ALGEBRAIC FIBER SPACES OVER ABELIAN VARIETIES: AROUND A RECENT THEOREM BY CAO AND PĂUN

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To Lawrence Ein, on his 60th birthday.

Abstract. We present a simplified proof for a recent theorem by Junyan Cao and Mihai Păun, confirming a special case of Iitaka's $C_{n,m}$ conjecture: if $f\colon X \to Y$ is an algebraic fiber space, and if the Albanese mapping of Y is generically finite over its image, then we have the inequality of Kodaira dimensions $\kappa(X) \geq \kappa(Y) + \kappa(F)$, where F denotes a general fiber of f. We include a detailed survey of the main algebraic and analytic techniques, especially the construction of singular hermitian metrics on pushforwards of adjoint bundles (due to Berndtsson, Păun, and Takayama).

A. Introduction

1. Main result. In the classification of algebraic varieties up to birational equivalence, the most fundamental invariants of a smooth projective variety X are the spaces of global sections of the pluricanonical bundles $\omega_X^{\otimes m}$. The rate of growth of the plurigenera $P_m(X) = \dim H^0(X, \omega_X^{\otimes m})$ is measured by the *Kodaira dimension*

$$\kappa(X) = \limsup_{m \to +\infty} \frac{\log P_m(X)}{\log m} \in \{-\infty, 0, 1, \dots, \dim X\}.$$

The following conjecture by Iitaka and Viehweg predicts the behavior of the Kodaira dimension in families. Recall that an *algebraic fiber space* is a surjective morphism with connected fibers between two smooth projective varieties.

Conjecture. *Let $f\colon X \to Y$ be an algebraic fiber space with general fiber F. Provided that $\kappa(Y) \geq 0$, the Kodaira dimension of X satisfies the inequality*

$$\kappa(X) \geq \kappa(F) + \max\{\kappa(Y), \mathrm{var}(f)\},$$

where $\mathrm{var}(f)$ measures the birational variation in moduli of the fibers.

Using analytic techniques, Cao and Păun [CP15] have recently proved the conjectured subadditivity of the Kodaira dimension in the case where Y is an abelian variety; as $\kappa(Y) = 0$, this amounts to the inequality

$$\kappa(X) \geq \kappa(F).$$

With very little extra work, one can deduce the subadditivity of the Kodaira dimension in any algebraic fiber space whose base Y has maximal Albanese dimension, meaning that the Albanese mapping $Y \to \mathrm{Alb}(Y)$ is generically finite over its image. This includes of course the case where Y is a curve of genus ≥ 1, where the following result was first proved by Kawamata [Kaw82, Theorem 2].

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Theorem 1.1. *Let $f\colon X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.*

Remark. A proof of this result is also claimed in [CH11], however the proof given there is incomplete because of a serious mistake in §4.

The purpose of this paper is to explain a simplified proof of the Cao-Păun theorem that combines both analytic and algebraic techniques. We first reduce to the case when $\kappa(X) = 0$ and Y is an abelian variety, where we then prove a more precise statement (Theorem 5.2). This is done in Chapter B. We then take the opportunity to provide a detailed survey of the results that are used in the proof, for the benefit of those readers who are more familiar with one or the other side of the story.

In Chapter C, we discuss the main algebraic tools, contained mostly in the papers [CH04, Hac04, Lai11, PP11a, PS14], namely results from generic vanishing theory. The upshot of the discussion is that when $f\colon X \to A$ is a fiber space over an abelian variety, with $\kappa(X) = 0$, then for all m sufficiently large and divisible, $f_*\omega_X^{\otimes m}$ is a unipotent vector bundle on A, meaning a successive extension of copies of $\mathscr{O}_A$. This is as far as the algebraic techniques seem to go at present. While we recall the basic generic vanishing and Fourier-Mukai tools involved, as this topic is well-established in the literature, we provide fewer background details. Sources where a comprehensive treatment can be found include the lecture notes [Sch13], as well as [Par12, Pop12].

In Chapter D and Chapter E we discuss the main analytic tools, contained mostly in the papers [BP08, PT14, CP15], namely the existence of singular metrics with semi-positive curvature (in a suitable sense) on pushforwards of pluricanonical bundles, and a very surprising criterion for such a metric to be smooth and flat. This time, the upshot is that when $f\colon X \to A$ is a fiber space onto an abelian variety, with $\kappa(X) = 0$, then $f_*\omega_X^{\otimes m}$ is a vector bundle with a flat hermitian metric. Because a unipotent vector bundle with a flat hermitian metric must be trivial, the algebraic and analytic results together lead to the conclusion in Theorem 1.1. Since the analytic results are still new, and are likely to be less familiar to algebraic geometers, we decided to include as many details as possible. For another survey of these and related results, we recommend [Pău16].

Remark. For the sake of exposition, we present only the simplest version of the result by Cao and Păun. One can tweak the proof of Theorem 1.1 to show that the inequality in Theorem 1.1 still holds when X is replaced by a klt pair (X, Δ), and F by the pair (F, Δ_F); this is done in [CP15, Theorem 4.22].

2. What is new? The presentation in Chapter D contains various small improvements compared to the original papers [BP08, PT14, CP15]. We briefly summarize the main points here. Let $f\colon X \to Y$ be a projective and surjective holomorphic mapping between two complex manifolds. Given a holomorphic line bundle L on X, and a singular hermitian metric h on L with semi-positive curvature, we construct a singular hermitian metric on the torsion-free coherent sheaf

$$\mathscr{F} = f_*\bigl(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\bigr),$$

and show that this metric has semi-positive curvature (in the sense that the logarithm of the norm of every local section of the dual sheaf is plurisubharmonic).

In [PT14], Păun and Takayama constructed a singular hermitian metric with semi-positive curvature on the larger sheaf $f_*(\omega_{X/Y} \otimes L)$, under the additional assumption that the restriction of (L, h) to a general fiber of f has trivial multiplier ideal. Another difference with [PT14] is that we do not use approximation by smooth metrics or results about Stein manifolds; instead, both the construction of the metric, and the proof that it has semi-positive curvature, rely on the Ohsawa-Takegoshi extension theorem with sharp estimates, recently proved by Błocki and Guan-Zhou [Bło13, GZ15]. This approach was suggested to us by Mihai Păun.

Note. Berndtsson and Lempert [BL16] explain how one can use the curvature properties of pushforwards of adjoint bundles to get a relatively short proof of (one version of) the Ohsawa-Takegoshi theorem with sharp estimates. This suggests that the two results are not so far from each other. That said, we hope that using the Ohsawa-Takegoshi theorem as a black box will make the proof of the main result more accessible to algebraic geometers than it would otherwise be.

We introduce what we call the "minimal extension property" for singular hermitian metrics (see §20), and show that, as a consequence of the Ohsawa-Takegoshi theorem with sharp estimates, the singular hermitian metric on $\mathscr{F}$ always has this property. We then use the minimal extension property, together with some basic inequalities from analysis, to give an alternative proof for the following result by Cao and Păun: when Y is projective, $\mathscr{F}$ is a hermitian flat bundle if and only if the line bundle $\det \mathscr{F}$ has trivial first Chern class in $H^2(Y, \mathbb{R})$. The original argument in [CP15] relied on some results by Raufi about curvature tensors for singular hermitian metrics [Rau15]. We also show that when Y is projective, every nontrivial morphism of sheaves $\mathscr{F} \to \mathscr{O}_Y$ is split surjective; this result is new.

In Chapter E, we apply these results to construct canonical singular hermitian metrics with semi-positive curvature on the sheaves $f_*\omega_{X/Y}^{\otimes m}$ for $m \geq 2$. Here, one small improvement over [PT14] is the observation that these metrics are continuous on the Zariski-open subset of Y where $f\colon X \to Y$ is submersive.

Our discussion of generic vanishing theory in Chapter C is fairly standard, but includes (in §11) a new result relating the structure of the cohomological support loci $V^0(\omega_X^{\otimes m})$ for $m \geq 2$ to the Iitaka fibration of X. Here the main Theorem 5.2 is one of the crucial ingredients.

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B. Proof of the main statement

4. Main analytic and algebraic input. In this section we lay out the tools neeed to prove the main result. We also give a brief sketch of the proof, which is presented in more detail in §5. The rest of the paper will be devoted to a detailed survey of the results stated here.

We first note that one can reduce Theorem 1.1 to the special case when $\kappa(X) = 0$ and $Y = A$ is an abelian variety, with the help of the Iitaka fibration; the argument for this is recalled in §5 below. We will therefore make these assumptions in the remainder of this section. The condition $\kappa(X) = 0$ is equivalent to saying that $P_m(X) \leq 1$ for all $m \in \mathbb{N}$, with equality for m sufficiently large and divisible. Let F be the general fiber of $f\colon X \to A$. Our goal is to prove that $\kappa(F) = 0$. What we will actually show is that $P_m(F) = 1$ whenever $P_m(X) = 1$; this is enough to conclude that $\kappa(F) = 0$.

Fix now an integer $m \in \mathbb{N}$ such that $P_m(X) = 1$, and consider the pushforward of the m-th pluricanonical bundle

$$\mathscr{F}_m = f_* \omega_X^{\otimes m}.$$

This is a torsion-free coherent sheaf on A, whose rank at the generic point of A is equal to $P_m(F)$. (In fact, this holds for every smooth fiber of f, by invariance of plurigenera.) The space of global sections of $\mathscr{F}_m$ has dimension

$$h^0\big(A, f_*\omega_X^{\otimes m}\big) = \dim H^0(X, \omega_X^{\otimes m}) = P_m(X) = 1.$$

To obtain the conclusion, it is enough to show that $\mathscr{F}_m$ has rank 1 generically; we will in fact prove the stronger statement that $\mathscr{F}_m \simeq \mathscr{O}_A$. This uses both algebraic and analytic properties of $\mathscr{F}_m$.

Generic vanishing and unipotency. We first explain the algebraic input. We borrow an idea from generic vanishing theory, initiated in [GL87, GL91], namely to consider the locus

$$\begin{aligned} V^0(A, \mathscr{F}_m) &= \big\{ P \in \operatorname{Pic}^0(A) \;\big|\; H^0(A, \mathscr{F}_m \otimes P) \neq 0 \big\} \\ &= \big\{ P \in \operatorname{Pic}^0(A) \;\big|\; H^0\big(X, \omega_X^{\otimes m} \otimes f^*P\big) \neq 0 \big\} \subseteq \operatorname{Pic}^0(A). \end{aligned}$$

The following result by Chen-Hacon [CH04, §3], Lai [Lai11, Theorem 3.5] and Siu [Siu11, Theorem 2.2] describes the structure of $V^0(A, \mathscr{F}_m)$; it is a generalization of a famous theorem by Simpson [Sim93]. The proof by Simpson (which applies when $m = 0, 1$) relies on Hodge theory and the Gelfond-Schneider theorem in transcendence theory; the cited works use a construction with cyclic coverings, originally due to Viehweg, to reduce the general case to the case $m = 1$. We review the argument in §10.

Theorem 4.1. *Let X be a smooth projective variety. For each $m \in \mathbb{N}$, the locus*

$$\big\{ P \in \operatorname{Pic}^0(X) \;\big|\; H^0\big(X, \omega_X^{\otimes m} \otimes P\big) \neq 0 \big\} \subseteq \operatorname{Pic}^0(X)$$

is a finite union of abelian subvarieties translated by points of finite order.

This theorem implies that $V^0(A, \mathscr{F}_m)$ is also a finite union of abelian subvarieties translated by points of finite order. The reason is that, as $f\colon X \to A$ has connected fibers, the pullback morphism $f^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$ is injective. Since we are assuming that $P_m(X) = 1$, we have $\mathscr{O}_A \in V^0(A, \mathscr{F}_m)$; let $s_0 \in H^0(X, \omega_X^{\otimes m})$ be any nontrivial section. Now we observe that $\kappa(X) = 0$ forces

$$V^0(A, \mathscr{F}_m) = \{\mathscr{O}_A\}.$$

To see why, suppose that we had $P \in V^0(A, \mathscr{F}_m)$ for some nontrivial line bundle $P \in \operatorname{Pic}^0(A)$. By Theorem 4.1, we can assume that P has finite order $d \neq 1$. Let

$$s_1 \in H^0(A, \mathscr{F}_m \otimes P) = H^0\big(X, \omega_X^{\otimes m} \otimes f^* P\big)$$

be any nontrivial section; then $s_0^{\otimes d}$ and $s_1^{\otimes d}$ are two linearly independent sections of $\omega_X^{\otimes dm}$, contradicting the fact that $P_{dm}(X) = 1$.

Knowing the locus $V^0(A, \mathscr{F}_m)$ gives us a lot of information about $\mathscr{F}_m$, due to the following result [PS14, Theorem 1.10]. It is based on a vanishing theorem for pushforwards of pluricanonical bundles, which is again proved using Viehweg's cyclic covering construction; we review the argument in §8.

Theorem 4.2. *Let $f \colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. For every $m \in \mathbb{N}$, the sheaf $\mathscr{F}_m = f_* \omega_X^{\otimes m}$ is a GV-sheaf on A.*

Recall that a coherent sheaf $\mathscr{F}$ on an abelian variety A is called a *GV-sheaf* if its cohomology support loci

$$V^i(A, \mathscr{F}) = \big\{\, P \in \operatorname{Pic}^0(A) \;\big|\; H^i(A, \mathscr{F} \otimes P) \neq 0 \,\big\}$$

satisfy the inequalities $\operatorname{codim} V^i(A, \mathscr{F}) \geq i$ for every $i \in \mathbb{N}$. This property can be seen as a variant of (semi-)positivity on abelian varieties; in fact every GV-sheaf on A is nef, see [PP11b, Theorem 4.1].

Note. A more conceptual description involves the Fourier-Mukai transform

$$\mathbf{R}\Phi_P \colon D^b_{coh}(\mathscr{O}_A) \to D^b_{coh}\big(\mathscr{O}_{\operatorname{Pic}^0(A)}\big),$$

which is an equivalence between the bounded derived categories of coherent sheaves on A and the dual abelian variety $\operatorname{Pic}^0(A)$. In terms of the Fourier-Mukai transform, $\mathscr{F}$ is a GV-sheaf if and only if the complex of sheaves

$$\mathbf{R}\mathcal{H}om\Big(\mathbf{R}\Phi_P(\mathscr{F}), \mathscr{O}_{\hat{A}}\Big) \in D^b_{coh}\big(\mathscr{O}_{\operatorname{Pic}^0(A)}\big)$$

is concentrated in degree 0, and is therefore again a coherent sheaf $\hat{\mathscr{F}}$ on $\operatorname{Pic}^0(A)$. By the base change theorem, the support of $\hat{\mathscr{F}}$ is precisely the locus $V^0(A, \mathscr{F})$.

In the case at hand, we have $V^0(A, \mathscr{F}_m) = \{\mathscr{O}_A\}$; consequently, $\hat{\mathscr{F}}_m$ is a successive extension of skyscraper sheaves supported at the origin in $\operatorname{Pic}^0(A)$. We will use this via the following elementary consequence; see §9 for details. Recall first from [Muk81] that a vector bundle U on A is called *unipotent* if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

such that $U_i/U_{i-1} \simeq \mathscr{O}_A$ for all $i = 1, \dots, n$. Note in particular that $\det U \simeq \mathscr{O}_A$. More generally, U is called *homogeneous* if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

such that U_i/U_{i-1} is isomorphic to a line bundle in $\operatorname{Pic}^0(A)$ for all $i = 1, \dots, n$. A homogeneous vector bundle U is called *decomposable* if $U = U_1 \oplus U_2$, where the U_i are non-zero vector bundles, and *indecomposable* if this is not the case.

Corollary 4.3. *Let X be a smooth projective variety with $\kappa(X) = 0$, and let $f \colon X \to A$ be an algebraic fiber space over an abelian variety.*

(a) If $\mathscr{F}_m \neq 0$ for some $m \in \mathbb{N}$, then the coherent sheaf $\mathscr{F}_m$ is an indecomposable homogeneous vector bundle.

(b) *If $H^0(X, \omega_X^{\otimes m}) \neq 0$ for some $m \in \mathbb{N}$, then the coherent sheaf $\mathscr{F}_m$ is an indecomposable unipotent vector bundle.*

Singular hermitian metrics on pushforwards of pluricanonical bundles. We now come to the analytic input. To motivate it, recall that the space of global sections of $\mathscr{F}_m$ has dimension $P_m(X) = 1$. In order to show that $P_m(F) = 1$, we therefore need to argue that the unipotent vector bundle $\mathscr{F}_m$ is actually the trivial bundle $\mathscr{O}_A$. For the moment this seems quite hopeless with algebraic methods, so it is at this point that the analytic methods take over.

The crucial development that allows us to proceed is recent work on the notion of a singular hermitian metric on a torsion-free sheaf; the highlight of this study is the following remarkable result by Păun and Takayama [PT14, Theorem 3.3.5]. In order to state it, recall that to a singular hermitian metric h on a line bundle L, one associates the multiplier ideal sheaf $\mathcal{I}(h) \subseteq \mathscr{O}_X$, consisting of those functions that are locally square-integrable with respect to h.

Theorem 4.4. *Let $f\colon X \to Y$ be a projective morphism of smooth varieties, and let (L, h) be a line bundle on X with a singular hermitian metric of semi-positive curvature. Then the torsion-free sheaf $f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big)$ has a canonical singular hermitian metric with semi-positive curvature.*

The relevant definitions and the proof are described in Chapter D and Chapter E, where we also present another key statement. Indeed, Cao and Păun [CP15, Corollary 2.9 and Theorem 5.23] show that their singular hermitian metrics behave much like smooth metrics with Griffiths semi-positive curvature: if the determinant line bundle $\det \mathscr{F}$ has trivial first Chern class, then $\mathscr{F}$ is actually a hermitian flat bundle. This is (a) below; part (b) is new.

Theorem 4.5. *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties. Let (L, h) be a line bundle on X with a singular hermitian metric of semi-positive curvature, and define $\mathscr{F} = f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big)$.*

(a) *If $c_1(\det \mathscr{F}) = 0$ in $H^2(Y, \mathbb{R})$, then the torsion-free sheaf $\mathscr{F}$ is locally free, and the singular hermitian metric in Theorem 4.4 is smooth and flat.*
(b) *Every nonzero morphism $\mathscr{F} \to \mathscr{O}_Y$ is split surjective.*

The application of these results to Theorem 1.1 stems from the fact that the bundles $\mathscr{F}_m$ with $m \geq 2$ naturally fit into this framework. Let us briefly summarize how this works when $f\colon X \to Y$ is an algebraic fiber space with general fiber F. For every $m \in \mathbb{N}$ such that $P_m(F) \neq 0$, the spaces of m-canonical forms on the smooth fibers of f induce in a canonical way a singular hermitian metric with semi-positive curvature on $\omega_{X/Y}$, called the *m-th Narasimhan-Simha metric.* (For $m = 1$, the Narasimhan-Simha metric is of course just the Hodge metric.) Denote by h the induced singular hermitian metric on the line bundle $L = \omega_{X/Y}^{\otimes(m-1)}$. Pretty much by construction, the inclusion

$$f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big) \subseteq f_*\big(\omega_{X/Y} \otimes L\big) = f_*\omega_{X/Y}^{\otimes m}$$

is generically an isomorphism, and so Theorem 4.4 and Theorem 4.5 apply.

Corollary 4.6. *Let $f\colon X \to Y$ be an algebraic fiber space.*

(a) *For any $m \in \mathbb{N}$, the torsion-free sheaf $f_*\omega_{X/Y}^{\otimes m}$ has a canonical singular hermitian metric with semi-positive curvature.*

(b) *If* $c_1(\det f_*\omega_{X/Y}^{\otimes m}) = 0$ *in* $H^2(Y,\mathbb{R})$*, then* $f_*\omega_{X/Y}^{\otimes m}$ *is locally free, and the singular hermitian metric on it is smooth and flat.*
(c) *Every nonzero morphism* $f_*\omega_{X/Y}^{\otimes m} \to \mathscr{O}_Y$ *is split surjective.*

In our case, $\mathscr{F}_m = f_*\omega_X^{\otimes m}$ is a unipotent vector bundle by Corollary 4.3, and so the hypothesis in (b) is satisfied; after this point, the proof of Theorem 1.1 becomes straightforward.

5. Proof of Theorem 1.1. We now explain how Theorem 1.1 follows quickly by combining the results outlined in the previous section. Recall that we are starting with an algebraic fiber space $f\colon X \to Y$, where X is a smooth projective variety, and Y is of maximal Albanese dimension. Let us note right away that one can perform a useful reduction, following in part the argument in [CH02, Theorem 4.9].

Lemma 5.1. *To prove Theorem 1.1, it is enough to assume that* $\kappa(X) = 0$ *and that* Y *is an abelian variety.*

Proof. We begin by showing that if $\kappa(X) = -\infty$, then $\kappa(F) = -\infty$. If this were not the case, then we could pick some $m > 0$ such that $P_m(F) > 0$ and hence $f_*\omega_X^{\otimes m} \neq 0$. Let $Y \to A$ be the Albanese morphism of Y, and $g\colon X \to A$ the induced morphism. Since F is an irreducible component of the general fiber of $X \to g(X) \subseteq A$, it follows that $g_*\omega_X^{\otimes m} \neq 0$. By Theorem 4.2, $g_*\omega_X^{\otimes m}$ is a GV-sheaf, and in particular by the general Lemma 7.4 below, the set

$$V^0(g_*\omega_X^{\otimes m}) = \left\{ P \in \operatorname{Pic}^0(A) \;\middle|\; H^0(A, g_*\omega_X^{\otimes m} \otimes P) \neq 0 \right\}$$

is non-empty. Now by Theorem 4.1 and the comments immediately after, $V^0(g_*\omega_X^{\otimes m})$ contains a torsion point $P \in \operatorname{Pic}^0(A)$, i.e. there is an integer $k > 0$ such that $P^{\otimes k} \simeq \mathscr{O}_A$. But then $h^0(X, \omega_X^{\otimes m} \otimes g^*P) = h^0(A, g_*\omega_X^{\otimes m} \otimes P) \neq 0$ and so

$$h^0(X, \omega_X^{\otimes km}) = h^0\big(X, (\omega_X^{\otimes m} \otimes P)^{\otimes k}\big) \neq 0.$$

This contradicts the assumption $\kappa(X) = -\infty$.

