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Primitive cohomology and the tube mapping

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Abstract Let X be a smooth complex projective variety of dimension d. We show that its primitive cohomology in degree d is generated by certain "tube classes," constructed from the monodromy in the family of all hyperplane sections of X. The proof makes use of a result about the group cohomology of certain representations that may be of independent interest.

Keywords Primitive cohomology · Lefschetz pencil · Monodromy · Group cohomology

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1 Introduction

Let X be a complex projective manifold. When X is embedded into projective space, there is a close relationship between the cohomology of X and that of any smooth hyperplane section $S = X \cap H$; this is the content of the Lefschetz Hyperplane Theorem. In fact, the only piece of the cohomology of X that cannot be inferred from that of S is the *primitive cohomology* in degree $d = \dim X$,

$$H_0^d(X, \mathbb{Q}) = \ker \left(H^d(X, \mathbb{Q}) \to H^d(S, \mathbb{Q}) \right),$$

which consists of those dth cohomology classes on X that restrict to zero on any smooth hyperplane section.

By definition, it is not possible to obtain the primitive cohomology of X from a single smooth hyperplace section; on the other hand, a consequence of Nori's famous Connectivity Theorem [7, Corollary 4.4 on p. 364] is the isomorphism

$$H_0^d(X, \mathbb{Q}) \simeq H^1 \left(P^{sm}, R_{van}^{d-1} \pi_*^{sm} \mathbb{Q} \right), \tag{1}$$

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which describes the primitive cohomology using the family $\pi^{sm}\colon \mathscr{S}^{sm} \to P^{sm}$ of *all* smooth hyperplane sections of X. For this to be true, the degree of the embedding of X into projective space has to be sufficiently large.

In this paper, we explain another very concrete way to obtain the primitive cohomology of X from the monodromy in the family of its smooth hyperplane sections. As above, let P^{sm} be the Zariski-open subset of the space of hyperplanes (in the ambient projective space), consisting of hyperplanes H such that $X \cap H$ is smooth. Since any two smooth hyperplane sections are diffeomorphic to each other, it is possible to transport homology classes among nearby ones; this gives rise to an action of the fundamental group $G = \pi_1(P^{sm}, H_0)$ on the homology groups of any smooth hyperplane section $S_0 = X \cap H_0$ (see [9, Chapitre 15]).

The flat transport of homology classes can also be used to produce elements of $H_d(X, \mathbb{Q})$. Namely, suppose a homology class $\alpha \in H_{d-1}(S_0, \mathbb{Q})$ is invariant under the action of an element $g \in G$. When α is transported along a closed path representing g, it moves through a one-dimensional family of hyperplane sections, and in the process, traces out a d-chain on X. This d-chain is a d-cycle, because $g \cdot \alpha = \alpha$. Taking the ambiguities in the construction into account, we get a well-defined element of the quotient $H_d(X, \mathbb{Q})/H_d(S_0, \mathbb{Q})$; we shall call it the tube class determined by g and α. Under Poincaré duality, the quotient $H_d(X, \mathbb{Q})/H_d(S_0, \mathbb{Q})$ is isomorphic to the primitive cohomology of X, and we obtain the *tube mapping*

$$\bigoplus_{g \in G} \left\{ \alpha \in H^{d-1}_{van}(S_0, \mathbb{Q}) \;\middle|\; g \cdot \alpha = \alpha \right\} \to H^d_0(X, \mathbb{Q}). \tag{2}$$

Here $H^{d-1}_{van}(S_0, \mathbb{Q})$ is the vanishing cohomology of the hypersurface S_0 (see Sect. 3 for basic definitions). We shall prove that the tube mapping is surjective, provided that the left-hand side is nontrivial.

Theorem 1 *Let X be a smooth complex projective variety of dimension d, with a given embedding into projective space. As above, let P^{sm} be the set of hyperplanes H such that the scheme-theoretic intersection $X \cap H$ is smooth. Let $S_0 = X \cap H_0$ be the hypersurface corresponding to some base point $H_0 \in P^{sm}$, and write $G = \pi_1(P^{sm}, H_0)$ for the fundamental group of P^{sm}. If $H^{d-1}_{van}(S_0, \mathbb{Q}) \neq 0$, then the tube mapping in (2) is surjective.*

This gives a positive answer to a question by H. Clemens. Although there are examples of smooth projective varieties for which $H^{d-1}_{van}(X \cap H_0, \mathbb{Q}) = 0$ (e.g., any smooth even-dimensional quadric), the condition is almost always satisfied: when the dimension of X is odd, it holds for essentially any embedding of X into projective space; when the dimension is even, it holds as long as the degree of the embedding is sufficiently high (Dimca and Saito [4, Theorem 6] have recently proved that an embedding by the third power of a very ample line bundle is sufficient).

2 Proof of the main theorem

We now give the proof of Theorem 1, referring to later sections for details.

2.1 Dual formulation

Generally speaking, it is easier to prove that a map is injective than to prove that it is surjective. With this in mind, we consider the mapping dual to (2), i.e.,

$$\mathrm{Hom}_{\mathbb{Q}}\left(H_0^d(X,\mathbb{Q}),\mathbb{Q}\right) \to \prod_{g\in G} \mathrm{Hom}_{\mathbb{Q}}\left(\ker(g-\mathrm{id}),\mathbb{Q}\right),$$

where each g acts on the space $V_{\mathbb{Q}} = H_{van}^{d-1}(S_0,\mathbb{Q})$. We now put this map into a more understandable form by using the intersection pairings on X and S_0. To begin with, the vanishing cohomology is self-dual under the intersection pairing on S_0; since the pairing is moreover G-invariant, we then have

$$\mathrm{Hom}_{\mathbb{Q}}\left(\ker(g-\mathrm{id}),\mathbb{Q}\right) \simeq \mathrm{coker}(g-\mathrm{id}) \simeq V_{\mathbb{Q}}/(g-\mathrm{id})V_{\mathbb{Q}}$$

for every $g\in G$. Similarly, we get

$$\mathrm{Hom}_{\mathbb{Q}}\left(H_0^d(X,\mathbb{Q}),\mathbb{Q}\right) \simeq H_0^d(X,\mathbb{Q})$$

by using the intersection pairing on X. The dual of the tube mapping is therefore

$$H_0^d(X,\mathbb{Q}) \to \prod_{g\in G} V_{\mathbb{Q}}/(g-\mathrm{id})V_{\mathbb{Q}}. \tag{3}$$

A simple linear algebra argument shows that surjectivity of the tube mapping is equivalent to the injectivity of the map in (3).

The main advantage to this point of view is that the map (3) can be factored into three simpler maps, given in (4), (5), and (7) below. We now discuss each of the three in turn.

2.2 The first map

The first step is to look at the topology of the family of all smooth hyperplane sections $\pi^{sm}\colon \mathscr{S}^{sm} \to P^{sm}$. From the projection $\mathscr{S}^{sm}\to X$, we have a pullback map $H^d(X,\mathbb{Q}) \to H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right)$. Now consider the Leray spectral sequence for π^{sm}, whose E_2-page is

$$E_2^{p,q} = H^p\left(P^{sm}, R^q\pi_*^{sm}\mathbb{Q}\right) \Longrightarrow H^{p+q}\left(P^{sm},\mathbb{Q}\right).$$

Here $R^q\pi_*^{sm}\mathbb{Q}$ is the local system on P^{sm} with fiber $H^q(S_0,\mathbb{Q})$. The spectral sequence degenerates at E_2 by Deligne's theorem [9, p. 379], because π^{sm} is smooth and projective. Letting $L^\bullet H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right)$ be the induced filtration on the cohomology of $\mathscr{S}^{sm}$, we see in particular that

$$H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right)/L^1H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right) \simeq E_2^{0,d} = H^0\left(P^{sm}, R^d\pi_*^{sm}\mathbb{Q}\right)$$

and

$$L^1H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right)/L^2H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right) \simeq E_2^{1,d} = H^1\left(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}\right).$$

By definition, primitive cohomology classes on X restrict to zero on every fiber of π^{sm}, and therefore go to zero in $E_2^{0,d}$. This means that $H_0^d(X,\mathbb{Q})$ is mapped into $L^1H^d\left(\mathscr{S}^{sm},\mathbb{Q}\right)$. Composing with the projection to $E_2^{1,d}$, we obtain a map

$$H_0^d(X,\mathbb{Q}) \to H^1\left(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}\right).$$

From the decomposition (11), we have $R^{d-1}\pi_*^{sm}\mathbb{Q} = H^{d-1}(X,\mathbb{Q}) \oplus R_{van}^{d-1}\pi_*^{sm}\mathbb{Q}$, where the first summand is constant. Now $H^1\left(P^{sm},\mathbb{Q}\right)=0$, because it is Poincaré dual to the space $H_{2N-1}\left(P, P-P^{sm},\mathbb{Q}\right)$, which vanishes because $H_{2N-1}(P,\mathbb{Q})=0$ and because $P\setminus P^{sm}$ is irreducible (here $N=\dim P$). Thus we find that

$$H^1\left(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}\right) \simeq H^1\left(P^{sm}, R_{van}^{d-1}\pi_*^{sm}\mathbb{Q}\right),$$

and so we obtain the first map in its final form as

$$H_0^d(X, \mathbb{Q}) \to H^1\left(P^{sm}, R_{van}^{d-1}\pi_*^{sm}\mathbb{Q}\right). \tag{4}$$

We shall prove in Sect. 3 that (4) is injective, provided $H_{van}^{d-1}(S_0, \mathbb{Q}) \neq 0$. This is a simple consequence of the topology of Lefschetz pencils on X. If one is willing to assume that the embedding of X into projective space is of sufficiently high degree, it also follows directly from the isomorphism in (1).

