1. (a) State (without proof) hypotheses under which it is justifiable to move a derivative past a summation sign. (Work on a real interval $I = (a, b)$; assume the summation is infinite.)

(b) Suppose $U \subseteq \mathbb{R}^n$ is open. Let $\| \|$ denote the sup norm on $U$. For $f \in C^1(U)$, let

$$\| f \|_{C^1} = \| f \| + \sum_{k=1}^n \| \frac{\partial f}{\partial x_k} \|$$

(Here $f : U \to \mathbb{R}$.) Let $C^1_b(U) = \{ f \in C^1(U) : \| f \|_{C^1} < \infty \}$. It is easy to see that $C^1_b(U)$ is a normed vector space with norm $\| \|_{C^1}$. Show that this normed vector space is complete.

(Hint: use (a)).

Solution (a) For example, if the $F_k$ are continuously differentiable on $I$, and if $\sum F_k$ and $\sum F_k'$ both converge uniformly on $I$, then $(\sum F_k)' = \sum F_k'$. (Of course, one can weaken these hypotheses, but this is all we need for (b).)

(b) Suppose $\sum m f_m$ converges absolutely in $C^1(U)$; we need only show that this series converges in $C^1_b(U)$. Since $C^1_b(U)$ is complete, and since the series $\sum m f_m$ converges absolutely in $C^1(U)$, it converges uniformly to a continuous function on $U$. Similarly, for any $k$, $\sum m \frac{\partial f_m}{\partial x_k}$ converges uniformly to a continuous function on $U$. Restricting all the $f_m$ to a line segment in a coordinate direction, and using (a), we see now that $\frac{\partial \sum m f_m}{\partial x_k} = \sum m \frac{\partial f_m}{\partial x_k}$. Thus $\sum m f_m$ is in $C^1(U)$, with the series converging in $C^1_b(U)$, since as $N \to \infty$,

$$\| \sum_{m=N+1}^\infty f_m \| + \sum_{k=1}^n \| \frac{\partial \sum_{m=N+1}^\infty f_m}{\partial x_k} \| \to 0.$$

2. Let $P$ denote the orthogonal projection onto a closed subspace $E$ of a Hilbert space $\mathcal{H}$. Assume $E \neq \{0\}$.

(a) Show that $\| P \| = 1$.

(b) Let $Q$ denote the orthogonal projection onto another closed subspace $F$ of $\mathcal{H}$, such that $E \cap F = \{0\}$. Suppose also that $\mathcal{H}$ is finite dimensional. Show that $\| PQ \| < 1$.

Solution (a) For any $x \in \mathcal{H}$, since $Px$ and $(I - P)x$ are orthogonal, we have

$$\| Px \|^2 + \| (I - P)x \|^2 = \| x \|^2.$$

Accordingly, for all $x \in \mathcal{H}$, $\| Px \| \leq \| x \|$ (with equality if and only if $Px = x$); so $\| P \| = \sup_{x \neq 0} \| Px \|/\| x \| \leq 1$. On the other hand, if $0 \neq x \in E$, then $Px = x$, so $\| Px \| = \| x \|$; so
3. Suppose that $Y_0 \in \mathbb{R}$. Let

$$S = [t_0 - h, t_0 + h] \times [Y_0 - R, Y_0 + R].$$

Suppose $F_1, F_2 : S \to \mathbb{R}$. (The domain of $F_1$ and of $F_2$ is precisely $S$.) Suppose that for $i = 1, 2, F_i$ is continuous, and that for some $K > 0$ we have

$$|F_i(t, Y_1) - F_i(t, Y_2)| \leq K|Y_1 - Y_2|$$

for all $(t_1, Y_1), (t_2, Y_2) \in S$.

Show that for some $P > 0$, there is a unique continuous function $y : (t_0 - P, t_0 + P) \to \mathbb{R}$ satisfying

$$y(t) = Y_0 + \left[ \int_{t_0}^{t} F_1(s, y(s))ds \right] \left[ \int_{t_0}^{t} F_2(u, y(u))du \right].$$

**Proof** Choose $M > 0$ so that $|F_i(t, Y)| \leq M$ for all $(t, Y) \in S$, $i = 1, 2$. We may assume $K > 0$. We claim that we may take $P = \min(\sqrt{\frac{M}{K}}, \frac{1}{2MK}, h)$. Say $0 < r < P$; we first solve the equation on $(t_0 - r, t_0 + r)$. Let $I = (t_0 - r, t_0 + r)$. Let

$$V_0 = \{ \text{continuous functions } f : I \to \mathbb{R} : \|f - Y_0\| \leq R, \text{ and } f(t_0) = Y_0 \}.$$ 

Since $F_1, F_2$ are only defined on $S = [t_0 - h, t_0 + h] \times [Y_0 - R, Y_0 + R]$, it follows that any solution of the equation on $I$ must lie in $V_0$. So we shall look for our solution within $V_0$; we shall find it by using the Contraction Mapping Principle.

