1. (a) Let \((Y, d)\) be a metric space. Say \(\varepsilon > 0\). Let us say that a sequence \(\{z_n\}\) in \(Y\) is eventually \(\varepsilon\)-close to \(z \in Y\) provided that, for some \(N > 0\), one has \(d(z_n, z) < \varepsilon\) whenever \(n > N\).

Prove that a sequence \(y_n \to y\) in \(Y\) if and only if, for every \(\varepsilon > 0\), every subsequence of \(\{y_n\}\) has a subsequence which is eventually \(\varepsilon\)-close to \(y\).

(b) Let \((X, \mathcal{M}, \mu)\) be a measure space. Suppose that \(g, f, f_1, f_2, \ldots\) are all measurable functions from \(X\) to \(C\). Suppose that \(|f_n| \leq g\) for all \(n\), and that \(\int_X g \, d\mu < \infty\). In the homework you showed that \(f_n \to f\) in \(L^1(X)\) if and only if \(f_n \to f\) in measure, provided that \(\mu(X) < \infty\). Show that this conclusion remains true, if \(X\) is only known to be \(\sigma\)-finite.

**Solution**

(a) Say \(y_n \to y\). Any fixed subsequence approaches \(y\), so for every \(\varepsilon > 0\), that subsequence itself is eventually \(\varepsilon\)-close to \(y\), so it surely has a subsequence (itself) which is eventually \(\varepsilon\)-close to \(y\).

Conversely, say every subsequence of \(\{y_n\}\) has a subsequence which is eventually \(\varepsilon\)-close to \(y\). If \(y_n\) does not approach \(y\), then, for some \(\varepsilon > 0\), \(\{y_n\}\) must have a subsequence, each term of which is more than \(\varepsilon\) away from \(y\). No subsequence of that subsequence could be eventually \(\varepsilon\)-close to \(y\), contradiction.

(b) (Sorry – this works for any measure space \(X\) (it doesn’t have to be \(\sigma\)-finite), and the actual homework problem said so.) Say \(f_n \to f\) in \(L^1\). Then \(f_n \to f\) in measure by Tchebychev’s inequality

\[
\mu(\{x : |f_n - f| > \varepsilon\}) \leq \frac{\|f_n - f\|_1}{\varepsilon}.
\]

On the other hand, say \(f_n \to f\) in measure. Since \(L^1\) is a metric space, it is enough to show that every subsequence of \(\{f_n\}\) has a subsequence which approaches \(f\) in \(L^1\). Let \(\{h_n\}\) be a subsequence of \(\{f_n\}\). Then \(h_n \to f\) in measure, so a subsequence of \(\{h_n\}\) approaches \(f\) a.e. By DCT, that subsequence of \(\{h_n\}\) approaches \(f\) in \(L^1\), as desired.

2. (a) For \(n \in \mathbb{Z}\), \(x \in [0, 2\pi]\), let \(e_n(x) = e^{inx}\). Prove that the sequence \(e_1, e_2, \ldots\) has no \(L^2\)-convergent subsequence. This obviously implies, in particular, that the closed unit ball of \(L^2([0, 2\pi])\) is not compact.

(b) Now \(1 \leq p \leq \infty\) is arbitrary. Show that the closed unit ball of \(L^p([0, 2\pi])\) is not compact.

**Solution** For both parts, it is enough to show that \(e_1, e_2, \ldots\) has no \(L^1\)-convergent subsequence. For if a subsequence converged in \(L^p\) for any \(p > 1\), then it would converge in \(L^1\).
Say then, to obtain a contradiction, that \( \{e_{n_j}\} \) is an \( L^1 \)-convergent subsequence of \( \{e_n\} \), and that in fact \( e_{n_j} \to f \) in \( L^1 \). On the one hand \( \|f\|_1 = \lim_{n \to \infty} \|e_{n_j}\|_1 = 1 \). On the other hand, for any \( m \in \mathbb{Z} \), \( \hat{f}(m) = \lim_{j \to \infty} e_{n_j}, e_m = 0 \), which implies \( f \equiv 0 \) by the injectivity of \( \hat{\cdot} \) on \( L^1(\mathbb{T}) \). This contradicts \( \|f\|_1 = 1 \), as desired. (The equation \( \hat{f}(m) = \lim_{j \to \infty} e_{n_j}, e_m \) follows from the hypothesis that \( e_{n_j} \to f \) in \( L^1 \) together with the fact that \( e_m \in L^\infty \), since \( (2\pi)^{-1} \int_0^{2\pi} e_{-m} - \int_0^{2\pi} e_{n_j} e_{-m} \leq \|f - e_{n_j}\|_1 \|e_{-m}\|_\infty \).)