2.3 The second map

The second step is to represent the cohomology of the local system $R_{van}^{d-1}\pi_*^{sm}\mathbb{Q}$ by group cohomology. The group in question is, of course, the fundamental group $G = \pi_1(P^{sm}, H_0)$, which acts on $H_{van}^{d-1}(S_0, \mathbb{Q})$ through monodromy.

In general, given a group G and a G-module M, the ith group cohomology is defined as

$$H^i(G, M) = \operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M)$$

in the category of $\mathbb{Z}G$-modules [10, p. 161]. In particular, $H^0(G, M) = M^G$ is the submodule of G-invariant elements. The first cohomology $H^1(G, M)$, which is all we shall use, can be described explicitly as a quotient $Z^1(G, M)/B^1(G, M)$, where

$$Z^1(G, M) = \left\{\phi\colon G \to M \;\middle|\; \phi(gh) = g \cdot \phi(h) + \phi(g) \text{ for all } g, h \in G\right\}$$

is the group of 1-cocyles, and

$$B^1(G, M) = \left\{\phi\colon G \to M \;\middle|\; \text{there is } x \in M \text{ such that } \phi(g) = g \cdot x - x\right\}$$

the group of 1-coboundaries for M.

There is a well-known correspondence between local systems and representations of the fundamental group [9, Corollaire 15.10 on p. 339]; similarly, there is a relationship between the cohomology of the local system and the group cohomology of the representation.

Lemma 1 *Let $\mathcal{M}$ be a local system on a connected topological space B. Let M be its fiber at some point $b_0 \in B$; it is the representation of the fundamental group $G = \pi_1(B, b_0)$ corresponding to $\mathcal{M}$. Assume that B has a universal covering space $\tilde{B} \to B$. Then there is a convergent spectral sequence*

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathbb{Z}G}\left(H_q(\tilde{B}, \mathbb{Z}), M\right) \Longrightarrow H^{p+q}(B, \mathcal{M}).$$

In particular, we have $H^1(B, \mathcal{M}) \simeq H^1(G, M)$.

Proof We sketch the simple proof. Let $S_\bullet(\tilde{B}, \mathbb{Z})$ be the singular chain complex of $\tilde{B}$; it is a complex of G-modules, because G acts on $\tilde{B}$ by deck transformations. According to Steenrod's original definition [8], the cohomology of the local system $\mathcal{M}$ is computed by the complex $\operatorname{Hom}_{\mathbb{Z}G}(S_\bullet(\tilde{B}, \mathbb{Z}), M)$. The spectral sequence in question comes from the double complex $\operatorname{Hom}_{\mathbb{Z}G}(S_\bullet(\tilde{B}, \mathbb{Z}), I^\bullet)$, where $I^\bullet$ is any injective resolution of M in the category of $\mathbb{Z}G$-modules. The second assertion follows immediately from the spectral sequence, because $H_0(\tilde{B}, \mathbb{Z}) \simeq \mathbb{Z}$, while $H_1(\tilde{B}, \mathbb{Z}) = 0$.

In our case, the local system $R^{d-1}_{van}\pi^{sm}_*\mathbb{Q}$ corresponds to the G-module $V_{\mathbb{Q}} = H^{d-1}_{van}(S_0, \mathbb{Q})$. Thus Lemma 1 gives us an isomorphism

$$H^1\left(P^{sm}, R^{d-1}_{van}\pi^{sm}_*\mathbb{Q}\right) \simeq H^1\left(G, V_{\mathbb{Q}}\right) \tag{5}$$

with the first group cohomology of $V_{\mathbb{Q}}$.

2.4 The third map

The third step is of a purely algebraic nature. Namely, for any G-module M, we have a restriction map

$$H^1(G, M) \to \prod_{g \in G} H^1\left(g^{\mathbb{Z}}, M\right),$$

where $g^{\mathbb{Z}}$ is the cyclic subgroup generated by g. From the explicit description, it is easy to see that $H^1\left(g^{\mathbb{Z}}, M\right) \simeq M/(g - \mathrm{id})M$. The resulting map

$$H^1(G, M) \to \prod_{g \in G} M/(g - \mathrm{id})M \tag{6}$$

takes the class of a 1-cocycle ϕ to the element with components $\phi(g) + (g - \mathrm{id})M$ in the product.

In the case at hand, where G is the fundamental group of P^{sm}, the G-module is $V_{\mathbb{Q}}$, and the restriction map becomes

$$H^1\left(G, V_{\mathbb{Q}}\right) \to \prod_{g \in G} V_{\mathbb{Q}}/(g - \mathrm{id})V_{\mathbb{Q}}. \tag{7}$$

Unfortunately, (6) fails to be injective for general M (an example is given below, in Example 1); nevertheless, we shall prove its injectivity for certain G-modules, in particular for the vanishing cohomology $V_{\mathbb{Q}}$.

It should be noted that, as a representation of G, the nature of $V_{\mathbb{Q}}$ is very different for even and odd values of d. This is because the intersection pairing is symmetric when d is odd, but alternating when d is even. Consequently, the proof that (7) is injective has to be different in the two cases. When d is odd, it is a straightforward calculation, given in Sect. 4. When d is even, we show that $H^{d-1}_{van}(S_0, \mathbb{Z})$, modulo torsion, is a vanishing lattice [5]. We can then use results by W. Janssen about the structure of vanishing lattices to prove the injectivity. Details can be found in Sect. 5.

2.5 Conclusion of the proof

Composing the three maps in (4), (5), and (7), we finally obtain an injective map

$$H^d_0(X, \mathbb{Q}) \to \prod_{g \in G} V_{\mathbb{Q}}/(g - \mathrm{id})V_{\mathbb{Q}}. \tag{8}$$

It remains to verify that this map really is the dual of the tube mapping in (3). This is sufficient to complete the proof of the theorem, because the injectivity of (8) is then equivalent to the surjectivity of the tube mapping by simple linear algebra.

So let $g \in G$ be an arbitrary element of the fundamental group of P^{sm}, and let $\alpha \in H_{d-1}(S_0, \mathbb{Q})$ be any class invariant under the action by g. We write $\tau_g(\alpha) \in H_d(X, \mathbb{Q})$ for the tube class determined by α; as we saw, it is well-defined up to the addition of elements in $H_d(S_0, \mathbb{Q})$. Take any closed d-form ω on X, whose class lies in $H_0^d(X, \mathbb{Q})$. Under the mapping

$$H_0^d(X, \mathbb{Q}) \to \prod_{g \in G} H_{van}^{d-1}(S_0, \mathbb{Q})/(g - \mathrm{id}) H_{van}^{d-1}(S_0, \mathbb{Q}).$$

in (8), ω is sent to an element of the product with coordinates $\big(\lambda_g(\omega) + \mathrm{im}(g - \mathrm{id})\big)$. Of course, $\lambda_g(\omega)$ itself is not uniquely determined by ω; we choose this notation only because the ambiguity turns out not to matter.

To prove that the map in (8) really is the dual of the tube mapping, it suffices to establish the identity

$$\int_{\tau_g(\alpha)} \omega = \int_{\alpha} \lambda_g(\omega) \tag{9}$$

To do this, represent g by an immersion $\mathbb{S}^1 \to P^{sm}$, and let $f : Y \to \mathbb{S}^1$ be the pullback of the family $\pi^{sm} : \mathscr{S}^{sm} \to P^{sm}$. Then Y is a smooth manifold of dimension $m = 2d - 1$. We have the following diagram of maps:

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "f"'] & \mathscr{S}^{sm} \arrow[r, "q"] \arrow[d, "\pi^{sm}"'] & X \\ \mathbb{S}^1 \arrow[r, "g"] & P^{sm} & \end{tikzcd}$$

The fiber over the base point of $\mathbb{S}^1$ is $Y_0 = S_0$, in the notation used above. Then $\alpha \in H_{d-1}(S_0, \mathbb{Q})$ determines a tube class $\tau(\alpha)$ on Y, and by the definition of the tube mapping, we have

$$\tau_g(\alpha) \equiv (qh)_* \tau(\alpha) \mod H_d(S_0, \mathbb{Q}).$$

Since ω is primitive, its restriction to S_0 is trivial. Lemma 2 below, applied to the class $(qh)^*\omega$, shows that

$$\int_{\tau_g(\alpha)} \omega = \int_{\tau(\alpha)} (qh)^*\omega = \int_{\alpha} \lambda\left((qh)^*\omega\right).$$

The class $\lambda\left((qh)^*\omega\right)$ is determined by the Leray spectral sequence for the map f. On the other hand, the class $\lambda_g(\omega)$ is determined in exactly the same way by the Leray spectral sequence for π^{sm}. But both spectral sequences are compatible with each other, starting from the E_2-page, and so it has to be the case that

$$\lambda\left((qh)^*\omega\right) \equiv (qh)^*\lambda_g(\omega) \mod (g - \mathrm{id}) H^{d-1}(S_0, \mathbb{Q}).$$

Now α is g-invariant, and its integral against any element of $(g - \mathrm{id}) H^{d-1}(S_0, \mathbb{Q})$ is therefore zero. It follows that

$$\int_{\alpha} \lambda\left((qh)^*\omega\right) = \int_{\alpha} (qh)^*\lambda_g(\omega) = \int_{\alpha} \lambda_g(\omega).$$

After combining this with the other equality, we obtain (9).