Assume now that $\kappa(X) \geq 0$. We will first prove the statement in the case that $\kappa(Y) = 0$. By Kawamata's theorem [Kaw81, Theorem 1], since Y is of maximal Albanese dimension, it is in fact birational to its Albanese variety and so we may assume that Y is an abelian variety. Let $h\colon X \to Z$ the Iitaka fibration of X. Since we are allowed to work birationally, we can assume that Z is smooth. We denote by G its general fiber, so that in particular $\kappa(G) = 0$. By the same result of Kawamata, the Albanese map of G is surjective, so we deduce that $B = f(G) \subseteq Y$ is an abelian subvariety. If $G \to B' \to B$ is the Stein factorization, then $B' \to B$ is an étale map of abelian varieties. We thus have an induced fiber space

$$G \longrightarrow B'$$

over an abelian variety, with $\kappa(G) = 0$, and whose general fiber is $H = F \cap G$. Assuming that Theorem 1.1 holds for algebraic fibers spaces of Kodaira dimension zero over abelian varieties, we obtain $\kappa(H) = 0$. Note however that H is also an irreducible component of the general fiber of

$$h_{|F}\colon F \longrightarrow h(F).$$

Considering the Stein factorization of this morphism, the easy addition formula [Mor87, Corollary 2.3], implies that

$$\kappa(F) \leq \kappa(H) + \dim h(F) = \dim h(F).$$

(Note that we can assume that $g(F)$ is smooth, by passing to a birational model.) Since $\dim h(F) \leq \dim Z = \kappa(X)$, we obtain the required inequality $\kappa(F) \leq \kappa(X)$.

Finally we prove the general case. Since Y has maximal Albanese dimension, after replacing it by a resolution of singularities of an étale cover of its Stein factorization, and X by a resolution of the corresponding fiber product, by [Kaw81, Theorem 13] we may assume that $Y = Z \times K$ where Z is of general type and K is an abelian variety. In particular $\kappa(Y) = \dim Z = \kappa(Z)$. If E is the general fiber of the induced morphism $X \to Z$, then the induced morphism $E \to K$ has general fiber isomorphic to F. By what we have proven above, we deduce that $\kappa(E) \geq \kappa(F)$. We then have the required inequality

$$\kappa(X) = \kappa(Z) + \kappa(E) \geq \kappa(Y) + \kappa(F),$$

where the first equality is [Kaw81, Theorem 3], since Z is of general type. □

We may therefore assume that $f\colon X \to A$ is a fiber space onto an abelian variety, and $\kappa(X) = 0$. Note that this last condition means that we have $h^0(X, \omega_X^{\otimes m}) = 1$ for all sufficiently divisible integers $m > 0$. The task at hand is to show that $\kappa(F) = 0$. (It is a well known consequence of the easy addition formula [Mor87, Corollary 2.3] that if $\kappa(F) = -\infty$, then $\kappa(X) = -\infty$.) We show in fact the following more precise statement:

Theorem 5.2. *If $f\colon X \to A$ is an algebraic fiber space over an abelian variety, with $\kappa(X) = 0$, then we have*

$$\mathscr{F}_m = f_* \omega_X^{\otimes m} \simeq \mathscr{O}_A$$

for every $m \in \mathbb{N}$ such that $H^0(X, \omega_X^{\otimes m}) \neq 0$.

Proof. From Corollary 4.3, we know that $\mathscr{F}_m$ is an indecomposable unipotent vector bundle on A. In particular,

$$\det \mathscr{F}_m \simeq \mathscr{O}_A.$$

Corollary 4.6 implies then that $\mathscr{F}_m$ has a smooth hermitian metric that is flat. Thus $\mathscr{F}_m$ is a successive extension of trivial bundles $\mathscr{O}_A$ that can be split off as direct summands with the help of the flat metric. It follows that in fact $\mathscr{F}_m \simeq \mathscr{O}_A^{\oplus r}$, the trivial bundle of some rank $r \geq 1$. But then, since

$$h^0(A, f_* \omega_X^{\otimes m}) = h^0(X, \omega_X^{\otimes m}) = 1,$$

we obtain that $r = 1$, which is the statement of the theorem. □

In the remaining chapters, we explain the material in §4 in more detail.

C. Generic vanishing

6. Canonical bundles and their pushforwards. As explained above, the algebraic tools used in this paper revolve around the topic of generic vanishing. This study was initiated by Green and Lazarsfeld [GL87, GL91], in part as an attempt to provide a useful weaker version of Kodaira Vanishing for the canonical bundle, in the absence of twists by positive line bundles. An important addition was provided in work of Simpson [Sim93]. The results of Green-Lazarsfeld were extended to include higher direct images of canonical bundles in [Hac04]. From the point of view of this paper, the main statements to keep in mind are summarized in the following theorem. Recall that for any coherent sheaf $\mathscr{F}$ on an abelian variety A, we consider for all $k \geq 0$ the *cohomological support loci*

$$V^k(\mathscr{F}) = \{\, P \in \operatorname{Pic}^0(A) \mid H^k(X, \mathscr{F} \otimes P) \neq 0 \,\}$$

They are closed subsets of $\operatorname{Pic}^0(A)$, by the semi-continuity theorem for cohomology.

Theorem 6.1. *If $f\colon X \to A$ is a morphism from a smooth projective variety to an abelian variety, then for any $j, k \geq 0$ we have*

1. [Hac04] $\operatorname{codim}_{\operatorname{Pic}^0(A)} V^k(R^j f_* \omega_X) \geq k$.
2. [GL91, Sim93] *Every irreducible component of $V^k(R^j f_* \omega_X)$ is a translate of an abelian subvariety of A by a point of finite order.*

What we use in this paper are (partial) extensions of these results to push-forwards of pluricanonical bundles $f_* \omega_X^{\otimes m}$, for $m \geq 2$. We describe these in the following sections, beginning with an abstract study in the next.

7. The GV property and unipotency. Let A be an abelian variety of dimension g. The generic vanishing property (1) in the theorem above can be formalized into the following:

Definition 7.1. The sheaf $\mathscr{F}$ is called a *GV-sheaf* on A if it satisfies

$$\operatorname{codim}_{\operatorname{Pic}^0(A)} V^k(\mathscr{F}) \geq k \qquad \text{for all } k \geq 0.$$

We will identify $\operatorname{Pic}^0(A)$ with the dual abelian variety $\widehat{A}$, and denote by P the normalized Poincaré bundle on $A \times \widehat{A}$. It induces the integral transforms

$$\mathbf{R}\Phi_P\colon D^b_{coh}(\mathscr{O}_A) \longrightarrow D^b_{coh}\big(\mathscr{O}_{\widehat{A}}\big), \quad \mathbf{R}\Phi_P \mathscr{F} = \mathbf{R}p_{2*}(p_1^* \mathscr{F} \otimes P).$$

and

$$\mathbf{R}\Psi_P\colon D^b_{coh}\big(\mathscr{O}_{\widehat{A}}\big) \longrightarrow D^b_{coh}(\mathscr{O}_A), \quad \mathbf{R}\Psi_P \mathscr{G} = \mathbf{R}p_{1*}(p_2^* \mathscr{G} \otimes P).$$

These functors are known from [Muk81, Theorem 2.2] to be equivalences of derived categories, usually called the Fourier-Mukai transforms; moreover,

$$\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A)^*[-g] \qquad \text{and} \qquad \mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P = (-1_{\widehat{A}})^*[-g], \tag{7.2}$$

where $[-g]$ denotes shifting g places to the right.

Standard applications of base change (see e.g. [PP11b, Lemma 2.1] and [PP11a, Proposition 3.14]) lead to the following basic properties of GV-sheaves:

Lemma 7.3. *Let $\mathscr{F}$ be a coherent sheaf on A. Then:*

1. *$\mathscr{F}$ is a GV-sheaf if and only if*
$$\operatorname{codim}_{\widehat{A}} \operatorname{Supp} R^k \Phi_P \mathscr{F} \geq k \quad \text{for all} \quad k \geq 0.$$
2. *If $\mathscr{F}$ is a GV-sheaf, then*
$$V^g(\mathscr{F}) \subseteq \cdots \subseteq V^1(\mathscr{F}) \subseteq V^0(\mathscr{F}).$$

To give a sense of what is going on, here is a sketch of the proof of part (1): note that the restriction of ${p_1}^* \mathscr{F} \otimes P$ to a fiber $A \times \{\alpha\}$ of p_2 is isomorphic to the sheaf $\mathscr{F} \otimes \alpha$ on A, and so fiberwise we are looking at the cohomology groups $H^k(A, \mathscr{F} \otimes \alpha)$. A simple application of the theorem on cohomology and base change then shows for every $m \geq 0$ that

$$\bigcup_{k \geq m} \operatorname{Supp} R^k \Phi_P \mathscr{F} = \bigcup_{k \geq m} V^k(\mathscr{F}).$$

This implies the result by descending induction on k.

Lemma 7.4. *If $\mathscr{F}$ is a GV-sheaf on A, then $\mathscr{F} = 0$ if and only if $V^0(\mathscr{F}) = \emptyset$.*

Proof. By Lemma 7.3, we see that $V^0(\mathscr{F}) = \emptyset$ is equivalent to $V^k(\mathscr{F}) = \emptyset$ for all $k \geq 0$, which by base change is in turn equivalent to $\mathbf{R}\Phi_P\mathscr{F} = 0$. By Mukai's derived equivalence, this is equivalent to $\mathscr{F} = 0$. □

The following proposition is the same as [Hac04, Corollary 3.2(4)], since it can be seen that the assumption on $\mathscr{F}$ imposed there is equivalent to that of being a GV-sheaf. This is the main way in which generic vanishing is used in this paper; for the definition of a unipotent vector bundle see §4.

Proposition 7.5. *Let $\mathscr{F}$ be a GV-sheaf on an abelian variety A. If $V^0(\mathscr{F}) = \{0\}$, then $\mathscr{F}$ is a unipotent vector bundle.*

Proof. By [Muk81, Example 2.9], if $g = \dim A$, then $\mathscr{F}$ is a unipotent vector bundle if and only if

$$R^i\Phi_P\mathscr{F} = 0 \quad \text{for all } i \neq g, \qquad \text{and} \qquad R^g\Phi_P\mathscr{F} = \mathscr{G}, \tag{7.6}$$

where $\mathscr{G}$ is a coherent sheaf supported at the origin $0 \in \widehat{A}$. To review the argument, notice that if this is the case, then if $l = \text{length}(\mathscr{G}) > 0$, we have $h^0(\widehat{A}, \mathscr{G}) \neq 0$ and so there is a short exact sequence

$$0 \longrightarrow k(0) \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}' \longrightarrow 0$$

where $\mathscr{G}'$ is a coherent sheaf supported at the origin $0 \in \widehat{A}$, with $\text{length}(\mathscr{G}') = l - 1$. Applying $\mathbf{R}\Psi_P$ we obtain a short exact sequence of vector bundles on A

$$0 \longrightarrow \mathscr{O}_A \longrightarrow R^0\Psi_P\mathscr{G} \longrightarrow R^0\Psi_P\mathscr{G}' \longrightarrow 0,$$

and by (7.2) we have $R^0\Psi_P\mathscr{G} = (-1_A)^*\mathscr{F}$. It is then not hard to see that $\mathscr{F}' = R^0\Psi_P\mathscr{G}'$ also satisfies the hypotheses in (7.6) and so, proceeding by induction on l, we may assume that $\mathscr{F}'$ is a unipotent vector bundle. It follows that $\mathscr{F}$ is also a unipotent vector bundle as well (since it is an extension of a unipotent vector bundle by $\mathscr{O}_A$).

We now check that the two conditions in (7.6) are satisfied. By Lemma 7.3 the hypothesis implies that

$$V^i(\mathscr{F}) \subseteq \{0\} \qquad \text{for all } i \geq 0.$$

By base change one obtains that $R^i\Phi_P\mathscr{F}$ is supported at most at $0 \in \widehat{A}$ for $0 \leq i \leq g$. It remains to show that $R^i\Phi_P\mathscr{F} = 0$ for $i \neq g$. Note that

$$H^j(\widehat{A}, R^i\Phi_P\mathscr{F} \otimes \alpha) = 0 \qquad \text{for all } j > 0,\ 0 \leq i \leq g, \text{ and } \alpha \in \text{Pic}^0(\widehat{A}),$$

and so by base change we have

$$R^j\Psi_P(R^i\Phi_P\mathscr{F}) = 0 \qquad \text{for all } j > 0 \text{ and } 0 \leq i \leq g.$$

By an easy argument involving the spectral sequence of the composition of two functors, since $\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A)^*[-g]$, it then follows that $R^0\Psi_P(R^i\Phi_P\mathscr{F}) = \mathcal{H}^i\big((-1_A)^*\mathscr{F}[-g]\big)$, and so in particular

$$R^0\Psi_P(R^i\Phi_P\mathscr{F}) = 0 \ \text{ for } \ i < g.$$

But then $\mathbf{R}\Psi_P(R^i\Phi_P\mathscr{F}) = 0$ for $i < g$, and hence $R^i\Phi_P\mathscr{F} = 0$ for $i < g$. □

For later use, we note that a very useful tool for detecting generic vanishing is a cohomological criterion introduced in [Hac04, Corollary 3.1]. Before stating it, we recall that an ample line bundle N on an abelian variety B induces an isogeny

$$\varphi_N : B \longrightarrow \widehat{B}, \quad x \to t_x^* N \otimes N^{-1},$$

where t_x denotes translation by $x \in B$.

Theorem 7.7. *A coherent sheaf $\mathscr{F}$ on A is a GV-sheaf if and only if given any sufficiently large power M of an ample line bundle on $\widehat{A}$, one has*

$$H^i\big(A, \mathscr{F} \otimes R^g\Psi_P(M^{-1})\big) = 0 \quad \text{for all} \quad i > 0.$$

If $\varphi_M \colon \widehat{A} \to A$ is the isogeny induced by M, this is also equivalent to

$$H^i(\widehat{A}, \varphi_M^* \mathscr{F} \otimes M) = 0 \quad \text{for all} \quad i > 0.$$

Remark. Note that since M is ample, $H^i(\widehat{A}, M^{-1} \otimes \alpha) = 0$ for all $i < g$ and $\alpha \in \mathrm{Pic}^0(\widehat{A}) \simeq A$, and so $R^i\Psi_P(M^{-1}) = 0$ for $i \neq g$. If we denote $R^g\Psi_P(M^{-1}) = \widehat{M^{-1}}$, then by [Muk81, Proposition 3.11] we have $\varphi_M^* \widehat{M^{-1}} \simeq M^{\oplus h^0(M)}$, hence the second assertion.

8. Pushforwards of pluricanonical bundles. In this section we explain the proof of Theorem 4.2, following [PS14, §5]. In *loc. cit.* we noted that a very quick proof can be given based on the general effective vanishing theorem for pushforwards of pluricanonical bundles proved there. However, another more self-contained, if less efficient, proof using cyclic covering constructions is also given; we choose to explain this here, as cyclic covering constructions are in the background of many arguments in this article. We first recall Kollár's vanishing theorem [Kol86, Theorem 2.1].

Theorem 8.1. *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth. If L is an ample line bundle on Y, then*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \;\text{ for all } i \text{ and all } j > 0.$$

Proof of Theorem 4.2. Let $M = L^{\otimes d}$, where L is an ample and globally generated line bundle on $\widehat{A}$, and d is an integer that can be chosen arbitrarily large. Let $\varphi_M \colon \widehat{A} \to A$ be the isogeny induced by M. According to Theorem 7.7, it is enough to show that

$$H^i(\widehat{A}, \varphi_M^* f_* \omega_X^{\otimes m} \otimes M) = 0 \quad \text{for all } i > 0.$$

Equivalently, we need to show that

$$H^i(\widehat{A}, h_* \omega_{X_1}^{\otimes m} \otimes L^{\otimes d}) = 0 \quad \text{for all } i > 0,$$

where $h \colon X_1 \to \widehat{A}$ is the base change of $f \colon X \to A$ via φ_M, as in the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ \big\downarrow h & & \big\downarrow f \\ \widehat{A} & \xrightarrow{\varphi_M} & A \end{array}$$

We can conclude immediately if we know that there exists a bound $d = d(g, m)$, i.e. depending only on $g = \dim A$ and m, such that the vanishing in question holds for any morphism h. (Note that we cannot apply Serre Vanishing here, as construction depends on the original choice of M.) But Proposition 8.2 below shows that there exists a morphism $\varphi : Z \to \widehat{A}$ with Z smooth projective, and $k \le g+m$, such that $h_*\omega_{X_1}^{\otimes m} \otimes L^{\otimes k(m-1)}$ is a direct summand of $\varphi_*\omega_Z$. Applying Kollár vanishing, Theorem 8.1, we deduce that

$$H^i(\widehat{A}, h_*\omega_{X_1}^{\otimes m} \otimes L^{\otimes d}) = 0 \quad \text{for all} \quad i > 0 \quad \text{and all} \quad d \ge (g+m)(m-1)+1,$$

which finishes the proof. (The main result of [PS14] shows that one can in fact take $d \ge m(g+1) - g$.) □

Proposition 8.2. *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. Let L be an ample and globally generated line bundle on Y, and $m \ge 1$ an integer. Then there exists a smooth projective variety Z with a morphism $\varphi : Z \to Y$, and an integer $0 \le k \le n+m$, such that $f_*\omega_X^{\otimes m} \otimes L^{\otimes k(m-1)}$ is a direct summand in $\varphi_*\omega_Z$.*

Proof. The sheaf $f_*\omega_X^{\otimes m} \otimes L^{\otimes pm}$ is globally generated for some sufficiently large p. Denote by k the minimal $p \ge 0$ for which this is satisfied.

Consider now the adjunction morphism

$$f^* f_* \omega_X^{\otimes m} \to \omega_X^{\otimes m}.$$

After blowing up X, if necessary, we can assume that the image sheaf is of the form $\omega_X^{\otimes m} \otimes \mathscr{O}_X(-E)$ for a divisor E with normal crossing support. As $f_*\omega_X^{\otimes m} \otimes L^{\otimes km}$ is globally generated, the line bundle

$$\omega_X^{\otimes m} \otimes f^* L^{\otimes km} \otimes \mathscr{O}_X(-E)$$

is globally generated as well. It is therefore isomorphic to $\mathscr{O}_X(D)$, where D is an irreducible smooth divisor, not contained in the support of E, such that $D+E$ also has normal crossings. We have thus arranged that

$$(\omega_X \otimes f^* L^{\otimes k})^{\otimes m} \simeq \mathscr{O}_X(D+E).$$

We can now take the associated covering of X of degree m, branched along $D+E$, and resolve its singularities. This gives us a generically finite morphism $g \colon Z \to X$ of degree m, and we denote $\varphi = f \circ g : Z \to Y$.

By a well-known calculation of Esnault and Viehweg, see e.g. [EV92, Lemma 2.3], the direct image $g_*\omega_Z$ contains the sheaf

$$\begin{aligned}\omega_X \otimes \big(\omega_X \otimes f^* L^{\otimes k}\big)^{\otimes m-1} &\otimes \mathscr{O}_X\left(-\left\lfloor \tfrac{m-1}{m}(D+E)\right\rfloor\right) \\ \simeq \omega_X^{\otimes m} \otimes f^* L^{\otimes k(m-1)} &\otimes \mathscr{O}_X\left(-\left\lfloor \tfrac{m-1}{m} E\right\rfloor\right)\end{aligned}$$

as a direct summand. If we now apply f_*, we find that

$$f_*\Big(\omega_X^{\otimes m} \otimes \mathscr{O}_X\left(-\left\lfloor \tfrac{m-1}{m} E\right\rfloor\right)\Big) \otimes L^{\otimes k(m-1)}$$

is a direct summand of $\varphi_*\omega_Z$. Finally, E is the relative base locus of $\omega_X^{\otimes m}$, and so

$$f_*\Big(\omega_X^{\otimes m} \otimes \mathscr{O}_X\left(-\left\lfloor \tfrac{m-1}{m} E\right\rfloor\right)\Big) \simeq f_*\omega_X^{\otimes m}.$$

In other words, $f_*\omega_X^{\otimes m} \otimes L^{\otimes k(m-1)}$ is a direct summand in $\varphi_*\omega_Z$. By Theorem 8.1, the sheaf $f_*\omega_X^{\otimes m} \otimes L^{\otimes k(m-1)+n+1}$ is 0-regular in the sense of Castelnuovo-Mumford, and hence globally generated.[1] By our minimal choice of k, this is only possible if

$$k(m-1)+n+1 \geq (k-1)m+1,$$

which is equivalent to $k \leq n+m$. $\square$

9. Fiber spaces over abelian varieties. Let $f\colon X \to A$ be a fiber space over an abelian variety. For simplicity, for each $m \geq 0$ we denote

$$\mathscr{F}_m = f_*\omega_X^{\otimes m}.$$

Note that $\mathscr{F}_0 = \mathscr{O}_A$. Though this is not really necessary for the argument, we first remark that we can be precise about the values of $m \geq 1$ for which $\mathscr{F}_m \neq 0$.

Lemma 9.1. *We have $\mathscr{F}_m \neq 0$ if and only if there exists $P \in \operatorname{Pic}^0(A)$ such that $H^0(X, \omega_X^{\otimes m} \otimes f^*P) \neq 0$.*

Proof. By Theorem 4.2 we know that $\mathscr{F}_m$ is a GV-sheaf on A for all $m \geq 1$. We conclude from Lemma 7.4 that $\mathscr{F}_m \neq 0$ if and only if $V^0(\mathscr{F}_m) \neq \emptyset$, which by the projection formula is precisely the statement of the lemma. $\square$

The purpose of this subsection is to give the

Proof of Corollary 4.3. We will only prove the second statement, since the first one is similar. We fix an m such that $H^0(A, \mathscr{F}_m) = H^0(X, \omega_X^{\otimes m}) \neq 0$. In particular $\mathscr{F}_m$ is a non-trivial GV-sheaf on A. Since $\kappa(X) = 0$, we have $h^0(A, \mathscr{F}_m) = 1$, and in particular $0 \in V^0(\mathscr{F}_m)$. We claim that

$$V^0(\mathscr{F}_m) = \{0\},$$

which implies that $\mathscr{F}_m$ is unipotent by Proposition 7.5.

To see this, note first that by Theorem 4.1 and the comments immediately after, $V^0(\mathscr{F}_m)$ is a union of torsion translates of abelian subvarieties of $\operatorname{Pic}^0(A)$. Then, proceeding as in [CH01, Lemma 2.1], if there were two distinct points $P, Q \in V^0(\mathscr{F}_m)$ we could assume that they are both torsion of the same order k. Since f is a fiber space, the mapping

$$f^*\colon \operatorname{Pic}^0(A) \longrightarrow \operatorname{Pic}^0(X)$$

is injective, and so f^*P and f^*Q are distinct as well. Now if $P \in V^0(\mathscr{F}_m)$, then

$$H^0(X, \omega_X^{\otimes m} \otimes f^*P) \simeq H^0(A, \mathscr{F}_m \otimes P) \neq 0,$$

and similarly for Q. Let $D \in |mK_X + f^*P|$ and $G \in |mK_X + f^*Q|$, so that $kD, kG \in |mkK_X|$. Since $h^0(X, \omega_X^{\otimes mk}) = 1$, it follows that $kD = kF$, and hence $f^*P = f^*Q$. This is the required contradiction.

