that $x = 0$, and by replacing u by a multiple of u if necessary, we may assume that $\Delta u(0) = 1$. Choose $R > 0$ with $\overline{B(0,R)} \subseteq D$ and with $u > 1/2$ on $\overline{B(0,R)}$, and note that we again have (11) and (10). But (10) is impossible, since the left side is positive and the right side is zero.

This completes the proof of the converse if u is C^2, and we are tempted to now try to finish the proof by smoothing out a general u with an approximate identity. When we try to do this, however, something remarkable happens.

Choose a standard approximate identity $\{\psi_t\}$ on $\mathbf{R}^n$; thus, for for some nonnegative $\psi \in C_c^\infty(\mathbf{R}^n)$, with $\int \psi = 1$, we have $\psi_t(x) = t^{-n}\psi(x/t)$; let us require further that ψ be *radial*, (that is, a function of $|x|$ only), and supported in $\overline{B(0,1)}$. Say, in fact, that $\psi(x) = H(|x|)$.

If $x \in D$, we may select $R = R_x$ such that $\overline{B(x,R)} \subseteq D$, and we claim then that

$$u(x) = u * \psi_R(x). \tag{12}$$

Since we can choose $R = R_x$ to be independent of x for all x in some neighborhood of any fixed $x_0 \in D$, (12) will imply that u is actually *smooth* on D, and we will be done.

To prove (12) note that, by means of a translation, we may again assume that $x = 0$. Say that in fact $\psi_R(x) = H(|x|)$. We compute

$$\begin{aligned}
u * \psi_R(0) &= \int u(x)\psi_R(-x)dx \\
&= \int_0^R H(r)r^{n-1}\int_S u(r\omega)d\sigma(\omega) \\
&= u(0)\int_0^R H(r)r^{n-1}\int_S 1 d\sigma(\omega) \\
&= u(0)1 * \psi_R(0) \\
&= u(0),
\end{aligned}$$

as desired.

As we have indicated, (12) implies the following corollary:

Corollary 2.2 *Any harmonic function on an open set in* $\mathbf{R}^n$ *is smooth there.*

Equation (12) has another important corollary:

Corollary 2.3 *There is a constant* $C > 0$*, depending only on* n*, as follows.*

Suppose u *is harmonic on an open set* $D \subseteq \mathbf{R}^n$*, and that* u *is bounded on* D*. Then*

$$|\mathrm{grad} u(x)| \leq \frac{C\|u\|_\infty}{\mathrm{dist}(x, \partial D)} \tag{13}$$

Proof We are going to throw the derivative onto ψ_R in (12), so let us estimate its derivatives.

For $1 \leq j \leq n$, let $\psi^j = \frac{\partial \psi}{\partial x_j}$, and, for $t > 0$, let $\psi_t^j(x) = t^{-n}\psi^j(x/t)$. Then $\|\psi_t^j\|_1 = \|\psi^j\|_1$ for all t; let $C_j = \|\psi^j\|_1$. We have $|\frac{\partial \psi_t}{\partial x_j}| = |\psi^j|/r$.

Say now u, D, x are as in the statement of the corollary, and let $2r = \mathrm{dist}(x, \partial D)$. Then x has a neighborhood U such that for all $y \in U$, $\mathrm{dist}(y, \partial D) > r$. Accordingly $u(y) = u * \psi_r(y)$ for all $y \in U$. Thus, for $1 \leq j \leq n$,

$$|\frac{\partial u}{\partial x_j}(x)| = |(u * \psi^j)(x)|/r \leq C_j\|u\|_\infty/r$$

as desired.

3 Strategy for Finding Poisson Kernels

If the Dirichlet problem can be solved on a bounded domain D, we know the solution is unique. For $x \in D$, define $\Lambda_x : C(\partial D) \to \mathbf{C}$ by $\Lambda_x f = u(x)$, where u is the solution of the Dirichlet problem with boundary value f. By the maximum principle, Λ_x is a positive linear functional on $C(\partial D)$. Thus, by the Riesz representation theorem, there is a measure μ_x on ∂D such that $u(x) = \int_{\partial D} f(y) d\mu_x(y)$. Ordinarily (for instance if D is a C^2 domain), μ_x turns out to be absolutely continuous with respect to surface measure on ∂D. Thus, for some function $P_x \in L^1(\partial D)$, $u(x) = \int_{\partial D} P_x(y) f(y) dS(y)$. Let $P(x,y) = P_x(y)$; then P is called the *Poisson kernel* for D.

Computing P explicitly is usually impossible, but we next explain a strategy which we will be able to execute in full if D is a ball in $\mathbf{R}^n$ or the upper half space $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_n > 0\}$. (Of course $\mathbf{R}^n_+$ is not a bounded domain, so certain modifications will be necessary in this case.) Although this precise strategy cannot be carried out in general, it can be modified, with the aid of some functional analysis, to produce Poisson kernels on much more general domains. (Here we are thinking of the method of *layer potentials*, which you will see in later courses.)

