THE BOUNDARY BEHAVIOR OF COHOMOLOGY CLASSES AND SINGULARITIES OF NORMAL FUNCTIONS

DISSERTATION

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ABSTRACT

In a family of projective complex algebraic varieties, all nonsingular fibers are topologically equivalent; in particular, their cohomology groups are isomorphic. Near the "boundary," where the varieties acquire singular points, this is no longer the case. The theory of variations of Hodge structure provides strong tools to understand the local behavior near points on the boundary (Cattani, Kaplan, and Schmid, 1986); these have been used, for instance, to prove that the locus of Hodge classes is a union of algebraic varieties (Cattani, Deligne, and Kaplan, 1995).

Recently, there has been interest in global questions related to the behavior at the boundary, especially for the family of all hypersurfaces (of large degree) of a given smooth projective variety. Green and Griffiths (2007, 2006) introduced the concept of the "singularity" of a normal function; following their ideas, Brosnan, Fang, Nie, and Pearlstein (2007), and de Cataldo and Migliorini (2007) proved that the Hodge conjecture is equivalent to the existence of such singularities.

In this dissertation, we investigate the boundary behavior of cohomology classes in families (in the above sense), from several different points of view. We also obtain new interpretations for the singularity of a normal function in the family of hypersurface sections of sufficiently large degree.

To Atsuko

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INTRODUCTION

The integral cohomology ring $H^*(X, \mathbb{Z}) = \bigoplus_{k \geq 0} H^k(X, \mathbb{Z})$ is a basic invariant of every topological space. When X is a complex algebraic variety, especially one that is projective and nonsingular, there are interesting relations between the geometry of X and its cohomology. Here are two examples:

1. Many geometric operations (such as blowing up X along a subvariety) are reflected in modifications to the cohomology ring of X. On the other hand, varieties that are birationally equivalent have a certain part of their cohomology in common.

2. Any subvariety $Z \subseteq X$ has a corresponding fundamental class; if the codimension of Z is p, then this is a class $[Z] \in H^{2p}(X, \mathbb{Z})$.

What is more, the cohomology of a smooth projective variety X has additional structure, not present in the case of arbitrary topological spaces. The extra structure is a decomposition of the complex cohomology

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

into subspaces, with $H^{q,p}(X)$ equal to the complex conjugate of $H^{p,q}(X)$; cohomology classes in $H^{p,q}(X)$ are said to be *of type* (p,q). This is the famous Hodge decomposition; its existence is the starting point of what is nowadays called *Hodge theory*. Understanding the relationship between such decompositions on the one hand, and the geometry and topology of algebraic varieties on the other, is the main objective of Hodge theory.

One basic result is that the fundamental class of a codimension p subvariety Z is always of type (p,p). In other words, the class $[Z]$ belongs to the intersection $H^{2p}(X,\mathbb{Z}) \cap H^{p,p}(X)$; such classes are called (integral) *Hodge classes*. This circumstance leads to the following innocent-looking question, actually the most important open problem in Hodge theory.

Question. On a smooth complex projective variety, is every rational Hodge class a $\mathbb{Q}$-linear combination of fundamental classes of algebraic subvarieties?

Hodge (1952, p. 184) posed (a stronger form of) this question during his address at the International Congress of Mathematicians in 1950; it is nowadays known as the *Hodge conjecture*. Only few cases[1] of this conjecture have been proved; the main one being that Hodge classes of type $(1,1)$ are always fundamental classes of divisors (here, the statement is true even with integer coefficients; in general, it is necessary to use coefficients in $\mathbb{Q}$). This is the so-called $(1,1)$-Theorem, due to Lefschetz (1950); it was originally proved by studying Lefschetz pencils, i.e., certain one-parameter families of hyperplane sections of the given variety.

[1]Lewis (1999) summarizes what is known about the Hodge conjecture.

In recent years, it has come to be understood that one-parameter families are not sufficient; instead, one should look at the complete linear system $P = \mathbb{P}_{sub}\big(H^0(X, \mathscr{L})\big)$ of a very ample line bundle $\mathscr{L}$ on X, preferably one of high degree. The idea is to form the incidence variety (or *universal hypersurface*)

$$\mathfrak{X} = \left\{ (\mathbb{C} \cdot s, x) \in P \times X \;\middle|\; s(x) = 0 \right\} \subseteq P \times X,$$

and view the map $\pi \colon \mathfrak{X} \to P$ as a family of hypersurfaces of X, parametrized by the projective space P. The set of points $p \in P$, where the hypersurface $\mathfrak{X}_p = \pi^{-1}(p)$ is singular, is called the dual variety $X^\vee$. Its complement, $P^{sm} = P \setminus X^\vee$, is where the fibers of π are smooth.

Over P^{sm}, the cohomology of the hypersurfaces $\mathfrak{X}_p$ is locally constant, and therefore fits together into local systems $R^k\pi_*^{sm}\mathbb{Z}$, with fibers $\big(R^k\pi_*\mathbb{Z}\big)_p = H^k\big(\mathfrak{X}_p, \mathbb{Z}\big)$. Moreover, through a classical construction due to Griffiths, primitive Hodge classes on X give rise to normal functions[2] over P^{sm}. All of this breaks down "at the boundary," meaning near points of $X^\vee$. It is hoped that understanding the exact behavior of normal functions at the boundary will lead to progress on the Hodge conjecture, among other things.

In this dissertation, we investigate the behavior of cohomology classes (especially Hodge classes) and normal functions at the boundary. Three separate results are proved, representing different angles from which to look at the problem. At the same time, all three results developed from thinking about one specific example, namely the generalized Hodge conjecture for Calabi-Yau threefolds. To give the reader a

[2]See 7.3.1 for some background on normal functions.

sense for how they are connected, a discussion of this problem has been included in Chapter 2.

The general outline of the other parts of the dissertation is as follows: Chapter 3 is about the boundary behavior of local systems (for example, the ones originating from the universal hypersurface $\mathfrak{X}$). We regard local systems as certain complex manifolds, and describe the construction of a (mildly singular) analytic space that we call the *canonical extension* of a local system.

Chapter 4 concerns the relationship between the cohomology of a (smooth, projective) variety X, and that of its hypersurface sections. A *tube mapping*[3] is constructed, using monodromy-invariant cohomology classes, and we prove that its image generates the middle-dimensional primitive cohomology of X, at least when $\dim X$ is odd. A consequence of this result is that the topology of the local system $R^{\dim X - 1}\pi_*^{sm}\mathbb{Z}_{van}$ (actually, even that of its canonical extension) allows one to recover the entire primitive cohomology of X.

The longest part of the dissertation, Chapters 5–7, is about the interplay between the primitive cohomology of X and the geometry of the universal hypersurface $\mathfrak{X}$. In Chapter 5, we introduce a certain filtered holonomic $\mathcal{D}$-module $(\mathcal{M}, F)$, which serves to formalize the residue calculus for hypersurfaces of high degree (Griffiths, 1969; Green, 1985). In Chapters 5 and 6, we establish many properties of $\mathcal{M}$; almost all of them depend on having the line bundle $\mathscr{L}$ be sufficiently ample. In Chapter 7, we apply those results to the study of normal functions; in particular, we reinterpret—and,

[3]In the literature, similar maps are usually called "cylinder mappings."

at times, reprove—the results of Brosnan, Fang, Nie, and Pearlstein (2007) about singularities of normal functions. The $\mathcal{D}$-module $\mathcal{M}$ turns out to be a *minimal extension*; in general, these are not easy to describe explicitly. The concrete description that we give, in terms of residues, may therefore be of independent interest.

We now proceed to introduce the three principal results of this work in more detail.

1.1 The canonical extension of a local system

In his book about differential equations with regular singular points, Deligne (1970) introduced the so-called "canonical extension" of a vector bundle with flat connection. The most important case of his construction is the following. Suppose $\mathscr{V}$ is a holomorphic vector bundle on a complex manifold M, equipped with a flat connection ∇. The connection ∇ is a $\mathbb{C}$-linear map from $\mathscr{O}_X$ to $\Omega^1_X \otimes \mathscr{V}$, obeying the Leibniz rule; it is said to be *flat* if the curvature is zero (meaning that $\nabla \circ \nabla = 0$).

Now let M be an open subset in a bigger complex manifold $\overline{M}$, in such a way that

1. the complement $\overline{M} \setminus M$ is a divisor with normal crossing singularities, and

2. the local monodromy of ∇ near points of $\overline{M} \setminus M$ is unipotent.

In this situation, Deligne shows that $\mathscr{V}$ extends in a canonical manner to a vector bundle $\overline{\mathscr{V}}$ on $\overline{M}$, whose characteristic property is that in any local frame for $\overline{\mathscr{V}}$ near points of $\overline{M} \setminus M$, the connection matrix for ∇ has at worst logarithmic poles with nilpotent residues.[4]

[4]Information about Deligne's construction may be found in Section 3.1, together with an explanation of the terms "logarithmic poles" and "nilpotent residues."

Local systems with integer coefficients are one source for flat vector bundles; if $\mathcal{H}_{\mathbb{Z}}$ is a local system of (finitely generated, free) Abelian groups, then $\mathscr{V} = \mathscr{O}_M \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$, together with the natural connection induced by differentiation, is such a bundle. When $\mathcal{H}_{\mathbb{Z}}$ is *unipotent* along $\overline{M} \setminus M$, i.e., has unipotent monodromy near points of $\overline{M} \setminus M$, Deligne's construction applies, and there is a canonical extension $\overline{\mathscr{V}}$ for the vector bundle. It is then natural to ask whether the local system also has an extension.

In general, $\mathcal{H}_{\mathbb{Z}}$ cannot be extended to $\overline{M}$ as a local system, precisely because the local monodromy around $\overline{M} \setminus M$ is usually non-trivial. We return therefore to the more old-fashioned view of a local system as a space, instead of as a sheaf—the total space $T = T(\mathcal{H}_{\mathbb{Z}})$ is a covering space of M, typically infinite-sheeted and with countably many components. In Chapter 3, we answer the extension question by constructing a natural extension $\overline{T}$ for the space T, as an analytic space with mild singularities. We call $\overline{T}$ the *canonical extension* of the local system $\mathcal{H}_{\mathbb{Z}}$.

The construction is easy to describe: T is naturally embedded into the vector bundle $\mathscr{V}$, and we define $\overline{T}$ as the closure of T inside the total space of the canonical extension $\overline{\mathscr{V}}$. Perhaps surprisingly, this closure is still an analytic space with good properties. For instance, the normalization of $\overline{T}$ is locally toric (i.e., locally isomorphic to a toric variety), and therefore has only mild singularities. Explicit local analytic equations for $\overline{T}$ are found in Section 3.5.

It follows from the construction that points in $\overline{T} \setminus T$ correspond to monodromy-invariant elements in the fibers of the local system $\mathcal{H}_{\mathbb{Z}}$. In fact, the closure $\overline{T}$ includes all elements that are invariant under the local monodromy along any analytic arc. Its fibers over $\overline{M} \setminus M$ are therefore fairly big—while T itself is discrete over M, the fibers over boundary points are unions of affine spaces, of dimension as big as $\dim M - 1$. To justify calling $\overline{T}$ the "canonical extension" of the local system, we show in Section 3.7 that it has a certain universal mapping property. A second reason for the terminology is the relation between $\overline{T}$ and the canonical extension of the bundle.

1.2 Primitive cohomology and the tube mapping

Let X be a nonsingular complex projective variety. Almost all the cohomology of X can be obtained from that of any smooth hyperplane section; this is the content of the so-called Lefschetz Hyperplane Theorem. The only piece of the cohomology ring that is not determined in this way is the primitive cohomology in degree $n = \dim X$, in other words, the group

$$H^n(X, \mathbb{Q})_{prim} = \ker\big(H^n(X, \mathbb{Q}) \to H^n(H \cap X, \mathbb{Q})\big),$$

for $H \cap X$ an arbitrary smooth hyperplane section of X.

In Chapter 4, we show how to obtain the primitive cohomology from the topology of a Lefschetz pencil on X. To be more precise, say $B \subseteq \mathbb{P}^1$ is the smooth locus of the pencil. Whenever one has an element g in the fundamental group of B, and a g-invariant homology class $\alpha \in H_{n-1}(H \cap X, \mathbb{Z})$ on a hyperplane section, one gets a n-cycle on X by translating α along g and taking its trace in X. This construction defines what we shall call the *tube mapping* (although we use cohomology for a rigorous definition).

We prove (Theorem 4.1.1) that the image of the tube mapping generates the primitive cohomology of X with coefficients in $\mathbb{Q}$, at least when the dimension of X is odd.[5] The proof is based on an interesting result about the group cohomology of certain representations.

A very concrete interpretation of the main theorem is as follows. Consider the total space T of the local system on B, whose fibers are $H_{n-1}(X \cap H, \mathbb{Z})$. It is naturally a covering space of B, with countably many sheets and components; moreover, the set $[\mathbb{S}^1, T]$ of homotopy classes of loops in T is exactly the set of pairs (g, α), where g is a loop in B, and α is a g-invariant homology class in $H_{n-1}(X \cap H, \mathbb{Z})$. The tube mapping can be viewed as giving, for each element of $H^n(X, \mathbb{Q})_{prim}$, a cohomology class in $H^1(T, \mathbb{Q})$; in this setting, our result is that the map $H^n(X, \mathbb{Q})_{prim} \to H^1(T, \mathbb{Q})$ is injective.

1.3 Residues and $\mathcal{D}$-modules

If D is a smooth hypersurface in a smooth complex projective variety X, we have a long exact sequence in cohomology,

$$\cdots \longrightarrow H^k(X, \mathbb{Q}) \longrightarrow H^k(X \setminus D, \mathbb{Q}) \longrightarrow H^{k-1}(D, \mathbb{Q}) \longrightarrow H^{k+1}(X, \mathbb{Q}) \longrightarrow \cdots$$

The most interesting part of this sequence lies in degree $n = \dim X$; that part can be rewritten as a short exact sequence

$$0 \longrightarrow H^n(X, \mathbb{Q})_{prim} \longrightarrow H^n(X \setminus D, \mathbb{Q}) \xrightarrow{\text{Res}} H^{n-1}(D, \mathbb{Q})_{van} \longrightarrow 0, \tag{1.1}$$

[5] The case when X is of even dimension is at present unsolved; a brief discussion of its features may be found in Section 4.7.

in which $H^n(X, \mathbb{Q})_{prim}$ denotes the primitive n-th cohomology of X, the mapping Res is the residue mapping, and $H^{n-1}(D, \mathbb{Q})_{van}$ is the vanishing cohomology of the hypersurface. The sequence contains a large amount of nontrivial cohomological information about the pair (X, D).

The cohomology of $X \setminus D$ can be computed by rational forms with poles along D; if the line bundle $\mathscr{L} = \mathscr{O}_X(D)$ is taken sufficiently ample, then the Hodge filtration is determined by the order of pole. In combination with the exact sequence in (1.1), this gives a convenient presentation for the vanishing cohomology of the hypersurface D, at least when D has no singularities. We review this residue description of the cohomology in Section 5.1.

One purpose of Chapter 5 is to extend the above description to singular hypersurfaces; two basic ideas are used to accomplish this.

1. The use of the universal hypersurface; it has the advantage of being nonsingular, even though the individual hypersurfaces are not.

2. The use of filtered $\mathcal{D}$-modules to assemble the entire residue calculus for hypersurfaces of X into a single object.

In brief, we let P be the linear system of a sufficiently ample line bundle on X, and work with the universal hypersurface $\mathfrak{X} \subseteq P \times X$. For each $k \geq 1$, we define a coherent sheaf $F_k\mathcal{M}$ on P, whose sections are locally residues of rational n-forms on $P \times X$, with a pole of order at most k along $\mathfrak{X}$. Let $\mathcal{M}$ be the union of all those sheaves. We then show that $(\mathcal{M}, F)$ is a filtered holonomic $\mathcal{D}$-module; in fact, up to a shift in the filtration F, it underlies a mixed Hodge module. We also prove that $\mathcal{M}$

is the minimal extension of the variation of Hodge structure given by the vanishing cohomology of the hypersurfaces.

We make use of recent work of Brosnan, Fang, Nie, and Pearlstein (2007); in particular, the idea of employing M. Saito's theory of mixed Hodge modules to obtain information about the $\mathcal{D}$-module $(\mathcal{M}, F)$ comes from their preprint.

1.4 Properties of the sheaves in the filtration

In Chapter 6, we establish several results about the sheaves $F_k\mathcal{M}$ in the filtration. They are valid whenever the line bundle $\mathscr{L}$ is sufficiently ample; here are three examples of what we prove.

1. The higher cohomology of $F_k\mathcal{M}$ vanishes (Theorem 6.1.2).
2. There is a duality theorem for the graded quotients $Gr_k^F\mathcal{M}$ (Proposition 6.4.13).
3. In the range $1 \leq k \leq \dim X$, both $F_k\mathcal{M}$ and $Gr_k^F\mathcal{M}$ satisfy Serre's condition S_p for large values of p (Theorem 6.6.2).

The duality results include both local and global statements; they are proved by analyzing a spectral sequence arising from a Koszul complex. We also give a version in the derived category $D^b(P)$, based on an application of the duality theory for morphisms. A proof of the fact that the duality is induced by the intersection pairing has been included in 6.4.1.

In the interval $1 \leq k \leq n$, the sheaf $F_k\mathcal{M}$ is a natural extension of the Hodge bundle with fibers $F^{n-k}\mathcal{V}_{van}^{n-1}$. Provided $\mathscr{L}$ is sufficiently ample, it seems plausible that these extensions should be “nice” sheaves—they cannot be locally free, but should come close.

Serre’s condition S_p measures how far a coherent sheaf is from being locally free; for $p = 2$, it is equivalent to being reflexive; in general, a sheaf satisfying condition S_p is locally free in codimension at least p. To prove that $Gr_k^F \mathcal{M}$ has that property when $1 \leq k \leq n$, we first show (in Section 6.5) that the subset of $X^\vee$ where the hypersurfaces have “many” singularities is of high codimension, if $\mathscr{L}$ is sufficiently ample. Combined with the duality result from Section 6.3, this is enough to obtain condition S_p for large values of p.

1.5 Applications to the study of normal functions

In Chapter 7, those results are then used to study normal functions and their singularities. We give two constructions that represent primitive cohomology classes in $H^n(X, \mathbb{C})_{prim}$ by objects on the projective space P.

1. A class $\zeta \in F^k H^n(X)_{prim}$ has an associated holomorphic one-form ω_ζ with coefficients in $F_{n+1-k}\mathcal{M}$; this form is closed under the differential $\nabla_{\mathcal{M}}$ of the de Rham complex for $\mathcal{M}$.

2. For each $\zeta \in H^{k,n-k}(X)_{prim}$, there is an element of $\mathrm{Ext}^1_P\big(Gr^F_{k+1}\mathcal{M}, \mathscr{O}_P\big)$, represented by a natural extension of coherent sheaves on P.

In either case, the object is “globally” nontrivial if $\zeta \neq 0$; the form ω_ζ is not exact on all of P, and the extension does not split. Locally, however, the picture is quite different; indeed, in a small neighborhood of each point $p \in P^{sm}$, the form can be integrated, and the extension splits. This leaves the question of what happens at the boundary. We prove that local triviality at a point $p \in X^\vee$ is equivalent to ζ being in the kernel of the restriction map $H^n(X, \mathbb{C}) \to H^n\big(\mathfrak{X}_p, \mathbb{C}\big)$.

In the case when X is of even dimension $2m$, and when $\zeta \in H^{2m}(X, \mathbb{Z})_{prim} \cap H^{m,m}(X)$ is a primitive Hodge class, both constructions lead to a new interpretation for the *singularity* of the normal function ν_ζ associated to ζ.[6]

The normal function ν_ζ is only defined on P^{sm}; the usual construction breaks down at the boundary. To quantify this break-down, Green and Griffiths (2007) introduced the concept of a *singularity* of a normal function. Using ideas of Thomas (2005) and Clemens, they sketched a proof that the Hodge conjecture is equivalent to the existence of such singularities, once $\mathscr{L}$ is of sufficiently high degree. The proof was completed shortly afterwards by Brosnan, Fang, Nie, and Pearlstein (2007), and de Cataldo and Migliorini (2007).

Our results leads to the following two interpretations for the singularity of ν_ζ at points of $X^\vee$; the second one is not fully proved, however.

1. It is the obstruction to locally integrating the holomorphic one-form ω_ζ with coefficients in $F_{m+1}\mathcal{M}$.

2. Conjecturally, it is the obstruction to locally splitting a certain natural extension of coherent sheaves on P.

In either case, the Hodge class ζ itself is the global obstruction to the problem.

[6]Background material on normal functions and their singularities may be found in 7.3.1.

1.6 Notation

Throughout the dissertation, standard notation from algebraic geometry is used, but a few things deserve special notice.

Projective bundles If $\mathscr{E}$ is a locally free sheaf on a variety, then $\mathbb{P}(\mathscr{E})$ stands for the space of all invertible quotients of $\mathscr{E}$; the universal line bundle on it will be denoted by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. In one or two places, we also use the notation $\mathbb{P}_{sub}(\mathscr{E})$ in place of $\mathbb{P}\big(\mathscr{E}^{\vee}\big)$.

Covering spaces and fundamental groups Contrary to usual parlance, covering spaces are not assumed to be connected. We can thus think of the total space of a local system as being a covering space, even though it typically has countably many connected components. When the choice of a base point is not important, we may write $\pi_1(B)$ for the fundamental group of a space B. When B has several components, we write $[\mathbb{S}^1, B]$ for the set of homotopy classes of maps from $\mathbb{S}^1$ to B. By Hurewicz' Theorem, the natural map $[\mathbb{S}^1, B] \to H_1(B, \mathbb{Z})$ is surjective.

Complexes and cohomology If $\mathscr{F}^\bullet$ is a complex of sheaves on an algebraic variety, we write $\mathbb{H}^i\big(\mathscr{F}^\bullet\big)$ for its hypercohomology groups. On the other hand, $\mathcal{H}^i\mathscr{F}^\bullet$ denotes the i-th cohomology sheaf of the complex. Notation such as

$$\Big[A \longrightarrow B \dashrightarrow Y \longrightarrow Z\Big][k]$$

means that the first term of the complex (in this case, A) sits in degree $-k$.

CHAPTER 2

THE EXAMPLE OF CALABI-YAU THREEFOLDS

The example that motivated most of this dissertation is that of Calabi-Yau threefolds. In this chapter, we explain what the results in Chapters 3–7 have to say about the generalized Hodge conjecture for this type of variety. Recall that a smooth projective variety X is called *Calabi-Yau* if its canonical bundle $\mathscr{O}_X(K_X)$ is the trivial line bundle. We mostly consider three-dimensional smooth complex projective varieties.

The generalized Hodge conjecture

While the Hodge conjecture is already known for such varieties (because of their small dimension), they do provide an interesting class on which to test the *generalized* Hodge conjecture. This is the following open problem, sometimes referred to as GHC$(2k+l,k)$.

Question. Let X be a smooth complex projective variety of dimension $n+k$, and $H \subseteq H^{2k+l}(X,\mathbb{Q})$ a rational sub-Hodge structure contained in $F^kH^{2k+l}(X)$. Are there finitely many algebraic subvarieties $j_i\colon Y_i \hookrightarrow X$ of codimension k, such that H is contained in the subspace $\sum_i j_{i*}H_{2n-l}(Y_i,\mathbb{Q})$?

For a threefold X, the only interesting case of this conjecture is when $k=l=1$; by resolution of singularities, GHC$(3,1)$ is equivalent to the following question.

Question 1. Let X be a smooth projective threefold. Given a rational sub-Hodge structure $H \subseteq H^3(X, \mathbb{Q})_{prim}$ contained in $F^1H^3(X, \mathbb{C})$, does there exist a map $f\colon Y \to X$ from a smooth projective surface Y, such that $f_*H^1(Y, \mathbb{Q})$ intersects H non-trivially?

It is easy to see that a positive answer to this question for a threefold X implies $\mathrm{GHC}(3, 1)$ for the same variety.

Proof. Since the non-primitive part of $H^3(X, \mathbb{Q})$ is already accounted for through Lefschetz' Hyperplane Theorem, it suffices to treat sub-Hodge structures of the primitive cohomology. Consider first the case where H is simple, i.e., contains no smaller sub-Hodge structures except the trivial one. If we can find a map $f\colon Y \to X$ from a surface as in Question 1, then clearly $H \cap f_*H^1(Y, \mathbb{Q})$ has to be equal to all of H, if it is nontrivial. H is therefore contained in $f_*H^1(Y, \mathbb{Q})$. Because of the polarization, an arbitrary sub-Hodge structure of $H^3(X, \mathbb{Q})_{prim}$ is a direct sum of simple ones; therefore $\mathrm{GHC}(3, 1)$ holds for X. $\square$

Clemens' approach to the problem is through the study of Hodge loci for hypersurfaces in X. These had already been used very successfully by Voisin in her study of the Abel-Jacobi map for Calabi-Yau threefolds (Voisin, 1994). As usual, we let $\mathscr{L}$ be a very ample line bundle on X, and P its complete linear system. Over $P^{sm} = P \setminus X^\vee$, we have the family $\pi^{sm}\colon \mathfrak{X}^{sm} \to P^{sm}$ of smooth hypersurfaces, and the local system

$R^2\pi_*^{sm}\mathbb{Z}_{van}$ of its vanishing cohomology in degree two.[1] We let

$$T = T\big(R^2\pi_*^{sm}\mathbb{Z}_{van}\big)$$

be the total space of the local system; it is a covering space of P^{sm} with infinitely many sheets. A point on T can be viewed as a pair (S, γ) of a hypersurface $S \subseteq X$ and a cohomology class $\gamma \in H^2(S, \mathbb{Z})_{van}$. Following Clemens, we call such pairs *geometric cycles* on X.

The locus of Hodge classes

Of course, T is a complex manifold, but no longer an algebraic variety; this parallels the fact that most geometric cycles are not (fundamental classes of) algebraic cycles. Contained in T, however, is the *locus of Hodge classes*,

$$\mathrm{HL}(T) = \big\{\, (S, \gamma) \in T \;\big|\; \gamma \text{ is a Hodge class of type } (1,1) \text{ on } S \,\big\},$$

which is a countable union of algebraic varieties, each finite over P^{sm}.[2] Moreover, Voisin (1992) has shown that the zero-dimensional components of $\mathrm{HL}(T)$ map to a dense subset of P^{sm} when X is a Calabi-Yau threefold.

The existence of certain positive-dimensional components in $\mathrm{HL}(T)$, on the other hand, is related to the generalized Hodge conjecture; in fact, it is possible to rephrase Question 1 as a statement about the locus of Hodge classes.

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[1] We are taking the integral cohomology modulo torsion.

[2] This is true in general, not just for threefolds, by a a difficult theorem due to Cattani, Deligne, and Kaplan (1995).

Before doing so, we recall the definition of the *tube mapping* $\tau\colon H_1(T,\mathbb{Z}) \to H^3(X,\mathbb{Q})_{prim}$ from Chapter 4. Let (S,γ) be a geometric cycle, and let g be a loop in P^{sm}, based at the point corresponding to the hypersurface S. From this data, we can construct a class $\tau_g(\gamma) \in H^3(X,\mathbb{Q})_{prim}$; it is the Poincaré dual[3] of the 3-cycle swept out when translating $\mathrm{PD}(\gamma)$ flatly along the loop g.

As a matter of fact, each triple (S,γ,g) naturally defines an element of $[\mathbb{S}^1,T]$, since the condition $g\cdot\gamma=\gamma$ means exactly that g can be lifted to a closed loop in T, based at the point (S,γ). The tube mapping can therefore be viewed as a map $[\mathbb{S}^1,T] \to H^3(X,\mathbb{Q})_{prim}$; by Hurewicz' Theorem, we get an induced mapping

$$\tau\colon H_1(T,\mathbb{Z}) \to H^3(X,\mathbb{Q})_{prim}.$$

Because X is of odd dimension, Theorem 4.1.1 implies that τ is surjective.

Using the tube mapping, we can now rephrase Question 1 as a statement about the locus of Hodge classes.

Question 2. Let $H \subseteq H^3(X,\mathbb{Q})_{prim}$ be a rational sub-Hodge structure contained in $F^1H^3(X)$. If $\mathscr{L}$ is sufficiently ample, is there an algebraic curve $C \subseteq \mathrm{HL}(T)$ such that $\tau\big(H_1(C,\mathbb{Z})\big)$ meets H non-trivially?

Whenever we have an algebraic curve $C \subseteq T$, we get a map

$$H_1(C,\mathbb{Z}) \to H_1(T,\mathbb{Z}) \xrightarrow{\tau} H^3(X,\mathbb{Q})_{prim}.$$

When C is actually contained[4] in $\mathrm{HL}(T)$—so that each point $(S,\gamma)\in C$ is an algebraic

[3]More precisely, its projection to the primitive part of $H^3(X,\mathbb{Q})$.

[4]Whether every algebraic curve in T has to be contained in $\mathrm{HL}(T)$ is an open question.

cycle—the map is a morphism of (mixed) Hodge structures. Its image is therefore a sub-Hodge structure of $H^3(X,\mathbb{Q})_{prim}$, and Question 2 is asking whether one can find a curve C such that this image has nontrivial intersection with the given sub-Hodge structure H.

Let us now show why the two problems are equivalent. One implication is fairly easy to prove.

Lemma. *A positive answer to Question 2 implies the same for Question 1.*

Proof. Suppose $C \subseteq \mathrm{HL}(T)$ is the given curve. Each point $(S,\gamma) \in C$ is a Hodge class of type $(1,1)$ on a hypersurface $S \subseteq X$; by Lefschetz' Theorem, it is the fundamental class of an 1-dimensional algebraic cycle on S. It can be shown that, as (S,γ) ranges over all of C, these fit together into a 2-dimensional cycle; thus we get several surfaces Y_i, mapping both to C and to X. After compactifying the Y_i, and resolving their singularities, we have the following situation.

$$\begin{tikzcd} Y_i' \arrow[r, "f_i"] \arrow[d] & X \\ C & \end{tikzcd}$$

For each i, the map from $H_1(C,\mathbb{Z})$ to $H^3(X,\mathbb{Q})_{prim}$ now factors through $H^1(Y_i',\mathbb{Q})$. Since the image of $H_1(C,\mathbb{Z})$ in $H^3(X,\mathbb{Q})_{prim}$ meets H non-trivially, the same has to be true for one of the spaces $f_{i*}H^1(Y_i',\mathbb{Q})$. $\square$

The converse is also true, but the proof is more involved. We already know that a positive answer to Question 1 implies $\mathrm{GHC}(3,1)$; we complete the circle by proving that $\mathrm{GHC}(3,1)$, in turn, implies a strong form of the statement in Question 2.

