ON A THEOREM OF CAMPANA AND PĂUN

CHRISTIAN SCHNELL

Abstract. Let X be a smooth projective variety over the complex numbers, and $\Delta \subseteq X$ a reduced divisor with normal crossings. We present a slightly simplified proof for the following theorem of Campana and Păun: If some tensor power of the bundle $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, then (X, Δ) is of log general type. This result is a key step in the recent proof of Viehweg's hyperbolicity conjecture.

1. Introduction. The purpose of this paper is to present a slightly simplified proof for the following result by Campana and Păun [CP15, Theorem 7.6]. It is a crucial step in the proof of Viehweg's hyperbolicity conjecture for families of canonically polarized manifolds [CP15, Theorem 7.13], and more generally, for smooth families of varieties of general type [PS16, Theorem A].

Theorem 1.1. *Let X be a smooth projective variety, and $\Delta \subseteq X$ a reduced divisor with at worst normal crossing singularities. If some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, then $K_X + \Delta$ is big.*

The simplification is that I have substituted an inductive procedure for the arguments involving Campana's "orbifold cotangent bundle"; otherwise, the proof of Theorem 1.1 that I present here is essentially the same as in the one in [CP15]. My reason for writing this paper is that it gives me a chance to draw attention to some of the beautiful ideas involved in the proof by Campana and Păun: slope stability with respect to movable classes; a criterion for the leaves of a foliation to be algebraic subvarieties; and positivity results for relative canonical bundles.

2. Strategy of the proof. Let (X, Δ) be a pair, consisting of a smooth projective variety X and a reduced divisor $\Delta \subseteq X$ with at worst normal crossing singularities. We denote the logarithmic cotangent bundle by the symbol $\Omega_X^1(\log \Delta)$, and its dual, the logarithmic tangent bundle, by the symbol $\mathscr{T}_X(-\log \Delta)$. Recall that $\mathscr{T}_X(-\log \Delta)$ is naturally a subsheaf of the tangent bundle $\mathscr{T}_X$, and that it is closed under the Lie bracket on $\mathscr{T}_X$. Indeed, suppose that Δ is given, in suitable local coordinates $x_1, x_2, \dots, x_n$, by the equation $x_1 x_2 \cdots x_k = 0$; then $\mathscr{T}_X(-\log \Delta)$ is generated by the n commuting vector fields

$$x_1 \frac{\partial}{\partial x_1}, \dots, x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n},$$

and is therefore closed under the Lie bracket.

Suppose that $\Omega_X^1(\log \Delta)^{\otimes N}$ contains a subsheaf with big determinant, for some $N \geq 1$. The following observation reduces the problem to the case of line bundles.

Lemma 2.1. *If $\Omega_X^1(\log \Delta)^{\otimes N}$ contains a subsheaf of generic rank $r \geq 1$ and with big determinant, then $\Omega_X^1(\log \Delta)^{\otimes Nr}$ contains a big line bundle.*

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Proof. Let $\mathscr{B} \subseteq \Omega_X^1(\log \Delta)^{\otimes N}$ be a subsheaf of generic rank $r \geq 1$, with the property that $\det \mathscr{B}$ is big. After replacing $\mathscr{B}$ by its saturation, whose determinant is of course still big, we may assume that the quotient sheaf

$$\Omega_X^1(\log \Delta)^{\otimes N} \big/ \mathscr{B}$$

is torsion-free, hence locally free outside a closed subvariety $Z \subseteq X$ of codimension ≥ 2. On $X \setminus Z$, we have an inclusion of locally free sheaves

$$\det \mathscr{B} \hookrightarrow \mathscr{B}^{\otimes r} \hookrightarrow \Omega_X^1(\log \Delta)^{\otimes Nr},$$

which remains valid on X by Hartog's theorem. □

For the purpose of proving Theorem 1.1, we are therefore allowed to assume that $\Omega_X^1(\log \Delta)^{\otimes N}$ contains a big line bundle L as a subsheaf. Let $\mathscr{Q}$ denote the quotient sheaf, and consider the resulting short exact sequence

$$0 \to L \to \Omega_X^1(\log \Delta)^{\otimes N} \to \mathscr{Q} \to 0. \tag{2.2}$$