In brief, the strategy for finding P is to apply Green's theorem in a manner somewhat similar to how we applied it, in the proof of Theorem 11.4.3. But we are now in the situation where one is working on D and not on all of $\mathbf{R}^n$, so one of the two functions we shall use in Green's Theorem is not N but rather a modification of it suited to D. Of course, if we add any harmonic function h to N, we will still formally have $\Delta(N+h) = \delta$. The stategy is to choose h in a manner suited to analysis on D.

Say, then, that $x \in D$. We seek a function H_x which is harmonic in a neighborhood of $\overline{D}$, such that the function

$$G_x = \tau_x N + H_x \tag{14}$$

vanishes identically on ∂D. If such a function H_x can be found, one lets $G(x,y) = G_x(y)$ and calls G the *Green's function* for the domain D. (This is what we said we were going to do in the previous paragraph provided $x = 0$; otherwise we have had to translate everything.)

Formally $\Delta G_x = \delta_x$ (by which we mean $\tau_x \delta$). Thus, if u is C^2 in a neighborhood of $\overline{D}$ and harmonic on D, and we apply Green's Theorem to u and G_x, we obtain formally that

$$u(x) = \int_D (u\Delta G_x - G_x \Delta u) = \int_{\partial D} (u\frac{\partial G_x}{\partial n} - G_x \frac{\partial u}{\partial n}) dS = \int_{\partial D} u \frac{\partial G_x}{\partial n}(y) dS(y), \tag{15}$$

so that the Poisson kernel $P(x,y)$ should equal $\frac{\partial G_x}{\partial n}(y)$.

Two remarks are in order. First, if H_x exists, then by (14) we have that on ∂D, $H_x = -\tau_x N$. Thus, on D, H_x is the solution to the Dirichlet problem with boundary value $-\tau_x N|_{\partial D}$, and therefore, H_x is uniquely determined on D. This allows us to call G "the" Green's function (not "a" Green's function), as we did above.

Our second remark is that, to give a rigorous version of (15), we must actually apply Green's theorem on the set-theoretic difference of D and a ball of radius ϵ centered at x, and then let $\epsilon \to 0^+$. (This was the part of the strategy of Theorem 11.4.3.) This introduces an extra boundary term arising from integration over the sphere of radius ϵ centered at x. Before looking at our specific examples of situations where one can find Green's function, we explain what happens when we take this sort of limit:

Proposition 3.1 *Suppose $x \in \mathbf{R}^n$, and that f, h are C^2 functions defined in a neighborhood U of x. Then*

$$\lim_{\epsilon \to 0^+} \int_S ([\frac{\partial \tau_{-x} f}{\partial r}(N + \tau_{-x} h) - \tau_{-x} f \frac{\partial (N + \tau_{-x} h)}{\partial r}](\epsilon\omega)\epsilon^{n-1}) d\sigma(\omega) = f(x). \tag{16}$$

Proof We may assume $x = 0$. For if this case is known, we may apply the result for $x = 0$ to the functions $\tau_{-x} f$ and $\tau_{-x} h$ and so obtain the result for general x (noting that $f(x) = (\tau_{-x} f)(0)$).

We recall the reasoning of the last four paragraphs of Theorem 11.4.3. At at any point $x \in \mathbf{R}^n$, $\partial f/\partial r = \text{grad} f \cdot \hat{x}$, where $\hat{x} = (x_1/r, \ldots, x_n/r)$. Thus $\partial f/\partial r$ is a bounded quantity on any compact subset K of U; if $L = \max_K |\text{grad} f|$, then by Cauchy-Schwarz, $|\partial f/\partial r| \leq L$ always. Similarly $\partial h/\partial r$ is bounded on K as well. Thus

$$\lim_{\epsilon \to 0^+} \int_S ([\frac{\partial f}{\partial r} h - f \frac{\partial h}{\partial r}](\epsilon\omega)\epsilon^{n-1}) d\sigma(\omega) = 0.$$

Thus we only need show that

$$\lim_{\epsilon \to 0^+} \int_S ([\frac{\partial f}{\partial r} N - f \frac{\partial N}{\partial r}](\epsilon\omega)\epsilon^{n-1}) d\sigma(\omega) = f(0).$$

But this is just what we showed in (11.92) and the arguments which followed it.

4 The Poisson Kernel for the Upper Half Space

In this section, it will be more convenient to work in $\mathbf{R}^{n+1}$ than in $\mathbf{R}^n$. (Now $n \geq 1$.) We write points in $\mathbf{R}^{n+1}$ in the form (x, t) with $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$, and we define the *upper half space* $\mathbf{R}^{n+1}_+$ by

$$\mathbf{R}^{n+1}_+ = \{(x, t) \in \mathbf{R}^{n+1} : t > 0\}.$$

The boundary of this space may be naturally thought of as $\mathbf{R}^n$.