Lemma. *Assume that* GHC$(3,1)$ *is true for the variety* X. *Given any rational sub-Hodge structure* $H \subseteq H^3(X,\mathbb{Q})_{prim}$ *contained in* $F^1H^3(X)$, *there is very ample line bundle* $\mathscr{L}$ *on* X, *and a curve* $C \subseteq \mathrm{HL}(T)$, *such that the image of* $H_1(C,\mathbb{Q})$ *under the tube mapping is precisely* H.

Proof. We carry out the proof in five steps, to make it easier to follow.

Step 1 Let $J(X)$ be the intermediate Jacobian associated to the Hodge structure on $H^3(X,\mathbb{Q})_{prim}$. Since the Hodge structure is polarized, we can find an Abelian subvariety $A \subseteq J(X)$, such that the image of $H_1(A,\mathbb{Q})$ in $H_1\big(J(X),\mathbb{Q}\big) \simeq H^3(X,\mathbb{Q})_{prim}$ is exactly H. Let C be any smooth complete intersection curve in A; then the map $H_1(C,\mathbb{Q}) \to H_1(A,\mathbb{Q})$ is surjective. We now have a morphism

$$\phi\colon H^1(C,\mathbb{Q}) \to H^3(X,\mathbb{Q})_{prim}$$

of Hodge structures, whose image is exactly the given sub-Hodge structure. Since $H^1(C,\mathbb{Q}) \otimes H^3(X,\mathbb{Q}) \subseteq H^4(C \times X,\mathbb{Q})$, we may also view ϕ as a rational Hodge class of type $(2,2)$ on the product $C \times X$.

Step 2 Since we are assuming GHC$(3,1)$, we can find maps $f_j\colon Y_j \to X$ from smooth projective surfaces Y_j, such that the sub-Hodge structure H is contained in $\sum_j f_{j*}H^1(Y_j,\mathbb{Q})$. Let $\alpha_1,\dots,\alpha_{2g}$ be a basis for the space $H^1(C,\mathbb{Q})$; also, let $\alpha_1^\vee,\dots,\alpha_{2g}^\vee$ be the dual basis. For each i, we may write

$$\phi(\alpha_i) = \sum_j f_{j*}(\beta_{i,j})$$

for certain classes $\beta_{i,j} \in H^1(Y_j, \mathbb{Q})$. Now let Y be the disjoint union of all the Y_j, and define a class $\beta \in H^2(C \times Y, \mathbb{Q})$ by the formula

$$\beta = \sum_{i,j} pr_C^*(\alpha_i^\vee) \cup pr_{Y_j}^*(\beta_{i,j}).$$

Under the map $f\colon C \times Y \to C \times X$, we then have

$$f_*(\beta) = \sum_{i,j} pr_C^*(\alpha_i^\vee) \cup \phi(\alpha_i) = \phi \in H^4(C \times X, \mathbb{Q}).$$

Thus ϕ is a Hodge class of type $(2,2)$ in the image of f_*. Since the Hodge structures in question are polarized, it is possible to find a *Hodge class* of type $(1,1)$ on $C \times Y$ whose image is ϕ. By Lefschetz' Theorem, it is the class of an algebraic cycle of dimension two. We conclude that the map ϕ is the correspondence given by the image Γ of that algebraic cycle in $C \times X$.

Step 3 By a result of Kleiman (1969, Theorem 5.8 on p. 297), we can represent a positive multiple of the algebraic cycle Γ by a difference $Z - Z'$, where both Z and Z' are smooth surfaces in $C \times X$, meeting transversely at finitely many points, and Z' is a complete intersection of two very ample hypersurfaces in $C \times X$. It is easy to see that the correspondence given by Z' takes its image in the subspace $\operatorname{im}\bigl(H^1(X, \mathbb{Q}) \to H^3(X, \mathbb{Q})\bigr)$, which is perpendicular to the vanishing cohomology; thus we only need to consider Z from now on.

Step 4 Now choose a sufficiently ample line bundle $\mathscr{L}$ on X, and embed the curve C into the projective space $P = \mathbb{P}\bigl(H^0(X, \mathscr{L})^\vee\bigr)$, making sure that it meets $X^\vee$ in only finitely many points. We write $\mathscr{O}_C(1)$ for the corresponding line bundle on the curve C. We require that $\mathscr{L}$ be ample enough for the line bundle $pr_C^* \mathscr{O}_C(1) \otimes pr_X^* \mathscr{L}$ on $C \times X$ to have a section whose zero locus W contains Z, and is nonsingular except for finitely many nodes along Z. Let $C_0 \subseteq C \cap P^{sm}$ be the subset over which $W \to C$ is smooth; if W_0 is its preimage, we have the following diagram.

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d, "q_0"'] & X \\ C_0 & \end{tikzcd}$$

Step 5 Since $C_0 \subseteq P^{sm}$, each point on C_0 corresponds to a smooth hypersurface $S \subseteq X$; by construction, S is a fiber of q_0. We define a cohomology class $\gamma(S)$ by taking the projection of $[S \cap Z]$ into $H^2(S, \mathbb{Q})_{van}$. The map $S \mapsto \big(S, \gamma(S)\big)$ is then an embedding of the curve C_0 into the space of geometric cycles T; since $\gamma(S)$ is always of type $(1,1)$, we have $C_0 \subseteq \mathrm{HL}(T)$. By construction, the tube mapping τ from $H_1(C_0, \mathbb{Q})$ to $H^3(X, \mathbb{Q})_{prim}$ is exactly given by the correspondence $[Z]$, followed by projection to the primitive cohomology. Since $[Z']$ maps into the orthogonal complement of $H^3(X, \mathbb{Q})_{prim}$, the image of τ is the same as that of $[Z] - [Z'] = \phi$. But this image is the given sub-Hodge structure H, and so the lemma is proved. $\square$

In summary, proving GHC$(3,1)$ for the threefold X is essentially equivalent to finding curves in the Hodge loci for a sufficiently ample line bundle.

The Hodge loci as gradient schemes

So far, everything works for an arbitrary smooth projective threefold. But now, an important insight, motivated by the Witten superpotential from physics, but due to

Clemens (2005) in the algebro-geometric setting, comes into play: The Hodge loci on a *Calabi-Yau* threefold have a very special structure—they are gradient schemes for a (locally well-defined) holomorphic function on T, the so-called *potential function.* The potential function has a simple topological description. Because X is Calabi-Yau, it is possible to find a nowhere vanishing holomorphic 3-form ω (unique up to scalars). Let (S, γ) be an arbitrary point of T. The homology class $\mathrm{PD}(\gamma)$ becomes trivial in $H_2(X, \mathbb{Z})$, and is therefore the boundary of a 3-chain Γ in X. Clemens defines

$$\Phi_\omega(S, \gamma) = \int_\Gamma \omega.$$

Because Γ can be flatly translated to nearby points, the resulting function Φ_ω is locally well-defined and holomorphic. Globally, the ambiguity in the definition is exactly in the periods $\int_\alpha \omega$, for $\alpha \in H_3(X, \mathbb{Z})$. The holomorphic one-form

$$\Omega_\omega = d\Phi_\omega$$

is therefore well-defined on all of T. Clemens (2005, Theorem 6.1 on p. 719) proves that $\mathrm{HL}(T)$ is the zero locus of the section $\Omega_\omega \in \Gamma(T, \Omega_T^1)$; it follows that $\mathrm{HL}(T)$ is a gradient scheme, because it is locally given by the vanishing of the partial derivatives of the potential function Φ_ω.

The periods of the form Ω_ω are related to those of the original three-form ω through the tube mapping.

Lemma. *Let (S, γ, g) represent a closed loop in T, based at a geometric cycle (S, γ). Then we have*

$$\int_{(S,\gamma,g)} \Omega_\omega = \int_X \tau_g(\gamma) \cup \omega.$$

Proof. This follows immediately from the homological description of the tube mapping, given on pp. 82–85. Viewing g as a smooth map from $[0,1]$ to P^{sm} with $g(0) = g(1)$, we let γ_t be the flat translate of $\gamma = \gamma_0$ to the point $g(t)$; then $\gamma_t = \partial\Gamma_t$, for suitable three-chains Γ_t on X. Also, the invariance of γ means that $\gamma_1 - \gamma_0 = \partial\Gamma'$ for some three-chain Γ' on the surface S. According to (4.14), we then have

$$\int_X \tau_g(\gamma) \cup \omega = \int_{\Gamma_1 - \Gamma_0 - \Gamma'} \omega = \int_{\Gamma_1} \omega - \int_{\Gamma_0} \omega,$$

because the restriction of the holomorphic three-form ω to the surface S is zero. But the right-hand side is clearly the result of integrating $\Omega_\omega = d\Phi_\omega$ along the loop given by (S, γ, g). $\square$

It is also possible to define the form Ω_ω globally, without making any local choices. This uses the filtered $\mathcal{D}$-module $(\mathcal{M}, F)$ introduced in Chapter 5. In the case of a threefold X, sections of the coherent sheaf $F_k\mathcal{M}$ on P are locally residues of rational 3-forms on $P \times X$, with a pole of order at most k along the universal hypersurface $\mathfrak{X} \subseteq P \times X$. On P^{sm}, the sheaf $F_k\mathcal{M}$ restricts to the Hodge bundle with fibers $F^{3-k}H^2(S, \mathbb{C})_{van}$.

Using the construction explained in 7.1.1, we get from the class $\omega \in H^3(X, \mathbb{Q})_{prim}$ a unique[5] holomorphic one-form $\omega_{\mathcal{M}}$ on P with coefficients in the sheaf $F_1\mathcal{M}$. For a point $(S, \gamma) \in T$, we can pair this section of $\Omega^1_P \otimes F_1\mathcal{M}$ with γ, using the intersection

[5]Since $pr_X^*\omega$ is already holomorphic on $P \times X$, there is no ambiguity in the construction.

pairing on the surface S. We then get a holomorphic one-form on the space T; it is given by the rule

$$\xi \mapsto \int_S \omega_{\mathcal{M}}(\rho_* \xi) \cup \gamma,$$

where ξ is a tangent vector to T at the point (S, γ), and $\rho\colon T \to P$ is the projection. It is not hard to see that the one-form thus obtained is exactly Ω_ω; the following lemma is the precise statement.

Lemma. *Let ξ be a tangent vector to the space T, say based at a geometric cycle (S, γ). Then we have*

$$\Omega_\omega(\xi) = \int_S \omega_{\mathcal{M}}(\rho_* \xi) \cup \gamma,$$

and so both constructions define the same holomorphic one-form on T.

Proof. Let $\xi' = \rho_* \xi$ be the image of the tangent vector in P. Let $s_0 \in H^0(X, \mathscr{L})$ be a section of $\mathscr{L}$ defining the surface $S \subseteq X$. The tangent vector ξ' is then based at the point $\mathbb{C} \cdot s_0 \in P$, and is given (up to multiples of s_0) by another section $s \in H^0(X, \mathscr{L})$. We can think of ξ' both as a section σ of the normal bundle of S in X (determined by s), or as a rational function $f = s/s_0$ on X with a first-order pole along S.

The first interpretation is helpful when computing $\Omega_\omega(\xi)$. Indeed, we have

$$\Omega_\omega(\xi) = d_{\xi'} \Phi_\omega = d_{\xi'} \int_\Gamma \omega = \int_{\mathrm{PD}(\gamma)} \operatorname{adj}_\sigma \omega = \int_S \operatorname{adj}_\sigma \omega \cup \gamma,$$

where $\operatorname{adj}_\sigma \omega$ is the two-form on S determined from the section σ of $N_{S \subseteq X}$ by adjunction.

On the other hand, the construction of the form $\omega_{\mathcal{M}}$ in 7.1.1 specifies that $\omega_{\mathcal{M}}(\xi') = \operatorname{Res}_S(f\omega)$; therefore

$$\int_S \omega_{\mathcal{M}}(\rho_* \xi) \cup \gamma = \int_S \operatorname{Res}_S(f\omega) \cup \gamma.$$

But we clearly have $\operatorname{adj}_\sigma \omega = \operatorname{Res}_S(f\omega)$, and so the asserted identity follows. $\square$

Extending the space of geometric cycles

To continue, we need some information about how Φ_ω and Ω_ω behave at the boundary of the space of geometric cycles. Chapter 3 contains a general construction of an analytic space extending the total space of a local system with unipotent monodromy in the maximal possible way. We now apply this construction to the local system $R^2\pi_*^{sm}\mathbb{Z}_{van}$, whose total space is the space T of geometric cycles.

To meet the conditions of Theorem 3.2.1, the boundary divisor should have normal crossing singularities, and the local system should have unipotent monodromy there. On P^{sm}, neither of those conditions is satisfied, and so we need to pass to a finite cover and resolve singularities. By a theorem of Borel (Voisin, 2002, Théorème 15.15 on p. 342), the local monodromy of $R^2\pi_*^{sm}\mathbb{Z}_{van}$ near points of $X^\vee$ is already quasi-unipotent. Take a finite branched cover of P, étale over P^{sm}, which makes the global monodromy be trivial modulo 3. Resolving the singularities, we thus arrive at a map $\mu\colon \overline{M} \to P$, with $M = \mu^{-1}(P^{sm})$ étale over P^{sm}, and $\overline{M} \setminus M$ a divisor with normal crossings. The following lemma implies that the local monodromy of $\mathcal{H}_\mathbb{Z} = \mu^{-1}\big(R^2\pi_*^{sm}\mathbb{Z}_{van}\big)$ around $\overline{M} \setminus M$ is now unipotent.

Lemma. *Let A be a square matrix with integer entries, all of whose eigenvalues are roots of unity. If A is congruent to the identity matrix modulo a prime number $p \geq 3$, then A is a unipotent matrix.*

Proof. Let v be any integer vector such that $A^k v = v$ for some $k > 0$. To prove that

A is unipotent, we need to show that we have $Av = v$. Write $A = \mathrm{id} + pB$, for a certain integer matrix B; setting $u = Bv$, we may rescale v by a rational number to guarantee that u is an integer vector, whose components are relatively prime.
Now we have

$$0 = A^k v - v = \sum_{i=1}^{k} \binom{k}{i} p^i \cdot B^{i-1} u \equiv kp \cdot u \mod p^2,$$

and so k has to be divisible by p. It follows easily that k is actually a power of p; by induction, it suffices to show that $u = Bv = 0$ in the case $k = p$.
When $k = p$, we get

$$0 = A^p v - v = p^2 \cdot u + \binom{p}{2} p^2 \cdot Bu + \cdots + \binom{p}{p} p^p \cdot B^p u \equiv p^2 \cdot u \mod p^3,$$

since $p \geq 3$. But if the components of u are relatively prime, this is not possible unless $u = 0$. $\square$

We write T_μ for the total space of $\mathcal{H}_{\mathbb{Z}}$ over M, still considering it as a space of geometric cycles. Let $\overline{\mathscr{V}}$ be Deligne's canonical extension of the vector bundle $\mathcal{H}_{\mathbb{Z}} \otimes \mathscr{O}_M$ to $\overline{M}$, as described in Section 3.1. By Theorem 3.2.1, the closure of T_μ inside the total space of $\overline{\mathscr{V}}$ is a nice analytic space; we denote it by $\overline{T_\mu}$ in the sequel.
The closure of the locus of Hodge classes $\mathrm{HL}(T_\mu)$ is evidently contained in $\overline{T_\mu}$; from the more precise version of the theorem by Cattani, Deligne, and Kaplan (1995, Theorem 1.5 on p. 485), it follows that the closure itself is a countable union of projective algebraic varieties, each finite over $\overline{M}$. Thus the problem of finding algebraic curves in $\mathrm{HL}(T_\mu)$ (raised by Question 2) is equivalent to finding *complete* curves in the closure.

Extending the tube mapping and the potential function

Of course, we still have a tube mapping $\tau\colon H_1(T_\mu, \mathbb{Z}) \to H^3(X, \mathbb{Q})_{prim}$; the definition of Φ_ω and Ω_ω also generalizes in the obvious way to T_μ, because $T_\mu \to T$ is a finite covering space. But what makes the extended space $\overline{T_\mu}$ useful is that both τ and Ω_ω extend over the boundary.

For the tube mapping, this means that it is the *global* topology of the space T_μ that is relevant for detecting the primitive cohomology of X; classes that are in the kernel of $H_1(T_\mu, \mathbb{Z}) \to H_1(\overline{T_\mu}, \mathbb{Z})$ contribute nothing.

Lemma. *The tube mapping $\tau\colon H_1(T_\mu, \mathbb{Z}) \to H^3(X, \mathbb{Q})_{prim}$ factors through a map $\overline{\tau}\colon H_1(\overline{T_\mu}, \mathbb{Z}) \to H^3(X, \mathbb{Q})_{prim}$, which is still surjective.*

Proof. Since the complement of T_μ in $\overline{T_\mu}$ is a divisor, $\pi_1\bigl(\overline{T_\mu}\bigr)$ is a quotient of $\pi_1(T_\mu)$. Thus it suffices to prove that any sufficiently small loop in T_μ, that can be contracted in $\overline{T_\mu}$, has trivial image under the tube mapping τ. Say the loop in question is based at a geometric cycle (S, γ), and write g for its image in $\overline{M}$. Without loss of generality, we may assume that the loop is contained in an analytic arc passing through a point of $\overline{T_\mu} \setminus T_\mu$.

By the local description of the canonical extension in Proposition 3.4.1, we are thus reduced to the following situation: We have a small arc $f\colon \Delta \to \overline{M}$, such that the image of $\mathbb{S}^1 = \partial\Delta$ is the loop g. Moreover, we have a g-invariant cohomology class γ in one of the surfaces over $f(\Delta)$.

We can now pull the universal family $\mathfrak{X} \to P$ back to Δ; to simplify the situation, we apply the Semi-stable Reduction Theorem (Kempf et al., 1973, p. 53). This involves replacing f by a finite cover, ramified only at the origin, and therefore gives information about some multiple of g; since we are working with rational coefficients, that poses no problem.

We can thus assume that we have a family $p\colon \mathfrak{S} \to \Delta$, whose total space $\mathfrak{S}$ is smooth, and whose central fiber $p^{-1}(0)$ is a reduced divisor with simple normal crossings. By the Local Invariant Cycle Theorem, the g-invariant cohomology class γ is the restriction of a class $\tilde{\gamma} \in H^2(\mathfrak{S}, \mathbb{Q})$. Shrinking Δ, if necessary, we may view $\mathfrak{S}$ as a manifold with boundary $\partial\mathfrak{S} = p^{-1}(\partial\Delta)$; note that the boundary itself is a real 5-manifold, mapping submersively to $\partial\Delta$.

Let $q\colon \mathfrak{S} \to X$ be the projection to X. The cohomological description of the tube mapping, on pp. 82–85, shows that the image $\tau_g(\gamma)$ of the given loop under τ can be obtained in the following way: Restrict $\tilde{\gamma}$ to the 5-manifold $\partial\mathfrak{S}$, push the resulting class forward to X using the Gysin map q_*, and then project to the primitive cohomology $H^3(X, \mathbb{Q})_{prim}$.

For any class $\zeta \in H^3(X, \mathbb{C})_{prim}$, we thus have

$$\int_X \tau_g(\gamma) \cup \zeta = \int_X q_*\bigl(\tilde{\gamma}\big|_{\partial\mathfrak{S}}\bigr) \cup \zeta = \int_{\partial\mathfrak{S}} \tilde{\gamma} \cup q^*(\zeta) = 0$$

by Stokes' Theorem. Consequently, the loop in T_μ has trivial image under the tube mapping, and the lemma is proved. $\square$

The proof that the potential function Φ_ω extends to $\overline{T_\mu}$ is more involved. One approach is to prove that Φ_ω is remains bounded near points of $\overline{T_\mu} \setminus T_\mu$; the fact the τ extends to $\overline{T_\mu}$ shows that there is no local monodromy, and therefore the potential function extends over $\overline{T_\mu}$, still being locally well-defined.

We choose a different, global approach to the problem, using a result by Kawamata (see Proposition 6.2.2 below). It relates the lowest level of the Hodge filtration of the canonical extension to the direct image of the relative canonical bundle in a nice family.

Lemma. *The potential function Φ_ω extends to a (locally well-defined) function on $\overline{T_\mu}$; the holomorphic one-form Ω_ω extends to a global section of the sheaf of Kähler differentials on $\overline{T_\mu}$.*

Proof. By pulling the family $\mathfrak{X} \to P$ back to $\overline{M}$, and resolving the singularities of the resulting space, we get a new family $\tilde{\pi}\colon \tilde{\mathfrak{X}} \to \overline{M}$. All three assumptions of Proposition 6.2.2 are then satisfied—indeed, the restriction $\tilde{\pi}^M$ of $\tilde{\pi}$ is smooth over the subset M; the divisor $\overline{M} \setminus M$ has normal crossings; and the local system $R^2\tilde{\pi}^M_*\mathbb{Z}$ on M has unipotent monodromy around $\overline{M} \setminus M$. Let $\overline{\mathscr{V}}$ be the canonical extension of the vector bundle $R^2\tilde{\pi}^M_*\mathbb{C} \otimes \mathscr{O}_M$ to all of $\overline{M}$. Kawamata's result in Proposition 6.2.2 implies that

$$\tilde{\pi}_*\mathscr{O}_{\tilde{\mathfrak{X}}}\big(K_{\tilde{\mathfrak{X}}/\overline{M}}\big) \simeq F^2\overline{\mathscr{V}}$$

is a locally free sheaf on $\overline{M}$.

Next, we observe that the local system $R^2\tilde{\pi}^M_*\mathbb{Z}$ is the direct sum of the vanishing cohomology $R^2\tilde{\pi}^M_*\mathbb{Z}_{van}$, and the constant local system $H^2(X,\mathbb{Z}) \otimes \mathbb{Z}_M$. The canonical extension of the latter is the trivial vector bundle $H^2(X,\mathbb{Z}) \otimes \mathscr{O}_{\overline{M}}$, and so we have a natural decomposition

$$\tilde{\pi}_*\mathscr{O}_{\tilde{\mathfrak{X}}}\big(K_{\tilde{\mathfrak{X}}/\overline{M}}\big) \simeq F^2\overline{\mathscr{V}}_{van} \oplus H^2(X,\mathbb{Z}) \otimes \mathscr{O}_{\overline{M}}. \tag{2.1}$$

The holomorphic three-form ω on X now defines, in a natural way, a section of $\Omega^1_{\overline{M}} \otimes F^2\overline{\mathscr{V}}_{van}$, as follows. First of all, we have

$$\Omega^1_{\overline{M}} \otimes \tilde{\pi}_* \mathscr{O}_{\tilde{\mathfrak{X}}}\big(K_{\tilde{\mathfrak{X}}/\overline{M}}\big) \simeq \mathcal{H}om\big(\Omega^{d-1}_{\overline{M}}, \tilde{\pi}_* \Omega^{d+2}_{\tilde{\mathfrak{X}}}\big) \simeq \tilde{\pi}_* \mathcal{H}om\big(\tilde{\pi}^* \Omega^{d-1}_{\overline{M}}, \Omega^{d+2}_{\tilde{\mathfrak{X}}}\big),$$

because $\dim \tilde{\mathfrak{X}} = d+2$. Taking the wedge product with the pullback of ω to $\tilde{\mathfrak{X}}$ gives a global section of the sheaf on the right, and hence a one-form with coefficients in $\tilde{\pi}_* \mathscr{O}_{\tilde{\mathfrak{X}}}\big(K_{\tilde{\mathfrak{X}}/\overline{M}}\big)$. Its image in the second summand of the decomposition (2.1) is clearly zero, and so we have a section $\tilde{\omega}$ of $\Omega^1_{\overline{M}} \otimes F^2\overline{\mathscr{V}}_{van}$.

The polarization on the local system $R^2\tilde{\pi}^M_* \mathbb{Z}_{van}$ extends uniquely to a pairing on the fibers of the canonical extension; just as in the case of $\omega_{\mathcal{M}}$, $\tilde{\omega}$ can thus be used to define a holomorphic one-form on the total space $T(\overline{\mathscr{V}})$ of the canonical extension. Its restriction to the analytic subspace $\overline{T_\mu}$ is then an analytic Kähler differential.

Over M, this construction is obviously the same as the one involving the form $\omega_{\mathcal{M}}$; since we have already shown that the derivative Ω_ω of the potential function can be obtained from $\omega_{\mathcal{M}}$, it follows that $\tilde{\omega}$ defines an extension of Ω_ω to all of $\overline{T_\mu}$. This extension is a closed form, because this is true for Ω_ω on the open subset T_μ. $\square$

CHAPTER 3

THE CANONICAL EXTENSION OF A LOCAL SYSTEM

3.1 Deligne's canonical extension of a flat vector bundle

In this short section, we review Deligne's construction of a canonical extension for flat vector bundles with unipotent monodromy.

Conventions used in this chapter

Local systems can be viewed as representations of the fundamental group; in this context, we use the following conventions.

Fundamental group If X is a topological space, with base point $x \in X$, we write $\pi_1(X, x)$ for its fundamental group. Given two closed paths $\gamma, \delta \colon [0,1] \to X$, with $\gamma(0) = \gamma(1) = \delta(0) = \delta(1) = x$, representing two elements of $\pi_1(X, x)$, their product is defined as

$$\gamma\delta \colon [0,1] \to X, \qquad t \mapsto \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq 1/2, \\ \delta(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Writing Δ for the unit disk in $\mathbb{C}$, and $\Delta^* = \Delta \setminus \{0\}$ for the punctured disk, we shall take the generator of $\pi_1(\Delta^*) \simeq \mathbb{Z}$ to be a loop that goes around the origin once, counter-clockwise.

Action on the fiber For any covering space $p\colon Y \to X$ of X, the group $\pi_1(X, x)$ acts on the fiber $p^{-1}(x)$ by a left action; given $\gamma \in \pi_1(X, x)$ and a point $y \in Y$ with $p(y) = x$, one has $\gamma \cdot y = \tilde{\gamma}(0)$, where γ has been lifted to a path $\tilde{\gamma}\colon [0, 1] \to Y$ with $\tilde{\gamma}(1) = y$. Put more succinctly, γ acts by parallel translation along the path γ^{-1}.

Local systems A special case of this is the correspondence between local systems on X and representations of the fundamental group. Given a local system $\mathcal{H}_{\mathbb{Z}}$ of Abelian groups on X, say with fiber $H = \mathcal{H}_{\mathbb{Z},x}$, each connected component of the total space of $\mathcal{H}_{\mathbb{Z}}$ is a covering space of X. One obtains a representation

$$\rho\colon \pi_1(X, x) \to \mathrm{Aut}(H)$$

by letting the fundamental group act on the fiber. From the representation ρ, on the other hand, one can recover $\mathcal{H}_{\mathbb{Z}}$. Indeed, if $p\colon \tilde{X} \to X$ is the universal covering space of X (assuming its existence), the quotient of $\tilde{X} \times H$ by the group action

$$\gamma \cdot \big(\tilde{x}, h\big) = \big(\gamma \cdot \tilde{x}, \rho(\gamma)h\big)$$

is isomorphic to the total space of $\mathcal{H}_{\mathbb{Z}}$. From this description, it follows that the space of sections of $\mathcal{H}_{\mathbb{Z}}$ over an open set $U \subseteq X$ is given by

$$\big\{\, \tilde{s}\colon p^{-1}(U) \to H \;\big|\; \tilde{s}(\gamma \cdot y) = \rho(\gamma)\tilde{s}(y) \text{ for all } \gamma \in \pi_1(X, x),\ y \in p^{-1}(U) \,\big\}.$$

Deligne's canonical extension

Let $(\mathscr{V}, \nabla)$ be a flat holomorphic vector bundle on a complex manifold M. We assume that M is an open subset of a bigger complex manifold $\overline{M}$, in such a way that

1. $\overline{M} \setminus M$ has normal crossing singularities, and
2. ∇ is unipotent along $\overline{M} \setminus M$.

The second condition means that, near points of $\overline{M} \setminus M$, the local monodromy for the local system of ∇-flat sections should be unipotent. Deligne (1970, pp. 91–5) proves that $(\mathscr{V}, \nabla)$ admits a unique extension to a vector bundle $\overline{\mathscr{V}}$ on $\overline{M}$, whose defining property is that, in any local frame for $\overline{\mathscr{V}}$, the connection ∇ has only logarithmic poles along $\overline{M} \setminus M$ with nilpotent residues.

We now review the construction of $\overline{\mathscr{V}}$ in local coordinates; this will also explain the meaning of the two conditions above. Let $t_1, \dots, t_n$ be local holomorphic coordinates near a point of $\overline{M}$, say defined on a polydisk Δ^n, and assume that $\overline{M} \setminus M$ is locally defined by the equation $t_1 \cdots t_r = 0$. Restricting further, if necessary, it suffices to treat the case $r = n$; thus we may assume that $M \cap \Delta^n = (\Delta^*)^n$.

Let d be the rank of the bundle $\mathscr{V}$, and V its fiber at some base point in $(\Delta^*)^n$. The fundamental group $\mathbb{Z}^n$ of $(\Delta^*)^n$ acts on V, by parallel translation, and we let T_j be the operator corresponding to the j-th standard generator of $\mathbb{Z}^n$. By assumption, each T_j is a *unipotent* operator, and we can therefore define the nilpotent operators

$$N_j = -\log T_j = \sum_{n=1}^{\infty} \frac{1}{n} (\mathrm{id} - T_j)^n$$

as their logarithms.[1]

The vector bundle $\overline{\mathscr{V}}$ has distinguished trivializations of the form

$$\overline{\mathscr{V}} \simeq \mathscr{O}_{\Delta^n} s_1 \oplus \cdots \oplus \mathscr{O}_{\Delta^n} s_d, \tag{3.1}$$

for certain special sections $s_1, \ldots, s_d$ of $\mathscr{V}$ over $(\Delta^*)^n$. To obtain the sections in question, pull $(\mathscr{V}, \nabla)$ back to the universal covering space

$$p\colon \mathbb{H}^n \to (\Delta^*)^n, \qquad p(z_1, \ldots, z_n) = \left(e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}\right),$$

where it becomes trivial (by virtue of being flat). By our conventions, the fundamental group $\mathbb{Z}^n$ acts on $\mathbb{H}^n$ by the rule

$$(a_1, \ldots, a_n) \cdot (z_1, \ldots, z_n) = (z_1 - a_1, \ldots, z_n - a_n),$$

and so sections of $\mathscr{V}$ over $(\Delta^*)^n$ correspond to holomorphic maps $\tilde{s}\colon \mathbb{H}^n \to V$ with the property that

$$\tilde{s}(z - e_j) = T_j \tilde{s}(z)$$

for all $z \in \mathbb{H}^n$ and all $j = 1, \ldots, n$.