2.6 Smooth families over the circle

The given proof above depends on an analysis of the tube mapping in the case of a family of smooth manifolds over $\mathbb{S}^1$; we now go over the details of that step. As above, let $f: Y \to \mathbb{S}^1$ be a proper and submersive map of smooth manifolds. Let m be the real dimension of Y, and let Y_0 be the fiber of f over the point $1 \in \mathbb{S}^1$. The cohomology of Y and Y_0 can be represented by smooth differential forms, and this will be done throughout.

The fundamental group of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$, and acts by monodromy on the homology and cohomology of Y_0. To get an explicit description, note that the pullback of Y to the universal covering space $e: \mathbb{R} \to \mathbb{S}^1$, $t \mapsto \exp(2\pi i t)$, is diffeomorphic to $\mathbb{R} \times Y_0$; the following diagram shows the relevant maps.

$$\begin{tikzcd}
Y_0 \arrow[r, hook, "i_t"] \arrow[d] & Y_0 \times \mathbb{R} \arrow[r, "\Phi"] \arrow[d] & Y \arrow[d, "f"] \\
\{t\} \arrow[r, hook] & \mathbb{R} \arrow[r, "e"] & \mathbb{S}^1
\end{tikzcd}$$

The composition $\Phi \circ i_0$ is simply the inclusion of Y_0 into Y, while $\Phi \circ i_1$ gives a different embedding of Y_0 into Y. We write $F: Y_0 \to Y_0$ be the resulting diffeomorphism, so that $\Phi \circ i_1 = \Phi \circ i_0 \circ F$.

For a homology class $\alpha \in H_i(Y_0, \mathbb{Q})$, a flat translate of α to the fiber over $e(t)$ is given by $(\Phi \circ i_t)(\alpha)$. The monodromy action $T_i: H_i(Y_0, \mathbb{Q}) \to H_i(Y_0, \mathbb{Q})$ by the standard generator is therefore $T_i(\alpha) = (\Phi \circ i_1)_* \alpha = F_* \alpha$. Similarly, we have an action on cohomology, $T^i: H^i(Y_0, \mathbb{Q}) \to H^i(Y_0, \mathbb{Q})$.

Tube classes on Y are defined in the following way. Suppose that $\alpha \in \ker T_{k-1}$ is a monodromy-invariant homology class on Y_0. This means that there is a k-chain A on Y_0, such that $\partial A = F(\alpha) - \alpha$. Translating α flatly along $\mathbb{S}^1$ and taking the trace in Y gives the k-chain $\Gamma = \Phi(\alpha \times [0, 1])$. Then $\Gamma - A$ is closed, and its class $\tau(\alpha) \in H_k(Y, \mathbb{Q})$ is the tube class determined by α. Of course, $\tau(\alpha)$ is only defined up to elements of $H_d(Y_0, \mathbb{Q})$, because of the ambiguity in choosing A.

We now have to connect this topological construction with the one coming from the Leray spectral sequence for the map f. The latter degenerates at E_2, and gives us for each $k \geq 0$ a short exact sequence

$$0 \longrightarrow H^1\big(\mathbb{S}^1, R^{k-1} f_* \mathbb{Q}\big) \longrightarrow H^k(Y, \mathbb{Q}) \longrightarrow H^0\big(\mathbb{S}^1, R^k f_* \mathbb{Q}\big) \longrightarrow 0. \tag{10}$$

It is well-known that $H^1\left(\mathbb{S}^1, R^{k-1} f_* \mathbb{Q}\right) \simeq \operatorname{coker} T^{k-1}$, and $H^0\left(\mathbb{S}^1, R^k f_* \mathbb{Q}\right) \simeq \ker T^k$. Now suppose we are given a cohomology class in $H^k(Y, \mathbb{Q})$ whose restriction to the fibers of f is trivial. By virtue of (10), it defines a class in $H^1\left(\mathbb{S}^1, R^{k-1} f_* \mathbb{Q}\right)$, and hence in $\operatorname{coker} T^{k-1}$. The following lemma gives a formula for this class.

Lemma 2 *Let β be a smooth and closed k-form on Y, representing an element of* $\ker\left(H^k(Y, \mathbb{Q}) \to H^k(Y_0, \mathbb{Q})\right)$. *Choose any $(k-1)$-form γ on $Y_0 \times \mathbb{R}$ with $\Phi^* \beta = d\gamma$, and let $\gamma_t = i_t^* \gamma$.*

(i) *The element of* $\operatorname{coker} T^{k-1}$ *determined by β is $\lambda(\beta) = (F^{-1})^* \gamma_1 - \gamma_0$.*

(ii) *For every monodromy-invariant class* $\alpha \in H^{k-1}(Y_0, \mathbb{Q})$, *we have*

$$\int_{\tau(\alpha)} \beta = \int_{\alpha} \lambda(\beta).$$

where $\tau(\alpha)$ *is the tube class on* Y *coming from* α.

Proof Note that $\lambda(\beta)$ is a closed $(k-1)$-form on S_0. Let m be the dimension of the smooth manifold Y, and let $i \colon Y_0 \to Y$ be the inclusion map. For every closed $(m-k)$-form ω on Y, a simple calculation using Stokes' Theorem shows that

$$\begin{aligned}
\int_Y \beta \wedge \omega &= \int_{Y_0 \times [0,1]} \Phi^* \beta \wedge \Phi^* \omega = \int_{Y_0 \times [0,1]} d\gamma \wedge \Phi^* \omega \\
&= \int_{Y_0 \times [0,1]} d\left(\gamma \wedge \Phi^* \omega\right) = \int_{Y_0} i_1^* \left(\gamma \wedge \Phi^* \omega\right) - \int_{Y_0} i_0^* \left(\gamma \wedge \Phi^* \omega\right) \\
&= \int_{Y_0} \gamma_1 \wedge F^* \left(i^* \omega\right) - \int_{Y_0} \gamma_0 \wedge i^* \omega = \int_{Y_0} \left((F^{-1})^* \gamma_1 - \gamma_0\right) \wedge i^* \omega.
\end{aligned}$$

The assertion in (i) now follows by duality.

To prove the second half, we recall that the tube class is given by $\Gamma - A$, where $\partial A = F(\alpha) - \alpha$ on Y_0, and $\Gamma = \Phi\left(\alpha \times [0,1]\right)$. Again using Stokes' Theorem, we compute that

$$\begin{aligned}
\int_{\alpha} \lambda(\beta) &= \int_{\alpha} (F^{-1})^* \gamma_1 - \gamma_0 = \int_{\alpha} (\gamma_1 - \gamma_0) + \int_{F^{-1}(\alpha) - \alpha} \gamma_1 \\
&= \int_{\partial(\alpha \times [0,1])} \gamma - \int_{F^{-1}(\partial A)} i_1^* \gamma = \int_{\alpha \times [0,1]} d\gamma - \int_{F^{-1}(A)} i_1^*(d\gamma) \\
&= \int_{\alpha \times [0,1]} \Phi^* \beta - \int_{F^{-1}(A)} i_1^* \left(\Phi^* \beta\right) = \int_{\Gamma} \beta - \int_{F^{-1}(A)} F^* \beta.
\end{aligned}$$

This equals $\int_{\Gamma} \beta - \int_A \beta = \int_{\tau(\alpha)} \beta$, proving the identity in (ii).

3 Topology of the universal hypersurface

The main purpose of this section is to show that the map in (4) is injective, as long as $V_{\mathbb{Q}} = H_{van}^{d-1}(S_0, \mathbb{Q}) \neq 0$. As we have seen, this is the same as showing the injectivity of the map

$$H_0^d(X, \mathbb{Q}) \to H^1\left(P^{sm}, R^{d-1} \pi_*^{sm} \mathbb{Q}\right),$$

derived from the Leray spectral sequence. Along the way, we need to review several results about the vanishing cohomology of S_0 that are obtained by studying Lefschetz pencils on X. Throughout, we shall assume that $V_{\mathbb{Q}} \neq 0$.