Finally, since $h^0(A, \mathscr{F}_m) = 1$, it is clear that $\mathscr{F}_m$ is indecomposable. $\square$

---

[1]Recall that a sheaf $\mathscr{F}$ on Y is 0-*regular* with respect to an ample and globally generated line bundle L if

$$H^i(Y, \mathscr{F} \otimes L^{\otimes -i}) = 0 \quad \text{for all} \quad i > 0.$$

The Castelnuovo-Mumford Lemma says that every 0-regular sheaf is globally generated.

10. Cohomological support loci for pluricanonical bundles. In this section we explain an important ingredient used in Corollary 4.3, namely Theorem 4.1, the analogue of Simpson's theorem for the 0-th cohomological support locus of a pluricanonical bundle. We give a slight generalization, emphasizing again the ubiquitous cyclic covering trick.

For a coherent sheaf $\mathscr{F}$ on an abelian variety A, for each $k \geq 1$ we consider the following refinement of $V^0(\mathscr{F})$, namely

$$V^0_k(\mathscr{F}) = \{\, P \in \operatorname{Pic}^0(A) \mid h^0(X, \mathscr{F} \otimes P) \geq k \,\}.$$

Theorem 10.1. *Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety, and fix integers $m, k \geq 1$. Then every irreducible component of $V^0_k(f_* \omega_X^{\otimes m})$ is a torsion subvariety, i.e. a translate of an abelian subvariety of $\operatorname{Pic}^0(A)$ by a torsion point.*

To prove Theorem 10.1, we first collect a few lemmas. Given a smooth projective variety X, and a line bundle L on X with $\kappa(L) \geq 0$, recall that the *asymptotic multiplier ideal* of L is defined as

$$\mathcal{I}\big(\lVert L \rVert\big) = \mathcal{I}\Big(\tfrac{1}{p} D\Big) \subseteq \mathscr{O}_X,$$

where p is any sufficiently large and divisible integer, D is the divisor of a general section in $H^0(X, L^{\otimes p})$, and the ideal sheaf on the right is the multiplier ideal of the $\mathbb{Q}$-divisor $\frac{1}{p} D$; see [Laz04, Ch. 11]. It is easy to see that the ideal sheaf $\mathcal{I}\big(\lVert L \rVert\big)$ is independent of the choice of p and D. Further properties of asymptotic multiplier ideals appear in the proof of Lemma 10.4 below.

Lemma 10.2. *There exists a morphism $g \colon Y \to X$ with Y smooth and projective, such that the sheaf $\omega_X \otimes L \otimes \mathcal{I}\big(\lVert L \rVert\big)$ is a direct summand of $g_* \omega_Y$.*

Proof. Take D as above, and let $\mu \colon X' \to X$ be a log resolution of (X, D) such that X' is smooth and $\mu^* D$ plus the exceptional divisor of μ is a divisor with simple normal crossing support. Then

$$\mu^* L^{\otimes p} = \mathscr{O}_{X'}(\mu^* D),$$

and we let $f \colon Y \to X'$ be a resolution of singularities of the degree p branched covering of X' defined by $\mu^* D$. According to the calculation of Esnault and Viehweg recalled in the proof of Proposition 8.2, $f_* \omega_Y$ contains as a direct summand the sheaf

$$\omega_{X'} \otimes \mu^* L \otimes \mathscr{O}_{X'}\Big(-\big\lfloor \tfrac{1}{p} \mu^* D \big\rfloor\Big) \simeq \mu^*\big(\omega_X \otimes L\big) \otimes \mathscr{O}_{X'}\Big(K_{X'/X} - \big\lfloor \tfrac{1}{p} \mu^* D \big\rfloor\Big),$$

and so $\mu_* f_* \omega_Y$ contains as a direct summand the sheaf

$$\omega_X \otimes L \otimes \mu_* \mathscr{O}_{X'}\Big(K_{X'/X} - \big\lfloor \tfrac{1}{p} \mu^* D \big\rfloor\Big) = \omega_X \otimes L \otimes \mathcal{I}\big(\lVert L \rVert\big).$$

We can therefore take $g = \mu \circ f$. □

Lemma 10.3. *If $\mathscr{F}$ and $\mathscr{G}$ are two coherent sheaves on an abelian variety A, and $\mathscr{F}$ is a direct summand of $\mathscr{G}$, then every irreducible component of $V^0_k(\mathscr{F})$ is also an irreducible component of $V^0_\ell(\mathscr{G})$ for some $\ell \geq k$.*

Proof. Let $Z \subseteq V_k^0(\mathscr{F})$ be an irreducible component. We can assume without loss of generality that $k = \min\{ h^0(X, \mathscr{F} \otimes \alpha) \mid \alpha \in Z \}$. By assumption, we have a decomposition $\mathscr{G} \simeq \mathscr{F} \oplus \mathscr{F}'$. We define

$$\ell = k + \min\{ h^0(X, \mathscr{F}' \otimes \alpha) \mid \alpha \in Z \}.$$

By the semicontinuity of $h^0(A, \mathscr{F}' \otimes \alpha)$ and $h^0(A, \mathscr{F} \otimes \alpha)$, it follows that there is a neighborhood U of the generic point of Z such that $h^0(\mathscr{F}' \otimes \alpha) \leq \ell - k$ and $h^0(\mathscr{F} \otimes \alpha) \leq k$ for any $\alpha \in U$. Since $h^0(\mathscr{F} \otimes \alpha) < k$ for any $\alpha \in U \setminus (U \cap Z)$ it is easy to see that Z is an irreducible component of $V_\ell^0(\mathscr{G})$. □

Lemma 10.4. *If $V_k^0(f_*\omega_X^{\otimes m})$ contains a point, then it also contains a torsion subvariety through that point.*

Proof. Take any point in $V_k^0(f_*\omega_X^{\otimes m})$. Since $\operatorname{Pic}^0(X)$ is divisible, we may assume that our point is of the form $L_0^{\otimes m}$ for some $L_0 \in \operatorname{Pic}^0(X)$. This means that

$$h^0\big(X, \omega_X^{\otimes m} \otimes f^* L_0^{\otimes m}\big) \geq k.$$

For $r \geq 0$, set $\mathcal{I}_r = \mathcal{I}\big(\|\omega_X^{\otimes r} \otimes f^* L_0^{\otimes r}\|\big)$. According to Lemma 10.2, there exists a morphism $g\colon Y \to X$ such that

$$\omega_X \otimes (\omega_X \otimes f^* L_0)^{\otimes (m-1)} \otimes \mathcal{I}_{m-1} = \omega_X^{\otimes m} \otimes L_0^{\otimes (m-1)} \otimes \mathcal{I}_{m-1}$$

is a direct summand of $g_*\omega_Y$. Consequently, $f_*(\omega_X^{\otimes m} \otimes L_0^{m-1} \otimes \mathcal{I}_{m-1})$ is a direct summand of $h_*\omega_Y$, where $h = f \circ g\colon Y \to A$. By Simpson's theorem we know that, for any ℓ, every irreducible component of $V_\ell^0(h_*\omega_Y)$ is a torsion subvariety. Together with Lemma 10.3, this shows that every irreducible component of

$$V_k^0\big(f_*(\omega_X^{\otimes m} \otimes L_0^{\otimes (m-1)} \otimes \mathcal{I}_{m-1})\big)$$

is a torsion subvariety. We observe that this set contains L_0: the reason is that since $\mathcal{I}_m \subseteq \mathcal{I}_{m-1}$ (see [Laz04, Theorem 11.1.8]), we have

$$\begin{aligned} H^0\big(X, (\omega_X \otimes f^* L_0)^{\otimes m} \otimes \mathcal{I}_m\big) &\subseteq H^0\big(X, (\omega_X \otimes f^* L_0)^{\otimes m} \otimes \mathcal{I}_{m-1}\big) \\ &\subseteq H^0\big(X, (\omega_X \otimes f^* L_0)^{\otimes m}\big), \end{aligned}$$

and the two spaces on the outside are equal because the subscheme defined by $\mathcal{I}_m$ is contained in the base locus of $(\omega_X \otimes f^* L_0)^{\otimes m}$ by [Laz04, Theorem 11.1.8].

Now let W be an irreducible component of $V_k^0\big(f_*(\omega_X^{\otimes m} \otimes L_0^{\otimes (m-1)} \otimes \mathcal{I}_{m-1})\big)$ passing through the point L_0. For every $L \in W$, we have

$$h^0\big(X, \omega_X^{\otimes m} \otimes L_0^{\otimes (m-1)} \otimes L\big) \geq h^0\big(X, \omega_X^{\otimes m} \otimes L_0^{\otimes (m-1)} \otimes \mathcal{I}_{m-1} \otimes L\big) \geq k,$$

and so $L_0^{\otimes (m-1)} \otimes W \subseteq V_k^0(f_*\omega_X^{\otimes m})$. As noted above, $L_0^{\otimes (m-1)} \otimes W$ contains the point $L_0^{\otimes m}$; it is also a torsion subvariety, because W is a torsion subvariety and $L_0 \in W$. □

Proof of Theorem 10.1. Let $Z \subseteq V_k^0(f_*\omega_X^{\otimes m})$ be an irreducible component; we have to show that Z is a torsion subvariety. In case Z is a point, this follows directly from Lemma 10.4, so let us assume that $\dim Z \geq 1$. Let $Z_0 \subseteq Z$ denote the Zariski-open subset obtained by removing the intersection with the other irreducible components of $V_k^0(f_*\omega_X^{\otimes m})$. Then again by Lemma 10.4, every point of Z_0 lies on a torsion subvariety that is contained in Z. Because there are only countably many torsion subvarieties in $\operatorname{Pic}^0(X)$, Baire's theorem implies that Z itself must be a torsion subvariety. □

11. Iitaka fibration and cohomological support loci. In this section, we use Theorem 5.2 to give a precise description of the cohomological support loci

$$V^0(\omega_X^{\otimes m}) = \left\{ P \in \operatorname{Pic}^0(X) \;\middle|\; H^0(X, \omega_X^{\otimes m} \otimes P) \neq 0 \right\}$$

for all $m \geq 2$, in terms of the Iitaka fibration of X. After a birational modification of X, the Iitaka fibration can be realized as a morphism $f\colon X \to Y$, where Y is smooth projective of dimension $\kappa(X)$. By the universal property of the Albanese mapping, we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle a_f} \\ Y & \xrightarrow{a_Y} & A_Y \end{array}$$

where $A_X = \operatorname{Alb}(X)$ and $A_Y = \operatorname{Alb}(Y)$ are the two Albanese varieties. The following simple lemma appears in [CH04, Lemma 2.6].

Lemma 11.1. *With notation as above, the following things are true:*

(a) *The homomorphism a_f is surjective with connected fibers.*
(b) *Setting $K = \ker(a_f)$, we have a short exact sequence*
$$0 \to \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(K) \to 0.$$
(c) *If F is a general fiber of f, then the kernel of the natural homomorphism $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(F)$ is a finite union of torsion translates of $\operatorname{Pic}^0(Y)$.*

Using this lemma and the results of Green-Lazarsfeld [GL87, GL91] and Simpson [Sim93], one can prove the following results about the locus $V^0(\omega_X)$:

(1) There are finitely many quotient abelian varieties $\operatorname{Alb}(X) \to B_i$ and finitely many torsion points $\alpha_i \in \operatorname{Pic}^0(X)$ such that
$$V^0(\omega_X) = \bigcup_{i=1}^{n} \bigl(\alpha_i + \operatorname{Pic}^0(B_i)\bigr).$$
This is proved in [GL91, Theorem 0.1] and [Sim93]. Note that $V^0(\omega_X)$ may be empty; in that case, we take $n = 0$.
(2) We have $\operatorname{Pic}^0(B_i) \subseteq \operatorname{Pic}^0(Y)$, where $f\colon X \to Y$ is the Iitaka fibration; when X is of maximal Albanese dimension, then the union of the $\operatorname{Pic}^0(B_i)$ generates $\operatorname{Pic}^0(Y)$. This is proved in [CH04, CH07].
(3) At a general point P of the i-th irreducible component $\alpha_i + \operatorname{Pic}^0(B_i)$, one has $s \cup v = 0$ for all $s \in H^0(X, \omega_X \otimes P)$ and all $v \in H^1(B_i, \mathscr{O}_{B_i})$; conversely, if $s \in H^0(X, \omega_X \otimes P)$ is nonzero and $s \cup v = 0$ for some $v \in H^1(X, \mathscr{O}_X)$, then necessarily $v \in H^1(B_i, \mathscr{O}_{B_i})$.

Note. One can interpret property (3) as follows. If $P \in \operatorname{Pic}^0(X)$ is a general point of a component of $V^0(\omega_X)$, and we identify the tangent space to $\operatorname{Pic}^0(X)$ at the point P with the vector space $H^1(X, \mathscr{O}_X)$, then $s \cup v = 0 \in H^1(X, \omega_X \otimes P)$ if and only if s deforms to first order in the direction of v. Property (3) then says that if s deforms to first order in the direction of v, then it deforms to arbitrary order.

It turns out that the cohomology support loci $V^0(\omega_X^{\otimes m})$ for $m \geq 2$ are governed by the Iitaka fibration $f\colon X \to Y$: in contrast to the case $m = 1$, every irreducible component is now simply a torsion translate of $\operatorname{Pic}^0(Y)$.

Theorem 11.2. *Let X be a smooth complex projective variety, and let F be a general fiber of the Iitaka fibration $f\colon X \to Y$. Let $m \geq 2$.*

(a) For every torsion point $\alpha \in \operatorname{Pic}^0(X)$, and every $\beta \in \operatorname{Pic}^0(Y)$, we have
$$h^0(X, \omega_X^{\otimes m} \otimes \alpha) = h^0(X, \omega_X^{\otimes m} \otimes \alpha \otimes f^*\beta).$$

(b) There exist finitely many torsion points $\alpha_1, \ldots, \alpha_n \in \operatorname{Pic}^0(X)$ such that
$$V^0(\omega_X^{\otimes m}) = \bigcup_{i=1}^{n} \bigl(\alpha_i + \operatorname{Pic}^0(Y)\bigr).$$

(c) At every point $\alpha \in V^0(\omega_X^{\otimes m})$, one has $s \cup v = 0$ for all $s \in H^0(X, \omega_X^{\otimes m} \otimes \alpha)$ and all $v \in H^1(Y, \mathscr{O}_Y)$; conversely, if $s \in H^0(X, \omega_X^{\otimes m} \otimes \alpha)$ is nonzero and $s \cup v = 0$ for some $v \in H^1(X, \mathscr{O}_X)$, then necessarily $v \in H^1(Y, \mathscr{O}_Y)$.

Proof. We begin by proving (a), following Jiang's version [Jia11, Lemma 3.2] of the original argument in [HP02, Proposition 2.12]. Let $g\colon X \to A_Y$ be the morphism induced by composing f with the Albanese map of Y.

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ \big\downarrow{\scriptstyle f} & \overset{g}{\searrow} & \big\downarrow{\scriptstyle a_f} \\ Y & \xrightarrow{a_Y} & A_Y \end{array}$$

Let H be an ample divisor on A_Y. By construction, g factors through the Iitaka fibration of X, and so there is an integer $d \gg 0$ such that

$$dK_X \sim g^*H + E \tag{11.3}$$

for some effective divisor E on X. In particular, all sufficiently large and divisible powers of ω_X have nontrivial global sections.

Now consider the torsion-free coherent sheaf
$$\mathscr{F} = g_*\Bigl(\omega_X^{\otimes m} \otimes \alpha \otimes \mathcal{I}\bigl(\|\omega_X^{\otimes(m-1)}\|\bigr)\Bigr)$$
on the abelian variety A_Y. From our discussion of asymptotic multiplier ideals in §10, it is easy to see that we have inclusions
$$\mathcal{I}\bigl(\|\omega_X^{\otimes m} \otimes \alpha\|\bigr) = \mathcal{I}\bigl(\|\omega_X^{\otimes m}\|\bigr) \subseteq \mathcal{I}\bigl(\|\omega_X^{\otimes(m-1)}\|\bigr),$$
by choosing the integer $p \in \mathbb{N}$ in the definition of the asymptotic multiplier ideal as a multiple of the order of the torsion point $\alpha \in \operatorname{Pic}^0(X)$. Since
$$H^0\Bigl(X, \omega_X^{\otimes m} \otimes \alpha \otimes \mathcal{I}\bigl(\|\omega_X^{\otimes m} \otimes \alpha\|\bigr)\Bigr) = H^0\bigl(X, \omega_X^{\otimes m} \otimes \alpha\bigr),$$
this shows that $H^0(A_Y, \mathscr{F}) = H^0(X, \omega_X^{\otimes m} \otimes \alpha)$. For $p \in \mathbb{N}$ sufficiently large and divisible, we have
$$\mathcal{I}\bigl(\|\omega_X^{\otimes(m-1)}\|\bigr) = \mu_*\mathscr{O}_{X'}\Bigl(K_{X'/X} - \bigl\lfloor \tfrac{1}{p} F \bigr\rfloor\Bigr).$$
Here $\mu\colon X' \to X$ is a log resolution of the complete linear system of $\omega_X^{\otimes p(m-1)}$: the divisor $F + D$ has simple normal crossing support, the linear system $|D|$ is base point free, and

$$p(m-1)\mu^*K_X \sim F + D. \tag{11.4}$$

(see [Laz04, 9.2.10]). We may also assume that the larger divisor $F + D + \mu^*E$ has simple normal crossing support.

Now $\mathscr{F}$ is the pushforward, via the mapping $g \circ \mu \colon X' \to A_Y$, of the line bundle

$$\mu^*\alpha \otimes \mathscr{O}_{X'}\Big(K_{X'} + (m-1)\mu^*K_X - \big\lfloor \tfrac{1}{p} F \big\rfloor\Big),$$

and for any $\varepsilon \in \mathbb{Q}$, we have the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-divisors

$$(m-1)\mu^*K_X - \big\lfloor \tfrac{1}{p}F \big\rfloor \sim_{\mathbb{Q}} \tfrac{1-\varepsilon}{p}(F+D) + \tfrac{\varepsilon(m-1)}{d}\big(\mu^*g^*H + \mu^*E\big) - \big\lfloor \tfrac{1}{p}F \big\rfloor$$

by combining (11.3) and (11.4). This allows us to write

$$K_{X'} + (m-1)\mu^*K_X - \big\lfloor \tfrac{1}{p}F \big\rfloor \sim_{\mathbb{Q}} K_{X'} + \Delta + \tfrac{\varepsilon(m-1)}{d}\mu^*g^*H,$$

where Δ is the $\mathbb{Q}$-divisor on X' given by the formula

$$\Delta = \tfrac{1-\varepsilon}{p}D + \tfrac{\varepsilon(m-1)}{d}\mu^*E + \tfrac{1-\varepsilon}{p}F - \big\lfloor \tfrac{1}{p}F \big\rfloor.$$

By construction, the support of Δ is a divisor with simple normal crossings; and if we choose $\varepsilon > 0$ sufficiently small, then Δ is a boundary divisor, meaning that the coefficient of every irreducible component belongs to the interval $[0,1)$. To see that $\Delta \geq 0$ it suffices to observe that $p(m-1)\mu^*E \geq dF$ and to check that $\lfloor \Delta \rfloor = 0$ it suffices to observe that the coefficients of $\frac{1}{p}D + \{\frac{1}{p}F\}$ are < 1 and apply continuity. In particular, the pair (X', Δ) is klt. We can now apply the version for $\mathbb{Q}$-divisors of Kollár's vanishing theorem [Kol95, §10] and conclude that the pushforward of

$$\mu^*(\alpha \otimes g^*\beta) \otimes \mathscr{O}_{X'}\Big(K_{X'} + (m-1)\mu^*K_X - \big\lfloor \tfrac{1}{p}F \big\rfloor\Big)$$

under the map $g \circ \mu \colon X' \to A_Y$ has vanishing higher cohomology for all $\beta \in \operatorname{Pic}^0(Y)$. Together with the projection formula, this shows that

$$H^i\big(A_Y, \mathscr{F} \otimes \beta\big) = 0$$

for every $i > 0$ and every $\beta \in \operatorname{Pic}^0(Y)$. It follows that

$$h^0\big(A_Y, \mathscr{F} \otimes \beta\big) = \chi\big(A_Y, \mathscr{F} \otimes \beta\big)$$

has the same value for every $\beta \in \operatorname{Pic}^0(Y)$. But then

$$\begin{aligned} h^0\big(X, \omega_X^{\otimes m} \otimes \alpha\big) &= h^0\big(A_Y, \mathscr{F}\big) = h^0\big(A_Y, \mathscr{F} \otimes g^*\beta\big) \\ &= h^0\Big(X, \omega_X^{\otimes m} \otimes \alpha \otimes g^*\beta \otimes \mathcal{I}\big(\|\omega_X^{\otimes(m-1)}\|\big)\Big) \\ &\leq h^0\big(X, \omega_X^{\otimes m} \otimes \alpha \otimes g^*\beta\big), \end{aligned}$$

and by semicontinuity, we conclude that in fact

$$h^0\big(X, \omega_X^{\otimes m} \otimes \alpha\big) = h^0\big(X, \omega_X^{\otimes m} \otimes \alpha \otimes g^*\beta\big).$$

We next prove (b). Assuming that $V^0(\omega_X^{\otimes m})$ is nonempty, there are by Theorem 10.1 finitely many distinct torsion elements $\alpha_1, \dots, \alpha_s \in \operatorname{Pic}^0(X)$, and abelian subvarieties $B_i \subset \operatorname{Pic}^0(X)$, such that

$$V^0(\omega_X^{\otimes m}) = \bigcup_{i=1}^{s} \big(\alpha_i + B_i\big).$$

By (a) we know that $\operatorname{Pic}^0(Y) \subseteq B_i$; indeed, (a) implies that for every torsion point $\alpha \in V^0(\omega_X^{\otimes m})$ we have $\alpha + \operatorname{Pic}^0(Y) \subseteq V^0(\omega_X^{\otimes m})$, and this applies of course to α_i. To prove (b), it is therefore enough to show that $B_i \subseteq \operatorname{Pic}^0(Y)$. Take an arbitrary element $\alpha \in V^0(\omega_X^{\otimes m})$, and let $s \in H^0(X, \omega_X^{\otimes m} \otimes \alpha)$ be a nonzero global section.

The restriction of s to a general fiber F of the Iitaka fibration $f\colon X \to Y$ is then a nonzero global section of

$$\omega_F^{\otimes m} \otimes \alpha|_F,$$

and because $\kappa(F) = 0$, it follows that $\alpha|_F$ is torsion in $\operatorname{Pic}^0(F)$. According to Lemma 11.1, a nonzero multiple of α therefore belongs to $\operatorname{Pic}^0(Y)$. This is enough to conclude that $B_i \subseteq \operatorname{Pic}^0(Y)$, and so (b) is proved.