Now let $v_1, \ldots, v_d \in V$ be an arbitrary basis for V. The maps

$$\tilde{s}_i\colon \mathbb{H}^n \to V, \qquad \tilde{s}_i(z) = e^{\sum z_j N_j} v_i, \tag{3.2}$$

have the required invariance property, because $\tilde{s}_i(z - e_j) = e^{-N_j} \tilde{s}_i(z) = T_j \tilde{s}_i(z)$, and thus define a frame of sections $s_1, \ldots, s_d$ for $\mathscr{V}$ on $(\Delta^*)^n$. These sections give the special trivialization of $\overline{\mathscr{V}}$ in (3.1).

[1] The minus sign is there to stay with the conventions of other authors (Cattani et al., 1995).

In this local frame for $\overline{\mathscr{V}}$, the connection ∇ now has logarithmic poles with nilpotent residues. This is easy to see from (3.2); indeed, on $\mathbb{H}^n$, we have

$$\nabla \tilde{s}_i = \sum_{j=1}^{d} dz_j \otimes e^{\sum z_j N_j} N_j v_i,$$

and since $2\pi i \cdot dz_j = dt_j/t_j$, we conclude that

$$\nabla s_i = \frac{1}{2\pi i} \cdot \sum_{j=1}^{d} \frac{dt_j}{t_j} \otimes N_j s_i.$$

The poles in this expression are logarithmic; moreover, for each $j = 1, \dots, d$, the residue of the connection along the divisor $t_j = 0$ is the nilpotent matrix N_j.

3.2 The existence of a canonical extension

In this section, we describe the construction of the "canonical extension" of a local system (as an analytic space). Its properties are given in Theorem 3.2.1, whose proof occupies the remainder of this chapter.

Let M be a complex manifold of dimension n, embedded as an open subset into a larger complex manifold $\overline{M}$, in such a way that $\overline{M} \setminus M$ is a divisor with only normal crossing singularities. Every point in $\overline{M}$ thus has a neighborhood isomorphic to Δ^n, with holomorphic coordinates $t_1, \dots, t_n$, in which the divisor $\overline{M} \setminus M$ is defined by an equation of the form $t_1 \cdots t_r = 0$.

On M, we assume that we are given a local system $\mathcal{H}_{\mathbb{Z}}$, with fiber $H \simeq \mathbb{Z}^d$ a finitely generated free $\mathbb{Z}$-module. Up to isomorphism, it is determined by the corresponding monodromy representation $\rho\colon G \to \operatorname{Aut}_{\mathbb{Z}}(H)$, where G is the fundamental group of M (for some choice of base point).

We shall assume that the local system $\mathcal{H}_{\mathbb{Z}}$ is *unipotent*; that is to say, in a neighborhood Δ^n of each point, the fundamental group of $\Delta^n \cap M$ should act by unipotent transformations on the fiber of $\mathcal{H}_{\mathbb{Z}}$. It is then possible to extend the holomorphic vector bundle $\mathscr{V} = \mathcal{H}_{\mathbb{Z}} \otimes \mathscr{O}_M$ to a vector bundle $\overline{\mathscr{V}}$ on $\overline{M}$, using Deligne's construction. The total space $T = T(\mathcal{H}_{\mathbb{Z}})$ of the local system is naturally a subset of the total space $T(\overline{\mathscr{V}})$ of this canonical extension. We extend T in the maximal possible way, by taking its (topological) closure inside the total space of the vector bundle. A priori, it is not clear that this produces even an analytic set. Nevertheless, the following theorem shows that the closure is an analytic space, with surprisingly good properties.

Theorem 3.2.1. *Assume that the local system $\mathcal{H}_{\mathbb{Z}}$ is unipotent on M, and that $\overline{M} \setminus M$ is a divisor with normal crossings. Let $\overline{T}$ be the closure of the total space T of the local system, taken inside the total space of Deligne's canonical extension of $\mathscr{V} = \mathcal{H}_{\mathbb{Z}} \otimes \mathscr{O}_M$ over $\overline{M}$. Then $\overline{T}$ has the following three properties:*

(i) $\overline{T}$ is a reduced analytic subset of $T(\overline{\mathscr{V}})$.

(ii) The projection map $p \colon \overline{T} \to \overline{M}$ is holomorphic, and $p^{-1}(M) = T$.

(iii) The normalization of $\overline{T}$ is locally toric.

Proof. To show that $\overline{T}$ is an analytic subset, we need to show that it is locally defined by analytic equations inside the complex manifold $T(\overline{\mathscr{V}})$. For an arbitrary point of $\overline{M}$, take a neighborhood isomorphic to Δ^n in which $\overline{M} \setminus M$ is defined by the equation $t_1 \cdots t_r = 0$. Then Proposition 3.5.3 in Section 3.5 below states precisely that the closure of T over Δ^n is a reduced analytic subset, and this establishes (i).

The assertion in (ii) that $p^{-1}(M) = T$ follows from the corresponding statement over each coordinate neighborhood Δ^n, also proved below (see the discussion at the end of Section 3.4). Finally, the statement about the normalization of $\overline{T}$ may be found in Section 3.6. $\square$

3.3 Local description of the problem

Determining the closure $\overline{T}$ inside the total space of the canonical extension is really a local problem, and we may restrict our attention to what happens in a small polydisk around each point $P \in \overline{M}$. Let $\Delta^n \subseteq \overline{M}$ be such a neighborhood, with local holomorphic coordinates $t_1, \dots, t_n$ centered at the point P in question. We assume that, in these coordinates, $\overline{M} \setminus M$ is defined by the equation $t_1 \cdots t_r = 0$. When P is a boundary point, we get $r > 0$, but the case of a point in M is included by taking $r = 0$. In any case, we have $(\Delta^*)^n \subseteq M$.

To avoid having to treat various cases based on the value of r, we will restrict the local system $\mathcal{H}_{\mathbb{Z}}$ to the set $(\Delta^*)^n$, and compute the closure of only this piece inside the total space of the canonical extension over Δ^n. We shall argue later, at the end of Section 3.4, that the result is the same.

The fundamental group of $(\Delta^*)^n$ is isomorphic to $\mathbb{Z}^n$. Let $H \simeq \mathbb{Z}^d$ be the fiber of the local system at some point in $(\Delta^*)^n$; by assumption, the monodromy action of $\mathbb{Z}^n$ on $\mathbb{Z}^d$ is by unipotent matrices. Let $T_j \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ be the matrix corresponding to the j-th standard generator of $\mathbb{Z}^n$, and put

$$N_j = -\log T_j = \sum_{m=1}^{\infty} \frac{1}{m} (\mathrm{id} - T_j)^m.$$

This is well-defined because $(\mathrm{id} - T_j)^m = 0$ for large values of m. The matrices N_j are nilpotent, with rational entries, and commute with one another.

In the given system of coordinates, we now describe how the local system is embedded into the total space of the canonical extension. To begin with,

$$\mathbb{H}^n \to (\Delta^*)^n, \qquad (z_1, \dots, z_n) \mapsto \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}\right),$$

is the universal covering space of $(\Delta^*)^n$. The canonical extension[2] of $\mathcal{H}_{\mathbb{Z}} \otimes \mathscr{O}_{(\Delta^*)^n}$ over Δ^n is isomorphic to the trivial vector bundle

$$\mathscr{O}_{\Delta^n} s_1 \oplus \cdots \oplus \mathscr{O}_{\Delta^n} s_d;$$

here s_i is the section of $\mathcal{H}_{\mathbb{Z}} \otimes \mathscr{O}_{(\Delta^*)^n}$ on $(\Delta^*)^n$ whose pullback to $\mathbb{H}^n$ is given by the map

$$\tilde{s}_i \colon \mathbb{H}^n \to \mathbb{Z}^d, \qquad \tilde{s}_i(z) = e^{z_1 N_1 + \cdots + z_n N_n} e_i,$$

e_i being one of the standard basis elements of $\mathbb{Z}^d$. The total space of the canonical extension is thus isomorphic to $\Delta^n \times \mathbb{C}^d$, using this frame.

When the local system is pulled back to the universal covering space $\mathbb{H}^n$, it becomes trivial. At any given point $z = (z_1, \dots, z_n)$ of $\mathbb{H}^n$, a class $h \in \mathbb{Z}^d$ in the fiber of the trivial local system has coordinates $e^{-(z_1 N_1 + \cdots + z_n N_n)} h$ with respect to the given framing for the canonical extension. It follows that the point $(z, h) \in \mathbb{H}^n \times \mathbb{Z}^d$ has coordinates

$$\left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \cdots + z_n N_n)} h\right)$$

[2]See Section 3.1 for a description of Deligne's construction.

in $\Delta^n \times \mathbb{C}^d$. The total space of the local system, when embedded into that of the canonical extension, is thus the image of the holomorphic map

$$f\colon \mathbb{H}^n \times \mathbb{Z}^d \to \Delta^n \times \mathbb{C}^d,$$

defined by the rule

$$(z_1, \ldots, z_n, h) \mapsto \left(e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \cdots + z_n N_n)} h\right). \tag{3.3}$$

The closure of this image will be computed in the following section.

3.4 Set-theoretic description of the closure

We now determine which points inside the total space of the canonical extension belong to the closure of the local system. As explained in the previous section, this is a local question; we chose a neighborhood $\Delta^n \subseteq \overline{M}$ of an arbitrary point $P \in \overline{M}$, and study the closure over that neighborhood.

According to the description above, the total space of the local system over $(\Delta^*)^n$ is the image of the holomorphic map f given in (3.3). As it stands, that map is not one-to-one; when the real parts $x_j = \operatorname{Re} z_j$ are restricted to $0 \leq x_1, \ldots, x_n < 1$, however, every point in the image is parametrized only once.

The remainder of this section is devoted to proving the following proposition, which describes the points in the closure of the image of the map f. As written, it only makes a statement about points that lie over the origin in Δ^n, which is to say over the point $P \in \overline{M}$. But as we are free to place P wherever we please, we really get a description of all the points in the closure.

Proposition 3.4.1. *A point in $\Delta^n \times \mathbb{C}^d$ over $(0, \dots, 0) \in \Delta^n$ is in the closure of the image of f if, and only if, it is of the form*

$$\bigl(0, \dots, 0, e^{-(w_1 N_1 + \dots + w_n N_n)} h\bigr);$$

here $h \in \mathbb{Z}^d$ is such that $a_1 N_1 h + \dots + a_n N_n h = 0$ for some choice of positive integers $a_1, \dots, a_n$, while $w_1, \dots, w_n \in \mathbb{C}$ can be arbitrary complex numbers.
For each limit point, there is an arc (for suitably small $\varepsilon > 0$)

$$\Delta(\varepsilon) \to \Delta^n \times \mathbb{C}^d,$$

of the form

$$t \mapsto \bigl(t^{a_1} e^{2\pi i w_1}, \dots, t^{a_n} e^{2\pi i w_n}, e^{-(w_1 N_1 + \dots + w_n N_n)} h\bigr),$$

contained in the image of f for $t \neq 0$, and passing through the limit point at $t = 0$.

Proof. One half of this is easy to prove—if $h \in \mathbb{Z}^d$ satisfies $a_1 N_1 h + \dots + a_n N_n h = 0$ for positive integers $a_1, \dots, a_n$, then every point of the form

$$\bigl(0, \dots, 0, e^{-(w_1 N_1 + \dots + w_n N_n)} h\bigr)$$

is in the closure of the image of f. Indeed, taking the imaginary part of $z \in \mathbb{H}$ sufficiently large to have $\mathrm{Im}(a_j z + w_j) > 0$ for all j, we get

$$\begin{aligned} f(a_1 z + w_1, \dots, a_n z + w_n, h) &= \bigl(e^{2\pi i a_1 z} e^{2\pi i w_1}, \dots, e^{2\pi i a_n z} e^{2\pi i w_n}, e^{-\sum (a_j z + w_j) N_j} h\bigr) \\ &= \bigl(t^{a_1} e^{2\pi i w_1}, \dots, t^{a_n} e^{2\pi i w_n}, e^{-\sum w_j N_j} e^{-z \sum a_j N_j} h\bigr) \\ &= \bigl(t^{a_1} e^{2\pi i w_1}, \dots, t^{a_n} e^{2\pi i w_n}, e^{-\sum w_j N_j} h\bigr), \end{aligned}$$

having set $t = \exp(2\pi i z)$. For $t \neq 0$, these points are all in the image of f; as $t \to 0$, in other words, as $\operatorname{Im} z \to \infty$, they approach the point $\bigl(0, \ldots, 0, e^{-(w_1 N_1 + \cdots + w_n N_n)} h\bigr)$, which is consequently in the closure.

To prove the converse, we take a sequence of points in the image that converges to some point of $\{(0, \ldots, 0)\} \times \mathbb{C}^d$, and show that its limit is of the stated form. So let

$$\bigl(z(m), h(m)\bigr) = \bigl(z_1(m), \ldots, z_n(m), h(m)\bigr) \in \mathbb{H}^n \times \mathbb{Z}^d$$

be a sequence of points such that $f(z(m), h(m))$ converges to a point over $(0, \ldots, 0)$. This means that each $y_j(m) = \operatorname{Im} z_j(m)$ is tending to infinity, and that the sequence of vectors

$$e^{-\sum z_j(m) N_j} h(m) \in \mathbb{C}^d$$

is convergent as $m \to \infty$. Changing the values of $h(m)$, if necessary, we may in addition assume that the real parts $x_j(m) = \operatorname{Re} z_j(m)$ satisfy $0 \leq x_j(m) \leq 1$.

In the course of the argument, we shall frequently have to pass to a subsequence of $\bigl(z(m), h(m)\bigr)$. To avoid clutter, this will not be indicated in the notation—in each case, the subsequence will be denoted by the same letters $\bigl(z(m), h(m)\bigr)$ as the original sequence. Since it should not lead to any confusion, we shall avail ourselves of this convenient device.

Keeping this convention in mind, we now proceed in several steps.

Step 1 The sequence of real parts $x_j(m)$ is bounded, for each $j = 1, \ldots, n$, and we can thus pass to a subsequence where each $x_j(m)$ converges. The vectors

$$e^{\sum x_j(m) N_j} e^{-\sum z_j(m) N_j} h(m) = e^{-i \sum y_j(m) N_j} h(m)$$

still form a convergent sequence in this case, and so the $x_j(m)$ really play no role for the remainder of the argument.

Step 2 While all imaginary parts $y_j(m)$ are going to infinity, this may happen at greatly different rates. To make their behavior more tractable, we use the following technique, borrowed from the paper by Cattani, Deligne, and Kaplan (1995, p. 494). Let $y(m) = \big(y_1(m), \dots, y_n(m)\big)$. By taking a further subsequence, we can arrange that

$$y(m) = \tau_1(m)\theta^1 + \cdots + \tau_r(m)\theta^r + \eta(m),$$

where $\theta^1, \dots, \theta^r \in \mathbb{R}^n$ are constant vectors with nonnegative components, and where the ratios

$$\tau_1(m)/\tau_2(m), \dots, \tau_{r-1}(m)/\tau_r(m), \tau_r(m) \tag{3.4}$$

are all going to infinity. The remainder term $\eta(m)$, on the other hand, is convergent. We can even assume that

$$0 \leq \theta_j^1 \leq \theta_j^2 \leq \cdots \leq \theta_j^r$$

for all j; because $y_j(m) \to \infty$, all components of the last vector θ^r have to be positive real numbers.

Now define

$$N(m) = \sum_{j=1}^{n} \big(y_j(m) - \eta_j(m)\big) N_j.$$

As in Step 1, the convergence of the expression $e^{-i \sum \eta_j(m) N_j}$ makes the η_j essentially irrelevant to the rest of the argument—the sequence

$$e^{-iN(m)} h(m)$$

is still a convergent sequence.

Step 3 For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$N^\alpha = \prod_{j=1}^{n} N_j^{\alpha_j}.$$

Since the N_j are commuting nilpotent operators, $N^\alpha = 0$ whenever $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is sufficiently large.

We can thus let $p \geq 0$ be the smallest integer for which there is a subsequence of $\bigl(z(m), h(m)\bigr)$ with

$$N^\alpha h(m) = 0 \text{ for all multi-indices } \alpha \text{ with } |\alpha| \geq p + 1.$$

Passing to this subsequence, we find that when $|\alpha| = p$, the sequence

$$N^\alpha e^{-i \sum y_j(m) N_j} h(m) = N^\alpha h(m)$$

is convergent. However, it takes its values in a discrete set (in fact, there is an integer $M > 0$ such that each coordinate of $N^\alpha h(m)$ is in $\mathbb{Z}[1/M]$, and M depends only on α and the N_j), and so it has to be eventually constant. If we remove finitely many terms from the sequence, we can therefore achieve that

$$h^\alpha = N^\alpha h(m)$$

is constant whenever $|\alpha| = p$. Moreover, we have $N(m) h^\alpha = 0$ by the choice of p.

Step 4 At this point, we can use an inductive argument to get the conclusion of Step 3 for all multi-indices α with $|\alpha| \leq p$. Thus let us assume that we already have a subsequence $\bigl(z(m), h(m)\bigr)$ for which $h^\alpha = N^\alpha h(m)$ is constant and $N(m)h^\alpha = 0$, whenever α is a multi-index with $p' \leq |\alpha| \leq p$. If $p' > 0$, we now show how to get the same statement with p' replaced by $p' - 1$.

Consider a multi-index α with $|\alpha| = p' - 1$. Then

$$\begin{aligned} N^\alpha e^{-iN(m)} h(m) = N^\alpha h(m) - iN(m)N^\alpha h(m) \\ + \sum_{s=1}^{p-p'} (-i)^{s+1} N(m)^s \cdot N(m) N^\alpha h(m) \end{aligned} \tag{3.5}$$

is again convergent. Since $\alpha + e_j$ has length p', we see that

$$N(m)N^\alpha h(m) = \sum_{j=1}^n \bigl(y_j(m) - \eta_j(m)\bigr) N^{\alpha+e_j} h(m) = \sum_{j=1}^n \bigl(y_j(m) - \eta_j(m)\bigr) h^{\alpha+e_j};$$

by the inductive hypothesis, the last term in (3.5) is actually zero.

Thus the sequence $N^\alpha h(m) - iN(m)N^\alpha h(m)$ is itself convergent, implying convergence of its real and imaginary parts separately. As before, the sequence of real parts $N^\alpha h(m)$ has to be eventually constant, and after omitting finitely many terms, we can assume that it is constant. Let

$$h^\alpha = N^\alpha h(m)$$

be that constant value. Then the convergence of the imaginary part

$$N(m)h^\alpha = N(m)N^\alpha h(m) = \sum_{i=1}^r \tau_i(m) \sum_{j=1}^n \theta_j^i N^{\alpha+e_j} h(m) = \sum_{i=1}^r \tau_i(m) \sum_{j=1}^n \theta_j^i h^{\alpha+e_j},$$

together with the behavior of the $\tau_i(m)$ described in (3.4), shows that

$$\sum_{j=1}^n \theta_j^i h^{\alpha+e_j} = 0$$

for all i. But this says that, in fact, $N(m)h^\alpha = 0$. The statement is thus proved for all multi-indices α of length $|\alpha| = p' - 1$ as well.

Step 5 From Step 4, we conclude that, on a suitable subsequence, $h^\alpha = N^\alpha h(m)$ is constant for all α, and satisfies $N(m)h^\alpha = 0$. In particular, $h(m)$ is itself constant, equal to a certain element $h = h^{(0,\dots,0)} \in \mathbb{Z}^d$. Moreover, we have $N(m)h = 0$ for all m.

On the one hand, we now find that, along the subsequence we have chosen in the previous steps, our original convergent sequence simplifies to

$$e^{-\sum z_j(m)N_j} h(m) = e^{-\sum (x_j(m)+i\eta_j(m))N_j} e^{-iN(m)} h = e^{-\sum (x_j(m)+i\eta_j(m))N_j} h.$$

If we set $w_j = \lim_{m\to\infty}\big(x_j(m) + i\eta_j(m)\big)$, then the limit of the sequence is of the form $e^{-\sum w_j N_j} h$, which was part of the assertion in Proposition 3.4.1.

On the other hand, we conclude from

$$N(m)h = \sum_{i=1}^{r} \tau_i(m) \sum_{j=1}^{n} \theta_j^i N_j h = 0$$

that

$$\sum_{j=1}^{n} \theta_j^i N_j h = 0$$

for all $i = 1, \dots, r$.

Step 6 By Step 5, we know that the n vectors $N_j h$ are linearly dependent; the coefficients θ_j^r in the relation (for $i = r$) are positive real numbers. But as the vectors themselves are in fact in $\mathbb{Q}^d$, we can also find a relation with positive rational coefficients. Taking a suitable multiple, we then obtain positive integers $a_1, \dots, a_n$ satisfying

$$\sum_{j=1}^{n} a_j N_j h = 0.$$

The remaining assertion of the proposition is thereby established, and this finishes the proof. $\square$

For later use, we record the result of the six steps in the following proposition.

Proposition 3.4.2. *Let* $\big(z(m), h(m)\big) \in \mathbb{H}^n \times \mathbb{Z}^d$ *be a sequence of points with* $x_j(m) = \operatorname{Re} z_j(m) \in [0,1]$*, and assume that* $f\big(z(m), h(m)\big)$ *converges to a point in* $\Delta^n \times \mathbb{C}^d$ *over* $(0, \ldots, 0) \in \Delta^n$*. Then there is a subsequence, still denoted* $\big(z(m), h(m)\big)$*, for which* $h(m)$ *is constant.*

A technical point

For the sake of convenience, we had restricted the local system from the open set $U = \Delta^n \cap M$—whose complement in Δ^n is the normal crossing divisor with equation $t_1 \cdots t_r = 0$—to $(\Delta^*)^n$, and computed the closure of only this smaller piece. We now have to argue that this makes no difference for the final result. In the process, we also show that all points in $\overline{T} \setminus T$ lie over the boundary $\overline{M} \setminus M$, thereby proving the second assertion (ii) of Theorem 3.2.1.

So let $T_U \subseteq U \times \mathbb{C}^d$ be the total space of the local system over U; each connected component of T_U is a covering space of U, and therefore the subset $T_U \cap \big((\Delta^*)^n \times \mathbb{C}^d\big)$ is itself dense in T_U. But from this circumstance, it follows immediately that it has the same closure in $\Delta^n \times \mathbb{C}^d$ as T_U. Since T_U is already closed inside of $U \times \mathbb{C}^d$, we see in particular that all points of $\overline{T_U} \setminus T_U$ have to lie over the boundary divisor $t_1 \cdots t_t = 0$.

Note. This last fact can also be inferred from Proposition 3.4.1. For if we take P to be a point in M, the local system is already defined at P, and trivial on a neighborhood Δ^n of that point. Consequently, the local monodromy over $(\Delta^*)^n$ is also trivial, and Proposition 3.4.1 shows that taking the closure is only adding back a copy of $\mathbb{Z}^d$ over P. This means that we get back the original fiber $(\mathcal{H}_{\mathbb{Z}})_P$ of the local system.

3.5 Local analytic equations for the closure

Now that we know which points are in the closure, we need to show that $\overline{T}$ is an analytic space. We shall do this by finding explicit local equations, over the same coordinate neighborhoods that were used in Section 3.4. Consequently, $\Delta^n \subseteq \overline{M}$ will continue to denote a neighborhood of an arbitrary point in $\overline{M}$, with local holomorphic coordinates $t_1, \dots, t_n$.

It has already been pointed out that the total space of the local system over M has countably many connected components. Locally, over the much smaller open set Δ^n, those components may break up even further. The map

$$f\colon \mathbb{H}^n \times \mathbb{Z}^d \to \Delta^n \times \mathbb{C}^d,$$

parametrizing the total space of the local system over Δ^n, was defined above by the rule

$$(z_1, \dots, z_n, h) \mapsto \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \cdots + z_n N_n)} h\right).$$

If we take any element $h \in H = \mathbb{Z}^d$, the image of

$$f(\_\,, h)\colon \mathbb{H}^n \to \Delta^n \times \mathbb{C}^d,$$

denoted by $C(h)$, is one of the *local* connected components of the total space $T = T(\mathcal{H}_{\mathbb{Z}})$. Obviously, two such components $C(h_0)$ and $C(h_1)$ are either the same (which is the case if h_0 and h_1 lie in the same $\mathbb{Z}^n$-orbit), or disjoint; this means that, typically, each component is equal to $C(h)$ for infinitely many $h \in H$.

In Proposition 3.4.1, we have described the closure of

$$\bigcup_{h \in H} C(h)$$

inside of $\Delta^n \times \mathbb{C}^d$, but only as a set. To show that this closure is actually an analytic space, we need to give holomorphic equations that define it inside $\Delta^n \times \mathbb{C}^d$. We first observe that, as a matter of fact,

$$\overline{\bigcup_{h \in H} C(h)} = \bigcup_{h \in H} \overline{C(h)}.$$

This is because of the description of the closure given in Proposition 3.4.1—any point in the closure is already in the closure of one of the components $C(h)$.

Next, as would be expected if the closure is an analytic space, only finitely many of the $\overline{C(h)}$ can come together at the boundary. This is expressed in the following lemma.

Lemma 3.5.1. *At most finitely many distinct $\overline{C(h)}$ can meet at any given point in $\Delta^n \times \mathbb{C}^d$.*

Proof. Suppose, to the contrary, that infinitely many distinct $\overline{C(h)}$ met at a certain point Q of the closure. Such a point Q necessarily lies in the closure of infinitely many distinct sheets $C(h)$. Moving the center P of the coordinate system, if necessary, we may assume that Q is a point over $(0, \dots, 0) \in \Delta^n$. We can then find a sequence of points $\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times \mathbb{Z}^d$, with $0 \leq \operatorname{Re} z_j(m) \leq 1$ for all $j = 1, \dots, n$, such that $f\bigl(z(m), h(m)\bigr)$ converges to Q, but all $h(m)$ are distinct. But such a sequence cannot exist by Proposition 3.4.2. This contradiction proves that the number of components meeting at Q is indeed finite. $\square$

We are now ready to determine equations for each of the closed subsets $\overline{C(h)}$ of $\Delta^n \times \mathbb{C}^d$. Let $h \in H$ be an arbitrary element; we will give finitely many holomorphic equations defining $\overline{C(h)}$. As before, we break the argument down into several steps.

Step 1 Let $S_h \subseteq \mathbb{Z}^n$ be the subgroup of elements that leave h invariant. As a subgroup of a free group, S_h is itself free, say of rank $n - k$. If $k = n$, then h is not invariant under any element in $\mathbb{Z}^n$, and so $C(h)$ has to be already closed by Proposition 3.4.1. Since no points are added when taking the closure, there is nothing to prove in this case. We shall assume from now on that $k < n$; then the closure $\overline{C(h)}$ is potentially bigger than the original sheet $C(h)$.

Step 2 The quotient $\mathbb{Z}^n / S_h$ is a free Abelian group.

Proof. Since $\mathbb{Z}^n$ acts on $H = \mathbb{Z}^d$ by unipotent transformations, we have

$$T_1^{a_1} \cdots T_n^{a_n} h = h \qquad \text{if, and only if,} \qquad a_1 N_1 h + \cdots + a_n N_n h = 0.$$

This means that S_h is the kernel of the homomorphism

$$\mathbb{Z}^n \to \mathbb{Q}^d, \qquad (a_1, \dots, a_n) \mapsto a_1 N_1 h + \cdots + a_n N_n h,$$

and so the quotient $\mathbb{Z}^n/S_h$ embeds into $\mathbb{Q}^d$. Since $\mathbb{Q}^d$ is torsion-free, the same has to be true for $\mathbb{Z}^n/S_h$; being finitely generated, $\mathbb{Z}^n/S_h$ is then actually free. $\square$

Step 3 Because of Step 2, we can now find an $n \times n$ matrix A, with integer entries and $\det A = 1$, whose last $n-k$ columns give a basis for the subgroup S_h. We then introduce new coordinates $(w_1, \dots, w_n) \in \mathbb{C}^n$ by the rule

$$z_i = \sum_{j=1}^{n} a_{i,j} w_j. \tag{3.6}$$

Rewriting $z_1 N_1 + \cdots z_n N_n$ in the form $w_1 M_1 + \cdots + w_n M_n$, where each

$$M_j = \sum_{i=1}^{n} a_{i,j} N_i$$

is still nilpotent, we now have $M_{k+1}h = \cdots = M_n h = 0$, while the remaining k vectors $M_1 h, \dots, M_k h$ are linearly independent. Instead of f, we can then use the parametrization

$$g \colon V \to \Delta^n \times \mathbb{C}^d, \qquad (w_1, \dots, w_n) \mapsto \bigl(t_1, \dots, t_n, e^{-(w_1 M_1 + \cdots + w_k M_k)} h\bigr), \tag{3.7}$$

of the sheet $C(h)$ under consideration; here

$$t_j = \prod_{s=1}^{n} e^{2\pi i a_{j,s} w_s},$$

and the map g is defined on the open subset $V \subseteq \mathbb{C}^n$ where all $|t_j| < 1$.

Step 4 We now analyze the term $e^{-(w_1 M_1 + \cdots + w_k M_k)} h$ in the parametrization g. As a matter of fact, the map

$$\mathbb{C}^k \to \mathbb{C}^d, \qquad (w_1, \dots, w_k) \mapsto v = e^{-(w_1 M_1 + \cdots + w_k M_k)} h,$$

is a closed embedding, because the vectors $M_1h, \ldots, M_kh$ are linearly independent. We will prove this by constructing an inverse—we show that there are polynomials $p_1(v), \ldots, p_k(v)$ in $v = (v_1, \ldots, v_d)$, such that whenever v is in the image, one has

$$(w_1, \ldots, w_k) = \bigl(p_1(v), \ldots, p_k(v)\bigr).$$

Proof. We construct suitable polynomials by induction on the number k of variables. If $k = 0$, there is nothing to do. So let us assume that the existence of such polynomials is known for $k - 1 \geq 0$ variables, and let us establish it for k.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we write

$$M^\alpha = M_1^{\alpha_1} \cdots M_k^{\alpha_k};$$

these matrices are zero whenever $|\alpha|$ is sufficiently large. Among all multi-indices α with $M^\alpha h \neq 0$, select one of maximal length $|\alpha|$. Then $|\alpha| \geq 1$, because the vectors M_jh are in particular nonzero, and without loss of generality we may assume that $\alpha_k \geq 1$. We have

$$M^{\alpha - e_k} v = \bigl(\mathrm{id} - w_1 M_1 - \cdots - w_{k-1} M_{k-1}\bigr) M^{\alpha - e_k} h - w_k M^\alpha h.$$

Because at least one of the components of $M^\alpha h$ is non-zero, we can now solve for w_k in the form

$$w_k = c_1 w_1 + \cdots + c_{k-1} w_{k-1} + l(v),$$

with $c_1, \ldots, c_{k-1} \in \mathbb{Q}$, and $l(v)$ a degree-one polynomial in v. Substituting back, we obtain

$$e^{l(v) M_k} v = e^{-w_1(M_1 + c_1 M_k) - \cdots - w_{k-1}(M_{k-1} + c_{k-1} M_k)} h,$$

and, by the inductive hypothesis, $w_1, \ldots, w_{k-1}$ can now be expressed as polynomials in the coordinates of the vector $e^{l(v)M_k}v$, since the vectors $(M_i + c_i M_k)h$ are of course still linearly independent. It is thus possible to find polynomials in v such that

$$(w_1, \ldots, w_{k-1}) = \big(p_1(v), \ldots, p_{k-1}(v)\big).$$

Then $w_k = c_1 p_1(v) + \cdots + c_{k-1} p_{k-1}(v) + l(v)$ is also a polynomial in v, and the assertion is proved. □

Step 5 The result of Step 4 now gives us half of the equations for the closed subset $\overline{C(h)}$. Indeed, we have seen that if $(t, v) \in \Delta^n \times \mathbb{C}^d$ is a point of $C(h)$, then it is in the image of g, and so its v-coordinates satisfy the relation

$$v = e^{-(p_1(v)M_1 + \cdots + p_k(v)M_k)}h. \tag{3.8}$$

In components, these are d polynomial equations for $v = (v_1, \ldots, v_d)$. The same equations obviously have to hold for every point in the closure $\overline{C(h)}$.