3.1 Review of the Lefschetz theorems

A comprehensive discussion of the relationship between the cohomology of X and that of a smooth hyperplane section $S = X \cap H$ can be found, for instance, in the book by C. Voisin [9, Section 13]. We only give a very brief outline of the main points. Let us write $i: S \to X$ for the inclusion map; we also let $d = \dim X$ be the complex dimension of X. The complement $X \setminus S$ is a Stein manifold, and Morse theory shows that it has the homotopy type of a d-dimensional CW-complex. One consequence is the following result, known as the Lefschetz Hyperplane Theorem.

Theorem 2 *The restriction map $i^*: H^k(X, \mathbb{Z}) \to H^k(S, \mathbb{Z})$ is an isomorphism for $k < \dim S = d - 1$, and injective for $k = d - 1$. Moreover, the quotient group $H^{d-1}(S, \mathbb{Z})/H^{d-1}(X, \mathbb{Z})$ is torsion-free.*

Since Poincaré duality on X (resp. S) can be used to describe the cohomology groups in dimensions greater than d (resp. $d - 1$), there are only two pieces of the cohomology rings of X and S that are not covered by Lefschetz' theorem. One is the *primitive cohomology*

$$\begin{aligned} H_0^d(X, \mathbb{Z}) &= \ker\left(i^*: H^d(X, \mathbb{Z}) \to H^d(S, \mathbb{Z})\right) \\ &= \ker\left(L: H^d(X, \mathbb{Z}) \to H^{d+2}(X, \mathbb{Z})\right), \end{aligned}$$

where L is the Lefschetz operator, given by cup product with the fundamental class of S in $H^2(X, \mathbb{Z})$. The other is the *vanishing cohomology* of the hypersurface

$$H_{van}^{d-1}(S, \mathbb{Z}) = \ker\left(i_*: H^{d-1}(S, \mathbb{Z}) \to H^{d+1}(X, \mathbb{Z})\right).$$

The vanishing cohomology is dual to $\ker\,(i_*: H_{d-1}(S, \mathbb{Z}) \to H_{d-1}(X, \mathbb{Z}))$ under Poincaré duality, and it is known that the kernel is generated by the vanishing cycles of any Lefschetz pencil on X, thus explaining the name.

At least over $\mathbb{Q}$, one has direct sum decompositions

$$H^{d-1}(S, \mathbb{Q}) = i^* H^{d-1}(X, \mathbb{Q}) \oplus H_{van}^{d-1}(S, \mathbb{Q}) \tag{11}$$

and

$$H^d(X, \mathbb{Q}) = i_* H^{d-2}(S, \mathbb{Q}) \oplus H_0^d(X, \mathbb{Q}), \tag{12}$$

orthogonal with respect to the intersection pairings on S and X, respectively. This is part of the content of the Hard Lefschetz Theorem [9, Proposition 14.27 on p. 328]. With integer coefficients, the map

$$H_{van}^{d-1}(S, \mathbb{Z}) \to H^{d-1}(S, \mathbb{Z})/H^{d-1}(X, \mathbb{Z})$$

is unfortunately neither injective nor surjective in general.

3.2 Lefschetz pencils

Recall that P is the space of all hyperplanes (in the ambient projective space), and P^{sm} the subset of those H for which $X \cap H$ is smooth. The *dual variety* $X^\vee = P \setminus P^{sm}$ is the set of hyperplanes such that $X \cap H$ is singular. It is an irreducible subvariety of P; since we are assuming that the vanishing cohomology is nontrivial, it is actually a hypersurface, whose smooth points correspond to hyperplane sections of X with a single ordinary double point.

Lemma 3 *If $V_{\mathbb{Q}} \neq 0$, then $X^\vee$ is a hypersurface in P.*

Proof We will prove the converse: if $X^\vee$ is not a hypersurface, then necessarily $V_{\mathbb{Q}} = 0$. So let us suppose that the codimension of $X^\vee$ is at least two. Choose a line $\mathbb{P}^1 \subseteq P$ that does not meet $X^\vee$, and let $f : \tilde{X} \to \mathbb{P}^1$ be the restriction of the family of hyperplane sections to $\mathbb{P}^1$. Then f is smooth and projective, and since $\mathbb{P}^1$ is simply connected, all the local systems $R^q f_* \mathbb{Q}$ are constant, with fiber $H^q(S_0, \mathbb{Q})$. Now consider the Leray spectral sequence for the map $f : \tilde{X} \to \mathbb{P}^1$. By Deligne's theorem, it degenerates at E_2, and thus gives a short exact sequence

$$0 \longrightarrow H^2\big(\mathbb{P}^1, R^{d-3} f_* \mathbb{Q}\big) \longrightarrow H^{d-1}\big(\tilde{X}, \mathbb{Q}\big) \longrightarrow H^0\big(\mathbb{P}^1, R^{d-1} f_* \mathbb{Q}\big) \longrightarrow 0;$$

using that $R^q f_* \mathbb{Q}$ is constant, this amounts to the exactness of the first row in the following diagram. (All cohomology groups are with coefficients in $\mathbb{Q}$.)

$$\begin{tikzcd}
H^2\big(\mathbb{P}^1\big) \otimes H^{d-3}(S_0) \arrow[r, hook] & H^{d-1}\big(\tilde{X}\big) \arrow[r, two heads] & H^0\big(\mathbb{P}^1\big) \otimes H^{d-1}(S_0) \\
H^2\big(\mathbb{P}^1\big) \otimes H^{d-3}(X) \arrow[r, hook] \arrow[u, "\simeq"'] & H^{d-1}\big(\mathbb{P}^1 \times X\big) \arrow[r, two heads] \arrow[u, "\simeq"'] & H^0\big(\mathbb{P}^1\big) \otimes H^{d-1}(X) \arrow[u]
\end{tikzcd}$$

The two vertical maps are isomorphisms because of the Hyperplane Theorem. Indeed, we have already seen that $H^{d-3}(X, \mathbb{Q}) \simeq H^{d-3}(S_0, \mathbb{Q})$. On the other hand, $\tilde{X} \subseteq \mathbb{P}^1 \times X$ is itself a smooth very ample hypersurface of dimension d, and so we also have $H^{d-1}\big(\mathbb{P}^1 \times X, \mathbb{Q}\big) \simeq H^{d-1}(\tilde{X}, \mathbb{Q})$. It follows that $H^{d-1}(X, \mathbb{Q}) \simeq H^{d-1}(S_0, \mathbb{Q})$, which means that $V_{\mathbb{Q}} = H^{d-1}_{van}(S_0, \mathbb{Q})$ is reduced to zero.

Now take any Lefschetz pencil of hyperplane sections of X containing S_0; in other words, a line $\mathbb{P}^1 \subseteq P$ through the base point $H_0 \in P$ that meets $X^\vee$ transversely in finitely many points. Also let $B = \mathbb{P}^1 \cap P^{sm}$ be the smooth locus of the pencil, and let $0 \in B$ be the point whose image is H_0. We write $\tilde{X} \to \mathbb{P}^1$ for the restriction of the family $\mathscr{S} \to P$ to the line, and $U \subseteq \tilde{X}$ for the part that lies over B. The following diagram shows the relevant maps; all diagonal arrows are inclusions of open subsets.

$$\begin{tikzcd}
& \tilde{X} \arrow[rr] \arrow[dd] & & \mathscr{S} \arrow[rr, "q"] \arrow[dd, "\pi"] & & X \\
U \arrow[ur] \arrow[rr] \arrow[dd, "f"'] & & \mathscr{S}^{sm} \arrow[ur] \arrow[dd] & & & \\
& \mathbb{P}^1 \arrow[rr] & & P & & \\
B \arrow[ur] \arrow[rr] & & P^{sm} \arrow[ur] & & &
\end{tikzcd}$$

We know from Lemma 3 that $D = \mathbb{P}^1 \cap X^\vee$ is nonempty; say $D = \{t_1, \ldots, t_n\}$, with all t_i distinct and different from the base point 0. Let S_i be the hyperplane section of X corresponding to the point t_i; each S_i has a single ordinary double point (ODP). From a local analysis around an ODP singularity, it is known that S_0 contains an embedded $(d-1)$-sphere for each i, the so-called *vanishing cycle* for the singularity on S_i. Moreover, the homotopy type of S_i is that of S_0 with a d-cell attached along the vanishing cycle [9, p. 322].

The vanishing homology $\ker(i_*\colon H_{d-1}(S_0,\mathbb{Z}) \to H_{d-1}(X,\mathbb{Z}))$ is generated over $\mathbb{Z}$ by the classes of these spheres [9, Lemme 14.26 on p. 327]. Writing e_i for the cohomology class Poincaré dual to the ith vanishing cycle, the e_i thus generate the vanishing cohomology with integer coefficients

$$H^{d-1}_{van}(S_0,\mathbb{Z}) = \ker\left(i_*\colon H^{d-1}(S_0,\mathbb{Z}) \to H^{d+1}(X,\mathbb{Z})\right).$$

Since $X^\vee$ is irreducible, it is further known [9, Corollaire 15.24 on p. 353] that all the e_i lie in one orbit of the monodromy action of $\pi_1(B,0)$ on $H^{d-1}(S_0,\mathbb{Z})$. In particular, we have $e_i \neq 0$ in $H^{d-1}_{van}(S_0,\mathbb{Q})$, because we are assuming that the latter is nontrivial.