To prove (c), note first that standard arguments (see for instance [EV92, Lemma 12.6]) imply the first half, namely that if $v \in H^1(Y, \mathscr{O}_Y)$ then

$$s \cup v = 0 \qquad \text{for all } s \in H^0\bigl(X, \omega_X^{\otimes m} \otimes \alpha\bigr).$$

For the second half, suppose that $s \in H^0(X, \omega_X^{\otimes m} \otimes \alpha)$ is a nonzero global section such that $s \cup v = 0$ for some $v \in H^1(X, \mathscr{O}_X)$. Restricting to a general fiber F of the Iitaka fibration $f\colon X \to Y$, we get

$$0 = s|_F \cup v|_F \in H^1\bigl(F, \omega_F^{\otimes m} \otimes \alpha|_F\bigr),$$

where $s|_F \in H^0\bigl(F, \omega_F^{\otimes m} \otimes \alpha|_F\bigr)$ is nonzero, and $v|_F \in H^1(F, \mathscr{O}_F)$. Since $\alpha \in V^0(\omega_X^{\otimes m})$, we have $\alpha|_F = \alpha_i|_F$ for some $i = 1, \dots, s$, as a consequence of Lemma 11.1 and (b). In particular, $\alpha|_F$ is torsion, say of order k, and so

$$s^{\otimes k}|_F \in H^0\bigl(F, \omega_F^{\otimes km}\bigr).$$

Let $a_F\colon F \to A_F$ denote the Albanese mapping of F. Recalling that $\kappa(F) = 0$, we get from Theorem 5.2 that $(a_F)_* \omega_F^{\otimes km} = \mathscr{O}_{A_F}$. Under the isomorphism

$$H^0\bigl(F, \omega_F^{\otimes km}\bigr) = H^0\bigl(A_F, \mathscr{O}_{A_F}\bigr),$$

our nonzero section $s^{\otimes k}|_F$ therefore corresponds to a nonzero constant $\sigma \in \mathbb{C}$; likewise, under the isomorphism

$$H^1(F, \mathscr{O}_F) = H^1\bigl(A_F, \mathscr{O}_{A_F}\bigr),$$

the vector $v|_F$ corresponds to a vector $u \in H^1\bigl(A_F, \mathscr{O}_{A_F}\bigr)$. It is not hard to see that the two isomorphisms are compatible with cup product; consequently, $s^{\otimes k} \cup v = 0$ implies that $\sigma u = 0$, and hence that $u = 0$. By the infinitesimal version of Lemma 11.1, this means that $v \in H^1(Y, \mathscr{O}_Y)$, as asserted. □

D. Singular metrics on pushforwards of adjoint line bundles

12. Plurisubharmonic functions. Let X be a complex manifold. We begin our survey of the analytic techniques by recalling the following important definition; see for example [Dem12, I.5] for more details.

Definition 12.1. A function $\varphi\colon X \to [-\infty, +\infty)$ is called *plurisubharmonic* if it is upper semi-continuous, locally integrable, and satisfies the mean-value inequality

$$(\varphi \circ \gamma)(0) \leq \frac{1}{\pi} \int_\Delta (\varphi \circ \gamma)\, d\mu \tag{12.2}$$

for every holomorphic mapping $\gamma\colon \Delta \to X$ from the open unit disk $\Delta \subseteq \mathbb{C}$.

Suppose that φ is plurisubharmonic. From (12.2) one can deduce, by integrating over the space of lines through a given point, that the mean-value inequality

$$(\varphi \circ \iota)(0) \leq \frac{1}{\mu(B)} \int_B (\varphi \circ \iota)\, d\mu$$

also holds for any open embedding $\iota\colon B \hookrightarrow X$ of the open unit ball $B \subseteq \mathbb{C}^n$; here n is the local dimension of X at the point $\iota(0)$. In other words, every plurisubharmonic function is also *subharmonic*. Together with local integrability, this implies that φ is locally bounded from above.

Lemma 12.3. *Every plurisubharmonic function on a compact complex manifold is locally constant.*

Proof. Let φ be a plurisubharmonic function on a compact complex manifold X. As φ is upper semi-continuous and locally bounded from above, it achieves a maximum on every connected component of X. The mean-value inequality then forces φ to be locally constant. □

Observe that a plurisubharmonic function is uniquely determined by its values on any subset whose complement has measure zero. Indeed, the mean-value inequality provides an upper bound on the value at any point $x \in X$, and the upper semi-continuity a lower bound. One also has the following analogue of the Riemann and Hartogs extension theorems for holomorphic functions [Dem12, Theorem I.5.24]; by what we have just said, there can be at most one extension in each case.

Lemma 12.4. *Let $Z \subseteq X$ be a closed analytic subset, and let φ be a plurisubharmonic function on $X \setminus Z$.*

- *(a) If $\operatorname{codim} Z \geq 2$, then φ extends to a plurisubharmonic function on X.*
- *(b) If $\operatorname{codim} Z = 1$, then φ extends to a plurisubharmonic function on X if and only if it is locally bounded near every point of Z.*

A plurisubharmonic function φ determines a coherent sheaf of ideals $\mathcal{I}(\varphi) \subseteq \mathscr{O}_X$, called the *multiplier ideal sheaf*, whose sections over any open subset $U \subseteq X$ consist of those holomorphic functions $f \in H^0(U, \mathscr{O}_X)$ for which the function $|f|^2 e^{-\varphi}$ is locally integrable. We use the convention that the value of the product is 0 at points $x \in X$ where $f(x) = 0$ and $\varphi(x) = -\infty$.

Since plurisubharmonic functions are locally bounded from above, Montel's theorem in several variables implies the following compactness property.

Proposition 12.5. *Let $\varphi\colon B \to [-\infty, +\infty)$ be a plurisubharmonic function on the open unit ball $B \subseteq \mathbb{C}^n$. Consider the collection of holomorphic functions*

$$H_K(\varphi) = \left\{ f \in H^0(B, \mathscr{O}_B) \;\middle|\; \int_B |f|^2 e^{-\varphi}\, d\mu \leq K \right\}.$$

Any sequence of functions in $H_K(\varphi)$ has a subsequence that converges uniformly on compact subsets to an element of $H_K(\varphi)$.

Proof. The mean-value inequality for holomorphic functions implies that all functions in $H_K(\varphi)$ are uniformly bounded on every closed ball of radius $R < 1$. Let us briefly review the argument. Because φ is locally bounded from above, there is a constant $C \geq 0$ such that $\varphi \leq C$ on the closed ball of radius $(R+1)/2$. Fix a point $z \in \overline{B}_R(0)$ in the closed ball of radius R, and a holomorphic function $f \in H_K(\varphi)$. By the mean-value inequality,

$$|f(z)|^2 \leq \frac{1}{r^n \mu(B)} \int_{B_r(z)} |f|^2\, d\mu \leq \frac{e^C}{r^n \mu(B)} \int_{B_r(z)} |f|^2 e^{-\varphi}\, d\mu \leq \frac{K \cdot e^C}{r^n \mu(B)},$$

where $r = (1-R)/2$. By the n-dimensional version of Montel's theorem [GR09, Theorem I.A.12], this uniform bound implies that any sequence $f_0, f_1, f_2, \dots \in H_K(\varphi)$ has a subsequence that converges uniformly on compact subsets to a holomorphic function $f \in H^0(B, \mathscr{O}_B)$. By Fatou's lemma,

$$\int_B |f|^2 e^{-\varphi} \, d\mu \leq \liminf_{k \to +\infty} \int_B |f_k|^2 e^{-\varphi} \, d\mu \leq K,$$

which means that $f \in H_K(\varphi)$. □

Note. The example of an orthonormal sequence in $H_K(\varphi)$ shows that the convergence need not be with respect to the L^2-norm.

13. Singular hermitian metrics on line bundles. Many of the newer applications of analytic techniques in algebraic geometry – such as Siu's proof of the invariance of plurigenera – rely on the notion of *singular hermitian metrics* on holomorphic line bundles. The word "singular" here means two things at once: first, that the metric is not necessarily C^∞; second, that certain vectors in the fibers of the line bundle are allowed to have either infinite length or length zero.

Let X be a complex manifold, and let L be a holomorphic line bundle on X with a singular hermitian metric h. In any local trivialization of L, such a metric is represented by a "weight function" of the form $e^{-\varphi}$, where φ is a measurable function with values in $[-\infty, +\infty]$. More precisely, suppose that the restriction of L to an open subset $U \subseteq X$ is trivial, and that $s_0 \in H^0(U, L)$ is a nowhere vanishing holomorphic section. Then any other holomorphic section $s \in H^0(U, L)$ can be written as $s = f s_0$ for a unique holomorphic function f on U, and the length squared of s with respect to the singular hermitian metric h is

$$|s|_h^2 = |f|^2 e^{-\varphi}. \tag{13.1}$$

The points where φ is not finite correspond to singularities of the metric: at points where $\varphi(x) = -\infty$, the metric becomes infinite; at points where $\varphi(x) = +\infty$, the metric stops being positive definite.

Note. At points $x \in U$ where $\varphi(x) = -\infty$, we use the following convention: the product in (13.1) equals 0 if $f(x) = 0$; otherwise, it equals $+\infty$. With this rule in place, $|s|_h$ is a well-defined measurable function on U with values in $[0, +\infty]$.

We say that a singular hermitian metric h is *continuous* if the local weight functions φ are continuous functions with values in $[-\infty, +\infty]$. This is equivalent to asking that, for every open subset $U \subseteq X$ and every section $s \in H^0(U, L)$, the function $|s|_h \colon U \to [0, +\infty]$ should be continuous.

We say that the pair (L, h) has *semi-positive curvature* if the local weight functions φ are plurisubharmonic. In that case, φ is locally integrable, and the curvature current of (L, h) can be defined, in the sense of distributions, by the formula

$$\Theta_h = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi.$$

It is easy to see that Θ_h is a well-defined closed positive $(1,1)$-current on X; its cohomology class in $H^2(X, \mathbb{R})$ equals the first Chern class $c_1(L)$. Conversely, if the current Θ_h is positive, then one can make the local weight functions φ plurisubharmonic by modifying them on a set of measure zero [Dem12, Theorem I.5.8].

Note. Most authors include the condition of local integrability into the definition of a singular hermitian metric. We use a different convention, so as to be consistent with the definition of singular hermitian metrics on vector bundles later on.

A singular hermitian metric of semi-positive curvature is automatically positive definite at every point. Indeed, φ is locally bounded from above, and so the factor $e^{-\varphi}$ in the local expression for h may equal $+\infty$ at certain points, but has to be locally bounded from below by a positive constant. Moreover, φ is upper semi-continuous, and so the function $|s|_h \colon U \to [0, +\infty]$ is not just measurable, but even lower semi-continuous, for every holomorphic section $s \in H^0(U, L)$ on some open subset $U \subseteq X$.

Lemma 13.2. *Suppose that X is compact, and that h is a singular hermitian metric with semi-positive curvature on a holomorphic line bundle L. If $c_1(L) = 0$ in $H^2(X, \mathbb{R})$, then h is actually a smooth metric with zero curvature.*

Proof. The cohomology class of the closed positive $(1,1)$-current Θ_h equals zero in $H^2(X, \mathbb{R})$, and so there is a globally defined plurisubharmonic function ψ in X such that $\Theta_h = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi$. By Lemma 12.3, ψ is locally constant, and so $\Theta_h = 0$. Now all the local weight functions φ satisfy $\partial\bar{\partial}\varphi = 0$, and are therefore smooth functions; but this means exactly that h is a smooth metric. □

The curvature assumption implies that the *multiplier ideal sheaf* $\mathcal{I}(h) \subseteq \mathscr{O}_X$ is a coherent sheaf of ideals on X; in the notation from above, a holomorphic function $f \colon U \to \mathbb{C}$ is a section of $\mathcal{I}(h)$ if and only if the function $|f|^2 e^{-\varphi}$ is locally integrable. Consequently, the subspace

$$H^0\big(X, L \otimes \mathcal{I}(h)\big) \subseteq H^0(X, L)$$

consists of all global holomorphic sections of L for which the lower semi-continuous function $|s|_h^2 \colon X \to [0, +\infty]$ is locally integrable.

14. The Ohsawa-Takegoshi extension theorem. It is known that a line bundle on a projective complex manifold admits a singular hermitian metric with semi-positive curvature if and only if it is pseudo-effective. The power of the metric approach to positivity comes from fact that one can extend holomorphic sections from submanifolds with precise bounds on the norm of the extension. The most important result in this direction is the famous *Ohsawa-Takegoshi theorem.*[2]

Let X be a complex manifold of dimension n, and let (L, h) be a holomorphic line bundle with a singular hermitian metric of semi-positive curvature. What we actually need is the "adjoint version" of the Ohsawa-Takegoshi theorem, which is about extending sections of the adjoint bundle $\omega_X \otimes L$, or equivalently, holomorphic n-forms with coefficients in L. Before we can state the theorem, we first have to introduce some notation.

Given $\beta \in H^0(X, \omega_X \otimes L)$, we define a nonnegative measurable (n,n)-form $|\beta|_h^2$ as follows: view $\beta \wedge \overline{\beta}$ as a smooth (n,n)-form with coefficients in $L \otimes \overline{L}$, compose with

[2]We use the name "Ohsawa-Takegoshi theorem" for convenience only; in reality, there is a large collection of different L^2-extension theorems in complex analysis, of which Theorem 14.4 below is an important but nevertheless special case. For more on this topic, see for example [**?**].

the singular hermitian metric h, and then multiply by the factor $c_n = 2^{-n}(-1)^{n^2/2}$. Locally, we can write $\beta = f s_0 \otimes dz_1 \wedge \cdots \wedge dz_n$, and then

$$|\beta|_h^2 = |f|^2 e^{-\varphi} (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n), \tag{14.1}$$

where $z_1 = x_1 + y_1\sqrt{-1}, \dots, z_n = x_n + y_n\sqrt{-1}$ are local holomorphic coordinates on U. Using this notation, we have

$$H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big) \\ = \big\{ \beta \in H^0(X, \omega_X \otimes L) \bigm| |\beta|_h^2 \text{ is locally integrable} \big\}.$$

We also define the L^2-norm of the element $\beta \in H^0(X, \omega_X \otimes L)$ to be

$$\|\beta\|_h^2 = \int_X |\beta|_h^2 \in [0, +\infty]. \tag{14.2}$$

When X is compact, $\beta \in H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big)$ is equivalent to having $\|\beta\|_h^2 < +\infty$; in general, finiteness of the L^2-norm is a much stronger requirement.

Now suppose that $f \colon X \to B$ is a holomorphic mapping to the open unit ball $B \subseteq \mathbb{C}^r$. We assume that f is projective and that $0 \in B$ is a regular value of f; the central fiber $X_0 = f^{-1}(0)$ is therefore a projective complex manifold of dimension $n - r = \dim X - \dim B$. We denote by (L_0, h_0) the restriction of (L, h) to X_0. As long as h_0 is not identically equal to $+\infty$, it defines a singular hermitian metric with semi-positive curvature on L_0, and we have the space

$$H^0\big(X_0, \omega_{X_0} \otimes L_0 \otimes \mathcal{I}(h_0)\big) \tag{14.3}$$

of holomorphic $(n-r)$-forms with coefficients in L_0 that are square-integrable with respect to h_0; as before, the defining condition is that the integral

$$\|\alpha\|_{h_0}^2 = \int_{X_0} |\alpha|_{h_0}^2$$

should be finite; note that the definition of $|\alpha|_{h_0}^2$ involves the constant c_{n-r}.

The Ohsawa-Takegoshi theorem says that every section of $\omega_{X_0} \otimes L_0 \otimes \mathcal{I}(h_0)$ can be extended to a section of $\omega_X \otimes L \otimes \mathcal{I}(h)$ with finite L^2-norm – and, crucially, it provides a universal upper bound on the L^2-norm of the extension. (If $h_0 \equiv +\infty$, then the space in (14.3) is trivial and the extension problem is not interesting.) Here $\beta \in H^0(X, \omega_X \otimes L)$ is an extension of $\alpha \in H^0(X_0, \omega_{X_0} \otimes L_0)$ if

$$\beta|_{X_0} = \alpha \wedge df = \alpha \wedge (df_1 \wedge \cdots \wedge df_r),$$

where $f = (f_1, \dots, f_r)$. That said, the precise statement of the Ohsawa-Takegoshi extension theorem is the following.

Theorem 14.4. *Let $f \colon X \to B$ be a projective morphism such that $0 \in B$ is a regular value. Let (L, h) be a holomorphic line bundle with a singular hermitian metric of semi-positive curvature. Denote by (L_0, h_0) the restriction to the central fiber $X_0 = f^{-1}(0)$, and suppose that $h_0 \not\equiv +\infty$. Then for every $\alpha \in H^0\big(X_0, \omega_{X_0} \otimes L_0 \otimes \mathcal{I}(h_0)\big)$, there exists at least one $\beta \in H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big)$ with*

$$\beta|_{X_0} = \alpha \wedge df \quad \text{and} \quad \|\beta\|_h^2 \leq \mu(B) \cdot \|\alpha\|_{h_0}^2.$$

The special thing about this form of the extension theorem is that the constant $\mu(B) = \pi^r/r!$ in the estimate is the volume of the unit ball $B \subseteq \mathbb{C}^r$; the example of a product $X = B \times X_0$ shows that this is optimal. Earlier proofs of the Ohsawa-Takegoshi theorem, for example by Siu or Păun [Siu02, Pău07], only gave a weaker estimate, in which $\mu(B)$ had to be replaced by a certain constant $C_0 \leq 200$. The proof of the sharp estimate is due to Błocki and Guan-Zhou [Bło13, GZ15]. There is also a (weaker) version of the Ohsawa-Takegoshi theorem for the case where the fibers are compact Kähler manifolds, proved by Cao [Cao14].

Proof of Theorem 14.4. In [GZ15, §3.12], the result is stated only for "projective families", meaning in the case where $f\colon X \to B$ is smooth and everywhere submersive, but the same proof works as long as $0 \in B$ is a regular value. Guan and Zhou have $(2\pi)^r/r!$ as the constant, but the extra factor of 2^r goes away because our definition of $\|\beta\|_h^2$ and $\|\alpha\|_h^2$ involves dividing by 2^n and 2^{n-r}, respectively.

For the reader who wants to look up the result in [GZ15], we briefly explain how to deduce Theorem 14.4 from Guan and Zhou's main theorem. Choose an embedding $X \hookrightarrow B \times \mathbb{P}^N$, and let $H \subseteq X$ be the preimage of a sufficiently general hyperplane in $\mathbb{P}^N$. Then $X \setminus H$ is a Stein manifold and $X_0 \setminus X_0 \cap H$ a closed submanifold. We can now apply [GZ15, Theorem 2.2] to the pair (X, X_0), taking $A = 0$, $c_A(t) \equiv 1$, and $\Psi = r \log|f|^2 = r \log\bigl(|f_1|^2 + \cdots + |f_r|^2\bigr)$. □

Note. Observe that if we write the inequality in Theorem 14.4 in the form

$$\frac{1}{\mu(B)} \|\beta\|_h^2 \leq \|\alpha\|_{h_0}^2,$$

then it looks like a mean-value inequality; this fact will play a crucial role later or, when we construct singular hermitian metrics on pushforwards of adjoint bundles.

15. Coherent sheaves and Fréchet spaces. In this section, we briefly review some fundamental results about section spaces of coherent sheaves on complex manifolds. Recall that a *Fréchet space* is a Hausdorff topological vector space, whose topology is induced by a countable family of semi-norms, and which is complete with respect to this family of semi-norms. Most of the familiar theorems about Banach spaces, such as the open mapping theorem or the closed graph theorem, remain true for Fréchet spaces.

Example 15.1. On a complex manifold X, the vector space $H^0(X, \mathscr{O}_X)$ of all holomorphic functions on X is a Fréchet space, under the topology of uniform convergence on compact subsets. More precisely, each compact subset $K \subseteq X$ gives rise to a semi-norm

$$\|f\|_K = \sup_{x \in K} |f(x)|$$

on the space $H^0(X, \mathscr{O}_X)$; to get a countable family, write X as a countable union of compact subsets. The same construction works for any open subset $U \subseteq X$, and when $U \subseteq V$, the restriction mapping $H^0(V, \mathscr{O}_X) \to H^0(U, \mathscr{O}_X)$ is continuous.

In fact, the section spaces of *all* coherent sheaves on a given complex manifold can be made into Fréchet spaces in a consistent way; the construction is explained for example in [GR09, Ch. VIII, §A]. Let $\mathscr{F}$ be a coherent sheaf on a complex manifold X. Then for every open subset $U \subseteq X$, the space of sections $H^0(U, \mathscr{F})$ has the structure of a Fréchet space, in such a way that if $U \subseteq V$, the restriction mapping $H^0(V, \mathscr{F}) \to H^0(U, \mathscr{F})$ is continuous. Moreover, if $\phi\colon \mathscr{F} \to \mathscr{G}$ is any morphism between two coherent sheaves, then the induced mappings $\phi_U \colon H^0(U, \mathscr{F}) \to H^0(U, \mathscr{G})$ are all continuous. The Fréchet space topology has several other good properties, such as the following [GR09, Proposition VIII.A.2].

Proposition 15.2. *If $\mathscr{F} \subseteq \mathscr{G}$, then $H^0(U, \mathscr{F})$ is a closed subspace of $H^0(U, \mathscr{G})$.*

Let $f\colon X \to Y$ be a proper holomorphic mapping between complex manifolds, and let $\mathscr{F}$ be a coherent sheaf on X. By Grauert's coherence theorem, the push-forward sheaf $f_*\mathscr{F}$ is a coherent sheaf on Y. The vector space

$$H^0(Y, f_*\mathscr{F}) = H^0(X, \mathscr{F})$$

therefore has two (a priori different) Fréchet space topologies, one coming from Y, the other from X.

Proposition 15.3. *In the situation just described, the two Fréchet space topologies on $H^0(Y, f_*\mathscr{F}) = H^0(X, \mathscr{F})$ are equal.*

Proof. Since the problem is local, we may replace Y by a Stein open subset and assume that we have a surjective morphism

$$\mathscr{O}_Y^{\oplus m} \to f_*\mathscr{F}.$$

The induced mapping $H^0(Y, \mathscr{O}_Y)^{\oplus m} \to H^0(Y, f_*\mathscr{F})$ is continuous and surjective; by the open mapping theorem, the topology on $H^0(Y, f_*\mathscr{F})$ must be the quotient topology. We also get a morphism $\mathscr{O}_X^{\oplus m} \to \mathscr{F}$, and therefore a factorization

$$H^0(Y, \mathscr{O}_Y)^{\oplus m} \to H^0(X, \mathscr{O}_X)^{\oplus m} \to H^0(X, \mathscr{F}).$$

Both mappings are continuous: the first because, f being proper, uniform convergence on compact subsets of Y implies uniform convergence on compact subsets of X; the second because $\mathscr{O}_X^{\oplus m} \to \mathscr{F}$ is a morphism. It follows that the identity mapping

$$H^0(Y, f_*\mathscr{F}) \to H^0(X, \mathscr{F})$$

is continuous; by the open mapping theorem, it must be a homeomorphism. □

16. Singular hermitian inner products. Before we can talk about singular hermitian metrics on vector bundles, we first have to be clear about what we mean by a "singular" hermitian inner product on a vector space. The purpose of this section is to define this notion with some care. Throughout, we let V be a finite-dimensional complex vector space. There are two ways in which a hermitian inner product can be singular: there may be vectors whose length is $+\infty$, and others whose length is 0. The best way to formalize this is to work not with the inner product itself, but with the associated length function [BP08, §3].

