MAT 544 – Test 1

1. (a) State (without proof) hypotheses under which it is justifiable to move a derivative past a summation sign. (Work on a real interval $I = (a, b)$; assume the summation is infinite.)
(b) Suppose $U \subseteq \mathbf{R}^n$ is open. Let $\| \ \|$ denote the sup norm on U. For $f \in C^1(U)$, let

$$\|f\|_{C^1} = \|f\| + \sum_{k=1}^{n} \|\frac{\partial f}{\partial x_k}\|$$

(Here $f : U \to \mathbf{R}$.) Let $C_b^1(U) = \{f \in C^1(U) : \|f\|_{C^1} < \infty\}$. It is easy to see that $C_b^1(U)$ is a normed vector space with norm $\| \ \|_{C^1}$. Show that this normed vector space is complete. (Hint: use (a)).
Solution (a) For example, if the F_k are continuously differentiable on I, and if $\sum F_k$ and $\sum F_k'$ both converge uniformly on I, then $(\sum F_k)' = \sum F_k'$. (Of course, one can weaken these hypotheses, but this is all we need for (b).)
(b) Suppose $\sum_m f_m$ converges absolutely in $C^1(U)$; we need only show that this series converges in $C_b^1(U)$. Since $C_b(U)$ is complete, and since the series $\sum_m f_m$ converges absolutely in $C_b(U)$, it converges uniformly to a continuous function on U. Similarly, for any k, $\sum_m \frac{\partial f_m}{\partial x_k}$ converges uniformly to a continuous function on U. Restricting all the f_m to a line segment in a coordinate direction, and using (a), we see now that $\frac{\partial \sum_m f_m}{\partial x_k} = \sum_m \frac{\partial f_m}{\partial x_k}$. Thus $\sum_m f_m$ is in $C^1(U)$, with the series converging in $C_b^1(U)$, since as $N \to \infty$,

$$\| \sum_{m=N+1}^{\infty} f_m\| + \sum_{k=1}^{n} \|\frac{\partial \sum_{m=N+1}^{\infty} f_m}{\partial x_k}\| \to 0.$$

2. Let P denote the orthogonal projection onto a closed subspace E of a Hilbert space $\mathcal{H}$. Assume $E \neq \{0\}$.
(a) Show that $\|P\| = 1$.
(b) Let Q denote the orthogonal projection onto another closed subspace F of $\mathcal{H}$, such that $E \cap F = \{0\}$. Suppose also that $\mathcal{H}$ is finite dimensional. Show that $\|PQ\| < 1$.

Solution (a) For any $x \in \mathcal{H}$, since Px and $(I - P)x$ are orthogonal, we have

$$\|Px\|^2 + |(I - P)x\|^2 = \|x\|^2.$$

Accordingly, for all $x \in \mathcal{H}$, $\|Px\| \leq \|x\|$ (with equality if and only if $Px = x$); so $\|P\| = \sup_{x\neq 0} \|Px\|/\|x\| \leq 1$. On the other hand, if $0 \neq x \in E$, then $Px = x$, so $\|Px\| = \|x\|$; so

$\|P\| = 1$.

(b) Let $S = \{x : \|x\| = 1\}$. Since $\mathcal{H}$ is a finite-dimensional normed vector space, and S is closed and bounded, S is a compact set. Also the map taking x to $\|PQx\|$ is continuous from S to $\mathbf{R}$, and hence achieves a maximum on S. Thus $\|PQ\| = \sup_{x\in S}\|PQx\| = \max_{x\in S}\|PQx\|$. So it suffices to show that if $\|x\| = 1$, then $\|PQx\| < \|x\|$. But for any x, $\|PQx\| \le \|Qx\| \le \|x\|$, with equality if and only if $PQx = Qx = x$. In particular $\|PQx\| \le \|x\|$ for all $x \in S$; we could only have $\|PQx\| = \|x\|$, for some $x \in S$, if $PQx = Qx = x$. Since $Qx \in F$ and $PQx \in E$, this can happen only if $x = Qx = PQx \in E \cap F = \{0\}$. Thus, if $\|PQx\| = \|x\|$, then $x = 0$ and $\|x\|$ cannot be 1, as desired.