Since $K_X + \Delta$ represents the first Chern class of $\Omega_X^1(\log \Delta)$, we obtain

$$N \cdot (\dim X)^{N-1} \cdot (K_X + \Delta) = c_1(L) + c_1(\mathscr{Q})$$

in $N^1(X)_{\mathbb{R}}$, the $\mathbb{R}$-linear span of codimension-one cycles modulo numerical equivalence. By assumption, the class $c_1(L)$ is big; Theorem 1.1 will therefore be proved if we manage to show that the class $c_1(\mathscr{Q})$ is pseudo-effective. In fact, we are going to prove the following more general result, which is of course just a special case of [CP15, Theorem 7.6 and Theorem 1.2].

Theorem 2.3. *Let X be a smooth projective variety, and $\Delta \subseteq X$ a reduced divisor with at worst normal crossing singularities. Suppose that some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant. Then the first Chern class of* every *quotient sheaf of* every *tensor power of $\Omega_X^1(\log \Delta)$ is pseudo-effective.*

3. Slopes and foliations. To simplify the presentation, we will prove Theorem 2.3 by contradiction. Suppose then that, for some integer $N \geq 1$, and for some quotient sheaf $\mathscr{Q}$ of $\Omega_X^1(\log \Delta)^{\otimes N}$, the class $c_1(\mathscr{Q})$ was *not* pseudo-effective. Let $\mathscr{Q}_{tor} \subseteq \mathscr{Q}$ denote the torsion subsheaf. Since

$$c_1(\mathscr{Q}) = c_1(\mathscr{Q}_{tor}) + c_1\big(\mathscr{Q}/\mathscr{Q}_{tor}\big),$$

and since $c_1(\mathscr{Q}_{tor})$ is effective, we may replace $\mathscr{Q}$ by $\mathscr{Q}/\mathscr{Q}_{tor}$, and assume without any loss of generality that $\mathscr{Q}$ is torsion-free (and nonzero).

By the characterization of the pseudo-effective cone in [BDPP13, Theorem 2.2], there is a movable class $\alpha \in N_1(X)_{\mathbb{R}}$ such that $c_1(\mathscr{Q}) \cdot \alpha < 0$. As shown in [CP11, GKP16], there is a good theory of α-semistability for torsion-free sheaves, with almost all the properties that are familiar from the case of complete intersection curves. We use this theory freely in what follows. By assumption,

$$\mu_\alpha(\mathscr{Q}) = \frac{c_1(\mathscr{Q}) \cdot \alpha}{\operatorname{rk} \mathscr{Q}} < 0,$$

and so $\mathscr{Q}$ is a torsion-free quotient sheaf of $\Omega_X^1(\log \Delta)^{\otimes N}$ with negative α-slope. The dual sheaf $\mathscr{Q}^*$ is therefore a saturated subsheaf of $\mathscr{T}_X(-\log \Delta)^{\otimes N}$ with positive α-slope. At this point, we recall the following result about tensor products.

Theorem 3.1. *Let $\alpha \in N_1(X)_{\mathbb{R}}$ be a movable class. If $\mathscr{F}$ and $\mathscr{G}$ are torsion-free and α-semistable coherent sheaves on X, then their tensor product*

$$\mathscr{F} \hat{\otimes} \mathscr{G} = (\mathscr{F} \otimes \mathscr{G}) \big/ (\mathscr{F} \otimes \mathscr{G})_{tor},$$

modulo torsion, is again α-semistable, and $\mu_\alpha(\mathscr{F} \hat{\otimes} \mathscr{G}) = \mu_\alpha(\mathscr{F}) + \mu_\alpha(\mathscr{G})$.