Step 6 Next, we turn our attention to the remaining n coordinates $(t_1, \ldots, t_n)$ of the parametrization g in (3.7). Each is of the form

$$t_j = \prod_{s=1}^{n} e^{2\pi i a_{j,s} w_s}.$$

Letting $u_j = \exp(2\pi i w_j)$, for $j = k+1, \ldots, n$, we have

$$t_j = u_{k+1}^{a_{j,k+1}} \cdots u_n^{a_{j,n}} \cdot e^{2\pi i (a_{j,1} w_1 + \cdots + a_{j,k} w_k)}.$$

The shape of these products leads us to consider the algebraic map

$$(\mathbb{C}^*)^{n-k} \to \mathbb{C}^n, \qquad (u_{k+1}, \ldots, u_n) \mapsto (x_1, \ldots, x_n), \tag{3.9}$$

whose coordinates are given by

$$x_j = \prod_{i=k+1}^{n} u_i^{a_{j,i}}. \tag{3.10}$$

Because the map is given by polynomials, the (topological) closure of its image is actually a closed algebraic subvariety of $\mathbb{C}^n$, and as such defined by finitely many polynomial equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_e(x_1, \ldots, x_n) = 0.$$

In fact, because the original map is monomial, each $f_b(x)$ can be taken as a binomial in the variables $x_1, \ldots, x_n$.[3]

Step 7 From Step 6, we can now read off the remaining equations for $\overline{C(h)}$. Indeed, a point (t, v) in the image of g has to satisfy the equations

$$f_b\Big(t_1 e^{-2\pi i \sum_{s\leq k} a_{1,s} w_s}, \ldots, t_n e^{-2\pi i \sum_{s\leq k} a_{n,s} w_s}\Big) = 0$$

for $b = 1, \ldots, e$. From Step 4 we know, moreover, that $w_s = p_s(v)$; therefore

$$f_b\Big(t_1 e^{-2\pi i \sum_{s\leq k} a_{1,s} p_s(v)}, \ldots, t_n e^{-2\pi i \sum_{s\leq k} a_{n,s} p_s(v)}\Big) = 0 \tag{3.11}$$

is another set of e holomorphic equations satisfied by the closure $\overline{C(h)}$.

Step 8 It remains to see that the $d + e$ equations in (3.8) and (3.11) really define $\overline{C(h)}$, and not a bigger set. The trivial case is when h is not invariant under any part

[3]We shall have to say more later about the toric structure of the image.

of the monodromy; here $k = n$, and as pointed out in Step 1, $C(h)$ is then already a closed set, and there is nothing to show. In the remaining case, when $k < n$, we are free to place the point P (the center of our coordinate system $t_1, \ldots, t_n$) anywhere we like. A moment's thought shows that it therefore suffices to consider solutions of the equations over $(0, \ldots, 0) \in \Delta^n$, and to prove that those have to lie in the closure of $C(h)$.

So consider a point $(0, v) \in \Delta^n \times \mathbb{C}^d$ that satisfies the equations. On the one hand, the equations in (3.8) define the image of a closed embedding, as explained in Step 4; therefore, $v = e^{-(w_1 M_1 + \cdots + w_k M_k)} h$ for a unique point $(w_1, \ldots, w_k) \in \mathbb{C}^k$. Letting $w = (w_1, \ldots, w_k, 0, \ldots, 0)$, and going back to the original coordinates z in (3.6), we get a point $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that

$$v = e^{-(z_1 N_1 + \cdots + z_n N_n)} h.$$

On the other hand, the equations in (3.11) arose from the map defined in Step 6. Now (3.10) shows that the point $(0, \ldots, 0)$ can only be in the closure of the image when some linear combination of the exponent vectors $(a_{1,i}, \ldots, a_{n,i})$, for $i = k+1, \ldots, n$, has positive coordinates. Since these vectors generate the subgroup S, we thus get positive integers $a_1, \ldots, a_n$ with

$$a_1 N_1 h + \cdots + a_n N_n h = 0$$

But by the description in Proposition 3.4.1, this says exactly that the point $(0, v)$ belongs to $\overline{C(h)}$.

In summary, we have established the following two results. First, we can give local equations for each of the components $\overline{C(h)}$.

Proposition 3.5.2. *The closure* $\overline{C(h)}$ *of the sheet* $C(h)$ *in* $\Delta^n \times \mathbb{C}^d$ *is an analytic subset, defined by* $d+e$ *holomorphic equations in the coordinates* $(t_1, \ldots, t_n, v_1, \ldots, v_d)$. *These equations are, firstly,*

$$v = e^{-(p_1(v)M_1 + \cdots + p_k(v)M_k)}h.$$

and, secondly,

$$f_b\Big(t_1 e^{-2\pi i \sum_{s \le k} a_{1,s} p_s(v)}, \ldots, t_n e^{-2\pi i \sum_{s \le k} a_{n,s} p_s(v)}\Big) = 0$$

for $b = 1, \ldots, e$. *In particular, as the closure of the complex submanifold* $C(h)$, *the subset* $\overline{C(h)}$ *is itself a reduced and irreducible analytic space.*

Moreover, because only finitely many components can meet at any given point (by Lemma 3.5.1), we can conclude that the closure of the image of f is an analytic subset of $\Delta^n \times \mathbb{C}^d$ as well.

Proposition 3.5.3. *Let* T_U *be the total space of the local system over the open set* $U = M \cap \Delta^n$. *The closure of* T_U *inside of* $\Delta^n \times \mathbb{C}^d$ *is a reduced analytic subset with countably many irreducible components, each of the form* $\overline{C(h)}$ *for some* $h \in \mathbb{Z}^d$.

3.6 Singularities of the closure

The analytic space $\overline{T}$, described in Theorem 3.2.1, will in general be singular at points not in T. This is apparent from the discussion in the previous section—on the one hand, several of the local components $C(h)$ may be coming together at the boundary (see Lemma 3.5.1); on the other hand, the local equations of the closure are such that singularities have to be expected even for each $C(h)$ itself (see Steps 6 and 7 in the previous section). There are two possible approaches to this problem—normalization, and resolution of singularities.

Normalization

For various applications, it is desirable to have at least a normal space; mostly because it is then possible to extend holomorphic maps that are naturally defined on T to all of $\overline{T}$, by showing that they extend in codimension one. In addition, the normalization of $\overline{T}$ is an unexpectedly nice space.

According to the local description of $\overline{T}$ given in Proposition 3.5.3, the process of normalizing $\overline{T}$ has two effects. Firstly, it separates all the local components $\overline{C(h)}$ at points where they meet, making them disjoint. Secondly, it normalizes each $\overline{C(h)}$ itself. From Step 6 in Section 3.5, we see that $\overline{C(h)}$ is locally isomorphic to a (typically non-normal) toric variety. Indeed, the map g in (3.7), whose image is the sheet $C(h)$, is locally the product of a closed immersion and a map defined by monomials. As explained in the article by Cox (1995, p. 402), the closure of the image of a monomial map as in (3.9) is a non-normal toric variety, and after taking the normalization, one gets a toric variety in the usual sense. It follows that the normalization of each $\overline{C(h)}$ is locally (in the analytic topology) isomorphic to a toric variety.

It is known (Fulton, 1993, Proposition on p. 76) that toric varieties in general have only mild singularities; in particular, the singularities are always rational. Because the normalization of $\overline{T}$ is locally toric, it has the same property; we conclude that it has at worst rational singularities. Given that $\overline{T}$ itself was defined by taking a closure, this is an unexpected circumstance.

Resolution of singularities

A second possibility is to resolve the singularities of $\overline{T}$ completely. By construction, the total space T of the local system is a nonsingular dense open subset of $\overline{T}$. Its complement $\overline{T} \setminus T$, being the preimage of the divisor $\overline{M} \setminus M$ under the holomorphic projection map from $\overline{T}$ to $\overline{M}$, is a closed analytic subspace. According to the results of Bierstone and Milman (1997, p. 298), it is possible to resolve the singularities of $\overline{T}$ by blowing up, at the same time making the preimage of $\overline{T} \setminus T$ into a divisor with only normal crossing singularities. Since the centers of the blowups can be chosen to lie outside of T, the resulting complex manifold will still have T as a dense open subset. Of course, the space one gets is as "canonical" as the resolution process.
Since the normalization of $\overline{T}$ is locally toric, one can also normalize first, and then use the older results on desingularizing toroidal embeddings (Kempf et al., 1973, p. 94) to create a non-singular space from $\overline{T}$.

3.7 A universal property of the canonical extension

In this section, we shall give some justification for calling the space $\overline{T}$ the "canonical extension" of the local system $\mathcal{H}_{\mathbb{Z}}$. The following proposition is the main result in this direction.

Proposition 3.7.1. *Let $g\colon \overline{X} \to \overline{M}$ be an arbitrary holomorphic map from a reduced* *and* normal *analytic space* $\overline{X}$ *to* $\overline{M}$*. Assume that the open set* $X = g^{-1}(M)$ *is dense in* $\overline{X}$*, and that there is a factorization*

$$\begin{tikzcd} & T \arrow[d, "p"] \\ X \arrow[ur, "s"] \arrow[r, "g"] & M \end{tikzcd}$$

as in the diagram. Then s *extends uniquely to a holomorphic map* $s\colon \overline{X} \to \overline{T}$*, making*

$$\begin{tikzcd} & \overline{T} \arrow[d, "p"] \\ \overline{X} \arrow[ur, dashed, "s"] \arrow[r, "g"] & \overline{M} \end{tikzcd}$$

commute.

In order to prove this, we shall first reformulate the statement. Let $\overline{\mathscr{V}}$ be the canonical extension of the flat vector bundle $\mathscr{V} = \mathscr{O}_M \otimes \mathcal{H}_{\mathbb{Z}}$ to $\overline{M}$; then $\overline{T}$ is the closure of T inside the total space of $\overline{\mathscr{V}}$. The map $s\colon X \to T$ gives a section of the pullback of the local system $g^{-1}\mathcal{H}_{\mathbb{Z}}$—as well as of the bundle $g^*\overline{\mathscr{V}}$—over X, that we continue to denote by s. Now the statement of the proposition is equivalent to saying that s extends to a section of $g^*\overline{\mathscr{V}}$ over $\overline{X}$. Indeed, such an extension (clearly unique if it exists) gives a map from $\overline{X}$ to the total space of $\overline{\mathscr{V}}$, and since X is mapped into T, the image has to be contained in the closure $\overline{T}$. Thus extending the map s to $\overline{X}$ is equivalent to extending the corresponding section s of $g^*\overline{\mathscr{V}}$.

We now begin the proof by establishing the following special case.

Lemma 3.7.2. *The conclusion of Proposition 3.7.1 holds whenever* $\overline{X} = \Delta$ *is the unit disk, and* $X = \Delta^*$*.*

Proof. Let $g\colon \Delta \to \overline{M}$ be the given morphism. Since the original local system $\mathcal{H}_{\mathbb{Z}}$ is unipotent along $\overline{M} \setminus M$, its pullback $g^{-1}\mathcal{H}_{\mathbb{Z}}$ to Δ^* has unipotent monodromy around $0 \in \Delta$. Thus the vector bundle $g^*\overline{\mathscr{V}}$ is the canonical extension of $\mathscr{O}_{\Delta^*} \otimes g^{-1}\mathcal{H}_{\mathbb{Z}}$ to Δ. As we said, it suffices to show that the section $s \in H^0(\Delta^*, g^*\overline{\mathscr{V}})$ extends to Δ; but this follows very easily from the construction of the canonical extension. Indeed, if we let N be the logarithm of the monodromy of $g^{-1}\mathcal{H}_{\mathbb{Z}}$ on Δ^*, then the description on p. 37 shows that the total space of $g^{-1}\mathcal{H}_{\mathbb{Z}}$, inside that of the bundle $g^*\overline{\mathscr{V}}$, is given by the image of the map

$$\mathbb{H} \times \mathbb{Z}^d \to \Delta^* \times \mathbb{C}^d, \qquad (z, h) \mapsto \bigl(e^{2\pi i z}, e^{-zN} h\bigr),$$

in a suitable frame of $g^*\overline{\mathscr{V}}$. The section s corresponds to a monodromy-invariant element $h \in \mathbb{Z}^d$, satisfying $Nh = 0$, and so the whole sheet $\Delta \times \{h\}$ lies in the closure of the image. This shows that s extends over 0, proving the lemma. $\square$

We can now turn to proving Proposition 3.7.1 in general. Let $g \colon \overline{X} \to \overline{M}$ be the given map, and set $Z = \overline{X} \setminus X$. As before, $s \in H^0(X, g^*\overline{\mathscr{V}})$ denotes the section of the pullback bundle corresponding to the given factorization. In order to show that s extends holomorphically to all of $\overline{X}$, it suffices to show that it extends in codimension one, since $\overline{X}$ is normal (Narasimhan, 1966, p. 118). The singular locus of $\overline{X}$ has codimension at least two; it is therefore enough to prove that s extends across those points $P \in Z$ where both Z and $\overline{X}$ are nonsingular, and $\operatorname{codim}_P(Z, X) = 1$. This is a local question, and after choosing suitable local coordinates $z_1, \dots, z_m$ on a small neighborhood U of P in $\overline{X}$, we can assume that $Z \cap U$ is defined by the equation $z_m = 0$. Applying the lemma to maps of the form

$$\Delta \to U, \qquad t \mapsto (z_1, \dots, z_{m-1}, t),$$

we see that the section s extends across P. This completes the proof of the proposition.

Note. Since $\overline{T}$ itself is usually not a normal space, Proposition 3.7.1 is not quite as strong as one might like it to be. It is, however, easy to find examples of maps from non-normal spaces $\overline{X}$ to $\overline{M}$ (for instance, taking $\overline{X}$ to be a nodal curve) where the statement is false. This suggests that the normalization of $\overline{T}$ is the space that deserves to be called the "canonical extension" of the local system $\mathcal{H}_{\mathbb{Z}}$.

CHAPTER 4

PRIMITIVE COHOMOLOGY AND THE TUBE MAPPING

4.1 Surjectivity of the tube mapping

Let X be a nonsingular complex projective variety. In this section, we first review the Lefschetz theorems about the relationship between the cohomology of X and that of a smooth hyperplane section. We then give a precise definition of the tube mapping, which produces primitive cohomology classes on X from monodromy-invariant classes on a hypersurface. The main result, Theorem 4.1.1, is that *all* the primitive cohomology is generated in that way, at least when the dimension of X is odd. Whether this also holds for even-dimensional varieties is an open question.

Review of the Lefschetz theorems

Before giving the definition of the tube mapping, it will be helpful to review briefly the relationship between the cohomology of X and that of a smooth hyperplane section $S = H \cap X$; a more comprehensive discussion can be found in Voisin (2002, Section 13). Let us write $i\colon S \to X$ for the inclusion map; we also let $d = \dim X$ be the complex dimension of X. Then Lefschetz' Hyperplane Theorem states that the restriction map $i^*\colon H^k(X,\mathbb{Q}) \to H^k(S,\mathbb{Q})$ is an isomorphism for $k < \dim S = d-1$, and injective for $k = d-1$. One also has direct sum decompositions

$$H^{d-1}(S,\mathbb{Q}) = i^* H^{d-1}(X,\mathbb{Q}) \oplus H^{d-1}(S,\mathbb{Q})_{van}$$

and

$$H^d(X,\mathbb{Q}) = i_* H^{d-2}(S,\mathbb{Q}) \oplus H^d(X,\mathbb{Q})_{prim}$$

by using the vanishing cohomology of S and the primitive cohomology of X, respectively. The primitive cohomology is the subspace

$$\begin{aligned} H^d(X,\mathbb{Q})_{prim} &= \ker\bigl(i^*\colon H^d(X,\mathbb{Q}) \to H^d(S,\mathbb{Q})\bigr) \\ &= \ker\bigl(L\colon H^d(X,\mathbb{Q}) \to H^{d+2}(X,\mathbb{Q})\bigr), \end{aligned}$$

where we have denoted the Lefschetz operator $\omega \mapsto [S] \cup \omega$ by L. The vanishing cohomology, on the other hand, is given by

$$H^{d-1}(S,\mathbb{Q})_{van} = \ker\bigl(i_*\colon H^{d-1}(S,\mathbb{Q}) \to H^{d+1}(X,\mathbb{Q})\bigr).$$

Both decompositions above are orthogonal with respect to the intersection pairings on S and X; moreover, Poincaré duality gives isomorphisms $H^k(X,\mathbb{Q}) \simeq H^{2d-k}(X,\mathbb{Q})^\vee$ for all k. It follows that all the cohomology groups of X are determined by the cohomology of the hyperplane section S, with the exception of the primitive cohomology $H^d(X,\mathbb{Q})_{prim}$ in degree d.

The tube mapping

It should now be clear that some construction is needed that produces *primitive* cohomology classes on X from data on a hyperplane section $X \cap H$. The "tube mapping," alluded to in the introduction, does just that. We now give a precise definition of this mapping; this is most easily done in homology. Fix a Lefschetz pencil of hypersurfaces on X, and let $B \subseteq \mathbb{P}^1$ be the subset of the base over which the hypersurfaces are nonsingular. Also fix a base point $0 \in B$, and let $S_0 \subseteq X$ be the corresponding hypersurface. The fundamental group $G = \pi_1(B, 0)$ of B then acts by monodromy on the homology $H_{d-1}(S_0, \mathbb{Q})$ of the fiber.

Whenever a cycle $\alpha \in H_{d-1}(S_0, \mathbb{Q})$ is invariant under the action of some loop $g \in G$, we can use it to produce a homology class in $H_d(X, \mathbb{Q})$, as follows. The element g can be represented by an immersion $\mathbb{S}^1 \to B$. Moving α along the circle traces out a d-chain on X, which is actually closed because α is g-invariant. The resulting d-cycle is unique up to elements of $H_d(S_0, \mathbb{Q})$; we thus get the tube mapping τ as

$$\bigoplus_{g \in G} \left\{ \alpha \in H_{d-1}(S_0, \mathbb{Q}) \;\middle|\; g \cdot \alpha = \alpha \right\} \to H_d(X, \mathbb{Q}) \big/ H_d(S_0, \mathbb{Q}).$$

By using Poincaré duality, this construction can equally well be done on the level of cohomology; there, it takes its image in the primitive cohomology $H^d(X, \mathbb{Q})_{prim}$, which is dual to the quotient $H_d(X, \mathbb{Q}) \big/ H_d(S_0, \mathbb{Q})$.

The purpose of this chapter is to prove the following result.[1]

Theorem 4.1.1. *Let X be a smooth complex projective variety of* odd *dimension $(2n+1)$. As above, let $B \subseteq \mathbb{P}^1$ be the smooth locus of an arbitrary Lefschetz pencil*

[1] A rigorous definition of the tube mapping may be found in Section 4.6.

on X; let S_0 be the hypersurface corresponding to some base point $0 \in B$, and write $G = \pi_1(B, 0)$ for the fundamental group of B. Then the tube mapping

$$\tau\colon \bigoplus_{g \in G} \left\{ \alpha \in H^{2n}(S_0, \mathbb{Q})_{van} \;\middle|\; g \cdot \alpha = \alpha \right\} \to H^{2n+1}(X, \mathbb{Q})_{prim}$$

is surjective.

The theorem will be proved in the course of the following four sections. The major steps are contained in Proposition 4.2.4 and Proposition 4.5.1, and the proof is finally brought to an end in Section 4.6. The core of the argument is Proposition 4.5.1 about the group cohomology of a certain class of representations of G, the cohomology $H^{2n}(S_0, \mathbb{Q})_{van}$ being a special case. This result is also the reason why we have to restrict ourselves to the case of an odd-dimensional variety X; the essential differences that arise when $\dim X$ is even are discussed in Section 4.7 below.

4.2 Background on the cohomology of Lefschetz pencils

Before beginning the proof, we fix some notation. Let X be a nonsingular complex projective variety of dimension $(2n+1)$. Take an arbitrary Lefschetz pencil of hypersurface sections of X, i.e., a line $\mathbb{P}^1 \subseteq P$ transverse to $X^\vee$. The base locus of the pencil is a certain nonsingular $(2n-1)$-fold $Z \subseteq X$; blowing up along Z gives a family

$$\bar{\pi}\colon \bar{\mathfrak{S}} = \mathrm{Bl}_Z(X) \to \mathbb{P}^1$$

of hypersurfaces S_t, for $t \in \mathbb{P}^1$. Let $D \subseteq \mathbb{P}^1$ be the finite set of points over which the fibers are singular, and let $B = \mathbb{P}^1 \setminus D$ be the complement; then $\pi\colon \mathfrak{S} = \bar{\pi}^{-1}(B) \to B$ is a smooth and projective morphism.

We first review several results about the cohomology of $\mathfrak{S}$; these are well-known, and may for instance be found in Lewis (1999, Lecture 14). So let $S = S_t$, for some $t \in D$, be one of the singular fibers of the Lefschetz pencil; S has exactly one ordinary double point singularity. We write $\tilde{X}$ for the blow-up of X at the singular point of S; the strict transform $\tilde{S}$ of S is then nonsingular, and meets the exceptional divisor $E \simeq \mathbb{P}^{2n}$ in a nonsingular hypersurface of degree 2. Here is a diagram of all the relevant maps:

$$\begin{tikzcd} \tilde{S} \arrow[r, "p"] \arrow[drr, "q"'] & S \arrow[r, "j"] \arrow[dr, "i"] & \bar{\mathfrak{S}} \arrow[d, "\sigma"] \\ & & X \end{tikzcd}$$

Lemma 4.2.1. *The pullback map $q^* \colon H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(\tilde{S}, \mathbb{Q})$ is surjective.*

Proof. Applying Lemma 4.8.1 to the diagram

$$\begin{tikzcd} \tilde{S} \cup E \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\ S \arrow[r, "i"] & X \end{tikzcd}$$

and noting that each map $H^k(X, \mathbb{Q}) \to H^k(\tilde{X}, \mathbb{Q})$ is injective, we get a short exact sequence

$$0 \longrightarrow H^k(X, \mathbb{Q}) \longrightarrow H^k(S, \mathbb{Q}) \oplus H^k(\tilde{X}, \mathbb{Q}) \longrightarrow H^k(\tilde{S} \cup E, \mathbb{Q}) \longrightarrow 0$$

for every integer k. Because $\tilde{X}$ is the blow-up of X at a point, the pullback map $H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(\tilde{X}, \mathbb{Q})$ is an isomorphism. It follows that the map

$$H^{2n-1}(S, \mathbb{Q}) \to H^{2n-1}(\tilde{S} \cup E, \mathbb{Q}) \tag{4.1}$$

is an isomorphism, too.

On the other hand, $\tilde{S} \cup E$ has only normal crossing singularities, and as explained in the article by Griffiths and Schmid (1975, p. 71), its cohomology can be computed by a spectral sequence, on whose E_1-page the only non-zero entries are

$$E_1^{0,q} = H^q(\tilde{S}, \mathbb{Q}) \oplus H^q(E, \mathbb{Q}) \qquad \text{and} \qquad E_1^{1,q} = H^q(\tilde{S} \cap E, \mathbb{Q}).$$

Since $\tilde{S} \cap E$ is a nonsingular hypersurface of degree two in $E \simeq \mathbb{P}^{2n}$, one has $E_1^{1,2n-1} = H^{2n-1}(\tilde{S} \cap E, \mathbb{Q}) = 0$, which implies that $E_\infty^{0,2n-1} = H^{2n-1}(\tilde{S}, \mathbb{Q})$. Therefore the restriction map

$$H^{2n-1}(\tilde{S} \cup E, \mathbb{Q}) \to H^{2n-1}(\tilde{S}, \mathbb{Q}) \tag{4.2}$$

is surjective as well.

Lefschetz' Theorem, applied to the very ample hypersurface $S \subseteq X$, together with (4.1) and (4.2), now shows that the map q^*, which is the composition

$$H^{2n-1}(X, \mathbb{Q}) \xrightarrow{\simeq} H^{2n-1}(S, \mathbb{Q}) \to H^{2n-1}(\tilde{S}, \mathbb{Q})$$

is a surjection. $\square$

Lemma 4.2.2. *With respect to the intersection pairing, the image of the composed map*

$$H_S^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q}) \to H^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q}) \xrightarrow{\sigma_*} H^{2n+1}(X, \mathbb{Q})$$

is orthogonal to the primitive cohomology $H^{2n+1}(X, \mathbb{Q})_{prim}$.

Proof. Let us denote the map in the statement by λ. There is a perfect pairing

$$H_S^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q}) \otimes H^{2n+1}(S, \mathbb{Q}) \to \mathbb{Q},$$

because $\bar{\mathfrak{S}}$ is compact and nonsingular (Peters and Steenbrink, 2008, Corollary 6.28 on p. 157), and therefore

$$H_S^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q}) \simeq H^{2n+1}(S, \mathbb{Q})^\vee.$$

With this isomorphism, and with the intersection pairing on $H^{2n+1}(X, \mathbb{Q})$, the map λ is dual to the restriction map $i^*\colon H^{2n+1}(X, \mathbb{Q}) \to H^{2n+1}(S, \mathbb{Q})$; in particular, $\operatorname{im} \lambda$ is perpendicular, with respect to the intersection pairing, to the space

$$P = \ker\Big(i^*\colon H^{2n+1}(X, \mathbb{Q}) \to H^{2n+1}(S, \mathbb{Q})\Big).$$

To establish the lemma, it is thus sufficient to prove that $P = H^{2n+1}(X, \mathbb{Q})_{prim}$. Both $H^{2n+1}(S, \mathbb{Q})$ and $H^{2n+1}(\tilde{S}, \mathbb{Q})$ carry mixed Hodge structures, and

$$\ker\Big(p^*\colon H^{2n+1}(S, \mathbb{Q}) \to H^{2n+1}(\tilde{S}, \mathbb{Q})\Big) = W_{2n} H^{2n+1}(S, \mathbb{Q}).$$

Because the mixed Hodge structure on $H^{2n+1}(X, \mathbb{Q})$ is pure of weight $(2n+1)$, it follows that $P = \ker\big(p^* \circ i^*\big) = \ker q^*$.

Now let $\omega \in H^{2n+1}(X, \mathbb{Q})$ be an arbitrary element. For each $\alpha \in H^{2n-1}(X, \mathbb{Q})$, we compute

$$\int_{\tilde{S}} q^*\omega \cup q^*\alpha = \int_X q_*[\tilde{S}] \cup \omega \cup \alpha = \int_X [S] \cup \omega \cup \alpha = \int_X L(\omega) \cup \alpha,$$

where $L\colon H^{2n+1}(X, \mathbb{Q}) \to H^{2n+3}(X, \mathbb{Q})$ is the Lefschetz operator associated to the given projective embedding of X. Since we have $H^{2n-1}(\tilde{S}, \mathbb{Q}) = q^* H^{2n-1}(X, \mathbb{Q})$ by Lemma 4.2.1, we see that ω lies in $P = \ker q^*$ if, and only if, $L(\omega) = 0$. But this condition defines the primitive cohomology, and so $P = H^{2n+1}(X, \mathbb{Q})_{prim}$, which finishes the proof. $\square$

Lemma 4.2.3. *The pullback map* $H^{2n+1}(X, \mathbb{Q}) \to H^{2n+1}(\mathfrak{S}, \mathbb{Q})$ *is injective.*

Proof. Let K denote the kernel of the map from $H^{2n+1}(X, \mathbb{Q})$ to $H^{2n+1}(\mathfrak{S}, \mathbb{Q})$. We need to show that $K = 0$. Let $\sigma \colon \bar{\mathfrak{S}} \to X$ be the blow-up map; then $H^{2n+1}(X, \mathbb{Q})$ is a direct summand of $H^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q})$, via the two maps σ_* and σ^*. The exact sequence

$$\bigoplus_{t \in D} H^{2n+1}_{S_t}(\bar{\mathfrak{S}}, \mathbb{Q}) \longrightarrow H^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q}) \longrightarrow H^{2n+1}(\mathfrak{S}, \mathbb{Q})$$

shows that K is contained in the image of the composition

$$\bigoplus_{t \in D} H^{2n+1}_{S_t}(\bar{\mathfrak{S}}, \mathbb{Q}) \to H^{2n+1}(\bar{\mathfrak{S}}, \mathbb{Q}) \xrightarrow{\sigma_*} H^{2n+1}(X, \mathbb{Q}).$$

By Lemma 4.2.2 above, that image is perpendicular to the primitive cohomology $H^{2n+1}(X, \mathbb{Q})_{prim}$.

On the other hand, $\mathfrak{S}$ contains every nonsingular fiber S' of the pencil, and therefore

$$K \subseteq \ker\Bigl(H^{2n+1}(X, \mathbb{Q}) \to H^{2n+1}(S', \mathbb{Q})\Bigr) = H^{2n+1}(X, \mathbb{Q})_{prim}$$

is at the same time contained in the primitive cohomology of X. Thus we find that $K = 0$, proving the injectivity of the restriction map. $\square$

Finally, here is the result about the cohomology of $\mathfrak{S}$ that we are after. The injectivity of the map is the first step towards proving Theorem 4.1.1, because it shows that primitive cohomology classes on X are faithfully represented on the Lefschetz pencil.