Although the upper half space is unbounded, we will use the ideas of the previous section to construct a Poisson kernel for it. We will solve the following very natural Dirichlet problem there: given a continuous, bounded function f on $\mathbf{R}^n$, we will show that there is a unique continuous function u on $\overline{\mathbf{R}^{n+1}}$ such that $u(x, 0) = f(x)$ for all x and such that u is harmonic on the upper half space, and we will compute the function $P((x_0, t_0), x)$ for which $u(x_0, t_0) = \int_{\mathbf{R}^n} P((x_0, t_0), x) f(x) dx$ for all (x_0, t_0) in the upper half space.

We need Green's function for $\mathbf{R}^{n+1}_+$, and to this end it helps to think physically again. Let $p = (x_0, t_0) \in \mathbf{R}^{n+1}_+$, and let us look for G_p. We need to find a function h_p which is harmonic in a neighborhood of $\overline{\mathbf{R}^{n+1}_+}$, such that $\tau_p N + h_p$ vanishes on the boundary $\mathbf{R}^n$. Now $\tau_p N$ is the equilibrium temperature distribution if -1 calorie per second of heat is being produced at p. (Here, for the usual constants, we have set $\rho = s = \kappa = 1$.) If we want to cancel out the effect of this heat sink on the boundary, it would make sense to try to do that by adding a heat *source* at the *image point* $\tilde{p} = (x_0, -t_0)$, where we produce 1 calorie of heat per second. Thus we should try $h_p = -\tau_{\tilde{p}} N$ (which is harmonic in a neighborhood of $\overline{\mathbf{R}^{n+1}_+}$, since $\tilde{p} \notin \overline{\mathbf{R}^{n+1}_+}$). (This is called the *method of images.*)

Explicitly,

$$\begin{aligned} G_p(x,t) &= \frac{1}{(1-n)\omega_{n+1}}[(|x-x_0|^2+(t-t_0)^2)^{(1-n)/2} - (|x-x_0|^2+(t+t_0)^2)^{(1-n)/2}] \text{ if } n \neq 1; \\ &= \frac{1}{4\pi}[\log(|x-x_0|^2+(t-t_0)^2) - \log(|x-x_0|^2+(t+t_0)^2)] \text{ if } n = 1. \end{aligned}$$

This definition makes sense for all (x,t) and all $p = (x_0, t_0)$ in $\mathbf{R}^{n+1}$, provided that $(x_0, t_0) \neq (x, t)$ or $(x, -t)$. Put $G((x_0,t_0),(x,t)) = G_{(x_0,t_0)}(x,t)$; note that $G((x_0,t_0),(x,t)) = G((x,t),(x_0,t_0))$. In particular, since $G((x_0,t_0),(x,t))$ is harmonic in $(x,t) \in \mathbf{R}^{n+1}$ for fixed (x_0,t_0), as long as $(x,t) \neq (x_0,t_0)$ or $(x_0,-t_0)$, it is also harmonic in (x_0,t_0) for fixed (x,t), as long as $(x_0,t_0) \neq (x,t)$ or $(x,-t)$.

The considerations of the last section lead us to define $P((x_0,t_0),x) = -\frac{\partial}{\partial t} G((x_0,t_0),(x,t))|_{t=0}$. This function will be *harmonic* in $(x_0,t_0) \in \mathbf{R}^{n+1}_+$, since partial derivatives with respect to different variables commute. Explicitly,

$$P((x_0,t_0),x) = \frac{2}{\omega_{n+1}} \frac{t_0}{(|x-x_0|^2+t_0^2)^{(n+1)/2}} \tag{17}$$

The considerations of the last section would seem to imply that if u is bounded and continuous on $\overline{\mathbf{R}^{n+1}_+}$, and harmonic on $\mathbf{R}^{n+1}_+$, we have

$$u(x_0,t_0) = \int_{\mathbf{R}^n} P((x_0,t_0),x) u(x,0)\,dx.$$

This can be written even more simply if we again use the notion of *convolution.* For $(x,t) \in \mathbf{R}^{n+1}_+$, define the *Poisson kernel*

$$P_t(x) = \frac{2}{\omega_{n+1}} \frac{t}{(|x|^2+t^2)^{(n+1)/2}}.$$

We are now motivated to conjecture and prove:

Theorem 4.1 *(a) Suppose that u is bounded and continuous on $\overline{\mathbf{R}^{n+1}_+}$, and harmonic on $\mathbf{R}^{n+1}_+$. Let $f(x) = u(x,0)$. Then, for any $(x_0,t_0) \in \mathbf{R}^{n+1}_+$, we have*

$$u(x_0,t_0) = (f * P_{t_0})(x_0).$$

(b) Conversely, suppose that f is bounded and continuous on $\mathbf{R}^n$. For $(x_0,t_0) \in \mathbf{R}^{n+1}_+$, set $u(x_0,t_0) = (f * P_{t_0})(x_0)$. Also, for $x_0 \in \mathbf{R}^n$, set $u(x_0,0) = f(x_0)$. Then u is bounded and continuous on $\overline{\mathbf{R}^{n+1}_+}$, and harmonic on $\mathbf{R}^{n+1}_+$.