3. Say \( f \in L^1(\mathbb{T}) \). Show that \( f \in C^\infty(\mathbb{T}) \) if and only if: for every \( k > 0 \) there exists \( C_k > 0 \) such that for all \( n \in \mathbb{Z} \), \( |n|^k |\hat{f}(n)| \leq C_k \).

(Hint: for one direction, you can use Corollary 11.2.12 (copy attached).)

**Solution** Say first that \( f \in C^\infty(\mathbb{T}) \). Repeated integrations by parts show that for any positive integer \( k \), \( |\hat{f}^{(k)}(n)| = |n|^k |\hat{f}(n)| \). Let \( C_k = \|f^{(k)}\|_1 \); then for any \( n \), \( |n|^k |\hat{f}(n)| \leq C_k \), as desired.

Conversely, suppose \( f \in L^1(\mathbb{T}) \), and that for every \( k > 0 \) there is a \( C_k \) such that \( |n|^k |\hat{f}(n)| \leq C_k \) for all \( n \). Note three facts:
1. We have \( \sum |n|^k |\hat{f}(n)| < \infty \) for any positive integer \( k \). For in fact,

\[
\sum_{n \neq 0} |n|^k |\hat{f}(n)| \leq \sum |n|^k \frac{C_{k+2}}{|n|^{k+2}} < \infty.
\]

2. By Corollary 11.2.12 and fact 1, \( f = \sum \hat{f}(n)e_n \), where the series converges uniformly on \( \mathbb{T} \).

3. Suppose \( \sum_{n=-\infty}^{\infty} |nb_n| < \infty \) (so that also \( \sum_{n=-\infty}^{\infty} |b_n| < \infty \)). Then surely \( \sum b_ne_n \) and \( \sum inb_ne_n \) converge uniformly to functions in \( C(\mathbb{T}) \); and by our theorems on moving derivatives past summation signs, in fact \( \sum b_ne_n \in C^1(\mathbb{T}) \), and \( (\sum b_ne_n)' = \sum inb_ne_n \).

Combining these three facts and using induction, we see that \( f \in C^k \) for any \( k \), and in fact that \( f^{(k)} = \sum (in)^k b_ne_n \), where the series converges uniformly on \( \mathbb{T} \). So \( f \in C^\infty \).

4. Let \( \mu \) be a measure on \( \mathbb{R}^n \). Suppose that \( F \subseteq \mathbb{R}^n \) is closed, and that \( m(F) = 0 \). Finally suppose that for every \( f \in C_c(\mathbb{R}^n) \), with \( \text{supp} \ f \subseteq F^c \), one has that \( \int_{\mathbb{R}^n} f d\mu = 0 \). Show that the measures \( \mu \) and \( m \) are mutually singular.

**Solution** Let \( U = F^c \). Since \( m(F) = 0 \), it suffices to show that \( \mu(U) = 0 \).

First say \( K \subseteq U \) is compact; we show that \( \mu(K) = 0 \). Indeed, we may select \( f \in C_c \) with \( K \prec f \prec U \), and then

\[
0 \leq \mu(K) = \int \chi_K d\mu \leq \int f d\mu = 0.
\]

Thus \( \mu(K) = 0 \), as claimed.

Now, if \( I \in \mathcal{D}_U \), \( I \) is a countable increasing union of compact rectangles, so \( \mu(I) = 0 \). Finally \( \mu(U) = \sum_{I \in \mathcal{D}_U} \mu(I) = 0 \), as desired.

(The proof shows that the result remains true if, in the statement, we replace \( (\mathbb{R}^n, m) \) by \( (X, \nu) \), where \( \nu \) is any measure on the locally compact Hausdorff space \( X \), provided that \( U \) is \( \sigma \)-compact.)