Proposition 4.2.4. *The Leray spectral sequence gives an injective map*

$$H^{2n+1}(X, \mathbb{Q})_{prim} \to H^1(B, R^{2n}\pi_*\mathbb{Q}_{van}). \tag{4.3}$$

Proof. In the Leray spectral sequence for the map π,

$$E_2^{pq} = H^p(B, R^q\pi_*\mathbb{Q}) \Longrightarrow H^{p+q}(\mathfrak{S}, \mathbb{Q}),$$

all terms with $p \geq 2$ vanish because B has the homotopy type of a one-dimensional CW-complex. We thus have a short exact sequence

$$0 \longrightarrow H^1(B, R^{2n}\pi_*\mathbb{Q}) \longrightarrow H^{2n+1}(\mathfrak{S}, \mathbb{Q}) \longrightarrow H^0(B, R^{2n+1}\pi_*\mathbb{Q}) \longrightarrow 0.$$

Since the kernel of the composition

$$H^{2n+1}(X, \mathbb{Q}) \to H^{2n+1}(\mathfrak{S}, \mathbb{Q}) \to H^0(B, R^{2n+1}\pi_*\mathbb{Q})$$

is exactly the primitive cohomology $H^{2n+1}(X, \mathbb{Q})_{prim}$, we get an induced map

$$H^{2n+1}(X, \mathbb{Q})_{prim} \to H^1(B, R^{2n}\pi_*\mathbb{Q}),$$

and Lemma 4.2.3 implies that it is injective, because the original map to $H^{2n+1}(\mathfrak{S}, \mathbb{Q})$ was injective. The decomposition

$$H^{2n}(S', \mathbb{Q}) \simeq H^{2n}(X, \mathbb{Q}) \oplus H^{2n}(S', \mathbb{Q})_{van}$$

of the cohomology of any smooth hypersurface section $S' \subseteq X$ gives a corresponding decomposition

$$H^1(B, R^{2n}\pi_*\mathbb{Q}) = H^1(B, \mathbb{Q}) \otimes H^{2n}(X, \mathbb{Q}) \oplus H^1(B, R^{2n}\pi_*\mathbb{Q}_{van}).$$

Since any class in $H^{2n+1}(X, \mathbb{Q})_{prim}$ obviously goes to zero in the first summand, we finally have an injective map $H^{2n+1}(X, \mathbb{Q})_{prim} \to H^1(B, R^{2n}\pi_*\mathbb{Q}_{van})$ as claimed. $\square$

Note. It should be observed that Proposition 4.2.4, as stated, is no longer true if $\dim X$ is even. Indeed, as pointed out by Voisin, any even-dimensional quadric gives an example of this. For if $Q \subseteq \mathbb{P}^{2k+1}$ is a smooth hypersurface of degree two, then every smooth hyperplane section $Q \cap H$ satisfies $H^{2k-1}(Q \cap H, \mathbb{Q})_{van} = 0$. Since Q does have nontrivial primitive cohomology, the mapping in (4.3) fails to be injective. On the other hand, the proposition remains true if the hypersurface sections are sufficiently ample. This observation forms the content of the next section.

4.3 An application of Nori's Connectivity Theorem

Clemens observed that one can obtain the result of Section 4.2 more easily—and in greater generality—by using Nori's Connectivity Theorem, provided one is willing to assume that the degree of the hypersurface sections is large. We give a short proof of that fact in this section.

We shall let $d = \dim X \geq 2$ be the dimension of the smooth projective variety X; in this section, it is not necessary to assume that d is odd. We fix a very ample line bundle $\mathscr{L}$ on X, and let $P = \mathbb{P}\big(H^0(X, \mathscr{L})^\vee\big)$ be the parameter space for hypersurface sections of X by $\mathscr{L}$. As usual, $\pi \colon \mathfrak{X} \to P$ is the universal hypersurface, and $\pi^{sm} \colon \mathfrak{X}^{sm} \to P^{sm}$ its restriction to the open set $P^{sm} \subseteq P$ parametrizing smooth hypersurfaces.

Nori's Connectivity Theorem (Voisin, 2002, Section 20.1) is the statement that the restriction map

$$H^k\big(P^{sm} \times X, \mathbb{Q}\big) \to H^k\big(\mathfrak{X}^{sm}, \mathbb{Q}\big) \tag{4.4}$$

is an isomorphism for all $k \leq 2d-3$, and injective for $k = 2d-2$, provided the line bundle $\mathscr{L}$ is sufficiently ample. We assume from now on that $\mathscr{L}$ is ample enough for (4.4) to hold.

The following result is an easy consequence of Nori's theorem.

Proposition 4.3.1 (Clemens)**.** *The Leray spectral sequence for the map π^{sm} gives a natural map*

$$H^d(X,\mathbb{Q})_{prim} \to H^1\big(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}_{van}\big).$$

It is an isomorphism if $d \geq 3$, and injective if $d = 2$.

Proof. We consider the Leray spectral sequence for the smooth projective map π^{sm}, given by

$$E_2^{p,q} = H^p\big(P^{sm}, R^q\pi_*^{sm}\mathbb{Q}\big) \Longrightarrow H^{p+q}\big(\mathfrak{X}^{sm},\mathbb{Q}\big).$$

Writing L^k for the induced filtration on the cohomology of $\mathfrak{X}^{sm}$, we have (for each $p \geq 0$) a short exact sequence

$$L^{p+1}H^d\big(\mathfrak{X}^{sm},\mathbb{Q}\big) \hookrightarrow L^pH^d\big(\mathfrak{X}^{sm},\mathbb{Q}\big) \twoheadrightarrow H^p\big(P^{sm}, R^{d-p}\pi_*^{sm}\mathbb{Q}\big),$$

because the spectral sequence degenerates at the E_2-page by a theorem of Deligne (Voisin, 2002, p. 379).

A similar result is true for the map $pr_{P^{sm}} \colon P^{sm} \times X \to P^{sm}$; since the Leray spectral sequence is functorial, we get for all $p \geq 0$ a commutative diagram

$$\begin{tikzcd}
L^{p+1}H^d\big(P^{sm}\times X\big) \arrow[r, hook] \arrow[d, "r_{p+1}"'] & L^pH^d\big(P^{sm}\times X\big) \arrow[r, two heads] \arrow[d, "r_p"'] & H^p\big(P^{sm}\big)\times H^{d-p}(X) \arrow[d, "q_p"'] \\
L^{p+1}H^d\big(\mathfrak{X}^{sm}\big) \arrow[r, hook] & L^pH^d\big(\mathfrak{X}^{sm}\big) \arrow[r, two heads] & H^p\big(P^{sm}, R^{d-p}\pi_*^{sm}\mathbb{Q}\big)
\end{tikzcd}$$

with exact rows (all cohomology groups are with $\mathbb{Q}$-coefficients).

We now analyze this diagram. For $p \geq 2$, the map $H^{d-p}(X, \mathbb{Q}) \to H^{d-p}(H \cap X, \mathbb{Q})$ is an isomorphism for every smooth hypersurface section $H \cap H \subseteq X$, by Lefschetz' Theorem; thus q_p is an isomorphism for $p \geq 2$. An easy induction, combined with the Five Lemma, shows that r_p is therefore an isomorphism for $p \geq 2$ as well.

For $p = 1$, we have

$$\begin{tikzcd}
L^2H^d\big(P^{sm} \times X\big) \arrow[r, hook] \arrow[d, "\simeq"'] & L^1H^d\big(P^{sm} \times X\big) \arrow[r, two heads] \arrow[d, "r_1"'] & H^1\big(P^{sm}\big) \times H^{d-1}(X) \arrow[d, "q_1"'] \\
L^2H^d\big(\mathfrak{X}^{sm}\big) \arrow[r, hook] & L^1H^d\big(\mathfrak{X}^{sm}\big) \arrow[r, dashed, two heads] & H^1\big(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}\big)
\end{tikzcd}$$

and since $H^{d-1}(H \cap X, \mathbb{Q}) \simeq H^{d-1}(X, \mathbb{Q}) \oplus H^{d-1}(H \cap X, \mathbb{Q})_{van}$ whenever $H \cap X$ is nonsingular, we obtain an isomorphism

$$\operatorname{coker} r_1 \simeq \operatorname{coker} q_1 \simeq H^1\big(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}_{van}\big). \tag{4.5}$$

Finally, we consider the diagram for $p = 0$:

$$\begin{tikzcd}
L^1H^d\big(P^{sm} \times X\big) \arrow[r, hook] \arrow[d, "r_1"'] & H^d\big(P^{sm} \times X\big) \arrow[r, two heads] \arrow[d, "r_0"'] & H^0\big(P^{sm}\big) \times H^d(X) \arrow[d, "q_0"'] \\
L^1H^d\big(\mathfrak{X}^{sm}\big) \arrow[r, hook] & H^d\big(\mathfrak{X}^{sm}\big) \arrow[r, two heads] & H^0\big(P^{sm}, R^d\pi_*^{sm}\mathbb{Q}\big)
\end{tikzcd}$$

By Nori's theorem, the map r_0 is an isomorphism if $2d - 3 \geq d$, equivalently $d \geq 3$, and at least injective if $d = 2$. The Snake Lemma now gives an isomorphism

$$\ker q_0 \simeq \operatorname{coker} r_1. \tag{4.6}$$

Since we clearly have $\ker q_0 \simeq H^d(X, \mathbb{Q})_{prim}$, we obtain a natural map

$$H^d(X, \mathbb{Q})_{prim} \to H^1\big(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}_{van}\big),$$

which is injective if $d = 2$, and an isomorphism for $d \geq 3$. $\square$

The result of the previous section now follows easily, by noting that for a Lefschetz pencil with base $\mathbb{P}^1 \subseteq P$, the map $\pi_1(\mathbb{P}^1 \cap P^{sm}) \to \pi_1(P^{sm})$ on fundamental groups is surjective by Zariski's Theorem (Voisin, 2002, p. 351). Consequently, the composition

$$H^d(X, \mathbb{Q})_{prim} \to H^1\big(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}_{van}\big) \to H^1\big(\mathbb{P}^1 \cap P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}_{van}\big)$$

is injective for all $d \geq 2$.

Remark. Together with the comments made in Section 4.1, the proposition expresses the well-known fact that the cohomology ring $H^*(X, \mathbb{Q})$ of X can be reconstructed from the following data:

(i) The cohomology ring $H^*(D, \mathbb{Q})$ of a smooth hypersurface section $D = H \cap X$ of sufficiently high degree.

(ii) The cohomology class $[D]\big|_D \in H^2(D, \mathbb{Q})$.

(iii) The representation of the fundamental group $G = \pi_1(P^{sm})$ on the vector space $V = H^{d-1}(D, \mathbb{Q})_{van}$, more precisely its first group cohomology $H^1(G, V)$.

To see this, note that knowing the class in (ii) gives the decomposition of $H^{d-1}(D, \mathbb{Q})$ into vanishing cohomology and $H^{d-1}(X, \mathbb{Q})$. Thus (i) and (ii) together determine everything but the primitive cohomology $H^d(X, \mathbb{Q})_{prim}$. But according to Proposition 4.3.1, we have

$$H^d(X, \mathbb{Q})_{prim} \simeq H^1\big(P^{sm}, R^{d-1}\pi_*^{sm}\mathbb{Q}_{van}\big) \simeq H^1(G, V),$$

where the second isomorphism is explained in Section 4.4.

4.4 Translation of the problem into group cohomology

Returning to the original Lefschetz pencil from Section 4.2, we now choose the coordinates on $\mathbb{P}^1$ in such a way that the singular locus is given by $D = \{P_1, \ldots, P_m, \infty\}$, with all $P_i \neq 0$. The fundamental group $G = \pi_1(B, 0)$ is isomorphic to a free group on m generators; in fact, a set of generators is given by taking, for each $i = 1, \ldots, m$, a loop g_i that goes exactly once around P_i with positive orientation, but not around any of the other P_j, nor around ∞.

The local system $R^{2n}\pi_*\mathbb{Q}_{van}$ corresponds to a representation of the group G on the $\mathbb{Q}$-vector space $V = H^{2n}(S_0, \mathbb{Q})_{van}$. Now the first cohomology group of a local system can always be computed from the group cohomology of the representation V alone; it is a fact that

$$H^1(B, R^{2n}\pi_*\mathbb{Q}_{van}) \simeq H^1(G, V).$$

If we write

$$Z^1(G, V) = \left\{ \phi\colon G \to V \;\middle|\; \phi(gh) = g\phi(h) + \phi(g) \text{ for all } g, h \in G \right\}$$

for the space of 1-cocyles, and

$$B^1(G, V) = \left\{ \phi\colon G \to V \;\middle|\; \text{there is } v \in V \text{ with } \phi(g) = gv - v \text{ for all } g \in G \right\}$$

for the space of 1-coboundaries, then $H^1(G, V) = Z^1(G, V) \big/ B^1(G, V)$.

Thus Proposition 4.2.4 gives us an injective map

$$H^{2n+1}(X, \mathbb{Q})_{prim} \to H^1(G, V)$$

into the first group cohomology of the representation V. The original problem — whether all the primitive cohomology of X is generated by the tube mapping—is now equivalent to the following question.

Question. Is every nontrivial element in $H^1(G, V)$ detected by a single loop $g \in G$? In other words, given a nonzero element $\phi \in H^1(G, V)$, can we always find some $g \in G$, so that $\phi(g)$ is not in the image of $g - \mathrm{id}$?

If X is of odd dimension, we prove in Section 4.5 that the question has a positive answer. In Section 4.6, we show that this, in turn, implies Theorem 4.1.1.

For an arbitrary representation, the question has a negative answer (an example is given on p. 80 below). In our case, however, the representation V is very special: There are distinguished elements $u_1, \ldots, u_m \in V$—the vanishing cycles at m of the singular fibers of the pencil—for which the Picard-Lefschetz formula

$$g_i \cdot v = v + \big\langle u_i, v \big\rangle u_i$$

holds (Voisin, 2002, Théorème 15.16 on p. 345). The pairing $\big\langle \_, \_ \big\rangle$ that occurs in the formula is related to the intersection pairing on the hypersurface $S_0 \subseteq X$ by the identity

$$\langle u, v \rangle = (-1)^{n+1} \int_{S_0} u \cup v.$$

Moreover, each vanishing cycle u_i has self-intersection number $\big\langle u_i, u_i \big\rangle = -2$. In the following section, we prove a general statement about the first group cohomology of this type of representation, and use it to finish the proof of Theorem 4.1.1.

4.5 The group cohomology of certain representations

At this point, abstracting from the specific question in Section 4.4 leads to some simplification. If we retain only the essential features of the problem, we arrive at the following situation. Let V be a finite-dimensional $\mathbb{Q}$-vector space with a symmetric bilinear form $(u,v) \mapsto \langle u, v\rangle$. Let G be the free group on n generators $g_1, \dots, g_m$, and assume that G acts on V, in such a way that

$$g_i \cdot v = v + \big\langle u_i, v \big\rangle u_i \qquad\qquad (v \in V)$$

for certain distinguished elements $u_1, \dots, u_m \in V$ satisfying $\big\langle u_i, u_i \big\rangle = -2$. This implies that each g_i^2 acts trivially on V. If $g \in G$ is any element, we shall write $g^{\mathbb{Z}}$ for the subgroup of G generated by g.

For this particular class of representations, we have the following injectivity result.

Proposition 4.5.1. *Let V be a representation of the free group G, subject to the assumptions stated above. Then the restriction map*

$$R \colon H^1(G,V) \to \prod_{g \in G} H^1\big(g^{\mathbb{Z}}, V\big)$$

is injective.

Proof. Let $\phi \in Z^1(G,V)$ represent an arbitrary class in the kernel of R. This means that for every $g \in G$, there is some $v \in V$ (depending on g, of course), such that

$$\phi(g) = gv - v.$$

To prove that R is injective, we need to show that $\phi \in B^1(G,V)$.

We shall do this in two steps. Re-indexing the generators $g_1, \dots, g_m$ of G, if necessary, we may assume that the vectors $u_1, \dots, u_p$ are linearly independent, while $u_{p+1}, \dots, u_m$ are linearly dependent on $u_1, \dots, u_p$.

Step 1 We prove that ϕ may be modified, by subtracting a suitable element of $B^1(G, V)$, to get $\phi(g_i) = 0$ for $i = 1, \dots, p$.

By assumption, there is some vector $v \in V$ such that

$$\phi(g_p \cdots g_1) = g_p \cdots g_1 v - v;$$

subtracting from ϕ the element $(g \mapsto gv - v)$ of $B^1(G, V)$, we can arrange that $\phi(g_p \cdots g_1) = 0$. For each $i = 1, \dots, p$, there is now some $v_i \in V$ with

$$\phi(g_i) = g_i v_i - v_i = \langle u_i, v_i \rangle u_i = a_i u_i,$$

where $a_i = \langle u_i, v_i \rangle \in \mathbb{Q}$. By induction, one easily shows that

$$0 = \phi(g_p \cdots g_1) = \sum_{k=1}^{p} b_k u_k,$$

with coefficients b_k that satisfy the recursive relations

$$b_1 = a_1 \qquad \text{and} \qquad b_{k+1} = a_{k+1} + \sum_{i=1}^{k} \langle u_{k+1}, u_i \rangle b_i.$$

But $u_1, \dots, u_p$ are linearly independent, and therefore $b_1 = \cdots = b_p = 0$. This gives $a_1 = \cdots a_p = 0$, and so we obtain $\phi(g_i) = 0$ for $i = 1, \dots, p$.

Note. This can be proved more quickly, if one is willing to add the assumption that the pairing is non-degenerate. The argument goes as follows. For each $i = 1, \dots, p$, there is a vector $v_i \in V$ with $\phi(g_i) = g_i v_i - v_i = \langle u_i, v_i \rangle u_i = a_i u_i$, where $a_i = \langle u_i, v_i \rangle \in \mathbb{Q}$.

Since the vectors $v_1, \ldots, v_p$ are linearly independent, and since the pairing $\langle \_, \_ \rangle$ is non-degenerate, we can find some $v \in V$ with $\langle u_i, v \rangle = a_i$ for $i = 1, \ldots, p$. Subtracting from ϕ the element $(g \mapsto gv - v)$ of $B^1(G, V)$, we then get $\phi(g_1) = \cdots = \phi(g_p) = 0$.

Step 2 Having thus modified ϕ, we show that now $\phi = 0$. For this, we only need to prove that $\phi(g_i) = 0$ for $i = p+1, \ldots, n$, because all the g_i together generate G. By symmetry, it obviously suffices to consider just g_{p+1}. Since u_{p+1} is linearly dependent on $u_1, \ldots, u_p$, we can write

$$u_{p+1} = \sum_{i=1}^{p} a_i u_i$$

for certain coefficients $a_i \in \mathbb{Q}$; these are subject to the condition that

$$-2 = \langle u_{p+1}, u_{p+1} \rangle = \sum_{ij} a_i \langle u_i, u_j \rangle a_j.$$

If we let a be the column vector with coordinates a_i, and U the $p \times p$-matrix with entries $U_{ij} = -\langle u_i, u_j \rangle$, we can write that condition in the form

$$a^\dagger U a = 2. \tag{4.7}$$

As before, there is a vector $v \in V$ with $\phi(g_{p+1}) = g_{p+1}v - v = \langle u_{p+1}, v \rangle u_{p+1}$, and if we set $\eta = \langle u_{p+1}, v \rangle \in \mathbb{Q}$, we have

$$\phi(g_{p+1}) = \eta \cdot \sum_{k=1}^{p} a_k u_k.$$

We may also find $w \in V$ such that

$$\phi(g_1 \cdots g_p g_{p+1}) = g_1 \cdots g_p g_{p+1} w - w.$$

Now $\phi(g_i) = 0$ for $i = 1, \dots, p$, and so we get $\phi(g_1 \cdots g_p g_{p+1}) = g_1 \cdots g_p \phi(g_{p+1})$; this gives us

$$\begin{aligned}\phi(g_{p+1}) &= g_p \cdots g_1 \cdot \big(g_1 \cdots g_p g_{p+1} w - w\big) = \big(g_{p+1} w - w\big) - \big(g_p \cdots g_1 w - w\big) \\ &= \big\langle u_{p+1}, w \big\rangle u_{p+1} - \big(g_p \cdots g_1 w - w\big) = \sum_{i,j=1}^{p} a_i x_i a_j u_j - \big(g_p \cdots g_1 w - w\big),\end{aligned}$$

with the abbreviations $x_i = \big\langle u_i, w \big\rangle$. By induction, one proves that $g_p \cdots g_1 w - w = \sum c_j u_j$, where

$$c_1 = x_1 \qquad \text{and} \qquad c_{k+1} = x_{k+1} - \sum_{i=1}^{k} U_{k+1,i} \cdot c_i. \tag{4.8}$$

We thus get the relation

$$\eta \cdot \sum_{j=1}^{p} a_j u_j = \sum_{i,j=1}^{p} a_i x_i a_j u_j - \sum_{j=1}^{p} c_j u_j.$$

As $u_1, \dots, u_p$ are linearly independent, we deduce that

$$\eta a_j = \sum_{i=1}^{p} a_i x_i a_j - c_j$$

for all $j = 1, \dots, p$, which we can write as a vector equation

$$\eta \cdot a = a^\dagger x \cdot a - c. \tag{4.9}$$

The recursive relations in (4.8) for the c_j can be put into the form $x = Sc$, where S is a lower-triangular matrix with entries

$$S_{ij} = \begin{cases} U_{ij} & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

But now $U = S + S^\dagger$, because U is symmetric and has twos along the diagonal. From (4.7), we find that

$$2 = a^\dagger U a = a^\dagger S a + a^\dagger S^\dagger a = 2a^\dagger S a,$$

and so $a^\dagger S a = 1$. Now apply $a^\dagger S$ to the equation in (4.9) to get

$$\eta = a^\dagger S a \cdot \eta = a^\dagger x \cdot a^\dagger S a - a^\dagger S c = a^\dagger x - a^\dagger S c = a^\dagger (x - Sc) = 0.$$

This shows that $\phi(g_{p+1}) = \eta \cdot u_{p+1} = 0$, and we have our result. $\square$

An example where Proposition 4.5.1 fails

It should be pointed out that the restriction map is in general not injective without some assumptions on the representation of G on V. Here is a simple example of this phenomenon.[2] Let $G = \mathbb{Z}^2$ be the free Abelian group on two generators, acting on $V = \mathbb{Q}^3$ by the two commuting matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define $\phi \colon G \to V$ by the rule $\phi(a, b) = (a, 0, 0)$. One easily verifies that ϕ is a non-zero element in $H^1(G, V)$, but that it goes to zero under the restriction map

$$H^1(G, V) \to \prod_{g \in G} H^1\bigl(g^{\mathbb{Z}}, V\bigr).$$

Thus Proposition 4.5.1 does not remain true for arbitrary representations.

[2]Similarly, any singularity of an admissible normal function near a normal crossing divisor provides an example. See 7.3.1 for details about normal functions and admissibility.

4.6 Completing the proof of the main theorem

In this section, we finish the proof of Theorem 4.1.1 about the surjectivity of the tube mapping. After combining the results of Proposition 4.2.4 and Proposition 4.5.1, we find that there is an injective map

$$H^{2n+1}(X, \mathbb{Q})_{prim} \to \prod_{g \in G} H^1\bigl(g^{\mathbb{Z}}, V\bigr),$$

where $G = \pi_1(B, 0)$ is now again the fundamental group of the projective line minus $(m+1)$ points. Because $H^{2n+1}(X, \mathbb{Q})_{prim}$ is finite-dimensional, one easily sees that the dual map

$$\bigoplus_{g \in G} H^1\bigl(g^{\mathbb{Z}}, V\bigr)^{\vee} \to H^{2n+1}(X, \mathbb{Q})^{\vee}_{prim}$$

has to be surjective. If we use the explicit description for the first group cohomology, and the isomorphism between V and $V^{\vee}$ given by the intersection pairing, we find that

$$H^1\bigl(g^{\mathbb{Z}}, V\bigr)^{\vee} \simeq \bigl\{\, v \in V \bigm| g \cdot v = v \,\bigr\}$$

Moreover, the space $H^{2n+1}(X, \mathbb{Q})_{prim}$ is its own dual under the intersection pairing on X; in total, we therefore get a surjective map

$$\bigoplus_{g \in G} \bigl\{\, \alpha \in H^{2n}(S_0, \mathbb{Q})_{van} \bigm| g \cdot \alpha = \alpha \,\bigr\} \to H^{2n+1}(X, \mathbb{Q})_{prim} \tag{4.10}$$

that we shall denote by T for the time being. To complete the proof, we need to show that T is the tube mapping used in Theorem 4.1.1; because of the various isomorphisms in the construction, this is not entirely obvious.

Cohomological description of the tube mapping

We are now going to show that the map T in (4.10) is indeed the tube mapping, in its cohomological version. Let us revert to writing $d = \dim X$ during the proof. It is clear from the above that we always have a map[3]

$$T\colon \bigoplus_{g \in G} \left\{ \alpha \in H^{d-1}(S_0, \mathbb{Q})_{van} \;\middle|\; g \cdot \alpha = \alpha \right\} \to H^d(X, \mathbb{Q})_{prim}, \tag{4.11}$$

and we are now going to describe T very concretely.

We shall consider the effect of the map T on a single summand in (4.11). So fix a particular element $g \in G = \pi_1(B, 0)$; the corresponding component of T is then

$$T_g\colon \left\{ \alpha \in H^{d-1}(S_0, \mathbb{Q})_{van} \;\middle|\; g \cdot \alpha = \alpha \right\} \to H^d(X, \mathbb{Q})_{prim}.$$

Choose an immersion of manifolds $\mathbb{S}^1 \to B$ that represents the element $g \in \pi_1(B, 0)$ and sends the point $1 \in \mathbb{S}^1$ to the base point $0 \in B$. Let $\mathfrak{S}_g$ be the smooth real $(2d-1)$-manifold obtained by pulling the family $\mathfrak{S} \to B$ back to $\mathbb{S}^1$. We are going to use the following three maps:

$$\begin{tikzcd} S_0 \arrow[r, "i_g"] & \mathfrak{S}_g \arrow[r, "f_g"] \arrow[d, "\pi_g"] & X \\ & \mathbb{S}^1 & \end{tikzcd} \tag{4.12}$$

As we have seen in Proposition 4.2.4, the Leray spectral sequence for the map $\pi\colon \mathfrak{S} \to B$ degenerates at E_2, and gives rise to a short exact sequence for the cohomology of $\mathfrak{S}$. The same is true for the map $\pi_g\colon \mathfrak{S}_g \to \mathbb{S}^1$; because the spectral sequence is

[3]When $d = 2n+1$ is odd, we know that T is injective; but this plays no result in describing the map more concretely.

functorial for the inclusion $\mathfrak{S}_g \to \mathfrak{S}$, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1\bigl(B, R^{d-1}\pi_*\mathbb{Q}\bigr) & \longrightarrow & H^d(\mathfrak{S}, \mathbb{Q}) & \longrightarrow & H^0\bigl(B, R^d\pi_*\mathbb{Q}\bigr) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1\bigl(\mathbb{S}^1, R^{d-1}\pi_{g*}\mathbb{Q}\bigr) & \longrightarrow & H^d(\mathfrak{S}_g, \mathbb{Q}) & \longrightarrow & H^0\bigl(\mathbb{S}^1, R^d\pi_{g*}\mathbb{Q}\bigr) & \longrightarrow & 0
\end{array}$$

Topologically, π_g is a fiber bundle with fiber S_0; because of this, the bottom row in the diagram is isomorphic to

$$0 \longrightarrow \frac{H^{d-1}(S_0, \mathbb{Q})}{(g - \mathrm{id})H^{d-1}(S_0, \mathbb{Q})} \xrightarrow{i_{g*}} H^d(\mathfrak{S}_g, \mathbb{Q}) \xrightarrow{i_g^*} \bigl(H^d(S_0, \mathbb{Q})\bigr)^g \longrightarrow 0,$$

which is part of the usual exact sequence for a fiber bundle over $\mathbb{S}^1$.

Now T_g is dual to $T_g^\vee \colon H^d(X, \mathbb{Q})_{prim} \to H^{d-1}(S_0, \mathbb{Q})\big/(g - \mathrm{id})H^{d-1}(S_0, \mathbb{Q})$. Because of the commutativity of the above diagram, $T_g^\vee$ has the property that

$$i_{g*}T_g^\vee(\omega) = f_g^*(\omega)$$

for each $\omega \in H^d(X, \mathbb{Q})_{prim}$; as we have seen above, $T_g^\vee$ actually takes its image inside the vanishing cohomology. Whenever $\alpha \in H^{d-1}(S_0, \mathbb{Q})_{van}$ is a class with $g \cdot \alpha = \alpha$, we can now compute $T_g(\alpha)$. Indeed, for each ω as above, one has

$$\int_X T_g(\alpha) \cup \omega = \int_{S_0} \alpha \cup T_g^\vee(\omega).$$

Since α is invariant under g, one can choose $\tilde{\alpha} \in H^{d-1}(\mathfrak{S}_g, \mathbb{Q})$ with $i_g^*(\tilde{\alpha}) = \alpha$; then

$$\int_{S_0} i_g^*(\tilde{\alpha}) \cup T_g^\vee(\omega) = \int_{\mathfrak{S}_g} \tilde{\alpha} \cup i_{g*}T_g^\vee(\omega) = \int_{\mathfrak{S}_g} \tilde{\alpha} \cup f_g^*(\omega) = \int_X f_{g*}(\tilde{\alpha}) \cup \omega \tag{4.13}$$

Thus we find that

$$T_g(\alpha) = \text{primitive part of } f_{g*}(\tilde{\alpha});$$

in plain English, one first lifts the invariant class α to a class $\tilde{\alpha} \in H^{d-1}(\mathfrak{S}_g, \mathbb{Q})$, pushes it forward to X via the map f_g, and then takes the primitive part of the resulting class. This is the cohomological description of the tube mapping.

Homological description of the tube mapping

At times, it is useful to have another description of the tube mapping in terms of homology classes. We already sketched the construction in Section 4.1; now we make it more precise, and show that it agrees with the cohomological definition given above. Let $g \in G = \pi_1(B, 0)$ be a closed loop; for the time being, we consider g as a smooth immersion $g \colon [0,1] \to B$ with $g(0) = g(1) = 0$ equal to the chosen base point in B.

Now let $\alpha \in H^{d-1}(S_0, \mathbb{Q})_{van}$ be a cohomology class invariant under the action of g. We write $\gamma_0 = \mathrm{PD}(\alpha)$ for the homology class Poincaré dual to α; by translating γ_0 flatly along the path g, we obtain homology classes $\gamma_t \in H^{d-1}\big(S_{g(t)}, \mathbb{Q}\big)_{van}$. Here $S_{g(t)}$ is the smooth hypersurface corresponding to the point $g(t) \in B$.