3. Suppose that $Y_0 \in \mathbf{R}$. Let

$$S = [t_0 - h, t_0 + h] \times [Y_0 - R, Y_0 + R].$$

Suppose $F_1, F_2 : S \to \mathbf{R}$. (The domain of F_1 and of F_2 is *precisely* S.) Suppose that for $i = 1, 2$, F_i is continuous, and that for some $K > 0$ we have

$$|F_i(t, Y_1) - F_i(t, Y_2)| \le K|Y_1 - Y_2|$$

for all $(t_1, Y_1), (t_1, Y_2) \in S$.

Show that for some $P > 0$, there is a unique continuous function $y : (t_0 - P, t_0 + P) \to \mathbf{R}$ satisfying

$$y(t) = Y_0 + [\int_{t_0}^{t} F_1(s, y(s))ds][\int_{t_0}^{t} F_2(u, y(u))du].$$

Proof Choose $M > 0$ so that $|F_i(t, Y)| \le M$ for all $(t, Y) \in S$, $i = 1, 2$. We may assume $K > 0$. We claim that we may take $P = \min(\frac{\sqrt{R}}{M}, \frac{1}{2MK}, h)$. Say $0 < r < P$; we first solve the equation on $(t_0 - r, t_0 + r)$. Let $I = (t_0 - r, t_0 + r)$. Let

$V_0 = \{$continuous functions $f : I \to \mathbf{R} : \|f - Y_0\| \le R$, and $f(t_0) = Y_0\}$.

Since F_1, F_2 are only defined on $S = [t_0 - h, t_0 + h] \times [Y_0 - R, Y_0 + R]$, it follows that any solution of the equation on I **must** lie in V_0. So we shall look for our solution within V_0; we shall find it by using the Contraction Mapping Principle.

V_0 is a complete metric space (with the uniform metric). For $f \in V_0$, define new functions T_1f, T_2f, Tf on I by

$$(T_if)(t) = \int_{t_0}^{t} F(s, f(s))ds$$

for $i = 1, 2$ (we may do this, since $s \in I$ implies $(s, f(s)) \in S$ by the definition of V_0), and

$$(Tf)(t) = Y_0 + [(T_1f)(t)][(T_2f)(t)].$$

We are looking for $y \in V_0$ with

$$Ty = y.$$

Note that for any $t \in I$, if $i = 1, 2$, then

$|(T_i f)(t)| = |\int_{t_0}^t F_i(s, f(s))ds| \leq |\int_{t_0}^t |F_i(s, f(s))|ds|$
$\leq |\int_{t_0}^t M ds| = M|t - t_0| < MP \leq \sqrt{R};$

so $\|T_i f\| \leq \sqrt{R}$. Also, if $y_1, y_2 \in V_0$, then for $i = 1, 2$, $t \in I$, we have

$$\begin{aligned}
|(T_i y_2)(t) - (T_i y_1)(t)| &= |\int_{t_0}^t [F_i(s, y_2(s)) - F_i(s, y_1(s))]ds| \\
&= \leq |\int_{t_0}^t |F_i(s, y_2(s)) - F_i(s, y_1(s))|ds| \\
&= \leq |\int_{t_0}^t K|y_2(s) - y_1(s)|ds| \\
&= \leq \|y_2 - y_1\|| \int_{t_0}^t K ds| \\
&= \|y_2 - y_1\| K|t - t_0| \\
&\leq (rK)\|y_2 - y_1\|.
\end{aligned}$$

To show that there exists a unique $y \in V_0$ satisfying $Ty = y$, we need only show that the key hypothesis of the contraction mapping principle holds, namely, we must show:

$T : V_0 \to V_0$ is a contraction.