Proof. For the reflexive hull of the tensor product, this is proved in [GKP16, Theorem 4.2 and Proposition 4.4], based on analytic results by Toma [CP11, Appendix]. Since $\mathscr{F} \hat{\otimes} \mathscr{G}$ and its reflexive hull are isomorphic outside a closed subvariety of codimension ≥ 2, the assertion follows. □

Similarly, the fact that $\mathscr{T}_X(-\log \Delta)^{\otimes N}$ has a subsheaf with positive α-slope implies, again by [GKP16, Theorem 4.2 and Proposition 4.4], that $\mathscr{T}_X(-\log \Delta)$ must also contain a subsheaf with positive α-slope. Let $\mathscr{F}_\Delta \subseteq \mathscr{T}_X(-\log \Delta)$ be the maximal α-destabilizing subsheaf [GKP16, Corollary 2.24].

Lemma 3.2. *$\mathscr{F}_\Delta$ is a saturated, α-semistable subsheaf of $\mathscr{T}_X(-\log \Delta)$, of positive α-slope. Every subsheaf of $\mathscr{T}_X(-\log \Delta)/\mathscr{F}_\Delta$ has α-slope less than $\mu_\alpha(\mathscr{F}_\Delta)$.*

Proof. This is clear from the construction of the maximal destabilizing subsheaf in [GKP16, Corollary 2.4]. Note that $\mathscr{F}_\Delta$ is the first step in the Harder-Narasimhan filtration of $\mathscr{T}_X(-\log \Delta)$, see [GKP16, Corollary 2.26]. □

Recall that we have an inclusion $\mathscr{T}_X(-\log \Delta) \subseteq \mathscr{T}_X$. We define another coherent subsheaf $\mathscr{F} \subseteq \mathscr{T}_X$ as the saturation of $\mathscr{F}_\Delta$ in $\mathscr{T}_X$; then $\mathscr{T}_X/\mathscr{F}$ is torsion-free, and

$$\mathscr{F} \cap \mathscr{T}_X(-\log \Delta) = \mathscr{F}_\Delta. \tag{3.3}$$

We will see in a moment that $\mathscr{F}$ is actually a (typically, singular) foliation on X. Recall that, in general, a *foliation* on a smooth projective variety is a saturated subsheaf $\mathscr{F} \subseteq \mathscr{T}_X$ that is closed under the Lie bracket on $\mathscr{T}_X$. From the Lie bracket, one constructs an $\mathscr{O}_X$-linear mapping

$$N \colon \mathscr{F} \hat{\otimes} \mathscr{F} \to \mathscr{T}_X/\mathscr{F},$$

called the *O'Neil tensor* of $\mathscr{F}$; evidently, $\mathscr{F}$ is a foliation if and only if its O'Neil tensor vanishes.

Lemma 3.4. *The O'Neil tensor*

$$N \colon \mathscr{F} \hat{\otimes} \mathscr{F} \to \mathscr{T}_X/\mathscr{F}$$

vanishes, and $\mathscr{F}$ is therefore a foliation on X.

Proof. The Lie bracket of two sections of $\mathscr{T}_X(-\log \Delta)$ is a section of $\mathscr{T}_X(-\log \Delta)$, and so we get a logarithmic O'Neil tensor

$$N_\Delta \colon \mathscr{F}_\Delta \hat{\otimes} \mathscr{F}_\Delta \to \mathscr{T}_X(-\log \Delta)/\mathscr{F}_\Delta.$$

The key point is that $N_\Delta = 0$. Indeed, by Theorem 3.1, the tensor product $\mathscr{F}_\Delta \hat{\otimes} \mathscr{F}_\Delta$, modulo torsion, is again α-semistable of slope

$$\mu_\alpha(\mathscr{F}_\Delta \hat{\otimes} \mathscr{F}_\Delta) = 2 \cdot \mu_\alpha(\mathscr{F}_\Delta) > \mu_\alpha(\mathscr{F}_\Delta),$$

which is strictly greater than the slope of any nonzero subsheaf of $\mathscr{T}_X(-\log \Delta)/\mathscr{F}_\Delta$ by Lemma 3.2. This inequality among slopes implies that $N_\Delta = 0$, see for instance [GKP16, Proposition 2.16 and Corollary 2.17].