Proof (a) We first prove (a) under the assumption that u is C^2 in a neigborhood of $\overline{\mathbf{R}^{n+1}_+}$; this restriction will be removed at the end of the proof.

After a translation in x, we may evidently assume that $x_0 = 0$. Let $p = (x_0, t_0) = (0, t_0)$.

For any $R > 0$, we let $B_+(0, R)$ denote the hemisphere $\{(x, t) \in \underline{B(0, R)} : t > 0\}$. Note that if a is sufficiently small and R is sufficiently large, then the ball $\overline{B(p, a)}$ is contained in $B_+(0, R)$. (Precisely, this happens as long as $a < t_0$ and $R > t_0 + a$.) If this holds, we let $D_+ = B_+(0, b) \setminus \overline{B(p, a)}$, and apply Green's theorem (Proposition 1.4) to u and G_p on D_+. Since these functions are harmonic there, we find

$$0 = I_R - II_a - III_R, \tag{18}$$

where now, if we write $S = \{(x, t) \in \mathbf{R}^{n+1} : |(x, t)| = 1\}$ and $S_+ = \{(x, t) \in S : t > 0\}$, we have

$$I_R = \int_{S_+} ([\frac{\partial u}{\partial r} G_p - u\frac{\partial G_p}{\partial r}](R\omega)R^n)d\sigma(\omega),$$

$$II_a = \int_S ([\frac{\partial \tau_{-p} u}{\partial r} \tau_{-p} G_p - \tau_{-p} u \frac{\partial \tau_{-p} G_p}{\partial r}](a\omega)a^n)d\sigma(\omega),$$

and

$$III_R = \int_{|x| \le R} ([\frac{\partial u}{\partial t} G_p - u\frac{\partial G_p}{\partial t}])(x, 0)dx.$$

In (18) we may take the limit as $a \to 0^+$, and use Proposition 3.1 to find

$$u(x) = I_R - III_R. \tag{19}$$

Since $G_p(x, 0) = 0$ for all x, and since $P((0, t_0), x) = -\frac{\partial}{\partial t} G_p(x, t)|_{t=0}$, we evidently have that

$$-III_R = \int_{|x| \le R} u(x, 0)P((0, t_0), x)dx = \int_{|x| \le R} u(x, 0)P_{t_0}(x)dx.$$

It is evident from (17) and the p-test on $\mathbf{R}^n$ that $P_{t_0} \in L^1(\mathbf{R}^n)$, and hence that $\int_{\mathbf{R}^n} |u(x, 0)P_{t_0}(x)|dx < \infty$. Accordingly, by DCT,

$$\lim_{R \to \infty} (-III_R) = u * P_{t_0}(0).$$

Thus, by (19), in order to prove (a) (under our assumption on u), we need only show that $\lim_{R \to \infty} I_R = 0$.

Since u is bounded, by Corollary 2.3, there is a constant $C > 0$ such that, for all $(x, t) \in \mathbf{R}^{n+1}_+$,

$$|u(x, t)| \le C_0 \text{ and } |\frac{\partial u}{\partial r}(x, t)| \le \frac{C_0}{t}. \tag{20}$$

We are going to show that, for all $R > 2t_0$,

(*) For some $C > 0$, $|\frac{\partial G_p}{\partial r}(R\omega)| \le CR^{-n-1}$ and $|G_p(R\omega)| \le C \min(t, t_0)R^{-n}$, if $R\omega = (x, t)$ in Cartesian coordinates.

If (*) is known, we can complete the proof as follows. By (20) and (*), there exists $C_1 > 0$ with

$$\int_{S_+} |(u\frac{\partial G_p}{\partial r})(R\omega)|R^n d\sigma(\omega) \le \frac{C_1}{R} \to 0$$

as $R \to \infty$; and

$$|(\frac{\partial u}{\partial r}G_p)(R\omega)R^n| \le C_1$$

for all $R > 2t_0$ and all ω. Thus, to show that $I_R \to 0$ as $R \to \infty$, by DCT we need only show that $|(\frac{\partial u}{\partial r}G_p)(R\omega)R^n| \to 0$ for *each* $\omega \in S_+$. By (*) it is enough to show that for each $\omega = (x_1, t_1) \in S_+$, we have $|\frac{\partial u}{\partial r}(Rx_1, Rt_1)| \to 0$ as $R \to \infty$. But this is clear from (20).