Of course, if $f \in V_0$, then $(Tf)(t_0) = Y_0$. Moreover, for any $t \in I$,

$$|(Tf)(t) - Y_0| = | \leq |(T_1 f)(t)||(T_2 f)(t)| \leq \sqrt{R}\sqrt{R} = R,$$

so, in fact, $T : V_0 \to V_0$.

Moreover, if $y_1, y_2 \in V_0$; then for $t \in I$,

$$\begin{aligned}
|(Ty_1)(t) - (Ty_1)(t)| &= |(T_1 y_1)(t)(T_2 y_1)(t) - (T_1 y_2)(t)(T_2 y_2)(t)| \\
&= |(T_1 y_1)(t)[(T_2 y_1)(t) - (T_2 y_2)(t)] + [(T_1 y_1)(t) - ((T_1 y_2)(t)](T_2 y_2)(t)| \\
&\leq \|T_1 y_1\||(T_2 y_1)(t) - (T_2 y_2)(t)| + |(T_1 y_1)(t) - (T_1 y_2)(t)|\|T_2 y_2\| \\
&\leq (2MrK)\|y_2 - y_1\|.
\end{aligned}$$

Put $\tau = 2MrK$; then $\tau < 1$ (since r is strictly less than $1/2MK$), and $\|Ty_2 - Ty_1\| \leq \tau\|y_2 - y_1\|$ for all $y_1, y_2 \in V_0$. So T is a contraction, as desired.

We have now seen that that there's a unique solution on $(t_0 - r, t_0 + r)$ for any $r < P$. If $0 < r_1 < r_2 < P$, and y_1 is the solution on $(t_0 - r_1, t_0 + r_1)$, and y_2 is the solution on $(t_0 - r_2, t_0 + r_2)$, then $y_1 = y_2$ on $(t_0 - r_1, t_0 + r_1)$. Clearly, then, we may find the desired solution on $(t_0 - P, t_0 + P)$, simply by requiring that it be equal to the solution on $(t_0 - r, t_0 + r)$ for any $0 < r < P$. This completes the proof.

4. Suppose that $\mathcal{H}$ is a real Hilbert space, that $(A, B) \subseteq \mathbf{R}$, and that $v : (A, B) \to \mathcal{H}$ is differentiable. Assume also that v' is continuous. Suppose $[a, b] \subseteq (A, B)$. Show that, for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|h| < \delta$ and $t, t + h \in [a, b]$, then

$$\|v(t + h) - v(t) - v'(t)h\| \leq \epsilon|h|.$$

(Hint: first explain why v' is uniformly continuous on $[a, b]$.)
Solution The proof of Proposition 1.4.4 (c) in the book shows in fact that if V is a normed vector space and $f : [a, b] \to V$ is continuous, then f is uniformly continuous on $[a, b[$. (In fact this is true if V is merely known to be a Hausdorff space.) Select $\delta > 0$ such that if $t, t + h \in [a, b]$, and $|h| < \delta$, then $\|v'(t) - v'(t + h)\| < \epsilon$.

For any $u \in \mathcal{H}$, $\|u\| = \sup_{\|y\|=1} |(u, y)|$. Thus it suffices to show that for any *fixed* $y \in \mathcal{H}$ with $\|y\| = 1$, we have that

$$\|(v(t + h) - v(t) - v'(t)h, y)\| \leq \epsilon|h|$$

whenever $|h| < \delta$ and $t, t + h \in [a, b]$. Define $f : (A, B) \to \mathbf{R}$ by $f(t) = (v(t), y)$, so that $f'(t) = (v'(t), y)$. We then have that

$$\begin{aligned}
\|(v(t + h) - v(t) - v'(t)h, y)\| &= |f(t + h) - f(t) - f'(t)h| \\
&= |f'(t + k)h - f'(t)h| \\
&= |([v'(t + k) - v'(t)], y)|\ |h| \\
&\leq \|v'(t + k) - v'(t)\|\ |h| \\
&< \epsilon|h|
\end{aligned}$$

as desired. (In the second line, we used the Mean Value Theorem; k is some number between 0 and h. In the fourth line, we used Cauchy-Schwarz.)