The O'Neil tensor N and the logarithmic O'Neil tensor N_Δ are both induced by the Lie bracket on $\mathscr{T}_X$, and so we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{F}_\Delta \hat{\otimes} \mathscr{F}_\Delta & \xrightarrow{N_\Delta} & \mathscr{T}_X(-\log \Delta)/\mathscr{F}_\Delta \\ \big\downarrow & & \big\downarrow \\ \mathscr{F} \hat{\otimes} \mathscr{F} & \xrightarrow{\quad N \quad} & \mathscr{T}_X/\mathscr{F} \end{array}$$

The vertical arrow on the right is injective by (3.3). Now $N_\Delta = 0$ implies that N factors through the cokernel of the vertical arrow on the left; but the cokernel is a torsion sheaf, whereas $\mathscr{T}_X/\mathscr{F}$ is torsion-free. The conclusion is that $N = 0$. □

The next step in the proof is to show that the foliation $\mathscr{F}$ is actually algebraic. This is a simple consequence of the powerful algebraicity theorem of Campana and Păun [CP15, Theorem 1.1], which generalizes a well-known result by Bogomolov and McQuillan [BM01] from complete intersection curves to movable classes.

Theorem 3.5. *Let X be a smooth projective variety over the complex numbers, and let $\mathscr{F} \subseteq \mathscr{T}_X$ be a foliation. Suppose that there exists a movable class $\alpha \in N_1(X)_\mathbb{R}$, such that every nonzero quotient sheaf of $\mathscr{F}$ has positive α-slope. Then $\mathscr{F}$ is an algebraic foliation, and its leaves are rationally connected.*

To apply this in our setting, we observe that every quotient sheaf of $\mathscr{F}$ is, at least over the open subset $X \setminus \Delta$, also a quotient sheaf of $\mathscr{F}_\Delta$, because $\mathscr{F}$ and $\mathscr{F}_\Delta$ agree outside the divisor Δ. As $\mathscr{F}_\Delta$ is α-semistable with $\mu_\alpha(\mathscr{F}) > 0$, it follows easily that every quotient sheaf of $\mathscr{F}$ has positive α-slope. We can now invoke Theorem 3.5 and conclude that the foliation $\mathscr{F}$ is algebraic. In other words [CP15, §4], there exists a dominant rational mapping

$$p\colon X \dashrightarrow Z$$

to a smooth projective variety Z, such that

$$\mathscr{F} = \ker\bigl(dp\colon \mathscr{T}_X \to p^*\mathscr{T}_Z\bigr)$$

outside a subset of codimension ≥ 2. More precisely, let us follow [CKT16, Construction 2.29] and denote by the symbol $\mathscr{T}_{X/Z}$ the unique reflexive sheaf on X that agrees with $\ker\bigl(dp\colon \mathscr{T}_X \to p^*\mathscr{T}_Z\bigr)$ on the big open subset where p is a morphism. Using this notation, the algebraicity of $\mathscr{F}$ may be expressed as

$$\mathscr{F} = \mathscr{T}_{X/Z}; \tag{3.6}$$

indeed, $\mathscr{F}$ is reflexive, due to the fact that $\mathscr{T}_X/\mathscr{F}$ is torsion-free.

Note. Theorem 3.5 also says that the fibers of p are rationally connected, but we are not going to make any use of this extra information.

4. Pseudo-effectivity. Let us first convince ourselves that Z cannot be a point. This will later allow us to argue by induction on the dimension, because the general fiber of p has dimension less than $\dim X$.

Lemma 4.1. *With notation as above, we must have* $\dim Z \geq 1$.

Proof. If $\dim Z = 0$, then $\mathscr{F} = \mathscr{T}_X$ and $\mathscr{F}_\Delta = \mathscr{T}_X(-\log \Delta)$, and consequently, the logarithmic tangent bundle $\mathscr{T}_X(-\log \Delta)$ is α-semistable of positive slope. Since the tensor product of α-semistable sheaves remains α-semistable [GKP16, Proposition 4.4], this means that any tensor power of $\Omega^1_X(\log \Delta)$ is α-semistable of negative slope. But that contradicts the hypothesis of Theorem 2.3, namely that some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, because the α-slope of such a subsheaf is obviously positive. □

The only properties of $\mathscr{F}_\Delta$ that we are still going to use in the proof of Theorem 2.3 are the identity in (3.3), and the fact that $c_1(\mathscr{F}_\Delta) \cdot \alpha > 0$ for a movable class $\alpha \in N_1(X)_\mathbb{R}$. In return, we are allowed to assume that $p \colon X \to Z$ is a morphism.