The proof of (*) is somewhat technical. We need the following lemma:

Lemma 4.2 *Say $\alpha \in \mathbf{R}$. Then there exists a constant C, depending only on α, as follows: For all $u, v \in \mathbf{R}^n$ with $|v| \le |u|/2$, we have*

$$|\,|u+v|^\alpha - |u|^\alpha| \le C|v||u|^{\alpha-1}. \tag{21}$$

Proof It suffices to prove this for all u with $|u| = 1$. For otherwise, we may assume $u \ne 0$ and note that (21) holds with $u/|u|, v/|u|$ in place of u, v; but this is equivalent to what we want.

In that case, $|u| = 1$ and $1/2 \le |u+v| \le 3/2$. Now, the mean value theorem as applied to the function $f(x) = x^\alpha$ shows that there is a $C > 0$ such that for all $x \in [1/2, 3/2]$, one has $|x^\alpha - 1| \le C|x-1|$. Thus

$$|\,|u+v|^\alpha - |u|^\alpha| \le C|\,|u+v| - |u|\,| \le C|v|$$

as desired.

To prove (*), put $u = R\omega = (x,t)$. Recall $p = (0, t_0)$, and set $\tilde{p} = (0, -t_0)$. For some constants c, C, we have

$$|G_p(R\omega)| = c||u-p|^{1-n} - |u-\tilde{p}|^{1-n}| \le c(||u-p|^{1-n} - |u|^{1-n}| + ||u|^{1-n} - |u-\tilde{p}|^{1-n}|) \le Ct_0|u|^{-n} = Ct_0R^{-n}, \tag{22}$$

by Lemma 4.2. On the other hand, say $t \le t_0$, and now put $v = (x, t_0)$, $q = (0, t)$, and $\tilde{q} = (0, -t)$. Note that $|q| = |\tilde{q}| \le t_0 \le R/2 \le |v|/2$, so for some constants c, C, we have

$$|G_p(R\omega)| = c||v-q|^{1-n} - |v-\tilde{q}|^{1-n}| \le c(||v-q|^{1-n} - |v|^{1-n}| + ||v|^{1-n} - |v-\tilde{q}|^{1-n}|) \le Ct|v|^{-n} \le CtR^{-n},$$

proving the needed estimate for $|G_p(R\omega)|$. For $\frac{\partial G_p}{\partial r}$, we note that at the point $R\omega = (x,t)$,

$$\frac{\partial}{\partial r} = \frac{1}{R}[\sum_j x_j \frac{\partial}{\partial x_j} + t\frac{\partial}{\partial t}] = \frac{1}{R}[\sum_j x_j \frac{\partial}{\partial x_j} + (t-t_0)\frac{\partial}{\partial t}] + \frac{t_0}{R}\frac{\partial}{\partial t} = \frac{1}{R}[\sum_j x_j \frac{\partial}{\partial x_j} + (t+t_0)\frac{\partial}{\partial t}] - \frac{t_0}{R}\frac{\partial}{\partial t}.$$

Recalling that, if $w \in \mathbf{R}^{n+1}$, $|w|\frac{\partial N}{\partial r}(w) = |w|^{1-n}/\omega_{n+1}$, we find that for some constant c_1,

$$\frac{\partial G_p}{\partial r}(R\omega) = \frac{1}{\omega_{n+1}R}[|u-p|^{1-n} - |u-\tilde{p}|^{1-n}] + \frac{t_0(t-t_0)}{R\omega_{n+1}}|u-p|^{-1-n} - \frac{t_0(t+t_0)}{R\omega_{n+1}}|u-\tilde{p}|^{-1-n}.$$

Recalling (22), the fact that $|t/R| \leq 1$, our assumption that $|p| = |\tilde{p}| \leq R/2 = |u|/2$, and the fact that t_0 is fixed, we see that $|\frac{\partial G_p}{\partial r}(R\omega)| \leq CR^{-n-1}$, as desired. This proves (*).

We have now proved (a) under the assumption that u is C^2 in a neighborhood of $\overline{\mathbf{R}^{n+1}_+}$. To remove this restriction, say now that u is general, and for $\epsilon > 0$, define $u_\epsilon(x,t) = u(x, t+\epsilon)$. Then u_ϵ is C^2 in a neighborhood of $\overline{\mathbf{R}^{n+1}}$, bounded on $\overline{\mathbf{R}^{n+1}_+}$, and harmonic on $\mathbf{R}^{n+1}_+$. Let $f_\epsilon(x) = u_\epsilon(x, 0)$. By what we have seen, $u_\epsilon(x_0, t_0) = (f_\epsilon * P_{t_0})(x_0)$. By taking the limit as $\epsilon \to 0^+$ and using DCT, we see that $u(x_0, t_0) = (f * P_{t_0})(x_0)$. This proves (a).