Definition 16.1. A *singular hermitian inner product* on a finite-dimensional complex vector space V is a function $|-|_h\colon V \to [0, +\infty]$ with the following properties:

(1) $|\lambda v|_h = |\lambda| \cdot |v|_h$ for every $\lambda \in \mathbb{C} \setminus \{0\}$ and every $v \in V$, and $|0|_h = 0$.
(2) $|v + w|_h \le |v|_h + |w|_h$ for every $v, w \in V$.
(3) $|v + w|_h^2 + |v - w|_h^2 = 2|v|_h^2 + 2|w|_h^2$ for every $v, w \in V$.

Our convention is that an inequality is satisfied if both sides are equal to $+\infty$. It is easy to deduce from the axioms that both

$$V_0 = \{ v \in V \mid |v|_h = 0 \} \quad \text{and} \quad V_{\mathit{fin}} = \{ v \in V \mid |v|_h < +\infty \}$$

are linear subspaces of V. We say that h is *positive definite* if $V_0 = 0$; we say that h is *finite* if $V_{\mathit{fin}} = V$. Clearly, $|-|_h$ is a semi-norm on V_{fin}; it is a norm if and only if

$V_0 = 0$. The third axiom is the parallelogram law for this semi-norm. The formula

$$\langle v, w \rangle_h = \frac{1}{4} \sum_{k=0}^{3} (\sqrt{-1})^k \cdot \left| v + (\sqrt{-1})^k w \right|_h$$

therefore defines a semi-definite hermitian inner product on the subspace V_{fin}; it is positive definite if and only if $V_0 = 0$. We use the same notation for the induced hermitian inner product on the quotient space V_{fin}/V_0.

Given a singular hermitian inner product h on V, we obtain a singular hermitian inner product h^* on the dual space $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by setting

$$|f|_{h^*} = \sup \left\{ \frac{|f(v)|}{|v|_h} \;\middle|\; v \in V \text{ with } |v|_h \neq 0 \right\}$$

for any linear functional $f \in V^*$, with the understanding that a fraction with denominator $+\infty$ is equal to 0. (If $V_0 = V$, then we define $|f|_{h^*} = 0$ for $f = 0$, and $|f|_{h^*} = +\infty$ otherwise.) It is easy to see that $|f|_{h^*} = 0$ if and only if f annihilates the subspace V_{fin}, and that $|f|_{h^*} < +\infty$ if and only if f annihilates the subspace V_0. One then checks that h^* is again a singular hermitian inner product on V^*, and that the resulting hermitian inner product $\langle -, - \rangle_{h^*}$ on

$$\frac{\left\{ f \in V^* \;\middle|\; |f|_{h^*} < +\infty \right\}}{\left\{ f \in V^* \;\middle|\; |f|_{h^*} = 0 \right\}} \simeq \operatorname{Hom}_{\mathbb{C}}\bigl(V_{fin}/V_0, \mathbb{C}\bigr)$$

agrees with the one naturally induced by $\langle -, - \rangle_h$. Here is another way to think about h^*. From a nonzero linear functional $f \colon V \to \mathbb{C}$, we get an induced singular hermitian inner product on $\mathbb{C}$ by setting

$$|\lambda|_{h,f} = \inf \left\{ |v|_h \;\middle|\; v \in V \text{ satisfies } f(v) = \lambda \right\}$$

If $\lambda \neq 0$, this quantity is $+\infty$ unless the restriction of f to the subspace V_{fin} is nonzero; if $V_0 = V$, then $|\lambda|_{h,f} = 0$ for every $\lambda \in \mathbb{C}$. Taking into account various special cases, the following result is immediate from the definition.

Lemma 16.2. *Let $f \colon V \to \mathbb{C}$ be a nonzero linear functional. Then*

$$|\lambda|_{h,f} = \frac{|\lambda|}{|f|_{h^*}}$$

for every nonzero $\lambda \in \mathbb{C}$.

Let $r = \dim V$. Since the product of 0 and $+\infty$ is undefined, we do not get a singular hermitian inner product on

$$\det V = \bigwedge^r V$$

unless $V_0 = 0$ or $V_{fin} = V$. But when h is either positive definite or finite, there is a well-defined singular hermitian inner product $\det h$ on the one-dimensional vector space $\det V$. If $V_{fin} = V$, we declare that

$$|v_1 \wedge \dots \wedge v_r|_{\det h} = \det \begin{pmatrix} \langle v_1, v_1 \rangle_h & \cdots & \langle v_1, v_r \rangle_h \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle_h & \cdots & \langle v_r, v_r \rangle_h \end{pmatrix}.$$

If $V_{fin} \neq V$ and $V_0 = 0$, we let $|-|_{\det h}$ equal $+\infty$ on all nonzero elements of $\det V$.

17. Singular hermitian metrics on vector bundles. The purpose of this section is to extend the concept of singular hermitian metrics from holomorphic line bundles to holomorphic vector bundles of arbitrary rank. Let X be a complex manifold, and let E be a holomorphic vector bundle on X of some rank $r \geq 1$.

Definition 17.1. A *singular hermitian metric* on E is a function h that associates to every point $x \in X$ a singular hermitian inner product $|-|_{h,x} \colon E_x \to [0, +\infty]$ on the complex vector space E_x, subject to the following two conditions:

(1) h is finite and positive definite almost everywhere, meaning that for all x outside a set of measure zero, $|-|_{h,x}$ is a hermitian inner product on E_x.
(2) h is measurable, meaning that the function

$$|s|_h \colon U \to [0, +\infty], \quad x \mapsto \big|s(x)\big|_{h,x},$$

is measurable whenever $U \subseteq X$ is open and $s \in H^0(U, E)$.

In the case $r = 1$, this specializes to the definition of singular hermitian metrics on holomorphic line bundles. The requirement that h be measurable is extremely weak: the singular hermitian metrics that we will actually encounter below are at least semi-continuous. The advantage of the above definition is that it behaves well under duality. By applying the general construction from the previous section, we obtain on each fiber

$$E_x^* = \operatorname{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$$

of the dual bundle E^* a singular hermitian inner product $|-|_{h^*,x}$. The following result shows that these form a singular hermitian metric on E^*.

Proposition 17.2. *A singular hermitian metric h on a holomorphic vector bundle E induces a singular hermitian metric h^* on the dual bundle E^*.*

Proof. If $|-|_{h,x}$ is finite and positive definite, then $|-|_{h^*,x}$ is also finite and positive definite, and so the first condition in the definition is clearly satisfied. The second condition is of a local nature, and so we may assume without loss of generality that E is the trivial bundle of rank r. Denote by $s_1, \dots, s_r \in H^0(X, E)$ the global sections corresponding to a choice of trivialization. The expression

$$H_{i,j}(x) = \langle s_i(x), s_j(x) \rangle_{h,x}$$

is well-defined outside a set of measure zero, and the resulting function is measurable. Denote by $H \in \operatorname{Mat}_{r \times r}(\mathbb{C})$ the $r \times r$-matrix with entries $H_{i,j}$. Then h^* is represented by the transpose of the matrix H^{-1}, in the natural trivialization of E^*; the usual formula for the inverse of a matrix shows that all entries of this matrix are again measurable functions. □

Note. In more sheaf-theoretic terms, a singular hermitian metric on a holomorphic vector bundle E is a morphism of sheaves of sets

$$|-|_h \colon E \to \mathscr{M}_X$$

from E to the sheaf of measurable functions on X with values in $[0, +\infty]$. The following conditions need to be satisfied:

(1) One has $|fs|_h = |f| \cdot |s|_h$ for every $s \in H^0(U, E)$ and every $f \in H^0(U, \mathscr{O})$.
(2) If $s \in H^0(U, E)$ and $|s|_h = 0$ almost everywhere, then $s = 0$.
(3) For almost every point $x \in X$, the function $|-|_{h,x} \colon E_x \to [0, +\infty]$ is a singular hermitian inner product (in the sense of Definition 17.1).

Again, we use the convention that $|f| \cdot |s|_h = 0$ at points where f is zero.

18. Semi-positive curvature. Let h be a singular hermitian metric on a holomorphic vector bundle E, and denote by h^* the induced singular hermitian metric on the dual bundle E^*. Suppose for a moment that h is smooth, and denote by Θ_h the curvature tensor of the Chern connection; it is a $(1,1)$-form with coefficients in the bundle $\operatorname{End}(E)$. One says that (E,h) has *semi-positive curvature in the sense of Griffiths* if, for every choice of holomorphic tangent vector $\xi \in T_x X$, the matrix $\Theta_h(\xi, \overline{\xi})$ is positive semi-definite [Dem12, VII.6]. This is known to be equivalent to the condition that the function $\log|f|_{h^*}$ is plurisubharmonic for every local section $f \in H^0(U, E^*)$. In the singular case, we use this condition as the definition.

Definition 18.1. We say that the pair (E,h) has *semi-positive curvature* if the function $\log|f|_{h^*}$ is plurisubharmonic for every $f \in H^0(U, E^*)$.

The point of this definition is that it allows us to talk about the curvature of a singular hermitian metric without mentioning the curvature tensor: unlike in the case of line bundles, the curvature tensor of h does not in general make sense even as a distribution [Rau15, Theorem 1.3]. The following lemma gives an equivalent formulation of the definition.

Lemma 18.2. *Let h be a singular hermitian metric on E. Then (E,h) has semi-positive curvature if, and only if, for every open subset $U \subseteq X$ and every nonzero morphism $E|_U \to L$ to a line bundle, the induced singular hermitian metric on L has semi-positive curvature.*

Proof. The construction of the induced singular hermitian metric on L works as in Lemma 16.2. At each point $x \in U$, the linear mapping $E_x \to L_x$ between fibers induces a singular hermitian inner product on the one-dimensional complex vector space L_x: the length of a vector $\lambda \in L_x$ is the infimum of $|e|_{h,x}$ over all $e \in E_x$ that map to λ. (If $E_x \to L_x$ is zero, then the infimum equals $+\infty$ whenever $\lambda \neq 0$.)

Let us compute the curvature of the induced metric. After replacing X by an open neighborhood of a given point in U, we may assume that L is trivial; our morphism $E \to \mathscr{O}_X$ is then given by a linear functional $f \in H^0(X, E^*)$. Let $e^{-\varphi}$ be the weight function of the induced metric. The formula in Lemma 16.2 says that

$$e^{-\varphi(x)} = \frac{1}{|f(x)|^2_{h^*,x}}$$

for every $x \in X$. Taking logarithms, we get $\varphi = 2\log|f|_{h^*}$, which is plurisubharmonic because the pair (E,h) has semi-positive curvature. ∎

Suppose that (E,h) has semi-positive curvature. Since plurisubharmonic functions are locally bounded from above, the singular hermitian inner product $|-|_{h^*,x}$ on E_x^* must be finite for every $x \in X$; dually, every $|-|_{h,x}$ must be positive definite. The determinant line bundle $\det E$ therefore has a well-defined singular hermitian metric that we denote by the symbol $\det h$. We will prove later (in Proposition 25.1) that the pair $(\det E, \det h)$ again has semi-positive curvature.

When (E,h) has semi-positive curvature, the pointwise length of any holomorphic section of E^* is an upper semi-continuous function. Likewise, the pointwise length of any holomorphic section of E is a lower semi-continuous function.

Lemma 18.3. *If (E,h) has semi-positive curvature, then for any $s \in H^0(X,E)$, the function $|s|_h \colon X \to [0,+\infty]$ is lower semi-continuous.*

Proof. Since the question is local, we may assume without loss of generality that X is the open unit ball in $\mathbb{C}^n$, and E the trivial bundle of rank $r \geq 1$. We have

$$|s|_h \geq \frac{|f(s)|}{|f|_{h^*}}$$

for every $f \in H^0(X,E^*)$, and it is easy to see that $|s|_h$ is the pointwise supremum of the collection of functions on the right-hand side. Because $\log|f|_{h^*}$ is upper semi-continuous, each

$$\frac{|f(s)|}{|f|_{h^*}} = |f(s)| \cdot e^{-\log|f|_{h^*}}$$

is a lower semi-continuous function from X to $[0,+\infty]$; their pointwise supremum is therefore also lower semi-continuous. □

Example 18.4. The following example, due to Raufi [Rau15, Theorem 1.3], shows that the function $|s|_h$ can indeed have jumps. Let E be the trivial bundle of rank 2 on $\mathbb{C}$. We first define a singular hermitian metric h^* on the dual bundle E^*: at each point $z \in \mathbb{C}$, it is represented by the matrix

$$\begin{pmatrix} 1+|z|^2 & z \\ \bar{z} & |z|^2 \end{pmatrix}.$$

From this, one computes that the singular hermitian metric h on E is given by

$$\frac{1}{|z|^4}\begin{pmatrix} |z|^2 & -z \\ -\bar{z} & 1+|z|^2 \end{pmatrix}$$

as long as $z \neq 0$. Contrary to what this formula might suggest, one has

$$\big|(1,0)\big|_{h,0} = 1;$$

the length of the vector $(1,0)$ is thus $|z|^{-2}$ for $z \neq 0$, but 1 for $z = 0$.

19. Singular hermitian metrics on torsion-free sheaves. Let X be a complex manifold, and let $\mathscr{F}$ be a torsion-free coherent sheaf on X. Let $X(\mathscr{F}) \subseteq X$ denote the maximal open subset where $\mathscr{F}$ is locally free; then $X \setminus X(\mathscr{F})$ is a closed analytic subset of codimension ≥ 2. If $\mathscr{F} \neq 0$, then the restriction of $\mathscr{F}$ to the open subset $X(\mathscr{F})$ is a holomorphic vector bundle E of some rank $r \geq 1$.

Definition 19.1. A *singular hermitian metric* on $\mathscr{F}$ is a singular hermitian metric h on the holomorphic vector bundle E. We say that such a metric has *semi-positive curvature* if the pair (E,h) has semi-positive curvature.

Suppose that $\mathscr{F}$ has a singular hermitian metric with semi-positive curvature. Since $X \setminus X(\mathscr{F})$ has codimension ≥ 2, every holomorphic section of the dual bundle E^* extends to a holomorphic section of the reflexive coherent sheaf

$$\mathscr{F}^* = \mathcal{H}om(\mathscr{F}, \mathscr{O}_X),$$

and every plurisubharmonic function on $X(\mathscr{F})$ extends to a plurisubharmonic function on X (see Lemma 12.4). For every open subset $U \subseteq X$ and every holomorphic section $f \in H^0(U,\mathscr{F}^*)$, we thus obtain a well-defined plurisubharmonic function

$$\log|f|_{h^*} \colon U \to [-\infty,+\infty).$$

Note that the function $|f|_{h^*}$ is upper semi-continuous.

What about holomorphic sections of the sheaf $\mathscr{F}$ itself? For any $s \in H^0(U, \mathscr{F})$, the function $|s|_h$ is lower semi-continuous on $U \cap X(\mathscr{F})$. In a suitable neighborhood of every point in U, we can imitate the proof of Lemma 18.3 and take the pointwise supremum of the functions

$$|f(s)| \cdot e^{-\log|f|_{h^*}},$$

where f runs over all sections of $\mathscr{F}^*$. Since the pointwise supremum of a family of lower semi-continuous functions is again lower semi-continuous, we obtain in this manner a distinguished extension

$$|s|_h \colon U \to [0, +\infty]$$

to a lower semi-continuous function on U.

Definition 19.2. We say that a singular hermitian metric on $\mathscr{F}$ is *continuous* if, for every open subset $U \subseteq X$ and every holomorphic section $s \in H^0(U, \mathscr{F})$, the function $|s|_h \colon U \to [0, +\infty]$ is continuous.

Proposition 19.3. *Let $\phi \colon \mathscr{F} \to \mathscr{G}$ be a morphism between two torsion-free coherent sheaves that is generically an isomorphism. If $\mathscr{F}$ has a singular hermitian metric with semi-positive curvature, then so does $\mathscr{G}$.*

Proof. Let h denote the singular hermitian metric on $\mathscr{F}$. On the open subset of $X(\mathscr{F}) \cap X(\mathscr{G})$ where ϕ is an isomorphism, $\mathscr{G}$ clearly acquires a singular hermitian metric that we also denote by h for simplicity. Because the dual morphism $\phi^* \colon \mathscr{G}^* \to \mathscr{F}^*$ is injective, the function $\log|f|_{h^*}$ is plurisubharmonic for every $f \in H^0(U, \mathscr{G}^*) \subseteq H^0(U, \mathscr{F}^*)$. Consequently, h extends to a singular hermitian metric with semi-positive curvature on all of $X(\mathscr{G})$. □

Example 19.4. If $\mathscr{F}$ has a singular hermitian metric of semi-positive curvature, then the same is true for the reflexive hull $\mathscr{F}^{**}$.

20. The minimal extension property. The Ohsawa-Takegoshi theorem leads us to consider the following "minimal extension property" for singular hermitian metrics. To keep the statement simple, let us assume that X is a connected complex manifold of dimension n, and denote by $B \subseteq \mathbb{C}^n$ the open unit ball.

Definition 20.1. We say that a singular hermitian metric on $\mathscr{F}$ has the *minimal extension property* if there exists a nowhere dense closed analytic subset $Z \subseteq X$ with the following two properties:

(1) $\mathscr{F}$ is locally free on $X \setminus Z$, or equivalently, $X \setminus Z \subseteq X(\mathscr{F})$.
(2) For every embedding $\iota \colon B \hookrightarrow X$ with $x = \iota(0) \in X \setminus Z$, and every $v \in E_x$ with $|v|_{h,x} = 1$, there is a holomorphic section $s \in H^0\big(B, \iota^* \mathscr{F}\big)$ such that

$$s(0) = v \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |s|_h^2 \, d\mu \leq 1;$$

here (E, h) denotes the restriction to the open subset $X(\mathscr{F})$.

The point of the minimal extension property is the ability to extend sections over the "bad" locus Z, with good control on the norm of the extension. We will see later that pushforwards of adjoint line bundles always have this property, as a consequence of the Ohsawa-Takegoshi theorem.

Example 20.2. The minimal extension property rules out certain undesirable examples like the following. Let $Z \subseteq X$ be a closed analytic subset of codimension ≥ 2, and let $\mathcal{I}_Z \subseteq \mathscr{O}_X$ denote the ideal sheaf of Z. Then $\mathcal{I}_Z$ is trivial on $X \setminus Z$, and the constant hermitian metric on this trivial bundle is a singular hermitian metric with semi-positive curvature on $\mathcal{I}_Z$. But this metric does not have the minimal extension property, because a holomorphic function $f \colon B \to \mathbb{C}$ with $f(0) = 1$ and

$$\frac{1}{\mu(B)} \int_B |f|^2 d\mu \leq 1$$

must be constant.

21. Pushforwards of adjoint line bundles. Let X be a complex manifold of dimension n, and let (L, h) be a holomorphic line bundle with a singular hermitian metric of semi-positive curvature. If X is compact, the space $H^0(X, \omega_X \otimes L)$ is finite-dimensional, and the formula

$$\|\beta\|_h^2 = \int_X |\beta|_h^2$$

endows it with a positive definite singular hermitian inner product that is finite on the subspace $H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big)$. We are now going to analyze how this construction behaves in families.

Suppose then that $f \colon X \to Y$ is a projective surjective morphism between two connected complex manifolds, with $\dim X = n$ and $\dim Y = r$; the general fiber of f is a projective complex manifold of dimension $n - r$, but there may be singular fibers. Let (L, h) be a holomorphic line bundle with a singular hermitian metric of semi-positive curvature on X. The following important theorem was essentially proved by Păun and Takayama [PT14, Theorem 3.3.5], building on earlier results for smooth morphisms by Berndtsson and Păun [Ber09, BP08].

Theorem 21.1. *Let $f \colon X \to Y$ be a projective surjective morphism between two connected complex manifolds. If (L, h) is a holomorphic line bundle with a singular hermitian metric of semi-positive curvature on X, then the pushforward sheaf*

$$\mathscr{F} = f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big)$$

has a canonical singular hermitian metric H. This metric has semi-positive curvature and satisfies the minimal extension property.

The metric in the theorem is uniquely characterized by a simple property that we now describe. Recall from (14.2) and (14.1) that any $\beta \in H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big)$ gives rise to a locally integrable (n, n)-form $|\beta|_h^2$. Any such form can be integrated against compactly supported smooth functions, and therefore defines a current of bidegree (n, n) on X. If we use brackets to denote the evaluation pairing between (n, n)-currents and compactly supported smooth functions, then

$$\big\langle |\beta|_h^2, \phi \big\rangle = \int_X \phi \cdot |\beta|_h^2.$$

By the same token, any section $\beta \in H^0(Y, \omega_Y \otimes \mathscr{F})$ defines a current of bidegree (r, r) on Y that we denote by the symbol $|\beta|_H^2$. Now suppose that

$$\beta \in H^0(U, \omega_Y \otimes \mathscr{F}) \simeq H^0\big(f^{-1}(U), \omega_X \otimes L \otimes \mathcal{I}(h)\big).$$

The singular hermitian metric H is uniquely characterized by the condition that

$$\big\langle |\beta|_H^2, \phi \big\rangle = \big\langle |\beta|_h^2, f^* \phi \big\rangle$$

for every compactly supported smooth function $\phi \in A_c(U)$. Said differently, $|\beta|_H^2$ is the pushforward of the current $|\beta|_h^2$ under the proper mapping f.

Corollary 21.2. *In the situation of Theorem 21.1, suppose that the inclusion*

$$f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big) \hookrightarrow f_*(\omega_{X/Y} \otimes L)$$

is generically an isomorphism. Then $f_*(\omega_{X/Y} \otimes L)$ also has a singular hermitian metric with semi-positive curvature and the minimal extension property.

Proof. The existence of the metric follows from Proposition 19.3. The minimal extension property continues to hold because every section of $f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big)$ is of course also a section of $f_*(\omega_{X/Y} \otimes L)$. □

Example 21.3. If we apply Theorem 21.1 to the identity morphism $\mathrm{id}\colon X \to X$, we only get a singular hermitian metric on $L \otimes \mathcal{I}(h)$. To recover the singular hermitian metric on L that we started from, we can use Corollary 21.2.

The proof of Theorem 21.1 gives the following additional information about the singular hermitian metric on $\mathscr{F} = f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big)$ (see the end of §24).

Corollary 21.4. *In the situation of Theorem 21.1, suppose that $f\colon X \to Y$ is submersive and that the singular hermitian metric h on the line bundle L is continuous. Then the singular hermitian metric H on $\mathscr{F}$ is also continuous.*

The following three sections explain the proof of Theorem 21.1. In a nutshell, it is an application of the Ohsawa-Takegoshi extension theorem. We present the argument in three parts that rely on successively stronger versions of the extension theorem: first the ability to extend sections from a fiber; then the fact that there is a universal bound on the norm of the extension; and finally the optimal bound in Theorem 14.4.