Lemma 4.2. *Without loss of generality, $p \colon X \to Z$ is a morphism.*

Proof. Choose a birational morphism $f \colon \tilde{X} \to X$, for example by resolving the singularities of the closure of the graph of $p \colon X \dashrightarrow Z$ inside $X \times Z$, with the following properties: the rational mapping $p \circ f$ extends to a morphism $\tilde{p} \colon \tilde{X} \to Z$; both $K_{\tilde{X}/X}$ and $\tilde{p}^* \Delta$ are normal crossing divisors; and f is an isomorphism over the open subset where p is already a morphism.

Let $\tilde{\Delta}$ be the reduced normal crossing divisor whose support is equal to the preimage of Δ in $\tilde{X}$. Then

$$\Omega_{\tilde{X}}^1(\log \tilde{\Delta}) \cong \tilde{p}^* \Omega_X^1(\log \Delta),$$

and since the pullback of a big line bundle by $\tilde{p}$ stays big, it is still true that some tensor power of $\Omega_{\tilde{X}}^1(\log \tilde{\Delta})$ contains a big line bundle as a subsheaf. Now define

$$\tilde{\mathscr{F}} = \mathscr{T}_{\tilde{X}/Z} = \ker\bigl(\tilde{p}^* \colon \mathscr{T}_{\tilde{X}} \to \tilde{p}^* \mathscr{T}_Z\bigr),$$

which is a saturated subsheaf of $\mathscr{T}_{\tilde{X}}$. The intersection

$$\tilde{\mathscr{F}} \cap \mathscr{T}_{\tilde{X}}(-\log \tilde{\Delta})$$

is a saturated (and hence reflexive) subsheaf of $\mathscr{T}_{\tilde{X}}(-\log \tilde{\Delta})$, whose pushforward to X is isomorphic to $\mathscr{F}_\Delta$, by (3.3) and the fact that $\mathscr{F}_\Delta$ is reflexive. Consequently,

$$c_1\Bigl(\tilde{\mathscr{F}} \cap \mathscr{T}_{\tilde{X}}(-\log \tilde{\Delta})\Bigr) \cdot \tilde{\alpha} = c_1(\mathscr{F}_\Delta) \cdot \alpha > 0,$$

where the class $\tilde{\alpha} = \tilde{p}^* \alpha \in N_1(\tilde{X})_\mathbb{R}$ is of course still movable. Nothing essential is therefore changed if we replace the rational mapping $p \colon X \dashrightarrow Z$ by the morphism $\tilde{p} \colon \tilde{X} \to Z$; the divisor $\Delta \subseteq X$ by $\tilde{\Delta} \subseteq \tilde{X}$; the sheaf $\mathscr{F}_\Delta$ by the intersection

$$\mathscr{T}_{\tilde{X}/Z} \cap \mathscr{T}_{\tilde{X}}(-\log \tilde{\Delta}) \subseteq \mathscr{T}_{\tilde{X}}$$

and the movable class $\alpha \in N_1(X)_\mathbb{R}$ by its pullback $\tilde{\alpha} = \tilde{p}^* \alpha$. □

Let $R(p)$ denote the ramification divisor of the morphism $p \colon X \to Z$; see [CKT16, Definition 2.16] for the precise definition. Recall from [CKT16, Lemma 2.31] the following formula for the first Chern class of our foliation $\mathscr{F} \subseteq \mathscr{T}_X$, in $N^1(X)_\mathbb{R}$:

$$c_1(\mathscr{F}) = c_1(\mathscr{T}_{X/Z}) = -K_{X/Z} + R(p) \tag{4.3}$$

Computing the first Chern class of $\mathscr{F}_\Delta$ is a little tricky [CP15, Proposition 5.1], but at least we can use the fact that $\mathscr{F} = \mathscr{T}_{X/Z}$ to estimate the difference

$$c_1(\mathscr{F}) - c_1(\mathscr{F}_\Delta) = c_1(\mathscr{F}/\mathscr{F}_\Delta).$$

Recall that the *horizontal part* $\Delta^{hor} \subseteq \Delta$ is the union of all irreducible components of Δ that map onto Z; evidently, Δ^{hor} is again a reduced divisor on X with at worst normal crossing singularities.