For (b), note that if we apply (a) to the function $u \equiv 1$, we find that for all (x,t), $1 = 1 * P_t(x)$; in other words, that $\int P_t(y)dy = 1$. Also $P_t \geq 0$ and $P_t(x) = t^{-n}P(x/t)$. It follows readily that $\{P_t\}$ is an approximate identity; indeed, if $a > 0$, $\int_{|y|>a} P_t(y)dy = \int_{|y|>a/t} P_1(y)dy \to 0$ as $t \to 0^+$. Consequently, by the method of proof of Exercise 11.4.8 (b), u as defined in (b) is continuous on $\overline{\mathbf{R}^{n+1}}$. Finally, since

$$u(x_0, t_0) = \int_{\mathbf{R}^n} P((x_0, t_0), x)u(x, 0)dx.$$

and $P((x_0, t_0), x)$ is harmonic in $(x_0, t_0) \in \mathbf{R}^{n+1}_+$, easily-justified differentiations under the integral sign show that u is harmonic in $\mathbf{R}^{n+1}_+$ as well. This proves (b).

The proof of Theorem 4.1 was complicated by the necessity of proving that the integral $I_R \to \infty$ as $R \to \infty$. The estimates we needed to prove this may be finessed, by means of a different kind of argument (see Exercise 5.5 below). However, the methods used in the proof above are also useful in other contexts. An estimate finessed is a learning opportunity missed!

5 The Poisson Kernel for the Ball

In this section, we let $B = B(0, R)$, the ball of radius R about the origin in $\mathbf{R}^2$. We shall apply the method of images to find the Poisson kernel for B.

Using the conventions of Euclidean geometry for the moment, let O be the origin, and say $P \neq O$. The image point of P is then defined to be the point obtained from P by *inversion* in the boundary of B, namely the point $\tilde{P}$ on the ray from O through P with $OP \cdot O\tilde{P} = R^2$. This definition may appear to be quite different from the definition in the previous section; however, when $n = 2$, they are *conformally equivalent*. (That is: think of $\mathbf{R}^2$ as $\mathbf{C}$, and define $\varphi : \mathbf{C} \to \mathbf{C}$ by $\varphi(z) = R\frac{z-i}{z+i}$. Then φ maps the upper half space in $\mathbf{C}$ conformally onto $B(0, R)$. One readily computes that if $\varphi(a+bi) = P$, then $\varphi(a-bi) = \tilde{P}$, so that "images" in the sense of the previous section are mapped to "images" in the sense of this section.)

It is easy to see, from Euclidean geometry, that if $P, X \neq O$, and O, P, X are not collinear, then then $\Delta OP\tilde{X} \simeq \Delta OX\tilde{P}$. Indeed, we may work in the plane containing O, P and X, noting that $\tilde{P}$ and $\tilde{X}$, are also in this plane. Of course $\angle \tilde{X}OP = \angle \tilde{P}OX$. Note also that $OP \cdot O\tilde{P} = OX \cdot O\tilde{X} = R^2$, so that we have the proportion $OP/O\tilde{X} = OX/O\tilde{P}$. This implies the similarity of $\Delta OP\tilde{X}$ and $\Delta OX\tilde{P}$.)

In particular, $P\tilde{X}/OP = X\tilde{P}/OX$, or equivalently $X\tilde{P} = (OX/OP)P\tilde{X}$. (This holds even if O, X, P are collinear, since we may approach X by a sequence of points X_k which are not collinear with O and P, and take limits.) In other words, for $0 \neq p \in \mathbf{R}^n$, define the image point of p to be $\tilde{p} = \frac{R^2}{|p|}\frac{p}{|p|}$. Then, if $0 \neq p, x \in \mathbf{R}^n$,

$$|x - \tilde{p}| = \frac{|x|}{|p|}|\tilde{x} - p|. \tag{23}$$

Analytically, this follows at once from the identity $|u-v|^2 = |u|^2 - 2u\cdot v + |v|^2$. This implies that, if one squares the left side of (23), one obtains $|x|^2 - 2\frac{R^2}{|p|^2}x\cdot p + \frac{R^4}{|p|^2}$. If one squares the right side of (23), one obtains $(|x|^2/|p|^2)(|p|^2 - 2\frac{R^2}{|x|^2}p\cdot x + \frac{R^4}{|x|^2})$, which is the same.