Lemma 4 *Classes in $H^d_0(X,\mathbb{Q})$ have trivial restriction to each of the singular hyperplane sections S_i.*

Proof The singular hyperplane section S_i is homotopy-equivalent to S_0 with a d-cell attached along the ith vanishing cycle. From the Mayer–Vietoris sequence in cohomology, we thus get an exact sequence isomorphic to

$$H^{d-1}(S_0,\mathbb{Q}) \longrightarrow H^{d-1}\left(\mathbb{S}^{d-1},\mathbb{Q}\right) \longrightarrow H^d(S_i,\mathbb{Q}) \longrightarrow H^d(S_0,\mathbb{Q}) \longrightarrow 0.$$

Since $e_i \neq 0$, the first map in the sequence is nontrivial, and so $H^d(S_i,\mathbb{Q}) \simeq H^d(S_0,\mathbb{Q})$. In particular, every primitive cohomology class on X has trivial restriction to S_i.

Lemma 5 *The pullback map $H^d_0(X,\mathbb{Q}) \to H^d(U,\mathbb{Q})$ is injective.*

Proof The complement of U in $\tilde{X}$ is the disjoint union of the singular fibers S_i; thus we have an exact sequence

$$\cdots \longrightarrow H^d_c(U,\mathbb{Q}) \longrightarrow H^d\left(\tilde{X},\mathbb{Q}\right) \longrightarrow \bigoplus_{i=1}^{n} H^d(S_i,\mathbb{Q}) \longrightarrow H^{d+1}_c(U,\mathbb{Q}) \longrightarrow \cdots$$

for cohomology with compact support. As U is a manifold,

$$H^d_c(U,\mathbb{Q}) \simeq \operatorname{Hom}\left(H^d(U,\mathbb{Q}),\mathbb{Q}\right),$$

with the isomorphism given by integration over U.

Now let $\omega \in H^d_0(X,\mathbb{Q})$ be any class whose pullback $q^*\omega$ has trivial restriction to U. The functional $H^d(\tilde{X},\mathbb{Q}) \to \mathbb{Q}$, given by integrating against $q^*\omega$, is then zero on $H^d_c(U,\mathbb{Q})$, and thus factors through the image of $H^d(\tilde{X},\mathbb{Q}) \to \bigoplus_i H^d(S_i,\mathbb{Q})$. Let $\lambda\colon \bigoplus_i H^d(S_i,\mathbb{Q}) \to \mathbb{Q}$ be any extension to the entire direct sum. For each $\alpha \in H^d_0(X,\mathbb{Q})$ we then have

$$\int_X \omega \cup \alpha = \int_{\tilde{X}} q^*\omega \cup q^*\alpha = \lambda\left(\alpha|_{S_1},\ldots,\alpha|_{S_n}\right) = 0$$

by Lemma 4. But the intersection pairing is nondegenerate on the primitive cohomology, and so $\omega = 0$.

3.3 Injectivity of the map

Now consider the Leray spectral sequence for the map $f: U \to B$. Since B has the homotopy-type of a bouquet of circles, the spectral sequence degenerates, and we get a short exact sequence

$$0 \longrightarrow H^1\big(B, R^{d-1}f_*\mathbb{Q}\big) \longrightarrow H^d(U, \mathbb{Q}) \longrightarrow H^0\big(B, R^d f_*\mathbb{Q}\big) \longrightarrow 0.$$

By definition, classes in $H_0^d(X, \mathbb{Q})$ go to zero in the group on the right; Lemma 5 then lets us conclude that the induced map

$$H_0^d(X, \mathbb{Q}) \to H^1\left(B, R^{d-1}f_*\mathbb{Q}\right)$$

has to be injective. This immediately implies the injectivity of (4). To see this, note that the functoriality of the Leray spectral sequence gives a factorization

$$H_0^d(X, \mathbb{Q}) \to H^1\left(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}\right) \to H^1\left(B, R^{d-1}f_*\mathbb{Q}\right);$$

since the composition is injective, the first map has to be injective, proving our claim.

3.4 Vanishing cycles with intersection number one

We have seen that the vanishing cohomology $H_{van}^{d-1}(S_0, \mathbb{Z})$ is generated by the Poincaré duals e_i of the vanishing cycles for any Lefschetz pencil. More generally, we shall refer to any element in the orbit $\Delta = G \cdot \{e_1, \ldots, e_n\}$ as a vanishing cycle. As shown above, all $\delta \in \Delta$ are nontrivial even as elements of $V_{\mathbb{Q}} = H_{van}^{d-1}(S_0, \mathbb{Q})$.

The fundamental group $\pi_1(B, 0)$ is isomorphic to a free group on $(n-1)$ letters; in fact, a set of generators is given by taking, for each $i = 1, \ldots, n$, a loop g_i based at 0 that goes exactly once around the point t_i with positive orientation, but not around any of the other t_j. The only relation is the obvious one, namely that $g_1 \ldots g_n = 0$. By Zariski's theorem, G itself is also generated by the g_i.

The monodromy action of each g_i on $H^{d-1}(S_0, \mathbb{Z})$ is described explicitly by the Picard–Lefschetz formula [9, Théorème 15.16 on p. 345]

$$g_i \cdot \alpha = \alpha - \varepsilon_d(\alpha, e_i)e_i, \tag{13}$$

where $(-,-)$ is the intersection pairing on S_0, and $\varepsilon_d = (-1)^{d(d-1)/2}$. This has different consequences for odd and even values of d:

(i) When d is odd, S_0 has even dimension, and the intersection pairing is symmetric. Moreover, each vanishing cycle has self-intersection number $2\varepsilon_d$, and g_i^2 acts trivially on $H^{d-1}(S_0, \mathbb{Z})$.
(ii) When d is even, S_0 has odd dimension, and the intersection pairing is skew-symmetric. Consequently, the self-intersection of e_i is zero, and the element g_i is of infinite order.

The same formulas are of course true for every vanishing cycle $\delta \in \Delta$.

To analyze the structure of $V_{\mathbb{Q}}$ for even values of d, we will need the following lemma about the set Δ. It is the main step in showing that $H_{van}^{d-1}(S_0, \mathbb{Z})$ is a skew-symmetric vanishing lattice [5].

Lemma 6 *Assume that* $d = \dim X$ *is even. Then there are two vanishing cycles* $\delta_1, \delta_2 \in \Delta$ *with* $(\delta_1, \delta_2) = 1$.

Proof As observed in [5, p. 132], it suffices to show that there is a singular hyperplane section $S' \subseteq X$ with an isolated singularity that is not an ordinary double point. Indeed, the vanishing homology of the Milnor fiber F of such a singularity embeds into $H_{d-1}^{van}(S_0, \mathbb{Z})$ by [1, p. 9], in such a way that vanishing cycles map to vanishing cycles. The Milnor fiber has the homotopy type of a bouquet of $(d-1)$-spheres; the number of spheres is the Milnor number μ of the singular point. If the singularity is not an ordinary double point, then $\mu \geq 2$, and so there are (at least) two independent vanishing cycles on F with intersection number one. We can then take δ_1 and δ_2 to be their images in $H_{d-1}^{van}(S_0, \mathbb{Z})$.

To find such a hyperplane section S', let $\mathbb{P}^2 \subseteq P$ be a general plane containing the base point, and $C = \mathbb{P}^2 \cap X^\vee$. Since the dual variety is irreducible, the curve C is also irreducible, and its only singularities are nodes and cusps. A node of C corresponds to a hyperplane section of X with two ordinary double points; a cusp corresponds to a hyperplane section with one isolated singularity of Milnor number two. To prove the lemma, it is therefore enough to show that C has at least one cusp.

Since both $\mathbb{P}^2$ and P are simply connected, it follows from the Lefschetz theorem for fundamental groups [6, Theorem 3.1.21] that the fundamental group of $\mathbb{P}^2 \backslash C$ is isomorphic to G. If C had only nodes and no cusps, then this group would be abelian [3], and hence a finite cyclic group since C is irreducible. In particular, the action of each vanishing cycle would be of finite order. Since d is even, this possibility is ruled out by our assumption that $H_{van}^{d-1}(S_0, \mathbb{Q}) \neq 0$.

4 Detecting group cohomology classes: the odd case

In this section, we show that the restriction map (7) is injective when d is odd. As it happens, this can be proved by using very little of the structure of the vanishing cohomology, and so we shall treat the problem abstractly first.

4.1 Injectivity of the restriction map

We consider a finite-dimensional $\mathbb{Q}$-vector space $V_\mathbb{Q}$ with a symmetric bilinear form $B\colon V_\mathbb{Q} \otimes V_\mathbb{Q} \to \mathbb{Q}$, and a finitely generated group G acting on $V_\mathbb{Q}$, subject to the following two assumptions:

1. There are distinguished elements $e_1, \ldots, e_n \in V_\mathbb{Q}$ with $B(e_i, e_i) = 2$.
2. There are generators $g_1, \ldots, g_n$ for G, such that

$$g_i \cdot v = v - B(v, e_i)e_i$$

   for all $v \in V_\mathbb{Q}$.