Lemma 4.4. *The class $c_1(\mathscr{F}) - c_1(\mathscr{F}_\Delta) - \Delta^{hor}$ is effective.*

Proof. It is easy to see from (3.3) that we have an inclusion of sheaves

$$\mathscr{F}/\mathscr{F}_\Delta \hookrightarrow \mathscr{T}_X/\mathscr{T}_X(-\log\Delta).$$

The sheaf on the right-hand side is supported on the divisor Δ, and a brief computation shows that

$$\mathscr{T}_X/\mathscr{T}_X(-\log\Delta) \cong \bigoplus_{D\subseteq\Delta} \mathscr{N}_{D|X}$$

is isomorphic to the direct sum of the normal bundles of the irreducible components of Δ. The rank of $\mathscr{F}/\mathscr{F}_\Delta$ at the generic point of D is thus either 0 or 1, and

$$c_1(\mathscr{F}/\mathscr{F}_\Delta) = \sum_{D\subseteq\Delta} a_D D,$$

where $a_D = 0$ if $\mathscr{F} = \mathscr{F}_\Delta$ at the generic point of D, and $a_D = 1$ otherwise. To prove that $c_1(\mathscr{F}/\mathscr{F}_\Delta) - \Delta^{hor}$ is effective, we only have to argue that $\mathscr{F} \neq \mathscr{F}_\Delta$ at the generic point of each irreducible component of Δ^{hor}. This is a consequence of the fact that $\mathscr{F} = \mathscr{T}_{X/Z}$, as we now explain.

Fix an irreducible component D of the horizontal part Δ^{hor}. At the generic point of D, the morphism $p\colon X \to Z$ is smooth. After choosing suitable local coordinates $x_1, \dots, x_n$ in a neighborhood of a sufficiently general point of D, we may therefore assume that p is locally given by

$$p(x_1, \dots, x_n) = (x_1, \dots, x_d),$$

where $d = \dim Z$, and that the divisor Δ is defined by the equation $x_n = 0$. In these local coordinates, $\mathscr{F} = \mathscr{T}_{X/Z}$ is the subbundle of $\mathscr{T}_X$ spanned by

$$\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \dots, \frac{\partial}{\partial x_{d+1}}.$$

On the other hand, the subsheaf $\mathscr{T}_X(-\log\Delta)$ is spanned by the vector fields

$$x_n\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \dots, \frac{\partial}{\partial x_{d+1}}, \dots, \frac{\partial}{\partial x_1},$$

and so it is clear from (3.3) that $\mathscr{F} \neq \mathscr{F}_\Delta$ in a neighborhood of the given point. ∎

From Lemma 4.4, we draw the conclusion that

$$-\big(K_{X/Z} + \Delta^{hor} - R(p)\big)\cdot\alpha = \big(c_1(\mathscr{F}) - \Delta^{hor}\big)\cdot\alpha \geq c_1(\mathscr{F}_\Delta)\cdot\alpha > 0, \tag{4.5}$$

where $\alpha \in N_1(X)_\mathbb{R}$ is the movable class from above. We will therefore reach the desired contradiction if we manage to prove that the divisor class $K_{X/Z}+\Delta^{hor}-R(p)$ is pseudo-effective. According to [CP15, Theorem 3.3] or to [CKT16, Theorem 7.1], it is actually enough to check that $K_F+\Delta_F$ is pseudo-effective for a general fiber F of the morphism p; and we can prove, by induction on the dimension, that $K_F+\Delta_F$ is not only pseudo-effective, but even big.

5. Induction on the dimension. In this section, we use induction on the dimension to finish the proof of Theorem 2.3 and Theorem 1.1.