In particular, if x is on the boundary of the ball B, so that $|x| = R$, we have

$$|x - \tilde{p}| = \frac{R}{|p|}|x - p|. \tag{24}$$

This implies that the temperature effects of a heat sink at p can be cancelled out, on ∂B, by an appropriate heat source at $\tilde{p}$. Specifically, if $p \in B$, the Green's function G_p is

$$\begin{aligned} G_p(x) &= \frac{1}{(2-n)\omega_n}(|x-p|^{(2-n)} - [\frac{|p|}{R}|x-\tilde{p}|]^{(2-n)}) \text{ if } n \neq 2; \\ &= \frac{1}{4\pi}[\log(|x-p|) - \log(\frac{|p|}{R}|x-\tilde{p}|)] \text{ if } n = 2. \end{aligned}$$

Then G_p is harmonic in x, and vanishes identically on ∂B. This definition of $G_p(x)$ makes sense for all $x, p \in \mathbf{R}^n$, provided that $p \neq 0$ and $x \neq p$ or $\tilde{p}$. By using (23), we find that

$$G_p(x) = G_x(p) \tag{25}$$

provided $0 \neq x, p$, and $x \neq p$ or $\tilde{p}$. With the restriction $x \neq p$ or $\tilde{p}$, $G_p(x)$ is defined as long as $p \neq 0$, while $G_x(p)$ is defined as long as $x \neq 0$. Thus (25) enables us to smoothly extend the definition of both $G_p(x)$ and $G_x(p)$ to the set $\{(x,p) \in \mathbf{R}^n \times \mathbf{R}^n : (x,p) \neq (0,0), x \neq p, x \neq \tilde{p}\}$. In particular, $G_p(x) = G_x(p)$ is harmonic in p for $p \neq x, \tilde{x}$.

Since the outward unit normal to ∂B is $\frac{\partial}{\partial r}$, for $p \in B$ and $x \in \partial B$, the considerations of section 3 now lead us to define the *Poisson kernel for the ball* $B(0,R)$ to be $P_p(x) = \frac{\partial}{\partial r}G_p(x)$. (Here, as usual, $\frac{\partial}{\partial r} = \frac{1}{R}\sum_j x_j \frac{\partial}{\partial x_j}$.) Put $P_p(x) = P(p,x)$. This function will be *harmonic* in $p \in B$, since $\frac{\partial}{\partial p_j}$ commutes with multiplication by x_k and with $\frac{\partial}{\partial x_k}$. We calculate that

$$P(p,x) = \frac{1}{\omega_n R}[|x-p|^{-n}\sum_j x_j(x_j - p_j) - \frac{|p|^{2-n}}{R^{2-n}}|x-\tilde{p}|^{-n}\sum_j x_j(x_j - \tilde{p}_j)].$$

Using (24) we find that

$$P(p,x) = \frac{1}{\omega_n R|x-p|^n}[|x|^2 - x\cdot p - \frac{|p|^2}{R^2}|x|^2 + \frac{|p|^2}{R^2}x\cdot\tilde{p}].$$

Recalling that $\frac{|p|^2}{R^2}\tilde{p} = p$ and that $|x|^2 = R^2$, we find finally that

$$P(p,x) = \frac{R^2 - |p|^2}{\omega_n R|x-p|^n}. \tag{26}$$

By using the methods of section 3, we see easily:

Theorem 5.1 *(a) Suppose that u is continuous on $\overline{B}$, and harmonic on B. For $x \in \partial B$, let $f(x) = u(x)$. Then, for any $p \in B$, we have*

$$u(p) = \int_S f(R\omega)P(p,R\omega)R^{n-1}d\sigma(\omega). \tag{27}$$

(b) Conversely, suppose that f is bounded and continuous on ∂B. For $p \in B$, define u by (27). Also, for $x \in \partial B$, set $u(x) = f(x)$. Then u is bounded and continuous on $\overline{B}$, and harmonic on B. In other words, u is the solution of the Dirichlet problem on B with boundary value f.

Proof (a) First suppose that u is C^2 in a neighborhood of $\overline{B}$. For $\epsilon > 0$, we apply Green's theorem (Proposition 1.3) on $D = B(0,b) \setminus \overline{B(p,\epsilon)}$ to the functions G_p and u, then we let $\epsilon \to 0^+$, invoking Proposition 3.1. Since G_p and u are harmonic on D, we obtain (27) at once, at least if u is C^2 in a neighborhood of $\overline{B}$. In general, for $0 < t < 1$, define $u_t(y) = u(ty)$ for $y \in B$, and $f_t(x) = u_t(x) = u(tx)$ for $x \in \partial B$. Then u_t is continuous on $\overline{B}$ and is harmonic in B. Thus (27) holds if we replace u, f by u_t, f_t respectively. Letting $t \to 1^-$, and using DCT, we find (27) in general. This proves (a).

For (b), note first that u as defined in (b) is harmonic in $p \in B$, since $P(p,R\omega)$ is harmonic in p for fixed ω, and we can differentiate under the intgral sign in (27). Thus we need only show that u is continuous on ∂B. To this end we need to show that P "acts like an approximate identity" in this situation. Note first then, that if we take $u \equiv 1$ in (a), we find that

$$\int_S P(p,R\omega)R^{n-1}d\sigma(\omega) = 1 \tag{28}$$

for all $p \in B$. Surely $P(p,R\omega) \geq 0$ always. These properties, together with the form of P (see (26)), enable one to complete the proof of (b).