22. Proof of the pushforward theorem, Part I. Our first goal is to define the singular hermitian metric on

$$\mathscr{F} = f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big),$$

and to establish a few basic facts about it. In this part of the proof, we only use the weakest version of the Ohsawa-Takegoshi extension theorem, namely the ability to extend sections from a fiber.

The idea is to construct the metric first over a Zariski-open subset $Y \setminus Z$ where everything is nice, and then to extend it over the bad locus Z. To begin with, choose a nowhere dense closed analytic subset $Z \subseteq Y$ with the following three properties:

(1) The morphism f is submersive over $Y \setminus Z$.
(2) Both $\mathscr{F}$ and the quotient sheaf $f_*(\omega_{X/Y} \otimes L)/\mathscr{F}$ are locally free on $Y \setminus Z$.
(3) On $Y \setminus Z$, the locally free sheaf $f_*(\omega_{X/Y} \otimes L)$ has the base change property.

By the base change theorem, the third condition will hold as long as the coherent sheaves $R^i f_*(\omega_{X/Y} \otimes L)$ are locally free on $Y \setminus Z$. The restriction of $\mathscr{F}$ to the open subset $Y \setminus Z$ is a holomorphic vector bundle E of some rank $r \geq 1$. The second and third condition together guarantee that

$$E_y = \mathscr{F}|_y \subseteq f_*(\omega_{X/Y} \otimes L)|_y = H^0\big(X_y, \omega_{X_y} \otimes L_y\big)$$

whenever $y \in Y \setminus Z$. As before, (L_y, h_y) denotes the restriction of (L, h) to the fiber $X_y = f^{-1}(y)$; it may happen that $h_y \equiv +\infty$. The Ohsawa-Takegoshi theorem gives us the following additional information about E_y.

Lemma 22.1. *For any $y \in Y \setminus Z$, we have inclusions*

$$H^0\big(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h_y)\big) \subseteq E_y \subseteq H^0\big(X_y, \omega_{X_y} \otimes L_y\big).$$

Proof. If $h_y \equiv +\infty$, then the subspace of the left is trivial, which means that the asserted inclusion is true by default. If h_y is not identically equal to $+\infty$, then given $\alpha \in H^0\big(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h_y)\big)$ and a suitable open neighborhood U of the point y, there is by Theorem 14.4 some

$$\beta \in H^0\big(U, \omega_Y \otimes \mathscr{F}\big) \simeq H^0\big(f^{-1}(U), \omega_X \otimes L \otimes \mathcal{I}(h)\big)$$

such that $\beta|_{X_y} = \alpha \wedge df$. Since ω_Y is trivial on U, this gives us a section of $\mathscr{F}$ in a neighborhood of the fiber X_y whose restriction to X_y agrees with α. □

Note. We will see in a moment that the two subspaces

$$H^0\big(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h_y)\big) \subseteq E_y$$

are equal for almost every $y \in Y \setminus Z$. But unless $\mathscr{F} = 0$, the two subspaces are different for example at points where h_y is identically equal to $+\infty$.

We can now define on each E_y with $y \in Y \setminus Z$ a singular hermitian inner product in the following manner. Given an element

$$\alpha \in E_y \subseteq H^0\big(X_y, \omega_{X_y} \otimes L_y\big),$$

we can integrate over the compact complex manifold X_y and define

$$|\alpha|^2_{H,y} = \int_{X_y} |\alpha|^2_{h_y} \in [0, +\infty].$$

It is easy to see that $|-|_{H,y}$ is a positive definite singular hermitian inner product. Clearly $|\alpha|_{H,y} < +\infty$ if and only if $\alpha \in H^0\big(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h_y)\big)$; in light of Lemma 22.1, our singular hermitian inner product $|-|_{H,y}$ is therefore finite precisely on the subspace $H^0\big(X_y, \omega_{X_y} \otimes L_y \otimes \mathcal{I}(h_y)\big) \subseteq E_y$.

Let us now analyze how the individual singular hermitian inner products $|-|_{H,y}$ fit together on $Y \setminus Z$. Fix a point $y \in Y \setminus Z$ and an open neighborhood $U \subseteq Y \setminus Z$ biholomorphic to the open unit ball $B \subseteq \mathbb{C}^r$; after pulling everything back to U, we may assume without loss of generality that $Y = B$ and $Z = \emptyset$ and $y = 0$. Denote by $t_1, \dots, t_r$ the standard coordinate system on B; then the canonical bundle ω_B is trivialized by the global section $dt_1 \wedge \dots \wedge dt_r$, and the volume form on B is

$$d\mu = c_r (dt_1 \wedge \dots \wedge dt_r) \wedge (d\bar{t}_1 \wedge \dots \wedge d\bar{t}_r).$$

Fix a holomorphic section $s \in H^0(B, E)$, and denote by

$$\beta = s \wedge (dt_1 \wedge \dots \wedge dt_r) \in H^0\big(B, \omega_B \otimes E\big) \simeq H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big)$$

the corresponding holomorphic n-form on X with coefficients in L. Since $f \colon X \to B$ is smooth, Ehresmann's fibration theorem shows that X is diffeomorphic to the product $B \times X_0$. After choosing a Kähler metric ω_0 on X_0, we can write

$$|\beta|^2_h = F \cdot d\mu \wedge \frac{\omega_0^{n-r}}{(n-r)!}, \tag{22.2}$$

where $F\colon B\times X_0 \to [0,+\infty]$ is lower semi-continuous and locally integrable; the reason is of course that the local weight functions for (L,h) are upper semi-continuous functions. At every point $y \in B$, we then have by construction

$$|s(y)|^2_{H,y} = \int_{X_0} F(y,-)\,\frac{\omega_0^{n-r}}{(n-r)!}. \tag{22.3}$$

By Fubini's theorem, the function $y \mapsto |s(y)|_{H,y}$ is measurable; moreover, since F is locally integrable and X_0 is compact, we must have $|s(y)|_{H,y} < +\infty$ for almost every $y \in B$. Being coherent, E is generated over B by a finite number of global sections; the singular hermitian inner product $|-|_{H,y}$ is therefore finite and positive-definite for almost every $y \in B$, hence for almost every $y \in Y \setminus Z$. In particular, the first inclusion in Lemma 22.1 is an equality for almost every $y \in Y \setminus Z$. We may summarize the conclusion as follows.

Proposition 22.4. *On $Y \setminus Z$, the singular hermitian inner products $|-|_{H,y}$ determine a singular hermitian metric on the holomorphic vector bundle E.*

While we are not yet ready to show that (E,H) has semi-positive curvature, we can already show that the function $|s|_H$ is always lower semi-continuous.

Proposition 22.5. *For any open subset $U \subseteq Y \setminus Z$ and any section $s \in H^0(U,E)$, the function $|s|_H\colon U \to [0,+\infty]$ is lower semi-continuous.*

Proof. As before, we may assume that $U = B$ is the open unit ball in $\mathbb{C}^m$; it is clearly sufficient to show that $|s|_H$ is lower semi-continuous at the origin. In other words, we need to argue that

$$|s(0)|_{H,0} \leq \liminf_{k\to+\infty} |s(y_k)|_{H,y_k}$$

holds for every sequence $y_0, y_1, y_2, \ldots \in B$ that converges to the origin. As in (22.2), the given section $s \in H^0(B,E)$ determines a lower semi-continuous function $F\colon B \times X_0 \to [0,+\infty]$ such that (22.3) is satisfied. By the lower semi-continuity of F and Fatou's lemma, we obtain

$$\begin{aligned}\int_{X_0} F(0,-)\frac{\omega_0^{n-r}}{(n-r)!} &\leq \int_{X_0} \liminf_{k\to+\infty} F(y_k,-)\frac{\omega_0^{n-r}}{(n-r)!}\\ &\leq \liminf_{k\to+\infty}\int_{X_0} F(y_k,-)\frac{\omega_0^{n-r}}{(n-r)!},\end{aligned}$$

which is the desired inequality up to taking square roots. □

23. Proof of the pushforward theorem, Part II. Having defined (E,H) on the open subset $Y\setminus Z$, our next task is to say something about the induced singular hermitian metric H^* on the dual vector bundle E^*. In particular, we need to prove that the norm of any local section of $\mathscr{F}^*$ is uniformly bounded in the neighborhood of any point in Z, and that its logarithm is an upper semi-continuous function. This part of the argument relies on the existence of a uniform bound in the Ohsawa-Takegoshi theorem, but not on the precise value of the constant. Let us start by reformulating the statement of the Ohsawa-Takegoshi in terms of the pair (E,H).

Lemma 23.1. *For every embedding $\iota\colon B \hookrightarrow Y$ with $y = \iota(0) \in Y \setminus Z$, and for every $\alpha \in E_y$ with $|\alpha|_{H,y} = 1$, there is a holomorphic section $s \in H^0(B, \iota^*\mathscr{F})$ with*

$$s(0) = \alpha \quad \textit{and} \quad \int_B |s|_H^2\, d\mu \leq C_0,$$

where C_0 is the same constant as in the Ohsawa-Takegoshi theorem.

Proof. After pulling everything back to B, we may assume that $Y = B$ and $y = 0$. Since $|\alpha|_{H,0} = 1$, by Theorem 14.4 there exists an element $\beta \in H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big)$ with

$$\beta|_{X_0} = \alpha \wedge df \quad \text{and} \quad \|\beta\|_h^2 = \int_X |\beta|_h^2 \leq C_0.$$

In fact, one can take $C_0 = \mu(B)$, but the exact value of the constant is not important here. Using $dt_1 \wedge \cdots \wedge dt_r$ as a trivialization of the canonical bundle ω_B, we may consider β as a holomorphic section $s \in H^0(B, \mathscr{F})$; the two conditions from above then turn into

$$s(0) = \alpha \quad \text{and} \quad \int_B |s|_H^2 \, d\mu \leq C_0,$$

due to the fact that $d\mu = c_r(dt_1 \wedge \cdots \wedge dt_r) \wedge (d\bar{t}_1 \wedge \cdots \wedge d\bar{t}_r)$. □

Fix an open subset $U \subseteq Y$ and a holomorphic section $g \in H^0(U, \mathscr{F}^*)$; after replacing Y by the open subset U, we may assume without loss of generality that $g \in H^0(Y, \mathscr{F}^*)$. Consider the measurable function

$$\psi = \log|g|_{H^*} \colon Y \setminus Z \to [-\infty, +\infty]. \tag{23.2}$$

Ultimately, our goal is to show that ψ extends to a plurisubharmonic function on all of Y. The following boundedness result is the crucial step in this direction.

Proposition 23.3. *Every point in Y has an open neighborhood $U \subseteq Y$ such that $\psi = \log|g|_{H^*}$ is bounded from above by a constant on $U \setminus U \cap Z$.*

Proof. Choose two sufficiently small open neighborhoods $U \subseteq V \subseteq Y$ of the given point, such that $\overline{V}$ is compact, $\overline{U} \subseteq V$, and for every point $y \in \overline{U}$, there is an embedding $\iota \colon B \hookrightarrow Y$ of the unit ball $B \subseteq \mathbb{C}^r$ with $\iota(0) = y$ and $\iota(B) \subseteq V$. We shall argue that there is a constant $C \geq 0$ such that $\psi \leq C$ on $U \setminus U \cap Z$.

Fix a point $y \in \overline{U} \setminus Z$. If $\psi(y) = -\infty$, there is nothing to prove, so let us suppose from now on that $\psi(y) \neq -\infty$. By definition of the metric on the dual bundle, we can then find a vector $\alpha \in E_y$ with $|\alpha|_{H,y} = 1$ such that

$$\psi(y) = \log\big|g(\alpha)\big|.$$

Choose an embedding $\iota \colon B \hookrightarrow Y$ such that $\iota(0) = y$ and $\iota(B) \subseteq V$. Using Lemma 23.1, we obtain a holomorphic section $s \in H^0(V, \mathscr{F})$ with $s(0) = \alpha$ and

$$\int_V |s|_H^2 \, d\mu \leq C_0;$$

the integrand is of course only defined on the subset $V \setminus V \cap Z$, but this does not matter because $V \cap Z$ has measure zero. It follows that $\psi(y)$ is equal to the value of $\log|g(s)|$ at the point y, and so the desired upper bound for ψ is a consequence of Lemma 23.4 below. □

Lemma 23.4. *Fix $K \geq 0$, and consider the set*

$$S_K = \left\{ s \in H^0(V, \mathscr{F}) \;\middle|\; \int_V |s|_H^2 \, d\mu \leq K \right\}.$$

There is a constant $C \geq 0$ such that, for every section $s \in S_K$, the holomorphic function $g(s)$ is uniformly bounded by C on the compact set $\overline{U}$.

Proof. Since $g(s)$ is holomorphic on V, it is clear that each individual function $g(s)$ is bounded on $\overline{U}$. To get an upper bound that works for every $s \in S_K$ at once, we use a compactness argument. Given a section $s \in H^0(V, \mathscr{F})$, we invert the process from above and define

$$\beta = s \otimes \big(dt_1 \wedge \cdots \wedge dt_r\big) \in H^0(V, \omega_Y \otimes \mathscr{F}) = H^0\big(f^{-1}(V), \omega_X \otimes L \otimes \mathcal{I}(h)\big).$$

If $s \in S_K$, then one has

$$\|\beta\|_h^2 = \int_V |s|_H^2 \, d\mu \leq K.$$

Because $\overline{V}$ is compact and f is proper, we can cover $f^{-1}(V)$ by finitely many open sets W that are biholomorphic to the open unit ball in $\mathbb{C}^n$, and on which L is trivial. Let $z_1, \dots, z_n$ be a holomorphic coordinate system on W, choose a nowhere vanishing holomorphic section $s_0 \in H^0(W, L)$, and write $|s_0|_h^2 = e^{-\varphi}$, with φ plurisubharmonic on W. Then $\beta|_W = b s_0 \otimes dz_1 \wedge \cdots \wedge dz_n$ for some holomorphic function $b \in H^0(W, \mathscr{O}_W)$, and

$$\int_W |b|^2 e^{-\varphi} (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n) = \int_W |\beta|_h^2 \leq K.$$

As we are dealing with finitely many open sets, Proposition 12.5 shows that every sequence in S_K has a subsequence that converges uniformly on compact subsets to some $\beta \in H^0\big(f^{-1}(V), \omega_X \otimes L \otimes \mathcal{I}(h)\big)$. This is all that we need.

Indeed, suppose that the assertion was false. Then we could find a sequence $s_0, s_1, s_2, \ldots \in S_K$ such that the maximum value of $|g(s_k)|$ on the compact set $\overline{U}$ was at least k. Let $\beta_0, \beta_1, \beta_2, \ldots$ denote the corresponding sequence of holomorphic sections of $\omega_X \otimes L \otimes \mathcal{I}(h)$ on the open set $f^{-1}(V)$; after passing to a subsequence, the β_k will converge uniformly on compact subsets to $\beta \in H^0\big(f^{-1}(V), \omega_X \otimes L \otimes \mathcal{I}(h)\big)$. Let $s \in H^0(V, \mathscr{F})$ be the unique section of $\mathscr{F}$ such that

$$\beta = s \otimes \big(dt_1 \wedge \cdots \wedge dt_r\big).$$

By Proposition 15.3, the s_k converge to s in the Fréchet space topology on $H^0(V, \mathscr{F})$. Since $g\colon \mathscr{F} \to \mathscr{O}_Y$ is a morphism, the holomorphic functions $g(s_k)$ therefore converge uniformly on compact subsets to $g(s)$. But then $|g(s_k)|$ must be uniformly bounded on $\overline{U}$, contradicting our initial choice. □

The next step is to show that the function $\psi = \log|g|_{H^*}$ is upper semi-continuous on $Y \setminus Z$. The proof is similar to that of Proposition 22.5.

Proposition 23.5. *For every $g \in H^0(Y, \mathscr{F}^*)$, the function $\psi = \log|g|_{H^*}$ is upper semi-continuous on $Y \setminus Z$.*

Proof. After restricting everything to a suitable open neighborhood of any given point $y \in Y \setminus Z$, we may assume without loss of generality that $Y = B$ and $Z = \emptyset$ and $y = 0$. Then $g \in H^0(B, E^*)$, and it will be enough to show that $\psi = \log|g|_{H^*}$ is upper semi-continuous at the origin. In other words, we need to argue that

$$\limsup_{k \to +\infty} \psi(y_k) \leq \psi(0) \tag{23.6}$$

for every sequence $y_0, y_1, y_2, \ldots \in B$ that converges to the origin. We may assume that $\psi(y_k) \neq -\infty$ for all $k \in \mathbb{N}$, and that the sequence $\psi(y_k)$ actually has a limit.

As we saw before, there is, for each $k \in \mathbb{N}$, a holomorphic section $s_k \in H^0(B, E)$ such that $\psi(y_k)$ equals the value of $\log|g(s_k)|$ at the point y_k; the Ohsawa-Takegoshi theorem allows us to choose these sections in such a way that

$$|s_k(y_k)|_{H,y_k} = 1 \quad \text{and} \quad \int_B |s_k|_H \, d\mu \leq K$$

for some constant $K \geq 0$. Passing to a subsequence, if necessary, we can arrange that the s_k converge uniformly on compact subsets to some $s \in H^0(B, E)$. Then the holomorphic functions $g(s_k)$ converge uniformly on compact subsets to $g(s)$, and (23.6) reduces to showing that the value at the origin of $\log|g(s)|$ is less or equal to $\psi(0)$. By definition of the dual metric H^*, we have

$$\psi \geq \log|g(s)| - \log|s|_H,$$

and so this is equivalent to proving that $|s(0)|_{H,0} \leq 1$. As in (22.2) and (22.3), each s_k determines a lower semi-continuous function $F_k \colon B \times X_0 \to [0, +\infty]$ with

$$1 = |s_k(y_k)|^2_{H,y_k} = \int_{X_0} F_k(y_k, -) \frac{\omega_0^{n-r}}{(n-r)!}.$$

Likewise, s determines a lower semi-continuous function $F \colon B \times X_0 \to [0, +\infty]$. Since the local weight functions $e^{-\varphi}$ of the pair (L, h) are lower semi-continuous, and since s_k converges uniformly on compact subsets to s, we get

$$F(0, -) \leq \liminf_{k \to +\infty} F_k(y_k, -).$$

We can now apply Fatou's lemma and conclude the proof in the same way as in Proposition 22.5. □

24. Proof of the pushforward theorem, Part III. In this section, we complete the proof of Theorem 21.1 by showing that the pair (E, H) has semi-positive curvature, and that H extends to a singular hermitian metric on $\mathscr{F}$ with the minimal extension property. The key point is that we can prove the required mean-value inequalities because the optimal value of the constant in the Ohsawa-Takegoshi theorem is exactly the volume of the unit ball. To illustrate how this works, let us first show that the singular hermitian metric H on $Y \setminus Z$ has the minimal extension property (see §20). For the statement, recall that $r = \dim Y$, and that $B \subseteq \mathbb{C}^r$ is the open unit ball.

Proposition 24.1. *For every embedding $\iota \colon B \hookrightarrow Y$ with $y = \iota(0) \in Y \setminus Z$, and for every $\alpha \in E_y$ with $|\alpha|_{H,y} = 1$, there is a holomorphic section $s \in H^0(B, \iota^*\mathscr{F})$ with*

$$s(0) = \alpha \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |s|^2_H \, d\mu \leq 1.$$

Proof. The proof is the same as that of Lemma 23.1; we only need to replace the constant C_0 by its optimal value $\mu(B)$. □

Now let us prove that H extends to a singular hermitian metric on $\mathscr{F}$ with semi-positive curvature. Keeping the notation from above, this amounts to proving that the function $\psi \colon Y \setminus Z \to [-\infty, +\infty)$ in (23.2) extends to a plurisubharmonic function on Y. We already know that ψ is upper semi-continuous (by Proposition 23.5) and bounded from above in a neighborhood of every point in Y (by Proposition 23.3). What we need to prove is the mean-value inequality along holomorphic arcs in

$Y \setminus Z$. The Ohsawa-Takegoshi theorem with sharp estimates renders the proof of the mean-value inequality almost a triviality.

Proposition 24.2. *For every holomorphic mapping $\gamma\colon \Delta \to Y \setminus Z$, the function $\psi = \log\lvert g\rvert_{H^*}$ satisfies the mean-value inequality*

$$(\psi \circ \gamma)(0) \leq \frac{1}{\pi}\int_{\Delta} (\psi \circ \gamma)\, d\mu.$$

Proof. If h is identically equal to $+\infty$ on the preimage of $\gamma(\Delta)$, the inequality is clear, so we may assume that this is not the case. Since $f\colon X \to Y$ is submersive over $Y \setminus Z$, we may then pull everything back to Δ and reduce the problem to the case $Y = \Delta$. If $\psi(0) = -\infty$, then the mean-value inequality holds by default. Assuming from now on that $\psi(0) \neq -\infty$, we choose an element $\alpha \in E_0$ with $\lvert\alpha\rvert_{H,0} = 1$, such that

$$\psi(0) = \log\lvert g\rvert_{H^*,0} = \log\lvert g(\alpha)\rvert.$$

Using the minimal extension property (in Proposition 24.1, with $m = 1$), there is a holomorphic section $s \in H^0(\Delta, E)$ such that

$$s(0) = \alpha \quad \text{and} \quad \frac{1}{\pi}\int_{\Delta} \lvert s\rvert_H^2 \, d\mu \leq 1.$$

The existence of this section is all that we need to prove the mean-value inequality. By definition of the metric H^* on the dual bundle, we have the pointwise inequality

$$\lvert g\rvert_{H^*} \geq \frac{\lvert g(s)\rvert}{\lvert s\rvert_H}$$

and therefore $2\psi \geq \log\lvert g(s)\rvert^2 - \log\lvert s\rvert_H^2$; here $g(s)$ is a holomorphic function on Δ, whose value at the origin equals $g(\alpha)$. Integrating, we get

$$\frac{1}{\pi}\int_{\Delta} 2\psi\, d\mu \geq \frac{1}{\pi}\int_{\Delta} \log\lvert g(s)\rvert^2\, d\mu - \frac{1}{\pi}\int_{\Delta} \log\lvert s\rvert_H^2\, d\mu$$

Now $\log\lvert g(s)\rvert^2$ satisfies the mean-value inequality, and so the first term on the right-hand side is at least $\log\lvert g(\alpha)\rvert^2 = 2\psi(0)$. Since the function $x \mapsto -\log x$ is convex, and since the function $\lvert s\rvert_H^2$ is integrable, the second term can be estimated by Jensen's inequality to be at least

$$-\log\left(\frac{1}{\pi}\int_{\Delta} \lvert s\rvert_H^2\, d\mu\right) \geq -\log 1 = 0.$$

Putting everything together, we obtain

$$\frac{1}{\pi}\int_{\Delta} 2\psi\, d\mu \geq 2\psi(0),$$

which is the mean-value inequality (up to a factor of 2). □

We have verified that ψ is plurisubharmonic on $Y \setminus Z$. We already know from Proposition 23.3 that ψ is locally bounded from above in a neighborhood of every point in Y; consequently, it extends uniquely to a plurisubharmonic function on all of Y, using Lemma 12.4. By duality, the singular hermitian metric H is therefore well-defined on the entire open set $Y(\mathscr{F})$ where the sheaf $\mathscr{F} = f_*\big(\omega_X \otimes L \otimes \mathcal{I}(h)\big)$ is locally free. We have already shown that H has the minimal extension property. This finishes the proof of Theorem 21.1. □

Proof of Corollary 21.4. Suppose that $f\colon X \to Y$ is submersive and that the singular hermitian metric h on the line bundle L is continuous. To prove that H is continuous, it suffices to show that for every locally defined section $s \in H^0(U, \mathscr{F})$, the function $|s|_H^2$ on $U \setminus U \cap Z$ admits a continuous extension to all of U. This is a local problem, and so we may assume that $Y = B$ is the open unit ball in $\mathbb{C}^r$, with coordinates $t_1, \dots, t_r$, and that $s \in H^0(B, \mathscr{F})$. Define

$$\beta = s \wedge (dt_1 \wedge \dots \wedge dt_r) \in H^0(B, \omega_B \otimes \mathscr{F}) = H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big).$$

By Ehresmann's fibration theorem, X is diffeomorphic to the product $B \times X_0$, and as in (22.2), we can write

$$|\beta|_h^2 = F \cdot d\mu \wedge \frac{\omega_0^{n-r}}{(n-r)!}$$

with $F\colon B \times X_0 \to [0, +\infty]$ continuous. Now

$$y \mapsto \int_{X_0} F(y, -) \frac{\omega_0^{n-r}}{(n-r)!}$$

defines a continuous function on B that agrees with $|s|_H^2$ on the complement of the bad set Z, due to (22.3). □

25. Positivity of the determinant line bundle. In this section, we show that if a holomorphic vector bundle E has a singular hermitian metric with semi-positive curvature, then the determinant line bundle $\det E$ has the same property. The proof in [Rau15, Proposition 1.1] relies on locally approximating a given singular hermitian metric from below by smooth hermitian metrics [BP08, Proposition 3.1].