$V_0$ is a complete metric space (with the uniform metric). For $f \in V_0$, define new functions $T_1f, T_2f, Tf$ on $I$ by

$$(T_i f)(t) = \int_{t_0}^{t} F(s, f(s))ds$$

for $i = 1, 2$ (we may do this, since $s \in I$ implies $(s, f(s)) \in S$ by the definition of $V_0$), and

$$(T f)(t) = Y_0 + [(T_1 f)(t)][(T_2 f)(t)].$$
We are looking for \( y \in V_0 \) with

\[ Ty = y. \]

Note that for any \( t \in I \), if \( i = 1, 2 \), then

\[
|(T_i f)(t)| = |\int_{t_0}^{t} F_i(s, f(s)) ds| \leq |\int_{t_0}^{t} F_i(s, f(s)) ds| \\
\leq |\int_{t_0}^{t} M ds| = M|t - t_0| < MP \leq \sqrt{R};
\]

so \( |T_i f| \leq \sqrt{R} \). Also, if \( y_1, y_2 \in V_0 \), then for \( i = 1, 2, t \in I \), we have

\[
|(T_i y_2)(t) - (T_i y_1)(t)| = |\int_{t_0}^{t} [F_i(s, y_2(s)) - F_i(s, y_1(s))] ds| \\
\leq |\int_{t_0}^{t} K |y_2(s) - y_1(s)| ds| \\
\leq \|y_2 - y_1\| \int_{t_0}^{t} K ds| \\
\leq \|y_2 - y_1\| |K| t - t_0| \\
\leq (rK)\|y_2 - y_1\|.
\]

To show that there exists a unique \( y \in V_0 \) satisfying \( Ty = y \), we need only show that the key hypothesis of the contraction mapping principle holds, namely, we must show:

\( T : V_0 \to V_0 \) is a contraction.

Of course, if \( f \in V_0 \), then \( (Tf)(t_0) = Y_0 \). Moreover, for any \( t \in I \),

\[
|(Tf)(t) - Y_0| = |(T_1f)(t)||T_2f)(t)| \leq \sqrt{R}\sqrt{R} = R;
\]

so, in fact, \( T : V_0 \to V_0 \).

Moreover, if \( y_1, y_2 \in V_0 \); then for \( t \in I \),

\[
|(Ty_1)(t) - (Ty_1)(t)| = |(T_1y_1)(t)(T_2y_1)(t) - (T_1y_2)(t)(T_2y_2)(t)| \\
\leq |(T_1y_1)(t)|(T_2y_1)(t) - (T_2y_2)(t) + [(T_1y_1)(t) - (T_1y_2)(t)](T_2y_2)(t)| \\
\leq \|T_1y_1\||T_2y_1)(t) - (T_2y_2)(t)| + |(T_1y_1)(t) - (T_1y_2)(t)||T_2y_2|| \\
\leq (2MrK)\|y_2 - y_1\|.
\]

Put \( \tau = 2MrK \); then \( \tau < 1 \) (since \( r \) is strictly less than \( 1/2MK \)), and \( \|Ty_2 - Ty_1\| \leq \tau\|y_2 - y_1\| \) for all \( y_1, y_2 \in V_0 \). So \( T \) is a contraction, as desired.
We have now seen that there’s a unique solution on \((t_0 - r, t_0 + r)\) for any \(r < P\). If \(0 < r_1 < r_2 < P\), and \(y_1\) is the solution on \((t_0 - r_1, t_0 + r_1)\), and \(y_2\) is the solution on \((t_0 - r_2, t_0 + r_2)\), then \(y_1 = y_2\) on \((t_0 - r_1, t_0 + r_1)\). Clearly, then, we may find the desired solution on \((t_0 - P, t_0 + P)\), simply by requiring that it be equal to the solution on \((t_0 - r, t_0 + r)\) for any \(0 < r < P\). This completes the proof.

4. Suppose that \(\mathcal{H}\) is a real Hilbert space, that \((A, B) \subseteq \mathbb{R}\), and that \(v : (A, B) \to \mathcal{H}\) is differentiable. Assume also that \(v'\) is continuous. Suppose \([a, b] \subseteq (A, B)\). Show that, for every \(\epsilon > 0\), there exists \(\delta > 0\) such that whenever \(|h| < \delta\) and \(t, t + h \in [a, b]\), then

\[
\|v(t + h) - v(t) - v'(t)h\| \leq \epsilon|h|.
\]

(Hint: first explain why \(v'\) is uniformly continuous on \([a, b]\).)

**Solution** The proof of Proposition 1.4.4 (c) in the book shows in fact that if \(V\) is a normed vector space and \(f : [a, b] \to V\) is continuous, then \(f\) is uniformly continuous on \([a, b]\). (In fact this is true if \(V\) is merely known to be a Hausdorff space.) Select \(\delta > 0\) such that if \(t, t + h \in [a, b]\), and \(|h| < \delta\), then \(\|v'(t) - v'(t + h)\| < \epsilon\).

For any \(u \in \mathcal{H}\), \(\|u\| = \sup_{\|y\|=1} |(u, y)|\). Thus it suffices to show that for any fixed \(y \in \mathcal{H}\) with \(\|y\| = 1\), we have that

\[
\|(v(t + h) - v(t) - v'(t)h, y)\| \leq \epsilon|h|
\]

whenever \(|h| < \delta\) and \(t, t + h \in [a, b]\). Define \(f : (A, B) \to \mathbb{R}\) by \(f(t) = (v(t), y)\), so that \(f'(t) = (v'(t), y)\). We then have that

\[
\|(v(t + h) - v(t) - v'(t)h, y)\| = \|f(t + h) - f(t) - f'(t)h\|
\]

\[
= \|f'(t + k)h - f'(t)h\|
\]

\[
= \|([v'(t + k) - v'(t)], y)\| |h|
\]

\[
\leq \|v'(t + k) - v'(t)\| |h|
\]

\[
< \epsilon|h|
\]

as desired. (In the second line, we used the Mean Value Theorem; \(k\) is some number between 0 and \(h\). In the fourth line, we used Cauchy-Schwarz.)