The image of each γ_t in $H^d(X, \mathbb{Q})$ is zero, and therefore $\gamma_t = \partial \Gamma_t$ for suitable d-chains on X with rational coefficients. We may assume that each Γ_t is also the flat translate of Γ_0 along the path g. Because γ_0 is g-invariant, both γ_0 and γ_1 represent the same homology class in $H_{d-1}(S_0, \mathbb{Q})_{van}$; thus we can find a rational d-chain Γ' on S_0 such that $\gamma_1 - \gamma_0 = \partial \Gamma'$. Note that Γ' is unique up to elements in $H_d(S_0, \mathbb{Q})$. The d-chain on X given by $\Gamma_1 - \Gamma_0 - \Gamma'$ is therefore closed; taking the ambiguity in the choice of Γ' into account, we get a well-defined element in $H_d(X, \mathbb{Q}) \big/ H_d(S_0, \mathbb{Q})$. Since this space is Poincaré dual to the primitive cohomology, we get a class

$$\tau_g(\alpha) = \mathrm{PD}\big(\Gamma_1 - \Gamma_0 - \Gamma'\big) \in H^d(X, \mathbb{Q})_{prim}.$$

Let us now see why $T = \tau$, i.e., why the cohomological and homological descriptions of the tube mapping agree with each other. We are going to show that

$$\int_X \tau_g(\alpha) \cup \omega = \int_X T_g(\alpha) \cup \omega$$

for every $\omega \in H^d(X, \mathbb{Q})_{prim}$. We use the same notation as in (4.12). Let $\tilde{\Gamma}$ be the $(d+1)$-chain in $\mathbb{S}^1 \times X$ consisting of the various Γ_t, and similarly, let $\tilde{\gamma}$ be the d-chain whose fibers over $\mathbb{S}^1$ are the cycles γ_t. Then clearly

$$\partial\tilde{\Gamma} = \Gamma_1 - \Gamma_0 - \tilde{\gamma},$$

and so the d-cycle $\Gamma_1 - \Gamma_0 - \Gamma'$ is homologous to $\tilde{\gamma} - \Gamma'$. Viewing the latter as a d-cycle on $\mathfrak{S}_g$, we thus get

$$\int_X \tau_g(\alpha) \cup \omega = \int_{\Gamma_1 - \Gamma_0 - \Gamma'} \omega = \int_{\tilde{\gamma} - \Gamma'} \omega = \int_{\tilde{\gamma} - \Gamma'} f_g^*(\omega). \tag{4.14}$$

From both constructions, it is clear that $\tilde{\gamma} - \Gamma'$ is the Poincaré dual of the class $\tilde{\alpha}$ (which satisfied $i_g^*(\tilde{\alpha}) = \alpha$). It follows that

$$\int_{\tilde{\gamma} - \Gamma'} f_g^*(\omega) = \int_{\mathfrak{S}_g} \tilde{\alpha} \cup f_g^*(\omega) = \int_X f_{g*}(\tilde{\alpha}) \cup \omega = \int_X T_g(\alpha) \cup \omega,$$

using (4.13), and so we have $\tau_g(\alpha) = T_g(\alpha)$ as claimed.

4.7 The case of an even-dimensional variety

For various reasons, it was necessary to assume that the dimension of the variety X is odd, in order to prove Theorem 4.1.1. In this concluding section, we briefly discuss this issue.

It was already pointed out (see p. 70) that the theorem becomes false for an even-dimensional variety X, because hypersurface sections of low degree might have trivial vanishing cohomology, as in the case of quadrics. With the help of Nori's Connectivity Theorem, however, this problem can be circumvented by using only hypersurface sections that are sufficiently ample. In the proof, this amounts to replacing Proposition 4.2.4 by Proposition 4.3.1, where it is no longer necessary to assume that $\dim X$ is odd.

The second problem lies in the result about group cohomology in Proposition 4.5.1. Its proof depends on the special nature of the G-representation $V = H^{2n}(S_0, \mathbb{Q})_{van}$; in particular, we used the fact that each g_i is an involution, and that the intersection pairing on the hypersurface $S_0 \subseteq X$ is symmetric. When X is of even dimension, the situation changes—each g_i is then of infinite order, and the intersection pairing becomes skew-symmetric. In other words, the representation of G on V is then of a markedly different kind.

Based on some calculations for small values of n, it seems likely that the result of Proposition 4.5.1 remains true; unfortunately, I am currently not able to prove this. The result is therefore not fully satisfactory; to complete it with the approach taken here, the following problem needs to be resolved.

Problem. Let V be a finite-dimensional $\mathbb{Q}$-vector space, endowed with a bilinear form $(u, v) \mapsto \langle u, v \rangle$ that is skew-symmetric and non-degenerate. Let G be the free group on m generators $g_1, \dots, g_m$. Assume that G acts on V in such a way that

$$g_i \cdot v = v + \langle u_i, v \rangle u_i \qquad (v \in V)$$

for certain distinguished elements $u_1, \ldots, u_m \in V$ that span the space V. Prove the group-cohomological result in Proposition 4.5.1 for this kind of representation!

4.8 A lemma about the cohomology of resolutions

In this section, we give the proof of an auxiliary lemma, used in Section 4.2. It refers to the following diagram of algebraic varieties:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{i}"] \arrow[d, "q"] & \tilde{X} \arrow[d, "p"] \\ Z \arrow[r, "i"] & X \end{tikzcd} \tag{4.15}$$

In the diagram, $p \colon \tilde{X} \to X$ is a map between projective algebraic varieties, and $Z \subseteq X$ is a closed subvariety. Also, $\tilde{Z} = p^{-1}(Z)$ is the preimage of Z.

Lemma 4.8.1. *Let a diagram as in* (4.15) *be given, and assume furthermore that the map p is an isomorphism over the open set $U = X \setminus Z$. Then there is a long exact sequence of cohomology groups*

$$\cdots \longrightarrow H^k(X) \xrightarrow{(i^*, p^*)} H^k(Z) \oplus H^k(\tilde{X}) \xrightarrow{q^* - \tilde{i}^*} H^k(\tilde{Z}) \longrightarrow H^{k+1}(X) \longrightarrow \cdots$$

over $\mathbb{Q}$.

Proof. The rows in the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^k(X) \arrow[r, "i^*"] \arrow[d, "p^*"] & H^k(Z) \arrow[r] \arrow[d, "q^*"] & H^{k+1}_{\mathrm{c}}(U) \arrow[r] \arrow[d, equal] & H^{k+1}(X) \arrow[r] \arrow[d, "p^*"] & \cdots \\ \cdots \arrow[r] & H^k(\tilde{X}) \arrow[r, "\tilde{i}^*"] & H^k(\tilde{Z}) \arrow[r] & H^{k+1}_{\mathrm{c}}(U) \arrow[r] & H^{k+1}(\tilde{X}) \arrow[r] & \cdots \end{tikzcd}$$

are long exact sequences of $\mathbb{Q}$-vector spaces. Now diagram chasing of the type used to prove the Mayer-Vietoris theorem gives the result. $\square$

CHAPTER 5

RESIDUES AND $\mathcal{D}$-MODULES

5.1 The cohomology of the complement of a smooth divisor

Let X be a smooth complex projective variety, of dimension n, and let $D \subseteq X$ be a smooth hypersurface. In this section, we review the description of the Hodge filtration on the cohomology of $X \setminus D$, using the algebraic de Rham complex. We shall look at this in two different ways, one elementary (going through forms with logarithmic poles), the other involving M. Saito's theory of mixed Hodge modules.

5.1.1 Description by forms with logarithmic poles

Both the cohomology, and the Hodge filtration, can be computed using forms with logarithmic poles along D; the relevant complex, starting in degree zero, is

$$\mathscr{O}_X \xrightarrow{d} \Omega_X^1(\log D) \xrightarrow{d} \Omega_X^2(\log D) \longrightarrow \cdots \longrightarrow \Omega_X^n(\log D), \tag{5.1}$$

and will be denoted by $\Omega_X^\bullet(\log D)$. Its hypercohomology is naturally isomorphic to the cohomology of $X \setminus D$; the isomorphism

$$H^*(X \setminus D, \mathbb{C}) \simeq \mathbb{H}^*\big(\Omega_X^\bullet(\log D)\big) \tag{5.2}$$

takes a logarithmic form to the cohomology class defined by its restriction to $X \setminus D$. Moreover, the complex has an obvious filtration by subcomplexes $F^p\Omega_X^\bullet(\log D)$, defined as

$$\Omega_X^p(\log D) \longrightarrow \Omega_X^{p+1}(\log D) \longrightarrow \cdots \longrightarrow \Omega_X^n(\log D),$$

with the first term in degree p. The following lemma describes the Hodge filtration in terms of these subcomplexes.

Lemma 5.1.1. *The Hodge filtration on the cohomology of $X \setminus D$ is given by*

$$F^pH^*(X \setminus D, \mathbb{C}) \simeq \mathbb{H}^*\big(F^p\Omega_X^\bullet(\log D)\big),$$

under the isomorphism in (5.2)*.*

Proof. By definition, we have

$$F^pH^*(X \setminus D, \mathbb{C}) \simeq \operatorname{im}\Big(\mathbb{H}^*\big(F^p\Omega_X^\bullet(\log D)\big) \to \mathbb{H}^*\big(\Omega_X^\bullet(\log D)\big)\Big).$$

The lemma will follow, if we can show that the mapping in parentheses is actually injective. This is clear when $p = 0$; from now on, we shall assume that $p \geq 1$. In that case, we have a residue mapping, and the sequence

$$0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log D) \xrightarrow{\text{Res}} \Omega_D^{p-1} \longrightarrow 0$$

is exact. Thus

$$0 \longrightarrow F^p\Omega_X^\bullet \longrightarrow F^p\Omega_X^\bullet(\log D) \longrightarrow F^{p-1}\Omega_D^{\bullet-1} \longrightarrow 0$$

is a short exact sequence of complexes. If we compare the resulting long exact sequence on hypercohomology to the cohomology sequence for $D \subseteq X$, we get a commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^k\big(F^p\Omega_X^\bullet\big) \arrow[r] \arrow[d] & \mathbb{H}^k\big(F^p\Omega_X^\bullet(\log D)\big) \arrow[r] \arrow[d] & \mathbb{H}^{k-1}\big(F^{p-1}\Omega_D^\bullet\big) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F^pH^k(X,\mathbb{C}) \arrow[r] & F^pH^k(X \setminus D,\mathbb{C}) \arrow[r] & F^{p-1}H^{k-1}(D,\mathbb{C}) \arrow[r] & \cdots
\end{tikzcd}
$$

with exact rows. Since X and D are nonsingular and projective, the first and third map are isomorphisms (because the Hodge–de Rham spectral sequence degenerates at the E_1-page). By the Five Lemma, the same has to be true for the second map, and this proves the lemma. $\square$

5.1.2 Description by rational forms

The cohomology of $X \setminus D$ can also be computed by rational differential forms on X, with poles along D. When the divisor D is sufficiently ample, this leads to a more convenient description of the Hodge filtration, avoiding hypercohomology. Let us write $\Omega_X^p(sD)$ for the sheaf of p-forms on X, with poles of order at most s along the divisor; similarly,

$$\Omega_X^p(*D) = \bigcup_{s=0}^{\infty} \Omega_X^p(sD)$$

will denote the sheaf of forms with poles of arbitrary order. These again make up a complex $\Omega_X^\bullet(*D)$, and since D is nonsingular, the inclusion

$$\Omega_X^\bullet(\log D) \subseteq \Omega_X^\bullet(*D)$$

is a quasi-isomorphism. Thus we have

$$H^*(X \setminus D, \mathbb{C}) \simeq \mathbb{H}^*\big(\Omega_X^\bullet(*D)\big) \tag{5.3}$$

as well. To describe the Hodge filtration in this context, we observe that the differential d increases the order of pole of a rational form by at most one. We can thus look at the subcomplex

$$F^p\Omega_X^\bullet(*D) = \Big[\Omega_X^p(D) \longrightarrow \Omega_X^{p+1}(2D) \longrightarrow \cdots \longrightarrow \Omega_X^n\big((n-p+1)D\big)\Big]$$

of $\Omega_X^\bullet(*D)$. It follows immediately from Voisin (2002, Lemme 18.6 on p. 419) that the inclusion of $F^p\Omega_X^\bullet(\log D)$ into this complex is a quasi-isomorphism. Now Lemma 5.1.1 implies that the Hodge filtration is also given by

$$F^pH^*(X \setminus D, \mathbb{C}) \simeq \mathbb{H}^*\big(F^p\Omega_X^\bullet(*D)\big). \tag{5.4}$$

If we assume that the divisor $D \subseteq X$ is sufficiently ample, we get the following nice conclusion.

Proposition 5.1.2. *Let X be a smooth complex projective variety of dimension n, and let $D \subseteq X$ be a smooth hypersurface. Assume that $\mathscr{O}_X(D)$ is ample enough to make the following cohomology groups vanish:*

$$H^q\big(X, \Omega_X^p(sD)\big) = 0 \textit{ for } q > 0 \textit{ and } s > 0.$$

Then the cohomology in degree k of the complex

$$\Gamma\big(X, \Omega_X^p(D)\big) \longrightarrow \Gamma\big(X, \Omega_X^{p+1}(2D)\big) \longrightarrow \cdots \longrightarrow \Gamma\big(X, \Omega_X^n\big((n-p+1)D\big)\big)$$

is isomorphic to $F^pH^k(X \setminus D, \mathbb{C})$. The isomorphism takes a d-closed rational form to the cohomology class defined by its restriction to $X \setminus D$.

Proof. This follows immediately from (5.4) by using the hypercohomology spectral sequence. Indeed, on the E_1-page, all but the bottom row is zero by our ampleness assumption, and the one nonzero row is exactly the indicated complex. $\square$

5.1.3 Background on mixed Hodge modules

Proposition 5.1.2 can also be proved via M. Saito's theory of mixed Hodge modules; this is explained, among other things, in Saito (1993). Since we are going to use mixed Hodge modules quite extensively later on, we now review some basic facts about them, and list several notational conventions, following Saito (1990). We shall then redo the proof of Proposition 5.1.2 in this framework.

Mixed Hodge modules If X is a (say, quasi-projective) complex algebraic variety, $\mathrm{MHM}(X)$ denotes the Abelian category of algebraic mixed Hodge modules on X. There is a fully faithful functor

$$\mathrm{rat}\colon \mathrm{MHM}(X) \to \mathrm{Perv}_{\mathbb{Q}}^{alg}(X)$$

from $\mathrm{MHM}(X)$ to the category of $\mathbb{Q}$-valued algebraic perverse sheaves on X ("algebraic" means that all stratifications should have algebraic strata). If M is a mixed Hodge module, then $\mathrm{rat}\, M$ is called the perverse sheaf underlying M.

Each mixed Hodge module M also has an underlying filtered holonomic $\mathcal{D}$-module $(\mathcal{M}, F)$, which corresponds to the perverse sheaf $\mathrm{rat}\, M$ under the Riemann-Hilbert correspondence. The comparison isomorphism

$$\alpha\colon \mathrm{rat}\, M \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} \mathrm{DR}_X(\mathcal{M})$$

is part of the data of M. In order for M to be a mixed Hodge module, several other strong conditions have to be satisfied (there has to be a weight filtration, and M has to behave well with respect to all nearby and vanishing cycle functors, to name just two). Details can be found in Saito (1988, §4 on pp. 314–29); suffice it to say that the definition of $\mathrm{MHM}(X)$ is highly inductive in nature.

The basic idea is that mixed Hodge modules are natural extensions of admissible variations of mixed Hodge structure. More precisely, an element $M \in \mathrm{MHM}(X)$ is called *smooth* if $\operatorname{rat} M$ is a local system of $\mathbb{Q}$-vector spaces. Saito proves (1990, Theorem 3.27 on p. 312) that a (polarized) smooth mixed Hodge module is the same as an admissible variation of mixed Hodge structure. Thus, given a mixed Hodge module M, there is an algebraic stratification of X, such that M is an admissible variation on each stratum. The indidvidual variations are not arbitrary, however, but are related to each other in a subtle way.

A special case is the space $pt = \operatorname{Spec} \mathbb{C}$ consisting of a single point. Here $\mathrm{MHM}(pt)$ is the category of polarized mixed Hodge structures defined over $\mathbb{Q}$.

Finally, when X is projective, we have an equivalence of categories

$$\mathrm{MHM}(X) \simeq \mathrm{MHM}\big(X^{an}\big)^{p},$$

between algebraic mixed Hodge modules and polarizable analytic mixed Hodge modules (Saito, 1990, Remark on p. 313). Thus we can equally well use the analytic topology when working with mixed Hodge modules on X.

Standard modules Given a smooth complex variety X, we write $d_X = \dim X$ for the dimension of X. Following Saito, we let $\mathbb{Q}^H \in \mathrm{MHM}(pt)$ be the unique Hodge structure on $\mathbb{Q}$ of weight 0. We let $a_X \colon X \to pt$ be the structural morphism, and define $\mathbb{Q}_X^H = a_X^* \mathbb{Q}^H \in D^b \operatorname{MHM}(X)$. Then

$$\mathbb{Q}_X^H[d_X] = \mathbb{Q}_X^H[\dim X]$$

is a mixed Hodge module on X whose underlying perverse sheaf is $\mathbb{Q}_X[d_X]$. The corresponding $\mathcal{D}$-module is $\mathscr{O}_X$, with filtration

$$F_s \mathscr{O}_X = \begin{cases} \mathscr{O}_X & \text{if } s \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In general, we shall use the suggestive notation $H^k(X) \otimes \mathbb{Q}_P^H$ to refer to the element $a_P^* H^k(X)$ of $D^b \operatorname{MHM}(P)$; this makes sense because $H^k(X) \in \operatorname{MHM}(pt)$.

Shifts For a complex $M = M^\bullet$ of mixed Hodge modules, we write $M[1]$ for the same complex, shifted to the left by one step; consequently, $M[1]^k = M^{k+1}$.

Tate twist The Tate twist of a mixed Hodge module M is defined as follows. Let $(\mathcal{M}, F_\bullet)$ be the underlying filtered $\mathcal{D}$-module, and K the underlying perverse sheaf, with comparison isomorphism $\alpha \colon K \otimes_{\mathbb{Q}} \mathbb{C} \to \operatorname{DR}_X(\mathcal{M})$. Then $M(s)$ has underlying $\mathcal{D}$-module $(\mathcal{M}, F_{\bullet - s})$, and underlying perverse sheaf K, but the comparison isomorphism is twisted by a factor of $(2\pi i)^s$, and becomes $(2\pi i)^s \alpha$.

Verdier duality Verdier duality on X will be denoted by $\mathbb{D}_X$; note that we have $\mathbb{D}_X\bigl(\mathbb{Q}_X^H[d_X]\bigr) = \mathbb{Q}_X^H(d_X)[d_X]$, since $\mathbb{D}_X$ preserves $\operatorname{MHM}(X)$. Thus, in general,

$$\mathbb{D}_X\bigl(\mathbb{Q}_X^H(s)[k]\bigr) = \mathbb{Q}_X^H(d_X - s)[2d_X - k].$$

Intermediate extension If $j \colon U \to X$ is the inclusion of an open subset, and M a mixed Hodge module on U, we write

$$j_{!*}M = \operatorname{im}\bigl(H^0 j_! M \to H^0 j_* M\bigr)$$

for the intermediate extension of M to X. If j is affine, then both $j_!$ and j_* are exact functors, and the H^0 can be omitted. By construction, $j_{!*}M$ has no nontrivial submodules or quotient modules that are supported on $X \setminus U$. By functoriality, the associated $\mathcal{D}$-module is the minimal extension $j_{!*}\mathcal{M}$.

Strictness The term *strictness* is used with two different meanings in the theory of mixed Hodge modules. On the one hand, morphisms of mixed Hodge modules are *strict*, in the sense that they strictly preserve the Hodge filtration. Thus if $\phi \colon M \to N$ is a morphism of mixed Hodge modules, the induced morphism $\phi \colon (\mathcal{M}, F) \to (\mathcal{N}, F)$ on the level of $\mathcal{D}$-modules satisfies

$$\phi\bigl(F_k\mathcal{M}\bigr) \subseteq F_k\mathcal{N} \qquad \text{and} \qquad \phi\bigl(\mathcal{M}\bigr) \cap F_k\mathcal{N} = \phi\bigl(F_k\mathcal{M}\bigr).$$

One consequence is that, for a short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of mixed Hodge modules, the induced sequence

$$0 \longrightarrow Gr_k^F \mathcal{M}' \longrightarrow Gr_k^F \mathcal{M} \longrightarrow Gr_k^F \mathcal{M}'' \longrightarrow 0$$

is exact for every value of k. This fact will be used very frequently below.

On the other hand, projective morphisms between varieties are *strict*, in the sense that the Hodge filtration on the direct image of a mixed Hodge module can be computed from a subcomplex. To understand what this means, let us consider the example of a smooth projective morphism $f\colon X \to Y$. Let $M \in \mathrm{MHM}(X)$ be a mixed Hodge module on X, with underlying filtered $\mathcal{D}$-module $(\mathcal{M}, F)$. Then $f_* M \in D^b \mathrm{MHM}(Y)$ is an object in the derived category, with cohomology objects $N^i = H^i f_* M \in \mathrm{MHM}(Y)$. The $\mathcal{D}$-modules underlying N^i can be found by using the relative de Rham complex. That is to say, the complex $(d = \dim X - \dim Y)$

$$\mathbf{R} f_* \, \mathrm{DR}_{X/Y}(\mathcal{M}) = \mathbf{R} f_* \Big[\mathcal{M} \longrightarrow \Omega^1_{X/Y} \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega^d_{X/Y} \otimes \mathcal{M} \Big][d]$$

is a complex of $\mathcal{D}$-modules on Y; its cohomology sheaf $\mathcal{N}^i = R^i f_* \, \mathrm{DR}_{X/Y}(\mathcal{M})$ is the holonomic $\mathcal{D}$-module underlying $H^i f_* M$.

The de Rham complex is itself filtered by subcomplexes

$$F_k \, \mathrm{DR}_{X/Y}(\mathcal{M}) = \Big[F_k \mathcal{M} \longrightarrow \Omega^1_{X/Y} \otimes F_{k+1} \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega^d_{X/Y} \otimes F_{k+d} \mathcal{M} \Big][d],$$

and we obtain a filtration on each $\mathcal{N}^i$ by setting

$$F_k \mathcal{N}^i = \mathrm{im}\Big(R^i f_* F_k \, \mathrm{DR}_{X/Y}(\mathcal{M}) \to R^i f_* \, \mathrm{DR}_{X/Y}(\mathcal{M}) \Big).$$

In this context, strictness means that the map in parentheses is *injective*; in other words, that $F_k R^i f_* \, \mathrm{DR}_{X/Y}(\mathcal{M}) = R^i f_* F_k \, \mathrm{DR}_{X/Y}(\mathcal{M})$. One of Saito's important results (Saito, 1990, Theorem 2.14 on p. 252) is that this is true when M is a mixed Hodge module, and $f\colon X \to Y$ is projective.

5.1.4 A proof using mixed Hodge modules

We now reprove Proposition 5.1.2, this time using mixed Hodge modules. The cohomology of the open set $U = X \setminus D$ is determined by the mixed Hodge module $g_* \mathbb{Q}_U^H[n]$

on X, where $g\colon U \to X$ is the inclusion map. The mixed Hodge module has an underlying perverse sheaf, which in this case is just $\operatorname{rat}\bigl(g_*\mathbb{Q}_U^H[n]\bigr) = g_*\mathbb{Q}_U[n]$; note that, in order to make the constant sheaf $\mathbb{Q}_U$ on U into a perverse sheaf, one has to put it in degree $-n = -\dim U$, which is what the indicated shift does. Moreover, there is an underlying filtered holonomic $\mathcal{D}$-module $(\mathcal{M}_U, F)$, whose de Rham complex

$$\mathrm{DR}_X(\mathcal{M}_U) = \Bigl[\mathcal{M}_U \longrightarrow \Omega_X^1 \otimes \mathcal{M}_U \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes \mathcal{M}_U\Bigr][n]$$

is isomorphic to $g_*\mathbb{C}_U[n]$; again, the complex has to start in degree $-n$ to make it perverse. Then we have, for each k, that

$$H^k(U,\mathbb{C}) \simeq \mathbb{H}^{k-n}\bigl(g_*\mathbb{C}_U[n]\bigr) \simeq \mathbb{H}^{k-n}\bigl(\mathrm{DR}_X(\mathcal{M}_U)\bigr).$$

Given the mixed Hodge module, it is equally easy to find the Hodge filtration on the cohomology of U. Since $(\mathcal{M}_U, F)$ is filtered,

$$F_s \,\mathrm{DR}_X(\mathcal{M}_U) = \Bigl[F_s\mathcal{M}_U \longrightarrow \Omega_X^1 \otimes F_{s+1}\mathcal{M}_U \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes F_{s+n}\mathcal{M}_U\Bigr][n]$$

is a subcomplex of the de Rham complex for each s. Then the Hodge filtration on $H^k(U,\mathbb{C})$, through the isomorphism above, is given by the following rule:

$$\begin{aligned} F^pH^k(U,\mathbb{C}) &\simeq F_{-p}\,\mathbb{H}^{k-n}\bigl(\mathrm{DR}_X(\mathcal{M}_U)\bigr) \\ &= \operatorname{im}\Bigl(\mathbb{H}^{k-n}\bigl(F_{-p}\,\mathrm{DR}_X(\mathcal{M}_U)\bigr) \to \mathbb{H}^{k-n}\bigl(\mathrm{DR}_X(\mathcal{M}_U)\bigr)\Bigr). \end{aligned}$$

In his paper, Saito points out that the $\mathcal{M}_U$ in question is the left $\mathcal{D}_X$-module $\mathscr{O}_X(*D)$ of rational functions with poles along D, whose $\mathcal{D}$-module structure is simply given by differentiation. It is naturally filtered, by the pole-order filtration

$$P_s \mathcal{M}_U = P_s \mathscr{O}_X(*D) = \begin{cases} \mathscr{O}_X\big((s+1)D\big) & \text{if } s \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This filtration is good, and $(\mathcal{M}_U, P)$ is regular and holonomic. Saito (1993, Corollary 4.3 on p. 71) proves that this filtration is the correct one for computing the Hodge filtration, in other words $F = P$, at least in the case when D is smooth.

The result in Proposition 5.1.2 now follows very easily. The de Rham complex of $\mathscr{O}_X(*D)$ is nothing but $\mathrm{DR}_X(\mathcal{M}_U) = \Omega_X^{\bullet+n}(*D)$, and we have for all k the isomorphism

$$H^k(U, \mathbb{C}) \simeq \mathbb{H}^{k-n}\big(\mathrm{DR}_X(\mathcal{M}_U)\big) \simeq \mathbb{H}^k\big(\Omega_X^\bullet(*D)\big).$$

Moreover, it is easily seen that $P_{-p} \mathrm{DR}_X(\mathcal{M}_U) = F^p \Omega_X^{\bullet+n}(*D)$, and so we conclude that the Hodge filtration is given by

$$\begin{aligned} F^p H^k(U, \mathbb{C}) &\simeq \operatorname{im}\Big(\mathbb{H}^{k-n}\big(P_{-p} \mathrm{DR}_X(\mathcal{M}_U)\big) \to \mathbb{H}^{k-n}\big(\mathrm{DR}_X(\mathcal{M}_U)\big)\Big) \\ &= \operatorname{im}\Big(\mathbb{H}^k\big(F^p \Omega_X^\bullet(*D)\big) \to \mathbb{H}^k\big(\Omega_X^\bullet(*D)\big)\Big) = \mathbb{H}^k\big(F^p \Omega_X^\bullet(*D)\big). \end{aligned}$$

The last equality is because $X \to \operatorname{Spec} \mathbb{C}$ is projective, and hence strict for the Hodge filtration (Saito, 1993, Remark 4.6 on p. 72). Thus we obtain exactly the same result as in Proposition 5.1.2.

Note. During the proof of Lemma 5.1.1, the injectivity of the map in parentheses followed from the degeneration of the Hodge-de Rham spectral sequence. That projective morphisms are strict for the Hodge filtration can thus be viewed as a relative version of this degeneration.

5.1.5 The cohomology of a smooth hypersurface

Using Proposition 5.1.2, together with the exact sequence of mixed Hodge structures

$$0 \longrightarrow H^n(X)_{prim} \longrightarrow H^n(X \setminus D) \xrightarrow{\mathrm{Res}} H^{n-1}(D)_{van}(-1) \longrightarrow 0,$$

we obtain the well-known residue representation for the vanishing cohomology of the hypersurface D and its Hodge filtration.

Proposition 5.1.3. *Under the same hypotheses as in Proposition 5.1.2, the vanishing cohomology $H^{n-1}(D,\mathbb{C})_{van}$ is generated by residues of rational n-forms on X with poles along D, and the Hodge filtration is determined by the order of the pole. More precisely,*

$$F^p H^{n-1}(D,\mathbb{C})_{van} = \mathrm{Res}\,\Gamma\big(X, \Omega_X^n\big((n-p)D\big)\big)$$

consists of residues of n-forms with a pole of order at most $n-p$.

5.2 Background on the universal hypersurface

In this section, we review some basic geometric properties of the universal hypersurface $\mathfrak{X}$. We also explain the relationship between $\mathfrak{X}$ and the cotangent bundle of the projective space P, and use it to describe characteristic varieties of filtered $\mathcal{D}$-modules on P.

5.2.1 The projectivized cotangent bundle of P

Let V be a finite-dimensional complex vector space, and $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ its dual space. The two projective spaces $P = \mathbb{P}_{sub}(V) = \mathbb{P}\big(V^\vee\big)$ and $Q = \mathbb{P}(V)$ are also dual to each other; points of P can be thought of as hyperplanes $H \subseteq Q$. We can then form the incidence variety

$$I = \left\{ (H, x) \in P \times Q \;\middle|\; x \in H \right\}$$

inside the product $P \times Q$. It is a smooth hypersurface, with line bundle $\mathscr{O}_{P\times Q}(I) \simeq pr_P^*\mathscr{O}_P(1) \otimes pr_Q^*\mathscr{O}_Q(1)$. This can be seen very easily by realizing that I is the projectivized cotangent bundle $\mathbb{P}_{sub}\big(\Omega_P\big)$ of P. To see why this should be the case, consider the Euler sequence

$$0 \longrightarrow \mathscr{O}_P \longrightarrow \mathscr{O}_P(1) \otimes V \longrightarrow T_P \longrightarrow 0.$$

on the projective space P. It implies that

$$\mathbb{P}_{sub}\big(\Omega_P\big) = \mathbb{P}\big(T_P\big) = \mathbf{Proj}_P\big(\mathrm{Sym}\, T_P\big)$$

is naturally a subvariety of $\mathbb{P}\big(\mathscr{O}_P(1) \otimes V\big) = P \times Q$, with line bundle $pr_P^*\mathscr{O}_P(1) \otimes pr_Q^*\mathscr{O}_Q(1)$. A moment's thought shows that this subvariety is exactly the incidence variety I.