Proposition 25.1. *If* (E, h) *has semi-positive curvature, so does* $(\det E, \det h)$.

Let us first analyze what happens over a point. Let V be a complex vector space of dimension r, and $|-|_h$ a positive definite singular hermitian inner product on V; in the notation of §16, we have $V_0 = 0$. Let $\mathbb{P}(V)$ be the projective space parametrizing one-dimensional quotient spaces of V, and denote by $\mathscr{O}(1)$ the universal line bundle on $\mathbb{P}(V)$. We have a surjective morphism $V \otimes \mathscr{O} \to \mathscr{O}(1)$, and so h induces a singular hermitian metric on $\mathscr{O}(1)$, with singularities along the subspace $\mathbb{P}(V/V_{\mathit{fin}}) \subseteq \mathbb{P}(V)$. To see this, choose a basis $e_1, \dots, e_r \in V$ such that $e_1, \dots, e_k$ form an orthonormal basis of V_{fin} with respect to the inner product $\langle -, - \rangle_h$, and denote by $[z_1, \dots, z_r]$ the resulting homogeneous coordinates on $\mathbb{P}(V)$. Then the local weight functions of the metric on $\mathscr{O}(1)$ are given by the formula

$$\log\big(|z_1|^2 + \dots + |z_k|^2\big),$$

with the convention that $z_i = 1$ on the i-th standard affine open subset.

Now the one-dimensional complex vector space $\det V = \bigwedge^r V$ is naturally the space of global sections of an adjoint bundle on $\mathbb{P}(V)$, because

$$\det V \simeq H^0\big(\mathbb{P}(V), \omega_{\mathbb{P}(V)} \otimes \mathscr{O}(r)\big).$$

The isomorphism works as follows. The element $e_1 \wedge \dots \wedge e_r \in \det V$ determines a holomorphic r-form $dz_1 \wedge \dots \wedge dz_r$ on the dual vector space V^*; after contraction with the Euler vector field $z_1 \partial/\partial z_1 + \dots + z_r \partial/\partial z_r$, we get a holomorphic $(r-1)$-form

$$\Omega = \sum_{i=1}^{r} (-1)^{i-1} z_i \, dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_r$$

on $\mathbb{P}(V)$ that is homogeneous of degree r, hence a global section of the holomorphic line bundle $\omega_{\mathbb{P}(V)} \otimes \mathscr{O}(r)$. Integration over $\mathbb{P}(V)$ therefore defines a positive definite singular hermitian inner product H on $\det V$. We have

$$|e_1 \wedge \cdots \wedge e_r|_H^2 = \int_{\mathbb{P}(V)} \frac{c_{r-1} \cdot \Omega \wedge \overline{\Omega}}{\bigl(|z_1|^2 + \cdots + |z_k|^2\bigr)^r},$$

which simplifies in the affine chart $z_1 = 1$ to

$$|e_1 \wedge \cdots \wedge e_r|_H^2 = \int_{\mathbb{C}^{r-1}} \frac{d\mu}{\bigl(1 + |z_2|^2 + \cdots + |z_k|^2\bigr)^r}.$$

Now there are two cases. If $V_{\mathit{fin}} \neq V$, then $k < r$, and the integral is easily seen to be $+\infty$. If $V_{\mathit{fin}} = V$, then $k = r$, and the integral evaluates to $\pi^{r-1}/(r-1)!$, the volume of the open unit ball in $\mathbb{C}^{r-1}$. In conclusion, we always have

$$|e_1 \wedge \cdots \wedge e_r|_H^2 = \frac{\pi^{r-1}}{(r-1)!} \cdot |e_1 \wedge \cdots \wedge e_r|_{\det h}^2.$$

With this result in hand, we can now prove Proposition 25.1.

Proof. Let $p \colon \mathbb{P}(E) \to X$ denote the associated $\mathbb{P}^{r-1}$-bundle, and let $\mathscr{O}_E(1)$ be the universal line bundle on $\mathbb{P}(E)$. We have a surjective morphism $p^*E \to \mathscr{O}_E(1)$, and by Lemma 18.2, the singular hermitian metric on E induces a singular hermitian metric on the line bundle $\mathscr{O}_E(1)$, still with semi-positive curvature. We have

$$\omega_{\mathbb{P}(E)/X} \simeq p^* \det E \otimes \mathscr{O}_E(-r),$$

and therefore $\det E \simeq p_*\bigl(\omega_{\mathbb{P}(E)/X} \otimes \mathscr{O}_E(r)\bigr)$ is the pushforward of an adjoint bundle. The calculation above shows that, up to a factor of $\pi^{r-1}/(r-1)!$, the resulting singular hermitian metric on $\det E$ agrees with $\det h$ pointwise. The assertion about the curvature of $(\det E, \det h)$ is therefore a consequence of Corollary 21.2. □

26. Consequences of the minimal extension property. In this section, we derive a few interesting consequences from the minimal extension property. All of the results below are true for smooth hermitian metrics with Griffiths semi-positive curvature on holomorphic vector bundles; the minimal extension property is what makes them work even in the presence of singularities.

Let $\mathscr{F}$ be a torsion-free coherent sheaf on X, of generic rank $r \geq 1$, and suppose that $\mathscr{F}$ has a singular hermitian metric with semi-positive curvature and the minimal extension property. Let E be the holomorphic vector bundle of rank r obtained by restricting $\mathscr{F}$ to the open subset $X(\mathscr{F})$; by assumption, the pair (E, h) has semi-positive curvature. Proposition 25.1 shows that $(\det E, \det h)$ also has semi-positive curvature. Let $\det \mathscr{F}$ be the holomorphic line bundle obtained as the double dual of $\bigwedge^r \mathscr{F}$; its restriction to $X(\mathscr{F})$ agrees with $\det E$. Since $X \setminus X(\mathscr{F})$ has codimension ≥ 2, the singular hermitian metric on $\det E$ extends uniquely to a singular hermitian metric on $\det \mathscr{F}$. The following result is due to Cao and Păun [CP15, Theorem 5.23], who proved it using results by Raufi [Rau15].

Theorem 26.1. *Suppose that X is compact and that $c_1(\det \mathscr{F}) = 0$ in $H^2(X, \mathbb{R})$. Then $\mathscr{F}$ is locally free, and (E, h) is a hermitian flat bundle on $X = X(\mathscr{F})$.*

Proof. Since X is compact, the singular hermitian metric on $\det \mathscr{F}$ is smooth and has zero curvature (by Lemma 13.2). Restricting to the open subset $X(\mathscr{F})$, we see that the same is true for $(\det E, \det h)$. Now the idea is to use the minimal extension property to construct, locally on X, a collection of r sections of $\mathscr{F}$ that are orthonormal with respect to h.

We can certainly cover X by open subsets that are isomorphic to the open unit ball $B \subseteq \mathbb{C}^n$ and are centered at points $x \in X \setminus Z$ where the singular hermitian inner product $|-|_{h,x}$ is finite and positive definite. After restricting everything to an open subset of this kind, we may assume that $X = B$, that the point $0 \in B$ lies in the subset $B \setminus Z$, and that $|-|_{h,0}$ is a genuine hermitian inner product on the r-dimensional complex vector space E_0. Choose an orthonormal basis $e_1, \dots, e_r \in E_0$. By the minimal extension property for $\mathscr{F}$, we can find r holomorphic sections $s_1, \dots, s_r \in H^0(B, \mathscr{F})$ such that

$$s_i(0) = e_i \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |s_i|_h^2 \, d\mu \leq 1.$$

Since the logarithm function is strictly concave, Jensen's inequality shows that

$$\frac{1}{\mu(B)} \int_B \log|s_i|_h^2 \, d\mu \leq \log\left(\frac{1}{\mu(B)} \int_B |s_i|_h^2 \, d\mu\right) \leq 0, \tag{26.2}$$

with equality if and only if $|s_i|_h = 1$ almost everywhere.

Now let us analyze the singular hermitian metric on $\det E$. The expression

$$H_{i,j}(x) = \langle s_i(x), s_j(x) \rangle_{h,x}$$

is well-defined outside a set of measure zero, and the resulting function is locally integrable. Denote by $H(x)$ the $r \times r$-matrix with these entries; it is almost everywhere positive definite, and we have

$$\left|s_1 \wedge \cdots \wedge s_r\right|_{\det h}^2 = \det H.$$

Since $\det h$ is actually smooth and flat, we can choose a nowhere vanishing section $\delta \in H^0(B, \det \mathscr{F})$ such that $|\delta|_{\det h} \equiv 1$. We then have $s_1 \wedge \cdots \wedge s_r = g \cdot \delta$ for a holomorphic function $g \in H^0(B, \mathscr{O}_B)$ with $g(0) = 1$, and

$$|g|^2 = \left|s_1 \wedge \cdots \wedge s_r\right|_{\det h}^2 = \det H.$$

From Hadamard's inequality for semi-positive definite matrices, we obtain

$$|g(x)|^2 = \det H(x) \leq \prod_{i=1}^r H_{i,i}(x) = \prod_{i=1}^r \left|s_i(x)\right|_{h,x}^2,$$

with equality if and only if the matrix $H(x)$ is diagonal. Taking logarithms, we get

$$\log|g(x)|^2 \leq \sum_{i=1}^r \log\left|s_i(x)\right|_{h,x}^2.$$

This inequality is valid almost everywhere; integrating, we find that

$$\frac{1}{\mu(B)} \int_B \log|g|^2 \, d\mu \leq \sum_{i=1}^r \frac{1}{\mu(B)} \int_B |s_i|_h^2 \, d\mu. \tag{26.3}$$

Now $\log|g|^2$ is plurisubharmonic, and so the mean-value inequality shows that the left-hand side in (26.3) is greater or equal to $\log|g(0)|^2 = 0$. At the same time, the right-hand side is less or equal to 0 by (26.2). The conclusion is that all our inequalities are actually equalities, and so $H(x)$ is almost everywhere equal to the identity matrix of size $r \times r$. In other words, the sections $s_1, \dots, s_r \in H^0(B, \mathscr{F})$ are almost everywhere orthonormal with respect to h.

For any holomorphic section $f \in H^0(B, \mathscr{F}^*)$, we therefore have

$$|f|_{h^*}^2 = \sum_{i=1}^{r} |f \circ s_i|^2$$

almost everywhere on B; because the logarithms of both sides are plurisubharmonic functions on B, the identity actually holds everywhere. The singular hermitian metric h^* is therefore smooth; but then h is also smooth, and the pair (E, h) is a hermitian flat bundle.

To conclude the proof, we need to argue that $\mathscr{F}$ is locally free on all of B. The sections $s_1, \dots, s_r \in H^0(B, \mathscr{F})$ give rise to a morphism of sheaves

$$\sigma \colon \mathscr{O}_B^{\oplus r} \to \mathscr{F}.$$

We already know that σ is an isomorphism on the open subset $B(\mathscr{F})$; by Hartog's theorem, its inverse extends to a morphism of sheaves

$$\tau \colon \mathscr{F} \to \mathscr{O}_B^{\oplus r}$$

with $\tau \circ \sigma = \mathrm{id}$. Because $\mathscr{F}$ is torsion-free, this forces σ to be an isomorphism. □

Note. Our proof gives a different interpretation for the fact that $(\det E, \det h)$ has semi-positive curvature. Indeed, without assuming that $\det h$ is smooth and flat, we have $\det H = |g|^2 e^{-\varphi}$, where $\varphi \colon B \to [-\infty, +\infty)$ is locally integrable and $\varphi(0) = 0$. The various inequalities above then combine to give

$$0 \leq \frac{1}{\mu(B)} \int_B \varphi \, d\mu,$$

which is exactly the mean-value inequality for φ.

The next theorem is a new result. It says that when X is compact, all global sections of the dual coherent sheaf $\mathscr{F}^*$ arise from trivial summands in $\mathscr{F}$. Equivalently, every nonzero morphism $\mathscr{F} \to \mathscr{O}_X$ has a section, which means that $\mathscr{F}$ splits off a direct summand isomorphic to $\mathscr{O}_X$.

Theorem 26.4. *Suppose that X is compact and connected. Then for every nonzero $f \in H^0(X, \mathscr{F}^*)$, there exists a unique global section $s \in H^0(X, \mathscr{F})$ such that $|s|_h$ is a.e. constant and $f \circ s \equiv 1$.*

Proof. Because the singular hermitian metric on $\mathscr{F}$ has semi-positive curvature, the function $\log|f|_{h^*}$ is plurisubharmonic on X, hence equal to a nonzero constant. After rescaling the metric, we may assume without loss of generality that $|f|_{h^*} \equiv 1$. As in the proof of Theorem 26.1, we cover X by open subsets that are isomorphic to $B \subseteq \mathbb{C}^n$ and are centered at points $x \in X \setminus Z$ where $|-|_{h,x}$ is finite and positive definite. We shall argue that there is a unique section of $\mathscr{F}$ with the desired properties on each open set of this type; by uniqueness, these sections will then glue together to give us the global section $s \in H^0(X, \mathscr{F})$ that we are looking for.

We may therefore assume without loss of generality that $X = B$, that the origin belongs to the subset $B \setminus Z$, and that $|-|_{h,0}$ is a hermitian inner product on the vector space E_0. It is easy to see from

$$\sup \left\{ \frac{|f(v)|}{|v|_{h,0}} \;\middle|\; v \in E_0 \text{ with } |v|_{h,0} \neq 0 \right\} = |f|_{h^*,0} = 1$$

that there exists a vector $v \in E_0$ with $f(v) = 1$ and $|v|_{h,0} = 1$. By the minimal extension property, there is a section $s \in H^0(B, \mathscr{F})$ such that

$$s(0) = v \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |s|_h^2 \, d\mu \leq 1.$$

Now $f \circ s$ is a holomorphic function on B, and by definition of h^*, we have

$$\frac{|f \circ s|}{|s|_h} \leq |f|_{h^*} = 1.$$

Taking logarithms and integrating, we get

$$\frac{1}{\mu(B)} \int_B \log|f \circ s|^2 \, d\mu \leq \frac{1}{\mu(B)} \int_B \log|s|_h^2 \, d\mu \leq \log\left(\frac{1}{\mu(B)} \int_B |s|_h^2 \, d\mu\right) \leq 0,$$

using Jensen's inequality along the way. By the mean-value inequality, the left-hand side is greater or equal to $\log(f \circ s)(0) = 0$, and so once again, all inequalities must be equalities. It follows that $f \circ s \equiv 1$, and that the measurable function $|s|_h$ is equal to 1 almost everywhere.

It remains to prove the uniqueness statement. Suppose that $s' \in H^0(B, \mathscr{F})$ is another holomorphic section with the property that $f \circ s' \equiv 1$ and $|s'|_h = 1$ almost everywhere. Outside a set of measure zero, we have

$$|s' - s|_h^2 + |s' + s|_h^2 = 2|s|_h^2 + 2|s'|_h^2 = 4,$$

and since $f(s' + s) = 2$, we must have $|s' + s|_h^2 \geq 4$. This implies that $|s' - s|_h^2 = 0$ almost everywhere, and hence that $s' = s$. □

E. Pushforwards of relative pluricanonical bundles

27. Introduction. In the previous chapter, we presented a general formalism for constructing singular hermitian metrics with semi-positive curvature on sheaves of the form $f_*(\omega_{X/Y} \otimes L)$. The applications to algebraic geometry come from the fact that the sheaves $f_* \omega_{X/Y}^{\otimes m}$ with $m \geq 2$ naturally fit into this framework. The main result is the following; see [BP08, Corollary 4.2], and also [Tsu11, Theorem 1.12], [PT14, Theorem 4.2.2].

Theorem 27.1. *Let $f \colon X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_* \omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$.*

(a) The line bundle $\omega_{X/Y}$ has a canonical singular hermitian metric with semi-positive curvature, called the m-th Narasimhan-Simha metric. *This metric is continuous on the preimage of the smooth locus of f.*

(b) If h denotes the induced singular hermitian metric on $L = \omega_{X/Y}^{\otimes(m-1)}$, then

$$f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big) \hookrightarrow f_* \omega_{X/Y}^{\otimes m}$$

is an isomorphism over the smooth locus of f.

We can therefore apply Corollary 21.2 and conclude that for any $m \geq 1$, the torsion-free sheaf $f_* \omega_{X/Y}^{\otimes m}$ has a singular hermitian metric with semi-positive curvature and the minimal extension property. Over the smooth locus of f, this metric is finite and continuous. The minimal extension property has the following remarkable consequences.

Corollary 27.2. *Suppose that Y is compact.*

(a) If $c_1\big(\det f_*\omega_{X/Y}^{\otimes m}\big) = 0$ *in* $H^2(Y,\mathbb{R})$*, then* $f_*\omega_{X/Y}^{\otimes m}$ *is locally free and the singular hermitian metric on it is smooth and flat.*
(b) Any nonzero morphism $f_*\omega_{X/Y}^{\otimes m} \to \mathscr{O}_Y$ *is split surjective.*

Proof. This follows from Theorem 26.1 and Theorem 26.4. □

Note. There are two or three points in the proof where we need to use invariance of plurigenera. This means that Theorem 27.1 cannot be used to give a new proof for the invariance of plurigenera.

28. The absolute case. Let us start by discussing the absolute case. Take X to be a smooth projective variety of dimension n. Fix an integer $m \geq 1$ for which the vector space

$$V_m = H^0(X, \omega_X^{\otimes m})$$

of all m-canonical forms is nontrivial. Our goal is to construct a singular hermitian metric on the line bundle ω_X, with singularities along the base locus of V_m, such that all elements of V_m have bounded norm. We can measure the *length* of an m-canonical form $v \in V_m$ by a real number $\ell(v) \in [0, +\infty)$, defined by the formula

$$\ell(v) = \left(\int_X (c_n^m v \wedge \overline{v})^{1/m} \right)^{m/2}. \tag{28.1}$$

The constant $c_n = 2^{-n}(-1)^{n^2/2}$ is there to make the expression in parentheses positive. A more concrete definition is as follows. In local coordinates $z_1, \dots, z_n$, we have an expression

$$v = g(z_1, \dots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m},$$

with g holomorphic; the integrand in (28.1) is then locally given by

$$|g|^{2/m} c_n (dz_1 \wedge \cdots \wedge dz_n) \wedge (d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n). \tag{28.2}$$

For $m \geq 2$, the length function ℓ is not a norm, because the triangle inequality fails to hold. On the other hand, ℓ is continuous on V_m, with $\ell(v) = 0$ iff $v = 0$; we also have $\ell(\lambda v) = |\lambda| \cdot \ell(v)$ for every $\lambda \in \mathbb{C}$.

We can now construct a singular hermitian metric h_m on the line bundle ω_X by using the length function ℓ. Given an element ξ in the fiber of ω_X at a point $x \in X$, we define

$$|\xi|_{h_m,x} = \inf\big\{ \ell(v)^{1/m} \;\big|\; v \in V_m \text{ satisfies } v(x) = \xi^{\otimes m} \big\} \in [0, +\infty].$$

In other words, we look for the m-canonical form of minimal length whose value at the point x is equal to the m-th power of ξ; if x belongs to the base locus of V_m, then $|\xi|_{h_m,x} = +\infty$ for $\xi \neq 0$. We obtain in this way a singular hermitian metric h_m on the line bundle ω_X, with singularities precisely along the base locus of the linear system V_m. The advantage of this construction is that it is completely canonical: there is no need to choose a basis for V_m.

Note. Following Păun and Takayama, we may call h_m the *m-th Narasimhan-Simha metric* on the line bundle ω_X, because Narasimhan and Simha [NS68] used this metric in the special case ω_X ample. A similar construction also appears in Kawamata's proof of Iitaka's conjecture over curves [Kaw82, §2].

Proposition 28.3. *The Narasimhan-Simha metric* h_m *on* ω_X *is continuous, has singularities exactly along the base locus of* V_m*, and has semi-positive curvature.*

Proof. We compute the local weights of h_m. Let $z_1, \dots, z_n$ be local holomorphic coordinates on a suitable open subset $U \subseteq X$, and set $s_0 = dz_1 \wedge \dots \wedge dz_n$, which is a nowhere vanishing section of ω_X on the subset U. Consider the function

$$\varphi_m = -\log|s_0|^2_{h_m} : U \to [-\infty, +\infty).$$

The definition of h_m shows that, for every $x \in U$,

$$\varphi_m(x) = \frac{2}{m} \sup \left\{ \log \frac{1}{\ell(v)} \;\middle|\; v \in V_m \text{ satisfies } v(x) = s_0(x)^{\otimes m} \right\}. \tag{28.4}$$

For each $v \in V_m$, there is a holomorphic function $g_v : U \to \mathbb{C}$ with $v|_U = g_v \cdot s_0^{\otimes m}$. If $g_v(x) \neq 0$, then the m-canonical form $v/g_v(x)$ contributes to the right-hand side of (28.4), and so we obtain

$$\varphi_m(x) = \frac{2}{m} \sup \Big\{ \log|g_v(x)| \;\Big|\; v \in V_m \text{ satisfies } \ell(v) \leq 1 \Big\}. \tag{28.5}$$

We will see in a moment that the supremum is actually a maximum, because the set of m-canonical forms $v \in V_m$ with $\ell(v) \leq 1$ is compact. Evidently, $\varphi_m(x) = -\infty$ if and only if $x \in U$ belongs to the base locus of V_m.