***Exercise 5.2** *Complete the proof of (b) above.*
(Hint: to show that P shares all the needed properties of an approximate identity, use (26) to argue that for any $\omega \in S$, $P(r\omega, x)$ becomes more and more concentrated near $x = R\omega$ as $r \to R^-$.)

Remarks 1. If we set $p = 0$ in (27), and use (26), we recover the mean value theorem

$$u(0) = \frac{1}{\omega_n}\int_S u(R\omega)d\sigma(\omega).$$

2. When $n = 2$, Theorem 5.1 reduces to Theorem 11.2.17.

Exercise 5.3 *In* $\mathbf{R}^{n+1}$*, use coordinates* (x,t) *where* $x \in \mathbf{R}^n$ *and* $t \in \mathbf{R}$*.*
(a) Let B *denote the ball of radius* R *centered at* 0 *in* $\mathbf{R}^{n+1}$*. Suppose* u *is continuous on* $\overline{B}$ *and harmonic in* B*, and that* $u(x,t) = -u(x,-t)$ *for all* $(x,t) \in \partial B$*. Show that* $u(x,0) = 0$ *whenever* $(x,0) \in B$*.*
(b) Suppose $U \subseteq \mathbf{R}^{n+1}$ *is open, and that* U *is symmetric across the plane* $\{(x,0)\}$*, by which we mean that* $(x,t) \in U \Rightarrow (x,-t) \in U$*. Let* $U_+ = \{(x,t) \in U : t > 0\}$ *and* $U_* = \{(x,t) \in U : t \geq 0\}$*. Suppose that* u *is continuous on* U_* *and harmonic on* U_+*, and that* $u(x,0) = 0$ *whenever* $(x,0) \in U$*. Extend* u *to all of* U *by setting* $u(x,t) = -u(x,-t)$ *for all* $(x,t) \in U$ *with* $t < 0$*. Show that* u *is harmonic on all of* U*.*
(Hint for (b): the argument that u *is harmonic at points of the form* $(x,0)$ *is somewhat indirect. It uses (a) and the maximum principle.)*

Exercise 5.4 *Show that any bounded harmonic function on* $\mathbf{R}^n$ *must be constant.*
(Hint: the value at any point is the average of the values over a huge ball centered at this point; why must that be independent of the point?)

Exercise 5.5 *Note that the proof of part (b) of Theorem 4.1 relies only upon the facts that* $\{P_t\}$ *is an approximate identity and the fact that* $P((x_0,t_0),x)$ *is harmonic in* $(x_0,t_0) \in \mathbf{R}^{n+1}_+$*. Assuming that Theorem 4.1 (b) is known, give another proof of Theorem 4.1 (a), based upon Exercises 5.3 and 5.4.*

Exercise 5.6 *(a) Suppose that* u *is continuous and nonnegative on* $\overline{B(0,R)}$*, and harmonic on* $B(0,R)$*. Say* $p \in B$*. Prove* Harnack's inequality*:*

$$\frac{R^{n-2}(R^2 - |p|^2)}{(R+|p|)^n} u(0) \leq u(p) \leq \frac{R^{n-2}(R^2 - |p|^2)}{(R-|p|)^n} u(0).$$

(b) Suppose U *is an open, connected subset of* $\mathbf{R}^n$*, and that* $p, q \in U$*. Show that for some* $C > 0$*, one has that* $u(q) \leq Cu(p)$ *for every function* u *which is harmonic and nonnegative on* U*.*
(c) Again suppose U *is an open, connected subset of* $\mathbf{R}^n$*, and that* $p \in U$*, and now suppose* K *is a compact subset of* U*. Show that for some* $C > 0$*, one has that* $\max_K u \leq C \min_K u$*, for every function* u *which is harmonic and nonnegative on* U*.*
(d) Again suppose U *is an open, connected subset of* $\mathbf{R}^n$*, and now suppose that* $\{u_k\}$ *is a non-decreasing sequence of nonnegative harmonic functions on* U*. Show that either* $u_k(x) \to \infty$ *at every point* p *of* U*, or* $\{u_k\}$ *converges uniformly on compact subsets of* U *to a function* u *which is also nonnegative and harmonic on* U*. This is called* Harnack's principle.

Exercise 5.7 *Suppose* u *is twice differentiable on an open subset* U *of* $\mathbf{R}^n$*, and that* $\Delta u = 0$ *on* U*. Show that* u *is* C^∞ *on* U*.*
(Note that we are not assuming that u *is* C^2 *on* U*; that is to be shown. See the first paragraph of section 2. Use the maximum principle and the Poisson kernel for the ball.)*