Now the line bundle $\mathscr{O}_{P\times Q}(I)$ is very ample, and embeds $P\times Q$ into a bigger projective space; we let

$$S = \bigoplus_{k=0}^{\infty} H^0\Big(I, \mathscr{O}_{P\times Q}(kI)\big|_I\Big)$$

be the homogeneous coordinate ring for I under this embedding. The ring is related to the symmetric algebra of the tangent bundle T_P, in the following way. On $P \times Q$, we have the exact sequence

$$0 \longrightarrow \mathscr{O}_{P\times Q}\big((k-1)I\big) \longrightarrow \mathscr{O}_{P\times Q}(kI) \longrightarrow \mathscr{O}_{P\times Q}(kI)\big|_I \longrightarrow 0,$$

which, after pushing forward to P, gives an isomorphism

$$\pi_* \mathscr{O}_{P\times Q}(kI)\big|_I \simeq \frac{\mathscr{O}_P(k)\otimes \operatorname{Sym}^k V}{\mathscr{O}_P(k-1)\otimes \operatorname{Sym}^{k-1} V}$$

for all $k \geq 0$; here $\pi\colon I \to P$ denotes the projection. But the quotient is naturally isomorphic to $\operatorname{Sym}^k T_P$, using the Euler sequence again, and so

$$S \simeq \bigoplus_{k=0}^{\infty} H^0\big(P, \operatorname{Sym}^k T_P\big) = H^0\big(P, \operatorname{Sym} T_P\big)$$

is a second description of the homogeneous coordinate ring.

5.2.2 Basic geometric facts about the universal hypersurface

From now on, let X be a smooth projective variety of dimension n, with a very ample line bundle $\mathscr{L}$ on X. We let $V = H^0(X, \mathscr{L})$ be the space of its global sections; we then get an embedding $i\colon X \to Q$ into the projective space $Q = \mathbb{P}(V)$.

In this setting, the dual projective space $P = \mathbb{P}\big(V^\vee\big)$ parametrizes hyperplane sections of X by $\mathscr{L}$. The *universal hypersurface* is the incidence variety

$$\mathfrak{X} = \big\{\, (H, x) \in P \times X \;\big|\; x \in H \cap X \,\big\}.$$

The reason for this name is that the fibers of the projection $\pi\colon \mathfrak{X} \to P$ are exactly the hypersurfaces $H \cap X$. We write $d = \dim P$; then $\mathfrak{X}$ is itself a smooth divisor in the product $P \times X$, of dimension $n + d - 1$, and with $\mathscr{O}_{P\times X}(\mathfrak{X}) \simeq pr_P^* \mathscr{O}_P(1) \otimes pr_X^* \mathscr{L}$. We shall also use the subvariety $\mathfrak{Z} \subseteq \mathfrak{X}$, defined by the condition

$$\mathfrak{Z} = \big\{\, (H, x) \in P \times X \;\big|\; \text{the hyperplane } H \text{ is tangent to } X \text{ at } x \,\big\}.$$

Its points are exactly the singular points in the fibers of π. Despite this, $\mathfrak{Z}$ is also smooth, and of dimension $d-1$. Its image in P is by definition the dual variety $X^\vee$. If there are points in P corresponding to hypersurfaces with just a single ordinary double point singularity, then $\delta\colon \mathfrak{Z} \to X^\vee$ is birational, hence a resolution of singularities of the dual variety.

The nonsingularity of both $\mathfrak{X}$ and $\mathfrak{Z}$ comes from the fact that they are again projective bundles. For $\mathfrak{X}$, this follows from the result about the incidence variety I in 5.2.1. Indeed, we clearly have $\mathfrak{X} = I \cap (P \times X)$, and so we get

$$\mathfrak{X} = \mathbb{P}_{sub}\big(i^*\Omega_Q\big) = \mathbb{P}\big(i^*T_Q\big). \tag{5.5}$$

For $\mathfrak{Z}$, we let $N_{X\subseteq Q}$ be the normal bundle of X in Q, and consider the exact sequence

$$0 \longrightarrow T_X \longrightarrow i^*T_Q \longrightarrow N_{X\subseteq Q} \longrightarrow 0. \tag{5.6}$$

The tangency condition in the definition of $\mathfrak{Z}$ means exactly that

$$\mathfrak{Z} = \mathbb{P}\big(N_{X\subseteq Q}\big).$$

In conjunction with $\mathfrak{X}$ and $\mathfrak{Z}$, the following four projection maps will be used.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & X \\ P & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathfrak{Z} \arrow[r, "\psi"] \arrow[d, "\delta"'] & X \\ P & \end{tikzcd}$$

As we said before, it is clear that $\mathfrak{X} = I \cap (P \times X)$; thus both $\mathfrak{X}$, and the smaller $\mathfrak{Z}$, are naturally subvarieties of the projectivized cotangent bundle $\mathbb{P}_{sub}\big(\Omega_P\big)$ of the projective space P.

5.2.3 Representing the cohomology of the complement

Since $\mathfrak{X}$ is a *smooth* hypersurface in the product $P \times X$, the result of Section 5.1 applies in this setting. Thus we get a nice description of the cohomology of its open complement by rational forms with poles along $\mathfrak{X}$.

Let us assume that $\mathscr{L}$ is sufficiently ample, so that we have

$$H^q\big(X, \Omega_X^p \otimes \mathscr{L}^s\big) = 0 \text{ for } q > 0 \text{ and } s > 0. \tag{5.7}$$

Since P is a projective space, we similarly know that

$$H^q\big(P, \Omega_P^p(s)\big) = 0 \text{ for } q > 0 \text{ and } s > 0$$

by Bott's Vanishing Theorem. Combining both facts, we find for $q > 0$ that

$$H^q\big(P \times X, \Omega_{P\times X}^p(s\mathfrak{X})\big) \simeq \bigoplus_{\substack{q=q_1+q_2\\ p=p_1+p_2}} H^{q_1}\big(P, \Omega_P^{p_1}(s)\big) \otimes H^{q_2}\big(X, \Omega_X^{p_2} \otimes \mathscr{L}^s\big) = 0,$$

and so Proposition 5.1.2 can be used. The cohomology groups of the complement $P \times X \setminus \mathfrak{X}$ can therefore be computed by rational forms with poles along $\mathfrak{X}$, and the Hodge filtration is determined by the order of pole. This is summarized in the following proposition.

Proposition 5.2.1. *Let X be a smooth complex projective variety of dimension n, and let $\mathscr{L}$ be a sufficiently ample line bundle on X. Let $\mathfrak{X} \subseteq P \times X$ be the universal hypersurface associated with $\mathscr{L}$. Then for all $k \geq 0$,*

$$H^k(X, \mathbb{C}) \simeq H^k(P \times X \setminus \mathfrak{X}, \mathbb{C}).$$

Moreover, the cohomology in degree k of the complex

$$\Gamma\big(P \times X, \Omega^p_{P\times X}(\mathfrak{X})\big) \longrightarrow \cdots \rightarrow \Gamma\big(P \times X, \Omega^{n+d}_{P\times X}\big((n+d-p+1)\mathfrak{X}\big)\big)$$

is isomorphic to $F^pH^k(X,\mathbb{C}) \simeq F^pH^k(P \times X \setminus \mathfrak{X}, \mathbb{C})$.

Proof. For the first assertion, note that $P \times X \setminus \mathfrak{X} \to X$ is a bundle of affine spaces; its cohomology is therefore isomorphic to that of X. The second assertion follows from Proposition 5.1.2, by virtue of the preceding discussion. □

5.2.4 Filtered $\mathcal{D}$-modules and their characteristic varieties

Lastly, we need to review a few facts about $\mathcal{D}$-modules and their characteristic varieties; a good reference is Borel (1987).

Left $\mathcal{D}$-modules and filtrations Let P be a complex manifold of dimension d; the example we have in mind is when P is projective space. Let $\mathcal{D}_P$ be the sheaf of differential operators on P, and $F^{ord}_s\mathcal{D}_P$ the subsheaf of those operators whose order is at most s (thus $F^{ord}_0 = \mathscr{O}_P$). Then $\mathcal{D}_P$ is a filtered, non-commutative $\mathscr{O}_P$-algebra, with

$$\bigoplus_{s=0}^{\infty} F^{ord}_s\mathcal{D}_P \big/ F^{ord}_{s-1}\mathcal{D}_P \simeq \bigoplus_{s=0}^{\infty} \operatorname{Sym}^s T_P = \operatorname{Sym} T_P.$$

A (left) $\mathcal{D}$-module is a quasi-coherent sheaf $\mathcal{M}$ on P, with a left action by $\mathcal{D}_P$. A *filtered* $\mathcal{D}$-module is a pair $(\mathcal{M}, F)$, consisting of a $\mathcal{D}$-module $\mathcal{M}$, together with a filtration F by $\mathscr{O}_P$-coherent subsheaves $F_k\mathcal{M}$. The filtration is required to be bounded from below, and to satisfy the condition

$$F^{ord}_s\mathcal{D}_P \cdot F_k\mathcal{M} \subseteq F_{s+k}\mathcal{M}$$

for all k and s.

Following standard practice, a shift in the (decreasing) filtration F will be indicated by writing

$$(\mathcal{M}, F)(k) = \big(\mathcal{M}, F[k]\big) = \big(\mathcal{M}, F_{\bullet - k}\big);$$

sometimes, this may be shortened to just $\mathcal{M}(k)$. When $(\mathcal{M}, F)$ underlies a mixed Hodge module, the indicated shift corresponds to a Tate twist by k steps.

Good filtrations and coherence A filtered $\mathcal{D}$-module $(\mathcal{M}, F)$ is coherent as a $\mathcal{D}_P$-module, if the filtration is *good*; this means that

$$F_s^{ord}\mathcal{D}_P \cdot F_k\mathcal{M} = F_{s+k}\mathcal{M}$$

for all sufficiently large values of k. An equivalent condition is that the graded sheaf

$$Gr^F\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

be finitely generated over $\operatorname{Sym} T_P$ (Borel et al., 1987, Theorem 4.4 on p. 121). Note that $\mathcal{M}$ is typically not coherent as an $\mathscr{O}_P$-module; in fact, $\mathcal{M}$ is $\mathscr{O}_P$-coherent if, and only if, it is locally free of finite rank. (Borel et al., 1987, Proposition 1.7 on p. 211).

Characteristic varieties The symmetric algebra $\operatorname{Sym} T_P$ is naturally the sheaf of functions on the cotangent bundle of P. Thus, given any filtered coherent $\mathcal{D}$-module $(\mathcal{M}, F)$ on P, we can consider $Gr^F\mathcal{M}$ as a coherent sheaf on Ω_P. The characteristic variety $SS(\mathcal{M}, F)$ is defined to be the support of this sheaf (Borel et al., 1987, p. 212–3). Again, $\mathcal{M}$ is a locally free sheaf of finite rank if, and only if, $SS(\mathcal{M}, F)$ is contained in the zero section of Ω_P. (This is because this latter condition is equivalent to $\mathcal{M}$ being $\mathscr{O}_P$-coherent.)

Because of the grading, $SS(\mathcal{M}, F)$ is a cone in Ω_P. It is known that each component of this cone has dimension at least d; this fact is called Bernstein's Inequality (Borel et al., 1987, Theorem 1.10 on p. 213).

Holonomic $\mathcal{D}$-modules If the characteristic variety is of pure dimension d, then the $\mathcal{D}$-module $\mathcal{M}$ is said to be *holonomic* (Borel et al., 1987, p. 213). For every holonomic module $\mathcal{M}$, there is a nonempty Zariski-open subset where $\mathcal{M}$ is a vector bundle, i.e., locally free of finite rank. (In fact, let Z be the image in P of those components of $SS(\mathcal{M}, F)$ that are not contained in the zero section of Ω_P. Then Z is a union of proper closed subvarieties of P, and one can take the open set to be $P \setminus Z$.)

The multiplicity of the sheaf $Gr^F\mathcal{M}$ at each component of $SS(\mathcal{M}, F)$ is an invariant of $\mathcal{M}$ alone (i.e., independent of the choice of good filtration). Since it is additive on short exact sequences, the category of all holonomic $\mathcal{D}$-modules is Artinian. Therefore each holonomic $\mathcal{D}$-module has a finite filtration with *simple* holonomic subquotients.

The Artinian property leads to the concept of a *minimal extension.* Suppose $U \subseteq P$ is on open subset of a complex manifold, and $\mathcal{M}$ is a holonomic $\mathcal{D}_U$-module. If there is any extension of $\mathcal{M}$ to a holonomic $\mathcal{D}$-module on P (and such an extension may not exist), then there is a unique minimal one. It is denoted by $j_{!*}\mathcal{M}$, for $j\colon U \to P$ the inclusion map. It can be characterized as the unique extension of $\mathcal{M}$ that has no proper submodules or quotient modules with support in $P \setminus U$.

Characteristic sheaves and modules We can also consider the projectivization of $SS(\mathcal{M}, F)$ inside of $\mathbb{P}_{sub}(\Omega_P)$; the resulting subscheme is now the support of the coherent sheaf $\mathscr{C}(\mathcal{M}, F)$ associated to the graded $H^0(P, \operatorname{Sym} T_P)$-module

$$C(\mathcal{M}, F) = \bigoplus_{k \in \mathbb{Z}} H^0\big(P, F_k\mathcal{M}/F_{k-1}\mathcal{M}\big) = H^0\big(P, Gr^F\mathcal{M}\big). \tag{5.8}$$

We shall call this module the *characteristic module* for $(\mathcal{M}, F)$; in a similar manner, $\mathscr{C}(\mathcal{M}, F)$ will be called the *characteristic sheaf* of the filtered $\mathcal{D}$-module. Note that it is fully determined by the homogeneous components of the characteristic module in degrees $k \gg 0$.

Note. By passing to the projectivization, we lose information about the zero section of Ω_P, which is typically contained in the characteristic variety. On the other hand, $C(\mathcal{M}, F)$ is sufficient to compute the multiplicity of the sheaf $SS(\mathcal{M}, F)$ at any other component, and to test for holonomicity. Indeed, $(\mathcal{M}, F)$ is holonomic precisely if each component of the support of $\mathscr{C}(\mathcal{M}, F)$ has dimension equal to $d-1 = \dim P - 1$, and then the original characteristic variety is the cone over that support, together with possibly the zero section.

5.3 Two $\mathcal{D}$-modules and their characteristic varieties

Studying residues of rational forms naturally leads to filtered $\mathcal{D}$-modules. In Section 5.1, we have already seen an example of this, in the case of a single smooth divisor $D \subseteq X$. To describe the cohomology of $X \setminus D$, we used rational n-forms with poles along D; these are sections of the sheaf

$$\Omega_X^n(*D) = \Omega_X^n \otimes \mathscr{O}_X(*D).$$

But $\mathscr{O}_X(*D)$ is naturally a filtered $\mathcal{D}_X$-module under differentiation. The reason is that applying a differential operator of order s to a form with a pole of order k produces a pole of order at most $s+k$.

To generalize that description to all hypersurfaces in the linear system of $\mathscr{L}$, we now introduce a filtered $\mathcal{D}$-module $(\mathcal{M}, F)$. Sections of $F_k\mathcal{M}$ are locally given by residues of rational n-forms on $P \times X$ with a pole of order at most k along $\mathfrak{X}$.

We also define an auxiliary $\mathcal{D}$-module $\mathcal{N}$, whose sections are rational n-forms on $P \times X$ with poles along $\mathfrak{X}$. We then determine the characteristic varieties of both $\mathcal{D}$-modules, through a direct computation, and show that $\mathcal{M}$ is the *minimal extension* of the $\mathcal{D}$-module $R^{n-1}\pi_*^{sm}\mathbb{C}_{van} \otimes_{\mathbb{C}} \mathscr{O}_{P^{sm}}$ from P^{sm} to all of P.

This section is not entirely self-contained—the computation of the characteristic variety of $(\mathcal{M}, F)$ relies on a vanishing theorem for the sheaves $F_k\mathcal{M}$, derived using mixed Hodge modules (see Theorem 6.1.2). Regrettably, this prevents the calculation from being fully elementary.

5.3.1 An auxiliary filtered $\mathcal{D}$-module

Let $\mathcal{N} = pr_{P*}\Omega^n_{P\times X/P}(*\mathfrak{X})$; this is a quasi-coherent sheaf on P, whose sections over an open set U are relative rational n-forms on $U \times X$ with poles along $\mathfrak{X}$. A natural filtration is given by looking at the order of the pole; thus we introduce the coherent subsheaves

$$F_k\mathcal{N} = pr_{P*}\Omega^n_{P\times X/P}(k\mathfrak{X}) \simeq H^0\big(X, \Omega^n_X \otimes \mathscr{L}^k\big) \otimes \mathscr{O}_P(k),$$

whose union is all of $\mathcal{N}$. To avoid dealing with special cases later on, we shall let $F_k\mathcal{N} = 0$ for all $k \leq 0$.

Now $\mathcal{N}$ is also a left $\mathcal{D}$-module, since the sheaf of differential operators $\mathcal{D}_P$ naturally acts on $\mathcal{N}$. This action is by differentiating the coefficients of the relative forms, themselves functions on P; to define this more precisely, let $\omega \in \Gamma\bigl(U, \mathcal{N}\bigr)$ be a local section of $\mathcal{N}$, i.e., a relative rational n-form on $U \times X$, and ξ a local section of the tangent bundle T_P. Then ξ can be lifted to a vector field ξ' on the product $U \times X$, in the obvious way. Viewing ω as a form on $U \times X$ that has all its differentials in the X-direction, we can let

$$\xi \cdot \omega = \mathcal{L}_{\xi'}(\omega) = \xi' \lrcorner\, d\omega,$$

which is again a rational n-form. Using the properties of the Lie derivative $\mathcal{L}$, it is not hard to show the following:

- For a holomorphic function f on U, we have $(f\xi) \cdot \omega = f(\xi \cdot \omega)$.
- Moreover, $\xi \cdot (f\omega) = (\xi f) \cdot \omega + f(\xi \cdot \omega)$.
- For a second vector field η, we have $[\xi, \eta] \cdot \omega = \xi \cdot (\eta \cdot \omega) - \eta \cdot (\xi \cdot \omega)$.

The action by vector fields thus extends to an action by $\mathcal{D}_P$, and $\mathcal{N}$ has the structure of a left $\mathcal{D}$-module.

If ω is a local section of $F_k\mathcal{N}$, meaning if the rational form has a pole of order at most k, then $\xi \cdot \omega$ is clearly a section of $F_{k+1}\mathcal{N}$, because differentiation can increase the order of the pole at most by one. Thus we have in general $F_s^{ord}\mathcal{D}_P \cdot F_k\mathcal{N} \subseteq F_{s+k}\mathcal{N}$, where $F^{ord}\mathcal{D}_P$ is the order filtration on $\mathcal{D}$.

Lemma 5.3.1. $\bigl(\mathcal{N}, F\bigr)$ *is a filtered coherent* $\mathcal{D}$*-module; in other words, the filtration* F *is good.*

Proof. To prove that the filtration F is good, we need to show that $F_1^{ord}\mathcal{D}_P \cdot F_k\mathcal{N} = F_{k+1}\mathcal{N}$ for all sufficiently large k. This is equivalent to the surjectivity of the map

$$T_P \otimes F_k\mathcal{N} \to \frac{F_{k+1}\mathcal{N}}{F_k\mathcal{N}}, \tag{5.9}$$

a question which is local on P. Consider then an arbitrary point in P, and let $s_0, s_1, \ldots, s_d$ be a basis for the vector space $H^0(X, \mathscr{L})$, with $[s_0]$ equal to the chosen point. Evidently, the mapping $\bigl(t_1, t_2, \ldots, t_d\bigr) \mapsto [s_0 + t_1 s_1 + \cdots + t_d s_d]$ gives local coordinates on P. Any local section ω of $F_k\mathcal{N}$ can now be written in the form

$$\omega = \frac{\omega(t)}{\bigl(s_0 + \sum t_i s_i\bigr)^k},$$

for some holomorphic map $\omega(t)$ from $\mathbb{C}^d$ into $H^0\bigl(\Omega_X^n \otimes \mathscr{L}^k\bigr)$. Setting $\partial_i = \partial/\partial t_i$, we then have

$$\partial_i \cdot \omega\big|_{t=0} = -k \cdot \frac{s_i \omega(0)}{s_0^{k+1}} + \frac{\partial_i \omega(t)\big|_{t=0}}{s_0^k} \equiv -k \cdot \frac{s_i \omega(0)}{s_0^{k+1}} \mod F_k\mathcal{N}.$$

Thus the surjectivity of (5.9) on stalks is equivalent to the surjectivity of the product map

$$H^0(X, \mathscr{L}) \otimes H^0\bigl(\Omega_X^n \otimes \mathscr{L}^k\bigr) \to H^0\bigl(\Omega_X^n \otimes \mathscr{L}^{k+1}\bigr).$$

But since $\mathscr{L}$ is very ample, this holds for $k \gg 0$, and so the lemma is proved. $\square$

5.3.2 A filtered $\mathcal{D}$-module, defined by residues

From the auxiliary object $(\mathcal{N}, F)$, we now define a second $\mathcal{D}$-module $(\mathcal{M}, F)$ by taking residues. This operation really makes sense only over P^{sm}, and so we proceed as follows. On P^{sm}, we have the vector bundle

$$\mathcal{V}_{van}^{n-1} = R^{n-1}\pi_*^{sm}\mathbb{Q}_{van} \otimes_{\mathbb{Q}} \mathscr{O}_{P^{sm}},$$

obtained from the local system of vanishing cohomology. It is naturally a $\mathcal{D}$-module via the Gauss-Manin connection; moreover, it carries the Hodge filtration, re-indexed as $F_p\mathcal{V}_{van}^{n-1} = F^{-p}\mathcal{V}_{van}^{n-1}$ to make it increasing. Griffiths' transversality condition, that

$$\nabla\big(F_p\mathcal{V}_{van}^{n-1}\big) \subseteq \Omega_P^1 \otimes F_{p+1}\mathcal{V}_{van}^{n-1},$$

then says exactly that $\big(\mathcal{V}_{van}^{n-1}, F\big)$ is a filtered $\mathcal{D}$-module on P^{sm}.

Now let $j\colon P^{sm} \to P$ be the inclusion, and form the quasi-coherent sheaf $j_*\mathcal{V}_{van}^{n-1}$. By taking fiber-wise residue, we then have a map

$$\operatorname{Res}_{\mathfrak{X}/P}\colon \mathcal{N} \to j_*\mathcal{V}_{van}^{n-1};$$

explicitly, it is given by

$$\Gamma(U, \mathcal{N}) \to \Gamma\big(U \cap P^{sm}, \mathcal{V}_{van}^{n-1}\big), \qquad \omega \mapsto \operatorname{Res}_{\mathfrak{X}/P}(\omega).$$

We define $\mathcal{M}$ to be the image sheaf; if we let $F_k\mathcal{M}$ be the image of $F_k\mathcal{N}$, we also get an increasing and exhaustive filtration F. Thus $(\mathcal{M}, F)$ is a filtered $\mathcal{D}$-module.

Over P^{sm}, taking residue commutes with the action by vector fields (Voisin, 2002, p. 425–6), and we can therefore give $\mathcal{M}$ the induced $\mathcal{D}$-module structure. Then F is a good filtration on $\mathcal{M}$, and $\mathrm{Res}_{\mathfrak{X}/P} \colon \mathcal{N} \to \mathcal{M}$ is a map of filtered $\mathcal{D}$-modules. From Proposition 5.1.3, we know that all the vanishing cohomology of a sufficiently ample smooth hypersurface is given by residues of rational forms; therefore,

$$j^* \mathcal{M} = \mathcal{V}_{van}^{n-1},$$

and so $\mathcal{M}$ is an extension of $\mathcal{V}_{van}^{n-1}$ to a $\mathcal{D}$-module on P. We also have

$$j^* F_k \mathcal{M} = F_{k-n} \mathcal{V}_{van}^{n-1} = F^{n-k} \mathcal{V}_{van}^{n-1},$$

because the Hodge filtration is determined by the order of the pole (according to Proposition 5.1.3). We shall soon show that $\mathcal{M}$ is, in a sense, the smallest possible extension of $\mathcal{V}_{van}^{n-1}$.

5.3.3 The characteristic variety of $\mathcal{N}$

We now examine the characteristic varieties of the two $\mathcal{D}$-modules $\mathcal{N}$ and $\mathcal{M}$. Actually, only the second one is really of interest, but we shall do the first one as a sort of warm-up exercise. We begin by computing the characteristic module $C(\mathcal{N}, F)$ (see (5.8) for the definition), since it essentially determines the characteristic variety. To compute each summand $H^0\big(P, F_k\mathcal{N}/F_{k-1}\mathcal{N}\big)$, recall that

$$F_k \mathcal{N} = pr_{P*} \Omega^n_{P \times X/P}\big(k\mathfrak{X}\big) = H^0\big(X, \Omega^n_X \otimes \mathscr{L}^k\big) \otimes \mathscr{O}_P(k)$$

for $k \geq 1$ (and 0 otherwise). Thus $H^1(P, F_{k-1}\mathcal{N}) = 0$, and so

$$H^0\big(P, F_k\mathcal{N}/F_{k-1}\mathcal{N}\big) = \frac{H^0\big(P, F_k\mathcal{N}\big)}{H^0\big(P, F_{k-1}\mathcal{N}\big)} = \frac{H^0\big(P \times X, \Omega^n_{P\times X/P}(k\mathfrak{X})\big)}{H^0\big(P \times X, \Omega^n_{P\times X/P}((k-1)\mathfrak{X})\big)}.$$

Writing $\mathscr{O}_{\mathfrak{X}}(1)$ for the restriction of $\mathscr{O}_{P\times X}(\mathfrak{X})$ to $\mathfrak{X}$, we have the exact sequence

$$0 \longrightarrow \Omega^n_{P\times X/P}\big((k-1)\mathfrak{X}\big) \longrightarrow \Omega^n_{P\times X/P}\big(k\mathfrak{X}\big) \longrightarrow i^*\Omega^n_{P\times X/P}\otimes \mathscr{O}_{\mathfrak{X}}(k) \longrightarrow 0,$$

where $i\colon \mathfrak{X}\to P\times X$ is the inclusion. Let $\phi = i\circ pr_X$; then the third term in the sequence can be rewritten as $i^*\Omega^n_{P\times X/P}\otimes \mathscr{O}_{\mathfrak{X}}(k) = \phi^*\Omega^n_X\otimes \mathscr{O}_{\mathfrak{X}}(k)$. Since the higher cohomology of the first term is clearly zero, we thus get

$$H^0\big(P, F_k\mathcal{N}/F_{k-1}\mathcal{N}\big) = H^0\big(\mathfrak{X}, \phi^*\Omega^n_X\otimes \mathscr{O}_{\mathfrak{X}}(k)\big),$$

and hence

$$C(\mathcal{N},F) = \bigoplus_{k\geq 1} H^0\big(\mathfrak{X}, \phi^*\Omega^n_X\otimes \mathscr{O}_{\mathfrak{X}}(k)\big).$$

This means that the characteristic sheaf $\mathscr{C}(\mathcal{N},F)$ is precisely $\phi^*\Omega^n_X$, viewed as a sheaf on $I = \mathbb{P}_{sub}\big(\Omega_P\big)$ under the inclusion $\mathfrak{X}\subseteq I$.

We conclude that the characteristic variety $SS(\mathcal{N},F)$ is the cone over $\mathfrak{X}$; this contains the zero section, and so the characteristic variety has a single component of dimension $\dim\mathfrak{X}+1 = d+n$, and of multiplicity one (because the sheaf $f^*\Omega^n_X$ has rank one at a general point of $\mathfrak{X}$).

5.3.4 The characteristic variety of $\mathcal{M}$

Next, we compute the characteristic variety of the main $\mathcal{D}$-module $\mathcal{M}$, again by identifying the sheaf $\mathscr{C}(\mathcal{M},F)$. Since $\mathcal{M}$ is locally free on the open set P^{sm}, the zero section of Ω_P is going to be one component of $SS(\mathcal{M},F)$; but we expect at least one other component over the discriminant locus $X^\vee$, since $\mathcal{M}$ is typically not locally free on all of P.[1]

To state the precise result, we need to make use of some results from a later section. The reader may find the definition of the filtered $\mathcal{D}$-module $(\mathcal{N}^0, F)$ in 5.5.1; it is a quotient of our first $\mathcal{D}$-module $(\mathcal{N}, F)$, and is related to $(\mathcal{M}, F)$ through the exact sequence in Theorem 5.5.2. Note that the filtrations on $\mathcal{N}$ and $\mathcal{N}^0$ are offset by $n+1$ steps; this is because we chose to index the former by the order of pole, but the latter following the conventions for mixed Hodge modules. We account for this difference by using the shift $F[n+1] = F_{\bullet-(n+1)}$ in the statement of the lemma.

Lemma 5.3.2. *We have $\mathscr{C}\big(\mathcal{N}^0, F[n+1]\big) \simeq \Omega^n_{\mathfrak{X}/P} = \bigwedge^n \Omega^1_{\mathfrak{X}/P}$, viewed as a sheaf on I via the inclusion $\mathfrak{X} \subseteq I$.*

Proof. Just as before, we shall compute the characteristic module $C\big(\mathcal{N}^0, F\big)$ directly, at least in sufficiently high degrees $k \gg 0$. To simplify the notation, we introduce the following vector spaces:

$$W_k^p = H^0\big(P \times X, pr_X^* \Omega_X^p(k\mathfrak{X})\big) = H^0\big(X, \Omega_X^p \otimes \mathscr{L}^k\big) \otimes \operatorname{Sym}^k V^\vee \tag{5.10}$$

Recall also that $V = H^0(X, \mathscr{L})$; thus $H^0\big(P, \mathscr{O}_P(k)\big) = \operatorname{Sym}^k V^\vee$.

Let us first find the graded module corresponding to $\Omega^n_{\mathfrak{X}/P}$. Letting $\mathscr{O}_{\mathfrak{X}}(1)$ denote the restriction to $\mathfrak{X}$ of the line bundle $\mathscr{O}_{P\times X}(1) = pr_P^*\mathscr{O}_P(1) \otimes pr_X^*\mathscr{L}$, the graded module in question is

$$\bigoplus_{k=0}^{\infty} H^0\big(\mathfrak{X}, \Omega^n_{\mathfrak{X}/P} \otimes \mathscr{O}_{\mathfrak{X}}(k)\big).$$

[1] Indeed, as we shall see later, it is the irreducibility of the monodromy action on the vanishing cohomology that prevents this from happening.