It follows that the action of G preserves the bilinear form, and that each g_i^2 acts trivially. In this situation, the restriction map is injective.

Proposition 1 *Let $V_\mathbb{Q}$ be a finite-dimensional $\mathbb{Q}$-vector space with an action by a group G, subject to the assumptions just stated. Then the restriction map*

$$H^1\left(G, V_\mathbb{Q}\right) \to \prod_{g \in G} V_\mathbb{Q}/(g - \mathrm{id})V_\mathbb{Q}$$

is injective.

We begin with a simple observation (easily proved by induction on n). To keep it general, we do not assume anything about the bilinear form; in this way, it can also be applied to the even case in the next section.

Lemma 7 *Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space with a bilinear form $B\colon V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$, and with an action by a group G. Assume that there are elements $g_1, \dots, g_n$ of G, and vectors $e_1, \dots, e_n \in V_{\mathbb{Q}}$, such that $g_i v = v - B(v, e_i)e_i$ holds for every $v \in V_{\mathbb{Q}}$. Let $\phi \in Z^1\left(G, V_{\mathbb{Q}}\right)$ be a 1-cocycle satisfying $\phi(g_i) = a_i e_i$ for all i. Then we have*

$$\phi(g_n \cdots g_1) = \sum_{k=1}^{n} b_k e_k,$$

and the coefficients $b_k \in \mathbb{Q}$ are determined by the recursive relations

$$b_1 = a_1 \qquad \text{and} \qquad b_{k+1} = a_{k+1} - \sum_{i=1}^{k} B(e_i, e_{k+1}) b_i. \tag{14}$$

We now give the algebraic proof of Proposition 1.

Proof Let $\phi \in Z^1\left(G, V_{\mathbb{Q}}\right)$ represent an arbitrary class in the kernel of the restriction map. This means that for every $g \in G$, there is some $v \in V_{\mathbb{Q}}$ with the property that $\phi(g) = gv - v$. Of course, v is allowed to depend on g. To prove the asserted injectivity, we need to show that $\phi \in B^1\left(G, V_{\mathbb{Q}}\right)$.

We shall do this in two steps. Re-indexing the generators $g_1, \dots, g_n$ of G, if necessary, we may assume that the vectors $e_1, \dots, e_p$ are linearly independent, while $e_{p+1}, \dots, e_n$ are linearly dependent on $e_1, \dots, e_p$. The *first step* is to show that we can subtract from ϕ a suitable element of $B^1\left(G, V_{\mathbb{Q}}\right)$ to get $\phi(g_i) = 0$ for $i = 1, \dots, p$.

By assumption, there is a vector $v \in V_{\mathbb{Q}}$ such that

$$\phi(g_p \cdots g_1) = g_p \cdots g_1 v - v;$$

after subtracting from ϕ the element $(g \mapsto gv - v) \in B^1(G, V_{\mathbb{Q}})$, we have $\phi(g_p \cdots g_1) = 0$. Furthermore, for each $i = 1, \dots, p$, there is some $v_i \in V_{\mathbb{Q}}$ with

$$\phi(g_i) = g_i v_i - v_i = -B(v_i, e_i) e_i = a_i e_i,$$

where $a_i = -B(v_i, e_i) \in \mathbb{Q}$. According to Lemma 7 below, we can write

$$\phi(g_p \cdots g_1) = \sum_{k=1}^{p} b_k e_k,$$

with coefficients b_k that satisfy the recursive relations given in the lemma. But $e_1, \dots, e_p$ are linearly independent, and therefore $b_1 = \cdots = b_p = 0$. The relations imply that $a_1 = \cdots = a_p = 0$, and so we obtain $\phi(g_i) = 0$ for $i = 1, \dots, p$.

In the *second step*, we show that ϕ is now actually zero. For this, we only need to prove that $\phi(g_i) = 0$ for $i = p+1, \dots, n$, because all the g_i together generate G and ϕ is a cocycle. By symmetry, it obviously suffices to consider just g_{p+1}. Since e_{p+1} is linearly dependent on $e_1, \dots, e_p$, we can write $e_{p+1} = \sum_{i=1}^{p} c_i e_i$ for certain coefficients $c_i \in \mathbb{Q}$, subject to the condition that

$$2 = B(e_{p+1}, e_{p+1}) = \sum_{i,j=1}^{p} c_i B(e_i, e_j) c_j.$$

If we let c be the column vector with coordinates c_i, and E the symmetric $p \times p$-matrix with entries $E_{ij} = B(e_i, e_j)$, we can put the condition into the form

$$2 = c^\dagger Ec. \tag{15}$$

Again, there is a vector $v \in V_{\mathbb{Q}}$ with $\phi(g_{p+1}) = g_{p+1}v - v = -B(v, e_{p+1})e_{p+1}$, and if we set $\eta = -B(v, e_{p+1}) \in \mathbb{Q}$, we have

$$\phi(g_{p+1}) = \eta \cdot e_{p+1} = \eta \cdot \sum_{j=1}^{p} c_j e_j.$$

We may also find $w \in V_{\mathbb{Q}}$ such that

$$\phi(g_{p+1}g_p \cdots g_1) = g_{p+1}g_p \cdots g_1 w - w.$$

Now $\phi(g_i) = 0$ for $i = 1, \ldots, p$, and so we get $\phi(g_{p+1}g_p \cdots g_1) = \phi(g_{p+1})$ from the fact that ϕ is a cocycle. Since $g_{p+1}e_{p+1} = -e_{p+1}$, we calculate that

$$\begin{aligned} -\eta \cdot e_{p+1} &= g_{p+1} \cdot \phi(g_{p+1}) = g_{p+1} \cdot \big(g_{p+1}g_p \cdots g_1 w - w\big) \\ &= \big(g_p \cdots g_1 w - w\big) - \big(g_{p+1}w - w\big) \\ &= \big(g_p \cdots g_1 w - w\big) + B(w, e_{p+1})e_{p+1} \end{aligned}$$

Let $x_i = -B(w, e_i)$. An application of Lemma 7 to the cocycle $(g \mapsto gw - w)$ shows that $g_p \cdots g_1 w - w = \sum y_j e_j$, where

$$y_1 = x_1 \qquad \text{and} \qquad y_{k+1} = x_{k+1} - \sum_{i=1}^{k} E_{i,k+1} y_i. \tag{16}$$

From our calculation, we now obtain a linear relation between $e_1, \ldots, e_p$, namely

$$-\eta \cdot \sum_{j=1}^{p} c_j e_j = \sum_{j=1}^{p} y_j e_j - \sum_{i,j=1}^{p} c_i x_i c_j e_j.$$

But $e_1, \ldots, e_p$ are linearly independent, and we deduce that

$$\eta \cdot c_j = \sum_{i=1}^{p} c_i x_i c_j - y_j$$

for all $j = 1, \ldots, p$, which we can write as a vector equation

$$\eta \cdot c = c^\dagger x \cdot c - y. \tag{17}$$

The recursive relations in (16) for the y_j can be put into the form $x = Sy$, where S is a lower-triangular matrix with entries

$$S_{ij} = \begin{cases} E_{ij} & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

But now $E = S + S^\dagger$, because E is symmetric and its diagonal entries are all equal to 2. From (15), we find that

$$2 = c^\dagger Ec = c^\dagger Sc + c^\dagger S^\dagger c = 2c^\dagger Sc,$$

and so $1 = c^\dagger Sc$. Now apply $c^\dagger S$ to the equation in (17) to get

$$\eta = c^\dagger Sc \cdot \eta = c^\dagger x \cdot c^\dagger Sc - c^\dagger Sy = c^\dagger x - c^\dagger Sy = c^\dagger (x - Sy) = 0.$$

This shows that $\phi(g_{p+1}) = \eta \cdot e_{p+1} = 0$, and we have our result.

Example 1 It should be pointed out that the restriction map (6) need not be injective for an arbitrary representation of a group G on a vector space M. Here is a simple example, where the group is even abelian. Let $G = \mathbb{Z}^2$ be the free abelian group on two generators, acting on $M = \mathbb{Q}^3$ by the two commuting matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define $\phi\colon G \to M$ by the rule $\phi(a, b) = (a, 0, 0)$. One easily verifies that ϕ gives a non-zero element in $H^1(G, M)$, but that it goes to zero under the restriction map

$$H^1(G, M) \to \prod_{g \in G} M/(g - \mathrm{id})M.$$

Thus Proposition 1 does not remain true for arbitrary representations.

4.2 Conclusion of the argument

Proposition 1 applies to the vanishing cohomology $V_{\mathbb{Q}} = H^{d-1}_{van}(S_0, \mathbb{Q})$ and shows that (7) is injective when d is odd. Indeed, it is clear from the results in Sect. 3 that $V_{\mathbb{Q}}$ satisfies all the assumptions of the proposition, if we set $B(u, v) = \varepsilon_d(u, v)$. We can take for $e_1, \dots, e_n$ the vanishing cycles in an arbitrary Lefschetz pencil on X, and for $g_1, \dots, g_n$ the corresponding generators of the fundamental group. The identity $g_i v = v - B(v, e_i)e_i$ is then simply the Picard–Lefschetz formula (13). We conclude that (7) is injective when $V_{\mathbb{Q}}$ is the vanishing cohomology. This completes the proof that (8) is injective when the dimension of X is odd.