Proposition 5.1. *Suppose that Theorem 1.1 is true in dimension less than* $\dim X$*. If some tensor power of* $\Omega^1_X(\log\Delta)$ *contains a subsheaf with big determinant, then* $K_{X/Z} + \Delta^{hor}$ *is pseudo-effective.*

Proof. Let F be a general fiber of the morphism $p\colon X \to Z$; since $\dim Z \geq 1$, we have $\dim F \leq \dim X - 1$. Denote by Δ_F the restriction of Δ; since F is a general fiber, Δ_F is still a normal crossing divisor. Clearly

$$(K_{X/Z} + \Delta^{hor})\big|_F = K_F + \Delta_F,$$

and according to [CKT16, Theorem 7.3], the pseudo-effectivity of $K_{X/Z} + \Delta^{hor}$ will follow if we manage to show that $K_F + \Delta_F$ is pseudo-effective.

By hypothesis and by Lemma 2.1, there is a nonzero morphism

$$L \to \Omega_X^1(\log \Delta)^{\otimes k}$$

from a big line bundle L to some tensor power of $\Omega_X^1(\log \Delta)$. Since F is a general fiber of $p\colon X \to Z$, we can restrict this morphism to F to obtain a nonzero morphism

$$L_F \to \left(\Omega_X^1(\log \Delta)\big|_F\right)^{\otimes k}.$$

Here L_F denotes the restriction of L to the fiber; since L is big, L_F is also big.

The inclusion of F into X gives rise to a short exact sequence

$$0 \to \mathscr{N}_{F|X} \to \Omega_X^1(\log \Delta)\big|_F \to \Omega_F^1(\log \Delta_F) \to 0,$$

which induces a filtration on the k-th tensor power of the locally free sheaf in the middle. Since the normal bundle $\mathscr{N}_{F|X}$ is trivial of rank $\dim Z$, we find, by looking at the subquotients of this filtration, that there is a nonzero morphism

$$L_F \to \Omega_F^1(\log \Delta_F)^{\otimes j}$$

for some $0 \leq j \leq k$. Because L_F is big, we actually have $1 \leq j \leq k$. Since we are assuming that Theorem 1.1 is true for the pair (F, Δ_F), the class $K_F + \Delta_F$ is big on F, hence pseudo-effective. Appealing to [CKT16, Theorem 7.3], we deduce that the class $K_{X/Z} + \Delta^{hor}$ is pseudo-effective on X. □

By induction on the dimension, the two assumptions of Proposition 5.1 are met in our case, and the class $K_{X/Z} + \Delta^{hor}$ is therefore pseudo-effective. According to [CKT16, Theorem 7.1], this implies that $K_{X/Z} + \Delta^{hor} - R(p)$ is also pseudo-effective.[1] Going back to the inequality in (4.5), we find that

$$0 \geq -\big(K_{X/Z} + \Delta^{hor} - R(p)\big) \cdot \alpha \geq c_1(\mathscr{F}_\Delta) \cdot \alpha > 0,$$

and so we have reached the desired contradiction. The conclusion is that $c_1(\mathscr{Q})$ is indeed pseudo-effective, and so Theorem 2.3 and Theorem 1.1 are proved.

Note. Most of the argument, for example the proof of Lemma 4.1, goes through when some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with *pseudo-effective* determinant. But Theorem 2.3 is obviously not true under this weaker hypothesis: for example, on the product $E \times \mathbb{P}^1$ of an elliptic curve and $\mathbb{P}^1$, there are nontrivial one-forms, yet the canonical bundle is not pseudo-effective. What happens is that the last step in the proof of Proposition 5.1 breaks down: when L is not big, it may be that $j = 0$ (and L_F is then trivial).

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[1] As stated, both [CP15, Theorem 3.3] and [CKT16, Theorem 7.1] actually assume that $K_X + \Delta$ is pseudo-effective, but in the case of a morphism $p\colon X \to Z$, the proofs go through under the weaker hypothesis that $K_{X/Z} + \Delta^{hor}$ is pseudo-effective.

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651
E-mail address: cschnell@math.sunysb.edu