MAT 550 – Test 1 Solutions

1. (a) Let (Y, d) be a metric space. Say $\epsilon > 0$. Let us say that a sequence $\{z_n\}$ in Y is *eventually ϵ-close to $z \in Y$* provided that, for some $N > 0$, one has $d(z_n, z) < \epsilon$ whenever $n > N$.

   Prove that a sequence $y_n \to y$ in Y if and only if, for every $\epsilon > 0$, every subsequence of $\{y_n\}$ has a subsequence which is eventually ϵ-close to y.
   (b) Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $g, f, f_1, f_2, \ldots$ are all measurable functions from X to $\mathbf{C}$. Suppose that $|f_n| \leq g$ for all n, and that $\int_X g d\mu < \infty$. In the homework you showed that $f_n \to f$ in $L^1(X)$ if and only if $f_n \to f$ in measure, *provided that* $\mu(X) < \infty$. Show that this conclusion remains true, if X is only known to be σ-finite.
   Solution (a) Say $y_n \to y$. Any fixed subsequence approaches y, so for every $\epsilon > 0$, that subsequence itself is eventually ϵ-close to y, so it surely has a subsequence (itself) which is eventually ϵ-close to y.
   Conversely, say every subsequence of $\{y_n\}$ has a subsequence which is eventually ϵ-close to y. If y_n does not approach y, then, for some $\epsilon > 0$, $\{y_n\}$ must have a subsequence, each term of which is more that ϵ away from y. No subsequence of that subsequence could be eventually ϵ-close to y, contradiction.
   (b) (Sorry – this works for any measure space X (it doesn't have to be σ-finite), and the actual homework problem said so.) Say $f_n \to f$ in L^1. Then $f_n \to f$ in measure by Tchebyshev's inequality

$$\mu(\{x : |f_n - f| > \epsilon\}) \leq \frac{\|f_n - f\|_1}{\epsilon}.$$

   On the other hand, say $f_n \to f$ in measure. Since L^1 is a metric space, it is enough to show that every subsequence of $\{f_n\}$ has a subsequence which approaches f in L^1. Let $\{h_n\}$ be a subsequence of $\{f_n\}$. Then $h_n \to f$ in measure, so a subsequence of $\{h_n\}$ approches f a.e. By DCT, that subsequence of $\{h_n\}$ approaches f in L^1, as desired.

2. (a) For $n \in \mathbf{Z}$, $x \in [0, 2\pi]$, let $e_n(x) = e^{inx}$. Prove that the sequence $e_1, e_2, \ldots$ has no L^2-convergent subsequence. This obviously implies, in particular, that the closed unit ball of $L^2([0, 2\pi])$ is not compact.
   (b) Now $1 \leq p \leq \infty$ is arbitrary. Show that the closed unit ball of $L^p([0, 2\pi])$ is not compact.
   Solution For both parts, it is enough to show that $e_1, e_2, \ldots$ has no L^1-convergent subsequence. For if a subsequence converged in L^p for any $p > 1$, then it would converge in L^1.

Say then, to obtain a contradiction, that $\{e_{n_j}\}$ is an L^1-convergent subsequence of $\{e_n\}$, and that in fact $e_{n_j} \to f$ in L^1. On the one hand $\|f\|_1 = \lim_{n\to\infty} \|e_{n_j}\|_1 = 1$. On the other hand, for any $m \in \mathbf{Z}$, $\hat{f}(m) = \lim_{j\to\infty}(e_{n_j}, e_m) = 0$, which implies $f \equiv 0$ by the injectivity of $\hat{}$ on $L^1(\mathbf{T})$. This contradicts $\|f\|_1 = 1$, as desired. (The equation $\hat{f}(m) == \lim_{j\to\infty}(e_{n_j}, e_m)$ follows from the hypothesis that $e_{n_j} \to f$ in L^1 together with the fact that $e_m \in L^\infty$, since $(2\pi)^{-1}|\int_0^{2\pi} fe_{-m} - \int_0^{2\pi} e_{n_j}e_{-m}| \leq \|f - e_{n_j}\|_1\|e_{-m}\|_\infty$.)

3. Say $f \in L^1(\mathbf{T})$. Show that $f \in C^\infty(\mathbf{T})$ if and only if:
for every $k > 0$ there exists $C_k > 0$ such that for all $n \in \mathbf{Z}$, $|n|^k|\hat{f}(n)| \leq C_k$.
(Hint: for one direction, you can use Corollary 11.2.12 (copy attached).)
Solution Say first that $f \in C^\infty(\mathbf{T})$. Repeated integrations by parts show that for any positive integer k, $|\widehat{f^{(k)}}(n)| = |n|^k|\hat{f}(n)|$. Let $C_k = \|f^{(k)}\|_1$; then for any n, $|n|^k|\hat{f}(n)| \leq C_k$, as desired.
Conversely, suppose $f \in L^1(\mathbf{T})$, and that for every $k > 0$ there is a C_k such that $|n|^k|\hat{f}(n)| \leq C_k$ for all n. Note three facts:
1. We have $\sum |n|^k|\hat{f}(n)| < \infty$ for any positive integer k. For in fact,

$$\sum_{n\neq 0} |n|^k|\hat{f}(n)| \leq \sum |n|^k \frac{C_{k+2}}{|n|^{k+2}} < \infty.$$

2. By Corollary 11.2.12 and fact 1, $f = \sum \hat{f}(n)e_n$, where the series converges uniformly on $\mathbf{T}$.
3. Suppose $\sum_{n=-\infty}^{\infty} |nb_n| < \infty$ (so that also $\sum_{n=-\infty}^{\infty} |b_n| < \infty$). Then surely $\sum b_ne_n$ and $\sum inb_ne_n$ converge uniformly to functions in $C(\mathbf{T})$; and by our theorems on moving derivatives past summation signs, in fact $\sum b_ne_n \in C^1(\mathbf{T})$, and $(\sum b_ne_n)' = \sum inb_ne_n$.
Combining these three facts and using induction, we see that $f \in C^k$ for any k, and in fact that $f^{(k)} = \sum (in)^k b_ne_n$, where the series converges uniformly on $\mathbf{T}$. So $f \in C^\infty$.

4. Let μ be a measure on $\mathbf{R}^n$. Suppose that $F \subseteq \mathbf{R}^n$ is closed, and that $m(F) = 0$. Finally suppose that for every $f \in C_c(\mathbf{R}^n)$, with supp $f \subseteq F^c$, one has that $\int_{\mathbf{R}^n} f d\mu = 0$. Show that the measures μ and m are mutually singular.
Solution Let $U = F^c$. Since $m(F) = 0$, it suffices to show that $\mu(U) = 0$.
First say $K \subseteq U$ is compact; we show that $\mu(K) = 0$. Indeed, we may select $f \in C_c$ with $K \prec f \prec U$, and then

$$0 \leq \mu(K) = \int \chi_K d\mu \leq \int f d\mu = 0.$$

Thus $\mu(K) = 0$, as claimed.
Now, if $I \in \mathcal{D}_U$, I is a countable increasing union of compact rectangles, so $\mu(I) = 0$. Finally $\mu(U) = \sum_{I\in\mathcal{D}_U} \mu(I) = 0$, as desired.
(The proof shows that the result remains true if, in the statement, we replace $(\mathbf{R}^n, m)$ by (X, ν), where ν is any measure on the locally compact Hausdorff space X, provided that U is σ-compact.)