5 Detecting group cohomology classes: the even case

The purpose of this section is to prove that the restriction map

$$V_{\mathbb{Q}} \to \prod_{g \in G} V_{\mathbb{Q}}/(g - \mathrm{id})V_{\mathbb{Q}} \tag{18}$$

is also injective when $d = \dim X$ is even. This is more subtle than in the case of odd d, and we will need to use the fact that the vanishing cohomology and the monodromy action can all be defined over $\mathbb{Z}$. The integral vanishing cohomology $H^{d-1}_{van}(S_0, \mathbb{Z})$, modulo torsion, is an example of a skew-symmetric vanishing lattice. We therefore have to begin by reviewing some results about the structure of skew-symmetric vanishing lattices, due to W. Janssen [5].

5.1 Skew-symmetric vanishing lattices

Let V be a free $\mathbb{Z}$-module of finite rank, with an alternating bilinear form $B\colon V \otimes V \to \mathbb{Z}$. Let $\mathrm{Sp}(V)$ be the group of all automorphisms of V that preserve B. For every element $v \in V$, we can define a symplectic transvection $T_v \in \mathrm{Sp}(V)$ by the formula $T_v(x) = x - B(x, v)v$. With the monodromy representation on $H^{d-1}(S_0, \mathbb{Z})$ and the facts in Sect. 3 in mind, we are interested in subgroups of $\mathrm{Sp}(V)$ generated by transvections. Given a subset $\Delta \subseteq V$, we write Γ_Δ for the subgroup of $\mathrm{Sp}(V)$ generated by all T_δ, for $\delta \in \Delta$.

As a matter of fact, all transvections are contained in a (potentially smaller) group $\mathrm{Sp}^\sharp(V)$, which we now define. The form induces a linear map $j: V \to \mathrm{Hom}(V, \mathbb{Z})$, given by the rule $j(v) = B(v, -)$; in general, it is neither injective nor surjective without further assumptions on B. Now $\mathrm{Sp}(V)$ naturally acts on the dual module $\mathrm{Hom}(V, \mathbb{Z})$ as well, by setting $(g\lambda)(x) = \lambda(g^{-1}x)$ for $x \in V$ and $\lambda \in \mathrm{Hom}(V, \mathbb{Z})$, and the map j is equivariant. We let $\mathrm{Sp}^\sharp(V)$ be the subgroup of those $g \in \mathrm{Sp}(V)$ that act trivially on $\mathrm{Hom}(V, \mathbb{Z})/j(V)$. Concretely, this means that

$$\mathrm{Sp}^\sharp(V) = \Big\{ g \in \mathrm{Sp}(V) \ \Big|\ \text{for any } \lambda \in \mathrm{Hom}(V, \mathbb{Z}), \text{there exists } v \in V \\ \text{such that } \lambda(gx - x) = B(v, x) \text{ for all} x \in V \Big\}.$$

It is easy to see that $T_v \in \mathrm{Sp}^\sharp(V)$; each Γ_Δ is therefore a subgroup of $\mathrm{Sp}^\sharp(V)$.

We now come to the main definition, due to Janssen. A (skew-symmetric) *vanishing lattice* in V is a subset $\Delta \subseteq V$ with the following three properties:

1. The set Δ generates V.
2. Δ is a single orbit under the action of Γ_Δ.
3. There exist two elements $\delta_1, \delta_2 \in \Delta$ such that $B(\delta_1, \delta_2) = 1$.

In that case, Γ_Δ is called the *monodromy group* of the vanishing lattice.

Janssen has carried out a very detailed study of such vanishing lattices. One of his main technical results, obtained by a careful choice of generators of the lattice, is the following theorem.

Theorem 3 [5, Theorem 2.5] *Let $\Delta \subseteq V$ be a vanishing lattice. Then the monodromy group of Δ contains the congruence subgroup*

$$\mathrm{Sp}_2^\sharp(V) = \Big\{ g \in \mathrm{Sp}(V) \ \Big|\ g \textit{ acts trivially on } \mathrm{Hom}(V, \mathbb{Z})/j(2V) \Big\}.$$

In particular, Γ_Δ is itself of finite index in $\mathrm{Sp}^\sharp(V)$.

We shall now use Janssen's theorem to show that Γ_Δ contains a finite-index subgroup with a particularly convenient set of generators, similar to that used in Janssen's proof of Theorem 3. This is the crucial step in proving the injectivity of (18) in the even-dimensional case.

Lemma 8 *Let V be a free $\mathbb{Z}$-module of rank r, and let $\Delta \subseteq V$ be a vanishing lattice. Then it is possible to find r linearly independent elements $\delta_1, \ldots, \delta_r \in \Delta$, such that the group $\Gamma_{\{\delta_1, \ldots, \delta_r\}}$ has finite index in Γ_Δ.*

Proof Pick an arbitrary element $\delta_1 \in \Delta$. Since Δ is a vanishing lattice, a short calculation [5, Lemma 2.7] shows that V is already generated by the smaller set

$$\Delta_1 = \Big\{ \delta \in \Delta \ \Big|\ B(\delta_1, \delta) = 1 \text{or } \delta = \delta_1 \Big\}.$$

We can therefore find r linearly independent elements $\delta_1, \ldots, \delta_r \in \Delta$ that satisfy $B(\delta_1, \delta_i) = 1$ for $i \geq 2$. Let $V' \subseteq V$ be their span; then V' is free of rank r, and the quotient is V/V' is finite, say of order k. Set

$$\mathrm{Sp}^\sharp(V)_{V'} = \Big\{ g \in \mathrm{Sp}^\sharp(V) \ \Big|\ g(V') = V' \Big\};$$

as a stabilizer for the action of $\mathrm{Sp}^\sharp(V)$ on the finite group V/V', it has finite index in $\mathrm{Sp}^\sharp(V)$. Note that we may consider $\mathrm{Sp}^\sharp(V)_{V'}$ as a subgroup of $\mathrm{Sp}(V')$.

Let $T_i = T_{\delta_i}$ be the corresponding transvections, and $\Gamma' = \Gamma_{\{\delta_1,\dots,\delta_r\}}$ the group generated by them. By construction, each T_i preserves V', which means that $\Gamma' \subseteq \mathrm{Sp}^\sharp(V)_{V'}$. For $i \geq 2$, we have

$$T_i T_1(\delta_i) = T_i(\delta_i + \delta_1) = \delta_i + \delta_1 - B(\delta_i + \delta_1, \delta_i)\delta_i = \delta_1,$$

and so all the δ_i lie in one orbit of the group Γ'. It follows that $\Delta' = \Gamma' \cdot \{\delta_1, \dots, \delta_r\}$ is itself a vanishing lattice in V'. Theorem 3, applied to $\Delta' \subseteq V'$, shows that Γ' has finite index in the group $\mathrm{Sp}^\sharp(V')$. We can thus complete the proof by appealing to the following lemma.

Lemma 9 *Let $V' \subseteq V$ be a submodule with V/V' finite. Then $\mathrm{Sp}^\sharp(V') \cap \mathrm{Sp}^\sharp(V)_{V'}$ has finite index in $\mathrm{Sp}^\sharp(V)_{V'}$.*

Proof Let $Q = \mathrm{Sp}^\sharp(V)_{V'} / \mathrm{Sp}^\sharp(V') \cap \mathrm{Sp}^\sharp(V)_{V'}$ be the quotient group; we have to show that Q has finite order.

As above, we let $j \colon V \to \mathrm{Hom}(V, \mathbb{Z})$ be the map induced by the bilinear form; since $\mathrm{Hom}(V, \mathbb{Z})$ embeds into $\mathrm{Hom}(V', \mathbb{Z})$, we denote the corresponding map for V' by the same letter. By definition, the group $\mathrm{Sp}^\sharp(V)_{V'}$ consists of elements g that satisfy $g(V') = V'$ and act trivially on the quotient $\mathrm{Hom}(V, \mathbb{Z})/j(V)$. It follows that $\mathrm{Sp}^\sharp(V)_{V'}$ acts on every one of the five finitely generated $\mathbb{Z}$-modules in the following diagram:

$$\begin{tikzcd}
 & j(V)/j(V') \arrow[d, hook] & \\
 & Hom(V', \mathbb{Z})/j(V') \arrow[d] & \\
\mathrm{Hom}(V, \mathbb{Z})/j(V) \arrow[r, hook] & \mathrm{Hom}(V', \mathbb{Z})/j(V) \arrow[r, two heads] & Ext^1(V/V', \mathbb{Z})
\end{tikzcd}$$

Thus Q embeds into the group of automorphisms of $\mathrm{Hom}(V', \mathbb{Z})/j(V')$ that act compatibly on each module, and trivially on $\mathrm{Hom}(V, \mathbb{Z})/j(V)$. Since all modules in the diagram are finitely generated, and $j(V)/j(V')$ and $\mathrm{Ext}^1(V/V', \mathbb{Z})$ are finite, it is easy to see that the group of such automorphisms is finite. This implies that Q is also a finite group.