Now observe that the family of holomorphic functions

$$G_m = \big\{ g_v \in H^0(U, \mathscr{O}_X) \;\big|\; v \in V_m \text{ satisfies } \ell(v) \leq 1 \big\}$$

is uniformly bounded on compact subsets. Indeed, the fact that $\ell(v) \leq 1$ gives us a uniform bound on the $L^{2/m}$-norm of each g_v, and then we can argue as in the proof of Proposition 12.5, using the mean-value inequality. By the n-dimensional version of Montel's theorem, the family G_m is equicontinuous; due to (28.5), our φ_m is therefore continuous, as a function from U into $[-\infty, +\infty)$.

From (28.5), we can also determine the curvature properties of h_m. For each fixed $v \in V_m$, the function $\log|g_v|^{2/m}$ is continuous and plurisubharmonic, and equal to $-\infty$ precisely on the zero locus of g_v. As the upper envelope of an equicontinuous family of plurisubharmonic functions, φ_m is itself plurisubharmonic [Dem12, Theorem I.5.7]. This shows that the Narasimhan-Simha metric on ω_X has semi-positive curvature. □

Another good feature of the Narasimhan-Simha metric is that all m-canonical forms are bounded with respect to this metric. Indeed, if we also use h_m to denote the induced singular hermitian metric on $\omega_X^{\otimes m}$, then by construction, we have the pointwise inequality $|v|_{h_m} \leq \ell(v)$ for every $v \in V_m$. In order to fit the Narasimhan-Simha metric into the framework of Chapter D, we write

$$\omega_X^{\otimes m} = \omega_X \otimes \omega_X^{\otimes(m-1)},$$

and endow the line bundle $L = \omega_X^{\otimes(m-1)}$ with the singular hermitian metric h induced by h_m. This metric is continuous and has semi-positive curvature.

Lemma 28.6. *For every $v \in V_m$, we have $\|v\|_h \leq \ell(v)$.*

Proof. We keep the notation introduced during the proof of Proposition 28.3. The weight of h with respect to the section $s_0^{\otimes(m-1)}$ of the line bundle $\omega_X^{\otimes(m-1)}$ is

$$e^{-(m-1)\cdot\varphi_m},$$

where φ_m is the function defined in (28.5). Now fix an m-canonical form $v \in V_m$ with $\ell(v) = 1$. On the open set U, the integrand in the definition of $\|v\|_h$ is

$$|g_v|^2 e^{-(m-1)\cdot\varphi_m} \cdot c_n(dz_1 \wedge \cdots \wedge dz_n) \wedge (d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n).$$

Because of (28.5), we have $\varphi_m \geq \log|g_v|^{2/m}$, and therefore

$$|g_v|^2 e^{-(m-1)\cdot\varphi_m} \leq |g_v|^2 \cdot |g_v|^{-2(m-1)/m} = |g_v|^{2/m}.$$

Looking back at the definition of $\ell(v)$ in (28.2), this shows that $\|v\|_h \leq \ell(v)$. $\square$

Since ℓ is not itself a norm, the inequality will in general be strict. One useful consequence of Lemma 28.6 is the identity

$$H^0\big(X, \omega_X \otimes L \otimes \mathcal{I}(h)\big) = H^0(X, \omega_X \otimes L) = H^0\big(X, \omega_X^{\otimes m}\big). \tag{28.7}$$

Note that the multiplier ideal $\mathcal{I}(h)$ may well be nontrivial; nevertheless, it imposes no extra conditions on global sections of $\omega_X^{\otimes m}$.

29. The Ohsawa-Takegoshi theorem for pluricanonical forms. To analyze how the Narasimhan-Simha metric behaves in families, we will need a version of the Ohsawa-Takegoshi theorem for m-canonical forms. Suppose that $f \colon X \to B$ is a holomorphic mapping to the open unit ball $B \subseteq \mathbb{C}^r$, with f projective and $f(X) = B$, and such that the central fiber $X_0 = f^{-1}(0)$ is nonsingular. To simplify the discussion, let us also assume that f is the restriction of a holomorphic family over a ball of slightly larger radius. As in (28.1), we have length functions ℓ and ℓ_0 on X respectively X_0; because X is not compact, it may happen that $\ell(v) = +\infty$ for certain $v \in H^0(X, \omega_X^{\otimes m})$.

Theorem 29.1. *For each $u \in H^0\big(X_0, \omega_{X_0}^{\otimes m}\big)$, there is some $v \in H^0\big(X, \omega_X^{\otimes m}\big)$ with*

$$\ell(v) \leq \mu(B)^{m/2} \cdot \ell_0(u),$$

such that the restriction of v to X_0 is equal to $u \wedge (df_1 \wedge \cdots \wedge df_r)^{\otimes m}$.

Proof. Without loss of generality, we may assume that $\ell_0(u) = 1$. Since X_0 is a projective complex manifold, invariance of plurigenera tells us that the fiber of the coherent sheaf $f_*\omega_X^{\otimes m}$ at the point $0 \in B$ is equal to $H^0\big(X_0, \omega_{X_0}^{\otimes m}\big)$. Because B is a Stein manifold, we can then certainly find a section

$$v \in H^0\big(B, f_*\omega_X^{\otimes m}\big) = H^0(X, \omega_X^{\otimes m})$$

with the correct restriction to X_0. By assuming that f comes from a morphism to a ball of slightly larger radius, we can also arrange that the quantity

$$\ell(v) = \left(\int_X (c_n^m v \wedge \overline{v})^{1/m} \right)^{m/2}$$

is finite. Of course, v will not in general satisfy the desired inequality.

The way to deal with this problem is to consider $\omega_X^{\otimes m} = \omega_X \otimes \omega_X^{\otimes(m-1)}$ as an adjoint bundle and to apply the Ohsawa-Takegoshi theorem to get another extension of smaller length. The section $v \in H^0(X, \omega_X^{\otimes m})$ induces a singular hermitian metric on the line bundle $\omega_X^{\otimes m}$, whose curvature is semi-positive. With respect to a local trivialization

$$\varphi \colon \omega_X^{\otimes m}|_U \to \mathscr{O}_U,$$

the weight of this metric is given by $\log|\varphi \circ v|^2$. Endow the line bundle $\omega_X^{\otimes(m-1)}$ with the singular hermitian metric whose local weight is

$$\frac{m-1}{m} \log|\varphi \circ v|^2.$$

It is easy to see that the norm of u with respect to this metric is still equal to $\ell_0(u) = 1$. Theorem 14.4 says that there exists another section $v' \in H^0(X, \omega_X^{\otimes m})$, with the same restriction to X_0, whose norm squared is bounded by $\mu(B)$. To get a useful expression for the norm squared, write

$$v' = Fv,$$

with F meromorphic on X and identically equal to 1 on X_0; then the inequality in the Ohsawa-Takegoshi theorem takes the form

$$\int_X |F|^2 (c_n^m v \wedge \overline{v})^{1/m} \leq \mu(B).$$

We can use this to get an upper bound for the quantity

$$\ell(v') = \left(\int_X |F|^{2/m} (c_n^m v \wedge \overline{v})^{1/m} \right)^{m/2}.$$

To begin with, let us write $(c_n^m v \wedge \overline{v})^{1/m} = L\,d\mu$, where L is a nonnegative real-analytic function on X, and $d\mu$ is some choice of volume form. Using Hölder's inequality with exponents $1/m$ and $(m-1)/m$, we have

$$\ell(v')^{2/m} = \int_X |F|^{2/m} L\,d\mu \leq \left(\int_X |F|^2 L\,d\mu \right)^{1/m} \left(\int_X L\,d\mu \right)^{(m-1)/m},$$

and therefore $\ell(v') \leq \mu(B)^{1/2} \cdot \ell(v)^{(m-1)/m}$, which we may rewrite as

$$\frac{\ell(v')}{\mu(B)^{m/2}} \leq \left(\frac{\ell(v)}{\mu(B)^{m/2}} \right)^{(m-1)/m}.$$

Now we iterate this construction to produce an infinite sequence of m-canonical forms $v_0, v_1, v_2, \ldots \in H^0(X, \omega_X^{\otimes m})$, all with the correct restriction to X_0. The inequality from above shows that one of two things happens: either $\ell(v_k) \leq \mu(B)^{m/2}$ for some $k \geq 0$; or $\ell(v_k) > \mu(B)^{m/2}$ for every $k \in \mathbb{N}$, and

$$\lim_{k \to +\infty} \ell(v_k) = \mu(B)^{m/2}.$$

If the former happens, we are done. If the latter happens, we apply Lemma 29.2: it says that a subsequence converges uniformly on compact subsets to an m-canonical form $v \in H^0(X, \omega_X^{\otimes m})$. Now v satisfies $\ell(v) \leq \mu(B)^{m/2}$ (by Fatou's lemma), and its restriction to X_0 is of course still equal to $u \wedge (df_1 \wedge \cdots \wedge df_r)^{\otimes m}$. □

Lemma 29.2. *Let X be a complex manifold, and let $v_0, v_1, v_2, \ldots \in H^0(X, \omega_X^{\otimes m})$ be a sequence of m-canonical forms such that $\ell(v_k) \leq C$ for every $k \in \mathbb{N}$. Then a subsequence converges uniformly on compact subsets to a limit $v \in H^0(X, \omega_X^{\otimes m})$.*

Proof. With respect to a local trivialization of $\omega_X^{\otimes m}$, we have a sequence of holomorphic functions whose $L^{2/m}$-norm is uniformly bounded. Using the mean-value inequality, this implies that the sequence of functions is uniformly bounded on compact subsets; now apply Montel's theorem to get the desired conclusion. □

Note. One interesting thing about the proof of Theorem 29.1 is that it looks very similar to Viehweg's covering trick (which we used for example in the proof of Proposition 8.2). The advantage of the metric approach is that one can take a limit to obtain a solution with the same properties as in the case $m = 1$.

30. The relative case. With the help of Theorem 29.1, it is quite easy to analyze the behavior of the Narasimhan-Simha metric in families. Let us first consider the case of a smooth morphism $f\colon X \to Y$; as in the statement of Theorem 27.1, we assume that f is projective with connected fibers, and that $f(X) = Y$. Recall that by invariance of plurigenera, the dimension of the space of m-canonical forms on the fiber $X_y = f^{-1}(y)$ is the same for every $y \in Y$.

The restriction of the relative canonical bundle $\omega_{X/Y}$ to the fiber X_y identifies to the canonical bundle ω_{X_y} of the fiber. We can therefore apply the construction in §28 fiber by fiber to produce a singular hermitian metric h_m on $\omega_{X/Y}$, called the *m-th relative Narasimhan-Simha metric*; we shall give a more careful definition of h_m in a moment. The first result is that h_m is continuous.

Proposition 30.1. *Under the assumptions above, the relative Narasimhan-Simha metric on $\omega_{X/Y}$ is continuous.*

Proof. Once again, this is an application of the Ohsawa-Takegoshi theorem for pluricanonical forms, which allows us to extend m-canonical forms from the fibers of f, with a uniform upper bound on the length of the extension. After shrinking Y, we can assume that $Y = B$ is the open unit ball in $\mathbb{C}^r$, with coordinates $t_1, \dots, t_r$. We denote by $V_m = H^0(X, \omega_X^{\otimes m})$ the (typically infinite-dimensional) vector space of all m-canonical forms on X. Given $v \in V_m$ and a point $y \in Y$, we have

$$v|_{X_y} = v_y \otimes (dt_1 \wedge \dots \wedge dt_r)^{\otimes m}$$

for a unique m-canonical form $v_y \in H^0(X_y, \omega_{X_y}^{\otimes m})$. We denote by $\ell(v)$ the length of v on X, and by $\ell_y(v_y)$ the length of v_y on X_y. The Ohsawa-Takegoshi theorem for pluricanonical forms (in Theorem 29.1) implies that, possibly after shrinking Y, there is a constant $C \geq 0$ with the following property:

(30.2) For every $y \in Y$ and every m-canonical form u on X_y of length ≤ 1, there is an m-canonical form $v \in V_m$ such that $v_y = u$ and $\ell(v) \leq C$.

Now let $n = \dim X$. As the morphism f is smooth, every point in X has an open neighborhood U with coordinates $z_1, \dots, z_{n-r}, t_1, \dots, t_r$. Then $s_0 = dz_1 \wedge \dots \wedge dz_{n-r}$ gives a local trivialization of $\omega_{X/Y}$, and we consider the weight function

$$\varphi_m = -\log|s_0|^2_{h_m} \colon U \to [-\infty, +\infty)$$

of the relative Narasimhan-Simha metric h_m. On each fiber, φ_m is given by the formula in (28.5); we can use the Ohsawa-Takegoshi theorem to obtain a more uniform description. For each $v \in V_m$, we have

$$v|_U = g_v \cdot (s_0 \wedge dt_1 \wedge \dots \wedge dt_r)^{\otimes m}$$

for a unique holomorphic function $g_v \colon U \to \mathbb{C}$. By (28.5) and (30.2), we have

$$\varphi_m(x) = \frac{2}{m} \sup \Big\{ \log|g_v(x)| \;\Big|\; v \in V_m \text{ satisfies } \ell(v) \leq C \text{ and } \ell_y(v_y) \leq 1 \Big\};$$

where $y = f(x)$. We are going to prove that this defines a continuous function on U.

Fix a point $x \in U$, and let $x_0, x_1, x_2, \dots$ be any sequence in U with limit x. Set $y_k = f(x_k)$ and $y = f(x)$. For every $k \in \mathbb{N}$, choose an m-canonical form u_k of length $\ell_{y_k}(u_k) = 1$ on the fiber X_{y_k}, such that u_k computes $\varphi_m(x_k)$. Extend u_k to an m-canonical form v_k of length $\ell(v_k) \leq C$ on X by using (30.2); then

$$\varphi_m(x_k) = \frac{2}{m} \log\bigl|g_{v_k}(x_k)\bigr|.$$

After passing to a subsequence, $v_0, v_1, v_2, \dots$ converges uniformly on compact subsets to an m-canonical form $v \in H^0(X, \omega_X^{\otimes m})$. Since $\ell_{y_k}(v_{n,y_k}) = 1$, Fatou's lemma shows that $\ell_y(v_y) \leq 1$. Moreover, the holomorphic functions g_{v_k} converge uniformly on compact subsets to g_v, and therefore

$$\lim_{k \to +\infty} \varphi_m(x_k) = \frac{2}{m} \log\bigl|g_v(x)\bigr| \leq \varphi_m(x). \tag{30.3}$$

On the other hand, we can choose an m-canonical form u' of length $\ell_y(u') = 1$ on the fiber X_y, such that u' computes $\varphi_m(x)$. Extend u' to an m-canonical form v' of length $\ell(v') \leq C$ on X by using (30.2); then

$$\varphi_m(x) = \frac{2}{m} \log|g_{v'}(x)|.$$

Now it is easy to see from the definition of the length function that $\ell_{y_k}(v'_{y_k})$ tends to $\ell_y(v'_y)$ as $k \to +\infty$. In particular, the m-canonical form v'_{y_k} on X_{y_k} has nonzero length for $k \gg 0$, which means that

$$\frac{2}{m}\Bigl(\log|g_{v'}(x_k)| - \log \ell_{y_k}(v'_{y_k})\Bigr) \leq \varphi_m(x_k).$$

Since the left-hand side tends to $\varphi_m(x)$, we obtain

$$\varphi_m(x) \leq \liminf_{k \to +\infty} \varphi_m(x_k). \tag{30.4}$$

The two inequalities in (30.3) and (30.4) together say that φ_m is continuous. □

Next, we prove that h_m has semi-positive curvature – just as in the case of adjoint bundles, the proof of this fact is very short, because we know the optimal value of the constant in Theorem 29.1.

Proposition 30.5. *Under the assumptions above, the relative Narasimhan-Simha metric on $\omega_{X/Y}$ has semi-positive curvature.*

Proof. Keep the notation introduced during the proof of Proposition 30.1. Because the local weight function φ_m is continuous, it suffices to prove that φ_m satisfies the mean-value inequality for mappings from the one-dimensional unit disk Δ into U. If the image of Δ lies in a single fiber, this is okay, because we already know from Proposition 28.3 that φ_m is plurisubharmonic on each fiber. So assume from now on that the mapping from Δ to Y is non-constant. Since the morphism f is smooth, we can then make a base change and reduce the problem to the case where $Y = \Delta$ and where $i \colon \Delta \hookrightarrow X$ is a section of $f \colon X \to \Delta$.

Now let $x_0 = i(0)$ and $X_0 = f^{-1}(0)$, and choose some $u \in H^0(X_0, \omega_{X_0}^{\otimes m})$ with $\ell_0(u) = 1$ that computes $\varphi_m(x_0)$. By Theorem 29.1, there exists an m-canonical form $v \in H^0(X, \omega_X^{\otimes m})$ with $v|_{X_0} = u \wedge df^{\otimes m}$, whose length satisfies the inequality

$$\ell(v) \leq \mu(\Delta)^{m/2} \cdot \ell_0(u) = \pi^{m/2}.$$

In the notation introduced during the proof of Proposition 30.1, we then have

$$\varphi_m(x_0) = \frac{2}{m} \log\bigl|g_v(x_0)\bigr|.$$

If we define $v_y \in H^0(X_y, \omega_{X_y}^{\otimes m})$ by the formula $v|_{X_y} = v_y \wedge df^{\otimes m}$, then we have

$$\ell(v)^{2/m} = \int_X (c_n^m v \wedge \overline{v})^{1/m} = \int_\Delta \ell_y(v_y)^{2/m}\, d\mu.$$

Now we observe that for almost every $y \in \Delta$, the ratio $v_y/\ell_y(v_y)$ is an m-canonical form on X_y of unit length; by definition of the weight function φ_m, we have

$$\varphi_m(x) \geq \frac{2}{m}\Bigl(\log|g_v(x)| - \log \ell_y(v_y)\Bigr) = \frac{2}{m}\log|g_v(x)| - \log \ell_y(v_y)^{2/m}.$$

If we now compute the mean value of $\varphi_m \circ i$ over Δ, we find that

$$\frac{1}{\pi}\int_\Delta \varphi_m(i(y))\, d\mu \geq \frac{1}{\pi}\int_\Delta \frac{2}{m}\log\bigl|g_v(i(y))\bigr|\, d\mu - \frac{1}{\pi}\int_\Delta \log \ell_y(v_y)^{2/m}\, d\mu.$$

The first term on the right is greater or equal to $2/m \log|g_v(x_0)| = \varphi_m(x_0)$, because the function $g_v \circ i$ is holomorphic. To estimate the remaining integral, note that

$$\frac{1}{\pi}\int_\Delta \log \ell_y(v_y)^{2/m}\, d\mu \leq \log\left(\frac{1}{\pi}\int_\Delta \ell_y(v_y)^{2/m}\, d\mu\right) = \log\left(\frac{1}{\pi}\cdot \ell(v)^{2/m}\right) \leq 0,$$

by Jensen's inequality and the fact that $\ell(v) \leq \pi^{m/2}$. Consequently, φ_m does satisfy the required mean-value inequality, and h_m has semi-positive curvature. □

Note. Compare also Lemma 7 and Lemma 8 in [Kaw82].

After these preparations, we can now prove Theorem 27.1 in general.

Proof of Theorem 27.1. Suppose that $f\colon X \to Y$ is a projective morphism between two complex manifolds with $f(X) = Y$. Let $Z \subseteq Y$ denote the closed analytic subset where f fails to be submersive. We already know that the restriction of the line bundle $\omega_{X/Y}$ to $f^{-1}(Y \setminus Z)$ has a well-defined singular hermitian metric h_m that is continuous and has semi-positive curvature. To show that h_m extends to a singular hermitian metric with semi-positive curvature on all of X, all we need to prove is that the local weights of h_m remain bounded near $f^{-1}(Z)$; this is justified by Lemma 12.4. Păun and Takayama [PT14, Theorem 4.2.7] observed that this local boundedness again follows very easily from the Ohsawa-Takegoshi theorem for pluricanonical forms.

Fix a point $x_0 \in X$ with $f(x_0) \in Z$. Since the problem is local on Y, we may assume that $Y = B$ is the open unit ball in $\mathbb{C}^r$, with coordinates $t_1, \dots, t_r$, and that $f(x_0) = 0$. On a suitable neighborhood U of the point x_0, we have coordinates $z_1, \dots, z_n$; note that because f is most likely not submersive at x_0, we cannot assert that $t_1, \dots, t_r$ are part of this coordinate system. Let $s_0 \in H^0(U, \omega_{X/Y})$ be a local trivialization of $\omega_{X/Y}$, chosen so that

$$dz_1 \wedge \cdots \wedge dz_n = s_0 \wedge (dt_1 \wedge \cdots \wedge dt_r).$$

Denote by φ_m the weight function of h_m with respect to this local trivialization:

$$\varphi_m(x) = -\log|s_0|_{h_m}^2 \colon U \to [-\infty, +\infty)$$

For $v \in H^0(X, \omega_X^{\otimes m})$, we have $v|_U = g_v \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}$ for a holomorphic function $g_v\colon U \to \mathbb{C}$. As explained during the proof of Proposition 30.1, the

Ohsawa-Takegoshi theorem for pluricanonical forms implies that there is a constant $C \geq 0$ with the following property: for every $x \in U$, there is some $v \in H^0(X, \omega_X^{\otimes m})$ of length $\ell(v) \leq C$ such that

$$\varphi_m(x) = \frac{2}{m} \log\lvert g_v(x)\rvert.$$

For x sufficiently close to x_0, there is a positive number $R > 0$ such that U contains the closed ball of radius R centered at x. The mean-value inequality and the fact that $\ell(v) \leq C$ now combine to give us an upper bound for $\varphi_m(x)$ that depends only on C and R, but is independent of the point x. In particular, φ_m is uniformly bounded in a neighborhood of the point $x_0 \in f^{-1}(Z)$, and therefore extends uniquely to a plurisubharmonic function on all of U.

The Narasimhan-Simha metric on each fiber X_y with $y \notin Z$ satisfies (28.7); by the Ohsawa-Takegoshi theorem, this means that the inclusion

$$f_*\big(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\big) \hookrightarrow f_*\big(\omega_{X/Y} \otimes L\big) = f_*\omega_{X/Y}^{\otimes m}$$

is an isomorphism over $Y \setminus Z$. Due to Corollary 21.4, the singular hermitian metric on $f_*\omega_{X/Y}^{\otimes m}$ is therefore finite and continuous on $Y \setminus Z$. □

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