5.2 Injectivity of the restriction map

In the presence of a vanishing lattice, it is again possible to prove the injectivity of the restriction map by a fairly simple argument. The most convenient setting is the following. Let V be a free $\mathbb{Z}$-module of finite rank, with an alternating bilinear form $B \colon V \otimes V \to \mathbb{Z}$. Let G be a finitely generated group acting on V, and assume that there are generators $g_1, \dots, g_n$ for G, and distinguished elements $e_1, \dots, e_n$ of V, such that

$$g_i v = v - B(v, e_i)e_i = T_{e_i}(v)$$

for all i. Furthermore, assume that $\Delta = G \cdot \{e_1, \dots, e_n\}$ is a vanishing lattice in V. Of course, the image of G in $\mathrm{Sp}^\sharp(V)$ is then exactly the monodromy group Γ_Δ.

Proposition 2 *Let V be a free $\mathbb{Z}$-module of finite rank with a G-action, subject to the assumptions above. Let $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$. Then the restriction map*

$$H^1\left(G, V_{\mathbb{Q}}\right) \to \prod_{g \in G} V_{\mathbb{Q}}/(g - \mathrm{id})V_{\mathbb{Q}}$$

is injective.

Proof Let $\phi \in Z^1\left(G, V_{\mathbb{Q}}\right)$ represent an element of the kernel; we have to show that it belongs to $B^1\left(G, V_{\mathbb{Q}}\right)$. To simplify the notation, we shall write $\Gamma = \Gamma_\Delta$. We begin the proof by noting that ϕ is identically zero on the normal subgroup $N = \ker(G \to \Gamma)$. Indeed, given any $g \in G$, we can find some $v \in V_{\mathbb{Q}}$ such that $\phi(g) = gv - v$; thus any g that acts trivially on V automatically satisfies $\phi(g) = 0$. Consequently, ϕ descends to an element in $H^1\left(\Gamma, V_{\mathbb{Q}}\right)$. Since $H^1\left(\Gamma, V_{\mathbb{Q}}\right)$ is easily seen to inject into $H^1\left(G, V_{\mathbb{Q}}\right)$, we may assume from now on that we are dealing with an element $\phi \in Z^1\left(\Gamma, V_{\mathbb{Q}}\right)$.

Let $r = \dim V_{\mathbb{Q}}$. Using Lemma 8, we can find r linearly independent elements $\delta_1, \ldots, \delta_r \in V$, such that $\Gamma' = \Gamma_{\{\delta_1, \ldots, \delta_r\}}$ has finite index in Γ. Let $T_i = T_{\delta_i} \in \Gamma'$. As in the odd case, we can adjust ϕ by an element of $B^1\left(\Gamma, V_{\mathbb{Q}}\right)$ to make sure that $\phi(T_r \cdots T_1) = 0$. By assumption, we can also find vectors $v_i \in V_{\mathbb{Q}}$ such that

$$\phi(T_i) = T_i v_i - v_i = -B(v_i, \delta_i)\delta_i = a_i v_i,$$

for $a_i = -B(v_i, \delta_i)$. An application of Lemma 7 shows that

$$0 = \phi(T_r \cdots T_1) = \sum_{k=1}^{r} b_k \delta_k,$$

for coefficients $b_k \in \mathbb{Q}$ satisfying the relations in (14). Since the δ_i are linearly independent, we have $b_k = 0$ for all k, and thus $a_i = 0$ for all i. After the adjustment, the cocycle ϕ thus satisfies $\phi(T_i) = 0$ for all i. Since Γ' is generated by the transvections T_i, we conclude that $\phi(\Gamma') = 0$.

It is now easy to show that ϕ is identically zero. Let m be the index of Γ' in Γ. Take an arbitrary element $\delta \in \Delta$. As usual, there is a vector $w \in V_{\mathbb{Q}}$ such that

$$\phi(T_\delta) = T_\delta w - w = -B(w, \delta)\delta.$$

From this, one easily deduces that

$$\phi\left(T_\delta^m\right) = -mB(w, \delta)\delta = m\phi\left(T_\delta\right).$$

On the other hand, T_δ^m belongs to Γ', and so $\phi\left(T_\delta^m\right) = 0$. Since the T_δ together generate Γ, and ϕ is a cocycle, we then have $m\phi = 0$, and hence $\phi = 0$. This proves the assertion.

5.3 Conclusion of the argument

To conclude that (7) is injective, we now apply our general result to the vanishing cohomology $V_{\mathbb{Q}} = H^{d-1}_{van}(S_0, \mathbb{Q})$. All the assumptions are satisfied by Sect. 3, if we let $B(u, v) = \varepsilon_d(u, v)$ be a multiple of the intersection pairing on S_0.

In more detail, we set $V = H^{d-1}_{van}(S_0, \mathbb{Z})$ modulo torsion; it is generated by the vanishing cycles $e_1, \ldots, e_n$ of any Lefschetz pencil. Let $g_1, \ldots, g_n$ be the corresponding elements of the fundamental group G. The collection of all vanishing cycles $\Delta = G \cdot \{e_1, \ldots, e_n\}$ is then a vanishing lattice in V, because $e_1, \ldots, e_n$ all lie in one G-orbit, and because of Lemma 6. Proposition 2 now gives us the injectivity of the restriction map in (18), which finishes the proof that (8) is injective when the dimension of X is even.

6 An application to Clemens' potential function for Calabi-Yau threefolds

Let X be a Calabi-Yau threefold, and let $\omega \in H^0\left(X, \Omega_X^3\right)$ be a nowhere vanishing holomorphic three-form. In studying the deformation theory of curves on X, it is useful to consider the covering space T_{van} of P^{sm}, whose fiber over a point corresponding to the hyperplane section $S = X \cap H$ is the group $H^2_{van}(S, \mathbb{Z})$. Points of T_{van} can naturally be viewed as pairs (S, α), where $S \subseteq X$ is a smooth hyperplane section, and $\alpha \in H^2_{van}(S, \mathbb{Z})$. Of course, T_{van} has countably many sheets and countably many connected components, and is thus far from being an algebraic variety.

Motivated by physics, Clemens [2] has shown that the locus of Hodge classes (known to be a countable union of algebraic varieties)

$$\operatorname{Alg}\left(T_{van}\right) = \left\{(S, \alpha) \in T_{van} \;\middle|\; \alpha \in H^{1,1}(S) \cap H^2_{van}(S, \mathbb{Z})\right\}$$

is the zero locus of a closed holomorphic 1-form Ω on T_{van}, constructed from ω via membrane integrals. Local integrals of Ω are referred to as *potential functions* in [2]; as in many other situations, the points in the paramater space coming from geometric objects (namely, curves on X) are therefore given as the critical locus of potential functions.

It is a natural question whether one can find a globally defined potential function on all of T_{van}; in other words, whether Ω is an exact 1-form. The result in Theorem 1 gives a negative answer to this question. We now explain why.

Return, for a moment, to the general setting, where T_{van} is the étalé space of the local system with fibers $H^{d-1}_{van}(S, \mathbb{Z})$, and a point in T_{van} is a pair (S, α). If we let $T_{van}(\alpha)$ be the component containing the point (S_0, α), it is easy to see that

$$\pi_1\left(T_{van}(\alpha), (S_0, \alpha)\right) = \left\{\, g \in G \;\middle|\; g \cdot \alpha = \alpha \,\right\} = \operatorname{Stab}_G(\alpha).$$

consequently, we have the isomorphism $H^1\left(T_{van}(\alpha), \mathbb{Q}\right) \simeq \operatorname{Hom}_{\mathbb{Q}}\left(\operatorname{Stab}_G(\alpha), \mathbb{Q}\right)$. For any primitive cohomology class $\omega \in H^d_0(X, \mathbb{Q})$, we can use the tube mapping in homology to construct a first cohomology class on $T_{van}(\alpha)$. Indeed, the rule

$$\operatorname{Stab}_G(\alpha) \to \mathbb{Q}, \qquad g \mapsto \int\limits_{\tau_g(\alpha)} \omega,$$

defines a homomorphism from the fundamental group of $T_{van}(\alpha)$ to $\mathbb{Q}$, and thus an element of $H^1\left(T_{van}(\alpha), \mathbb{Q}\right)$. It is not hard to show that this class is independent of the choice of base point on $T_{van}(\alpha)$. Thus we have a well-defined map

$$F\colon H^d_0(X, \mathbb{Q}) \to H^1\left(T_{van}, \mathbb{Q}\right).$$

Theorem 1, in its dual formulation (3), is precisely the assertion that this map is injective. In particular, the topology of the space T_{van} is sufficiently complicated to detect primitive cohomology classes on X.

This fact implies that the form Ω constructed by Clemens cannot be globally integrated. Indeed, one easily sees from the description in [2, p. 735] that

$$[\Omega] = F(\omega) \in H^1\left(T_{van}, \mathbb{C}\right).$$

Since the map F is injective by Theorem 1, it follows that the 1-form Ω cannot be exact, and thus that there cannot be a global potential function on all of T_{van}.

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