Now, $\mathfrak{X}$ is a smooth subvariety of $P \times X$, and so the following sequence is exact:

$$0 \longrightarrow \mathscr{O}_{\mathfrak{X}}(-1) \longrightarrow i^* \Omega^1_{P \times X/P} \longrightarrow \Omega^1_{\mathfrak{X}/P} \longrightarrow 0$$

Of course, $\Omega^1_{P \times X/P} = pr_X^* \Omega^1_X$. Even though $\Omega^1_{\mathfrak{X}/P}$ is not locally free, we still get at least a presentation for $\Omega^n_{\mathfrak{X}/P}$, since

$$\phi^* \Omega^{n-1}_X \otimes \mathscr{O}_{\mathfrak{X}}(-1) \xrightarrow{d_X} \phi^* \Omega^n_X \twoheadrightarrow \Omega^n_{\mathfrak{X}/P} \tag{5.11}$$

remains exact. Tensoring with $\mathscr{O}_{\mathfrak{X}}(k)$ and taking global sections preserves the exactness, provided that $k \gg 0$, and so we obtain $H^0\big(\mathfrak{X}, \Omega^n_{\mathfrak{X}/P} \otimes \mathscr{O}_{\mathfrak{X}}(k)\big)$ as the cokernel of the map

$$H^0\big(\mathfrak{X}, \phi^* \Omega^{n-1}_X \otimes \mathscr{O}_{\mathfrak{X}}(k-1)\big) \to H^0\big(\mathfrak{X}, \phi^* \Omega^n_X \otimes \mathscr{O}_{\mathfrak{X}}(k)\big).$$

Using that $k \gg 0$, one easily shows that

$$H^0\big(\mathfrak{X}, \phi^* \Omega^n_X \otimes \mathscr{O}_{\mathfrak{X}}(k)\big) = \frac{H^0\big(P \times X, pr_X^* \Omega^n_X(k\mathfrak{X})\big)}{H^0\big(P \times X, pr_X^* \Omega^n_X((k-1)\mathfrak{X})\big)} = \frac{W^n_k}{W^n_{k-1}};$$

a similar result holds for the term involving $\phi^* \Omega^{n-1}_X$. We conclude that

$$H^0\big(\mathfrak{X}, \Omega^n_{\mathfrak{X}/P} \otimes \mathscr{O}_{\mathfrak{X}}(k)\big) \simeq \frac{W^n_k}{dW^{n-1}_{k-1} + W^n_{k-1}} \tag{5.12}$$

for all sufficiently large values of k.

Next, let us compute the characteristic module itself. Observe first that

$$H^0\big(P, F_k \mathscr{N}^0 / F_{k-1} \mathscr{N}^0\big) = \frac{H^0\big(P, F_k \mathscr{N}^0\big)}{H^0\big(P, F_{k-1} \mathscr{N}^0\big)},$$

because $H^1\big(P, F_k \mathscr{N}^0\big) = 0$ by Theorem 6.1.2. Referring to Lemma 6.1.3, we also have

$$H^0\big(P, F_k \mathscr{N}^0\big) \simeq W^n_{k+n+1} / dW^{n-1}_{k+n},$$

and going back to the quotient, we then find

$$H^0\big(P, F_k\mathcal{N}^0 / F_{k-1}\mathcal{N}^0\big) \simeq \frac{W^n_{k+n+1}}{dW^{n-1}_{k+n} + W^n_{k+n}}. \tag{5.13}$$

Upon comparing (5.13) and (5.12), it is clear that $C(\mathcal{M}, F)$ agrees with the graded module for $\Omega^n_{\mathfrak{X}/P}$ in large degrees, up to a shift in degree by $n+1$. This proves that the corresponding sheaves are isomorphic, and thus finishes the proof. □

Provided that $\mathscr{L}$ is sufficiently ample, we can then deduce a similar result for the characteristic sheaf of $(\mathcal{M}, F)$, the real object of interest.

Lemma 5.3.3. *Let $\mathscr{L}$ be sufficiently ample. Then we also have $\mathscr{C}(\mathcal{M}, F) \simeq \Omega^n_{\mathfrak{X}/P}$.*

Proof. This follows almost immediately from the calculations previous lemma. Indeed, consider the short exact sequence in Theorem 5.5.2. It shows that $\mathcal{M}$ and $\mathcal{N}^0$ differ only by the locally free sheaf $H^n(X, \mathbb{C})_{prim} \otimes \mathscr{O}_P$, and so we have the isomorphism

$$Gr^F_k \mathcal{M} \simeq Gr^F_{k-n-1} \mathcal{N}^0$$

once k is greater than zero. Thus we get

$$\mathscr{C}(\mathcal{M}, F) = \mathscr{C}\big(\mathcal{N}^0, F[n+1]\big) \simeq \Omega^n_{\mathfrak{X}/P},$$

as claimed. □

5.3.5 Holonomicity of $\mathcal{M}$, and the minimal extension property

Lemma 5.3.3 has two interesting consequences. On the one hand, the support of $\Omega^n_{\mathfrak{X}/P}$ is precisely the set $\mathfrak{Z}$ of singular points in the fibers of $\mathfrak{X} \to P$. As such, it is a smooth subvariety of $\mathfrak{X}$, of dimension $d-1$. The $\mathcal{D}$-module $\mathcal{M}$ is therefore holonomic, and its characteristic variety $SS(\mathcal{M}, F)$ has exactly two components:

1. The zero section of Ω_P.[2]

2. The cone over the set of singular points $\mathfrak{Z}$.

On the other hand, we can compute the multiplicities of both components. For the first one, it is the rank of $\mathcal{M}$ at a general point in P^{sm}, and thus equal to the dimension of the vanishing cohomology of the hypersurfaces. The second component, however, has multiplicity one, because $\Omega^n_{\mathfrak{X}/P}$ is of rank one at a general point of its support. Indeed, the general singular fiber $\mathfrak{X}_p$ has only one ordinary double point, with equation $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$ in suitable local coordinates. Using (5.11), we then have

$$\Omega^n_{\mathfrak{X}/P} \otimes \mathbb{C}(X) \simeq \Omega^n_{\mathfrak{X}_p} \simeq \frac{\mathbb{C}\{z_1, \dots, z_n\}}{(z_1^2 + \cdots + z_n^2, 2z_1, \dots, 2z_n)} \simeq \mathbb{C},$$

which is one-dimensional.

But the multiplicity along a component is an additive function on short exact sequences of holonomic $\mathcal{D}$-modules. Since there is only one component in $SS(\mathcal{M}, F)$ mapping to $P \setminus P^{sm}$, and since $\mathcal{M}$ has multiplicity one along that component, no nonzero holonomic $\mathcal{D}$-module with support in $P \setminus P^{sm}$ can be a submodule or quotient module of $\mathcal{M}$. Thus we have proved the following result.

---

[2]Unless the hypersurfaces have no vanishing cohomology, the zero section has to be part of the characteristic variety, since $\mathcal{M}$ is supported on all of P.

Proposition 5.3.4. *Provided that $\mathscr{L}$ is sufficiently ample, we have $\mathcal{M} = j_{!*}\mathcal{V}^{n-1}_{van}$. In other words, the $\mathcal{D}$-module $\mathcal{M}$ is the minimal extension of the flat vector bundle $\mathcal{V}^{n-1}_{van}$ to all of P.*

5.4 Applications of the theory of mixed Hodge modules

In this section, we use M. Saito's theory of mixed Hodge modules to prove several results about the $\mathcal{D}$-module $(\mathcal{M}, F)$. The main point is that, up to a shift in the filtration, it underlies a mixed Hodge module.

More precisely, we obtain another $\mathcal{D}$-module $\mathcal{M}_{van}$, as the filtered $\mathcal{D}$-module associated to a certain mixed Hodge module on the projective space P. Using the relative de Rham complex, we then show that $(\mathcal{M}_{van}, F) = (\mathcal{M}, F_{\bullet - n})$.

Note. The method we use is very similar[3] to the one introduced by Brosnan, Fang, Nie, and Pearlstein (2007). Since the emphasis in this dissertation is on $\mathcal{D}$-modules, not on perverse sheaves, alternative proofs for some of their results are included here.

5.4.1 Cohomology of the universal hypersurface

We borrow from Brosnan et al. (2007) the idea of studying the universal hypersurface $\mathfrak{X}$ by means of mixed Hodge modules. Here again, the crucial point is that $\mathfrak{X}$ is nonsingular; this makes it possible to use the Decomposition Theorem (due to Bernstein, Deligne, and Gabber in the context of perverse sheaves, and to Saito in the context

[3] More precisely, "dual to."

of mixed Hodge modules). For notational economy, we let $d = \dim P$, $n = \dim X$, and $d_{\mathfrak{X}} = n + d - 1$.

We recall (Brosnan et al., 2007, p. 10) that there is a decomposition

$$\pi_* \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq \bigoplus_{i,j} E_{i,j}[-i] \tag{5.14}$$

in $D^b \operatorname{MHM}(P)$. Indeed, since $\mathfrak{X}$ is smooth and projective, the mixed Hodge module $\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$ on $\mathfrak{X}$ is pure of weight $d_{\mathfrak{X}}$, and because the map π is projective, the same is then true for the image of $\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$ under π_*. Applying Saito's version of the Decomposition Theorem, we now have

$$\pi_* \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq \bigoplus_{i} E_i[-i],$$

where $E_i = H^i \pi_* \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$. We may further decompose each E_i into simple pieces; let $E_{i,j}$ be the direct sum of all those pieces whose codimension of strict support is equal to j. We then arrive at the isomorphism in (5.14), where each $E_{i,j}$ is pure of weight $d_{\mathfrak{X}} + i$, and supported on a closed subscheme of P of codimension j. Moreover, we see that

$$H^i \pi_* \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq \bigoplus_{j=0}^{d} E_{i,j} \tag{5.15}$$

for all $i \in \mathbb{Z}$. This decomposition is most interesting when $i = 0$, because it is then related to the vanishing cohomology of the hypersurfaces.

One general result about the decomposition, called Saito's Hard Lefschetz Theorem (Saito, 1990, Théorème 1 on p. 853), is that

$$E_{i,j} \simeq E_{-i,j}(-i); \tag{5.16}$$

it comes from the fact that π is a projective morphism, and that both $\mathfrak{X}$ and P are nonsingular. Essentially all the Hodge modules $E_{i,j}$ are known (Brosnan et al., 2007); most of them vanish, provided the line bundle $\mathscr{L}$ is sufficiently ample.

For the convenience of the reader, we recall two of the results about the modules $E_{i,j}$ from Brosnan et al. (2007); we shall re-derive both statements later, using a slightly different method.

Theorem 5.4.1 (Brosnan, Fang, Nie, and Pearlstein)**.** *If $k < 0$, then we have*

$$H^k \pi_* \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq H^{k+n-1}(X) \otimes \mathbb{Q}^H_P[d].$$

Proof. According to the Perverse Weak Lefschetz Theorem (Brosnan et al., 2007, Theorem 5.2 on p. 12),

$$E_{i,j} = 0 \text{ unless either } i = 0 \text{ or } j = 0;$$

moreover, for $j = 0$, we have

$$E_{i,0} \simeq H^{n+i-1}(X) \otimes \mathbb{Q}^H_P[d] \text{ for } i < 0.$$

Together with the decomposition in (5.15), this immediately gives the result. □

Before stating their second result, we introduce some notation. Let $P^{sm} = P \setminus X^{\vee}$ be the open set over which π is smooth; we write $j \colon P^{sm} \to P$ for the inclusion, and $\pi^{sm} \colon \mathfrak{X}^{sm} \to P^{sm}$ for the restriction of π. Then $R^{n-1}\pi^{sm}_* \mathbb{Q}$ is a local system on P^{sm}, and we shall write

$$V^{n-1} = H^{n-1}\pi^{sm}_* \mathbb{Q}^H_{\mathfrak{X}^{sm}}[d]$$

for the mixed Hodge module on P^{sm} whose underlying perverse sheaf is $R^{n-1}\pi_*^{sm}\mathbb{Q}[d]$. We have the direct sum decomposition

$$V^{n-1} = V_{van}^{n-1} \oplus H^{n-1}(X) \otimes \mathbb{Q}_{P^{sm}}^H[d], \tag{5.17}$$

where V_{van}^{n-1} corresponds to $R^{n-1}\pi_*^{sm}\mathbb{Q}_{van}[d]$. Of course, V_{van}^{n-1} is just the polarized variation of Hodge structure given by the vanishing cohomology of the fibers, but viewed as an element of $\mathrm{MHM}\big(P^{sm}\big)$.

Theorem 5.4.2 (Brosnan, Fang, Nie, and Pearlstein)**.** *We have $E_{0,0} \simeq j_{!*}V^{n-1}$. If $\mathscr{L}$ is sufficiently ample, $E_{0,j} = 0$ for all $j > 0$, and then*

$$H^0\pi_*\mathbb{Q}_{\mathfrak{X}}^H[d_{\mathfrak{X}}] \simeq j_{!*}V^{n-1} \simeq j_{!*}V_{van}^{n-1} \oplus H^{n-1}(X) \otimes \mathbb{Q}_P^H[d].$$

Proof. The first assertion follows from Brosnan et al. (2007, Proposition 4.8 on p. 11). If $\mathscr{L}$ is sufficiently ample, then the authors remark on p. 14 that $E_{0,j} = 0$ if $j > 0$. The remaining assertion is then an immediate consequence of the decomposition in (5.15), together with the definition of V_{van}^{n-1}. $\square$

5.4.2 An exact sequence of mixed Hodge modules

We now derive an exact sequence of mixed Hodge modules; later on, we show that $\mathcal{M}$ is the $\mathcal{D}$-module underlying one of the terms in the sequence. Some basic information about mixed Hodge modules, including the standard notational conventions, may be found in 5.1.3.

Let $U = P \times X \setminus \mathfrak{X}$ be the complement of the universal hypersurface. We denote by

g the inclusion of U into $P \times X$, and by i that of $\mathfrak{X}$ into $P \times X$, as shown in the following diagram.

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "i"] \arrow[dr, "\pi"'] & P \times X \arrow[d] & U \arrow[l, "g"'] \arrow[dl, "q"] \\
& P &
\end{tikzcd}$$

Our starting point is the distinguished triangle

$$i_* i^! \mathbb{Q}^H_{P\times X} \longrightarrow \mathbb{Q}^H_{P\times X} \longrightarrow g_* g^* \mathbb{Q}^H_{P\times X} \longrightarrow i_* i^! \mathbb{Q}^H_{P\times X}[1] \tag{5.18}$$

in the category $D^b \operatorname{MHM}(P\times X)$ (Saito, 1990, (4.4.1) on p. 321). Obviously, $g^*\mathbb{Q}^H_{P\times X} = \mathbb{Q}^H_U$; moreover, as $\mathfrak{X}$ is a smooth hypersurface in $P \times X$, the following is true.

Lemma 5.4.3. *Verdier duality gives an isomorphism* $i^!\mathbb{Q}^H_{P\times X} \simeq \mathbb{Q}^H_{\mathfrak{X}}(-1)[-2]$.

Proof. The functors $i^!$ and i^* are interchanged under Verdier duality, and so

$$\begin{aligned}
\mathbb{D}_{\mathfrak{X}} i^! \mathbb{Q}^H_{P\times X} &= i^* \mathbb{D}_{P\times X} \mathbb{Q}^H_{P\times X} \\
&= i^* \mathbb{Q}^H_{P\times X}(d_{P\times X})[2d_{P\times X}] = \mathbb{Q}^H_{\mathfrak{X}}(d_{P\times X})[2d_{P\times X}].
\end{aligned}$$

But we also have $\mathbb{D}_{\mathfrak{X}}\mathbb{Q}^H_{\mathfrak{X}} = \mathbb{Q}^H_{\mathfrak{X}}(d_{\mathfrak{X}})[2d_{\mathfrak{X}}]$, since $\mathfrak{X}$ is itself smooth of dimension $d_{\mathfrak{X}} = n + d - 1$. Applying the duality operator a second time, we then get

$$i^! \mathbb{Q}^H_{P\times X} = \mathbb{D}_{\mathfrak{X}}\Big(\mathbb{Q}^H_{\mathfrak{X}}(d_{P\times X})[2d_{P\times X}]\Big) = \mathbb{Q}^H_{\mathfrak{X}}(-1)[-2],$$

as asserted. $\square$

We can therefore rotate the triangle in (5.18) one step to the left, and apply the shift $[d_{P\times X}]$, to rewrite it as

$$\mathbb{Q}^H_{P\times X}[d_{P\times X}] \longrightarrow g_* \mathbb{Q}^H_U[d_U] \longrightarrow i_* \mathbb{Q}^H_{\mathfrak{X}}(-1)[d_{\mathfrak{X}}] \longrightarrow \mathbb{Q}^H_{P\times X}[d_{P\times X} + 1] \tag{5.19}$$

We then apply the functor pr_{P*}, and take cohomology. This gives a long exact sequence in $\mathrm{MHM}(P)$; for our purposes, the interesting part of this sequence is

$$\cdots \longrightarrow H^k pr_{P*}\mathbb{Q}^H_{P\times X}[d_{P\times X}] \longrightarrow H^k q_* \mathbb{Q}^H_U[d_U] \longrightarrow H^k \pi_* \mathbb{Q}^H_{\mathfrak{X}}(-1)[d_{\mathfrak{X}}] \longrightarrow \cdots \quad (5.20)$$

for arbitrary $k \in \mathbb{Z}$.

5.4.3 Computing the terms of the exact sequence

Ultimately, we would like to find the $\mathcal{D}$-module underlying the mixed Hodge module $H^0\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$ in the exact sequence (5.20). In order to do this, we first need to compute the other terms.

Note. Most of the terms in question have been determined in Brosnan et al. (2007), see Theorem 5.4.1 and Theorem 5.4.2 above. However, we shall redo the calculation here, from a slightly different point of view.

The description of the first term is the content of the following lemma.

Lemma 5.4.4. *For each $k \in \mathbb{Z}$, we have $H^k pr_{P*}\mathbb{Q}^H_{P\times X}[d_{P\times X}] \simeq H^{k+n}(X) \otimes \mathbb{Q}^H_P[d]$.*

Proof. By Proper Base Change (Saito, 1990, (4.4.3) on p. 323) for the diagram

$$\begin{tikzcd} P \times X \arrow[r, "pr_X"] \arrow[d, "pr_P"] & X \arrow[d, "a_X"] \\ P \arrow[r, "a_P"] & pt, \end{tikzcd}$$

we have $pr_{P*}\mathbb{Q}^H_{P\times X} = pr_{P*}pr_X^*\mathbb{Q}^H_X = a_P^* a_{X*}\mathbb{Q}^H_X$. From the decomposition

$$a_{X*}\mathbb{Q}^H_X \simeq \bigoplus_i H^i a_{X*}\mathbb{Q}^H_X[-i] = \bigoplus_i H^i(X)[-i]$$

in MHM(pt) (Saito, 1988, Corollaire 3 on p. 857), it follows that

$$pr_{P*}\mathbb{Q}^H_{P\times X}[d_{P\times X}] \simeq a_P^* \bigoplus_i H^i(X)[d_{P\times X} - i] = \bigoplus_i H^i(X) \otimes \mathbb{Q}^H_P[d_{P\times X} - i].$$

Now apply H^k to get $H^k pr_{P*}\mathbb{Q}^H_{P\times X}[d_{P\times X}] \simeq H^{k+n}(X) \otimes \mathbb{Q}^H_P[d]$, since $\mathbb{Q}^H_P[d_{P\times X} - i]$ sits in degree $d - (d_{P\times X} - i) = i - n$. ☐

Another useful fact is that, just as in the case of a smooth affine variety, the cohomology of $q_*\mathbb{Q}^H_U[d_U]$ vanishes in positive degrees (i.e., above the middle dimension). This is the content of the following lemma.

Lemma 5.4.5. *We have $H^k q_*\mathbb{Q}^H_U[d_U] = 0$ for all $k > 0$.*

Proof. We give a proof using $\mathcal{D}$-modules. The mixed Hodge module $g_*\mathbb{Q}^H_U[d_U]$ has underlying $\mathcal{D}$-module $\mathscr{O}_{P\times X}(*\mathfrak{X})$, because $\mathfrak{X}$ is nonsingular (Saito, 1993, Corollary 4.3 on p. 71). The $\mathcal{D}$-module associated to $q_*\mathbb{Q}^H_U[d_U] = pr_{P*}g_*\mathbb{Q}^H_U[d_U]$ is therefore the direct image of $\mathscr{O}_{P\times X}(*\mathfrak{X})$. It can be computed by using the relative de Rham complex

$$\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big) = \Big[\mathscr{O}_{P\times X}(*\mathfrak{X}) \longrightarrow \Omega^1_{P\times X/P}(*\mathfrak{X}) \longrightarrow \cdots \longrightarrow \Omega^n_{P\times X/P}(*\mathfrak{X})\Big][n].$$

Noting that each sheaf in the complex is acyclic for the functor pr_{P*}, the direct image is represented by the complex $pr_{P*}\,\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big)$. Since this is clearly supported in degrees $-n, \dots, 0$, the cohomology sheaf in degree $k > 0$ vanishes. We conclude that $H^k q_*\mathbb{Q}^H_U[d_U]$ is also zero. ☐

Finally, we borrow the following lemma from Brosnan et al. (2007) (it is Proposition 4.8 on p. 11 in the paper).

Lemma 5.4.6 (Brosnan, Fang, Nie, and Pearlstein)**.** *We have*

$$E_{0,0} \simeq j_{!*}V^{n-1} \simeq j_{!*}V^{n-1}_{van} \oplus H^{n-1}(X) \otimes \mathbb{Q}^H_P[d].$$

Proof. By definition, $E_{0,0}$ is the piece in the decomposition of $H^0\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$ that has strict support equal to all of P. By the Base Change Theorem, applied to the inclusion $j\colon P^{sm} \to P$, we have

$$j^*H^0\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq H^0\pi^{sm}_*\mathbb{Q}^H_{\mathfrak{X}^{sm}}[d_{\mathfrak{X}}] = H^{n-1}\pi^{sm}_*\mathbb{Q}^H_{\mathfrak{X}^{sm}}[d] = V^{n-1}.$$

Therefore, $j_{!*}V^{n-1}$ is a submodule of $H^0\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$. Since all other terms $E_{0,j}$ are supported in proper subvarieties, we conclude that $j_{!*}V^{n-1} \simeq E_{0,0}$. The second isomorphism is then an immediate consequence of (5.17). $\square$

5.4.4 Breaking the exact sequence into pieces

We can now look at the exact sequence in (5.20) one more time. For $k > 0$, each portion of the sequence simplifies to

$$0 \longrightarrow H^k\pi_*\mathbb{Q}^H_{\mathfrak{X}}(-1)[d_{\mathfrak{X}}] \longrightarrow H^{n+1+k}(X) \otimes \mathbb{Q}^H_P[d] \longrightarrow 0,$$

using the vanishing in Lemma 5.4.5, and the result of Lemma 5.4.4. In terms of the decomposition (5.15), we thus have

$$E_{k,0} \simeq H^{n+1+k}(X)(1) \otimes \mathbb{Q}^H_P[d] \qquad \text{for } k > 0.$$

Moreover, $E_{k,j} = 0$ for $j \neq 0$, because E_k is supported on all of P.

Using Saito's Hard Lefschetz Theorem (5.16), we then deduce a similar statement for $k < 0$; we compute

$$E_{k,0} \simeq E_{-k,0}(-k) \simeq H^{n+1-k}(X)(1-k) \otimes \mathbb{Q}_P^H[d] \simeq H^{n+k-1}(X) \otimes \mathbb{Q}_P^H[d],$$

where the last isomorphism is because of the usual Hard Lefschetz Theorem. Since both isomorphisms are induced by the polarization, and therefore compatible, it follows that the restriction map $H^{n+k-1}(X) \otimes \mathbb{Q}_P^H[d] \to E_{k,0}$ itself has to be an isomorphism. Again, we have $E_{k,j} = 0$ if $k < 0$ and $j \neq 0$.

Next, we look at the exact sequence in negative degrees. After incorporating the results from above, a typical portion simplifies to

$$\begin{tikzcd}
H^{n+k-2}(X)(-1) \otimes \mathbb{Q}_P^H[d] \arrow[d] & & \\
H^{n+k}(X) \otimes \mathbb{Q}_P^H[d] \arrow[r] & H^k q_* \mathbb{Q}_U^H[d_U] \arrow[r] & H^{n+k-1}(X)(-1) \otimes \mathbb{Q}_P^H[d] \arrow[d] \\
& & H^{n+k+1}(X) \otimes \mathbb{Q}_P^H[d].
\end{tikzcd}$$

Both vertical maps are injective (by the usual Hard Lefschetz Theorem), and so we find that

$$H^k q_* \mathbb{Q}_U^H[d_U] \simeq \frac{H^{n+k}(X)}{H^{n+k-2}(X)(-1)} \otimes \mathbb{Q}_P^H[d] \tag{5.21}$$

when $k < 0$.

Finally, the part of the exact sequence for $k = 0$ reads

$$\begin{tikzcd}
H^{n-2}(X)(-1) \otimes \mathbb{Q}_P^H[d] \arrow[d] & & \\
H^n(X) \otimes \mathbb{Q}_P^H[d] \arrow[r] & H^k q_* \mathbb{Q}_U^H[d_U] \arrow[r] & H^0 \pi_* \mathbb{Q}_{\mathfrak{X}}^H[d_{\mathfrak{X}}] \arrow[d] \\
& & H^{n+1}(X) \otimes \mathbb{Q}_P^H[d].
\end{tikzcd}$$

From (5.15) and Lemma 5.4.6, we have the decomposition

$$H^0 \pi_* \mathbb{Q}_{\mathfrak{X}}^H[d_{\mathfrak{X}}] \simeq j_{!*} V_{van}^{n-1} \oplus H^{n-1}(X) \otimes \mathbb{Q}_P^H[d] \oplus \bigoplus_{j>0} E_{0j}. \tag{5.22}$$

Combining this with the fact that the primitive cohomology of weight n satisfies

$$H^n(X)_{prim} \simeq \frac{H^n(X)}{H^{n-2}(X)(-1)},$$

we obtain from the above the short exact sequence

$$H^n(X)_{prim} \otimes \mathbb{Q}_P^H[d] \hookrightarrow H^0 q_* \mathbb{Q}_U^H[d_U] \twoheadrightarrow j_{!*} V_{van}^{n-1}(-1) \oplus R. \tag{5.23}$$

Here $R = \bigoplus_{j>0} E_{0j}(-1)$ is a sort of "error term," containing those pieces in the decomposition of $H^0 \pi_* \mathbb{Q}_{\mathfrak{X}}^H[d_{\mathfrak{X}}]$ that are supported in a subset of the dual variety $X^\vee$.

Note. Because of Theorem 5.4.2, we can always force $R = 0$ by assuming that the line bundle $\mathscr{L}$ is sufficiently ample; further on, in 5.5.4, we shall give a more precise condition for the vanishing of the error term.

5.5 From mixed Hodge modules back to $\mathcal{D}$-modules

We now translate the results in 5.4.4 back into the language of $\mathcal{D}$-modules, by passing to the underlying $\mathcal{D}$-modules in the exact sequence (5.20).

5.5.1 The underlying $\mathcal{D}$-modules

As pointed out already (during the proof of Lemma 5.4.5), the mixed Hodge module $g_*\mathbb{Q}_U^H[d_U]$ has associated $\mathcal{D}$-module $\mathscr{O}_{P\times X}(*\mathfrak{X})$, with filtration

$$F_s\mathscr{O}_{P\times X}(*\mathfrak{X}) = \begin{cases} \mathscr{O}_{P\times X}\big((s+1)\mathfrak{X}\big) & \text{if } s \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.24}$$

To compute the direct images under the projection $P \times X \to P$, the relative de Rham complex

$$\mathrm{DR}_{P\times X/P} = \Big[\mathscr{O}_{P\times X} \longrightarrow \Omega^1_{P\times X/P} \longrightarrow \cdots \rightarrow \Omega^n_{P\times X/P}\Big][n],$$

with differential $d_{P\times X/P}$, can be used. For $q > 0$, we have

$$R^q pr_{P*}\Omega^k_{P\times X/P} \otimes \mathscr{O}_{P\times X}(*\mathfrak{X}) = R^q pr_{P*}\Omega^k_{P\times X/P}(*\mathfrak{X}) = 0,$$

and so each of the sheaves $\Omega^k_{P\times X/P}(*\mathfrak{X})$ is acyclic for the push-forward map pr_{P*}. Thus the complex $pr_{P*}\,\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big)$, which looks like

$$\Big[pr_{P*}\mathscr{O}_{P\times X}(*\mathfrak{X}) \longrightarrow pr_{P*}\Omega^1_{P\times X/P}(*\mathfrak{X}) \longrightarrow \cdots \rightarrow pr_{P*}\Omega^n_{P\times X/P}(*\mathfrak{X})\Big][n], \tag{5.25}$$

represents the direct image $pr_{P+}\mathscr{O}_{P\times X}(*\mathfrak{X})$ in the derived category of filtered holonomic complexes on P. Note that each term in the complex is naturally a $\mathcal{D}$-module on P; moreover, the maps in the complex are $\mathscr{O}_P$-linear. We conclude that the k-th cohomology sheaf $\mathcal{N}^k$ of the complex,

$$\mathcal{N}^k = H^k pr_{P*}\,\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big),$$

is the holonomic $\mathcal{D}$-module underlying the mixed Hodge module $H^k q_* \mathbb{Q}_U^H[d_U] = H^k pr_{P*} g_* \mathbb{Q}_U^H[d_U]$. Since the relative de Rham complex is supported only in degrees $-n, \ldots, 0$, it follows in particular that the $\mathcal{N}^k$ vanish outside the range $-n \leq k \leq 0$. The pole-order filtration F on $\mathscr{O}_{P\times X}(*\mathfrak{X})$ in (5.24) defines a filtration on the relative de Rham complex; since we have the vanishing in (5.7), the induced filtration on the direct image, $F_s pr_{P*} \operatorname{DR}_{P\times X/P}\bigl(\mathscr{O}_{P\times X}(*\mathfrak{X})\bigr)$, is given by the subcomplex

$$\Bigl[pr_{P*}F_s \longrightarrow pr_{P*}\Omega^1_{P\times X/P} \otimes F_{s+1} \longrightarrow \cdots \longrightarrow pr_{P*}\Omega^n_{P\times X/P} \otimes F_{s+n}\Bigr][n], \tag{5.26}$$

where $F_s = F_s\mathscr{O}_{P\times X}(*\mathfrak{X})$. The map pr_P being projective, this filtration is strict by a result of Saito (1990, Theorem 2.14 on p. 252); note that we are using algebraic mixed Hodge modules, which are always polarizable. Thus the cohomology sheaves of the subcomplex inject into those of the whole complex. Each of the $\mathcal{N}^k$ is therefore filtered, and the filtration is given by the cohomology sheaves of the displayed subcomplex.

The most interesting among those $\mathcal{D}$-modules is the one for $k = 0$; we denote it by

$$\mathcal{N}^0 = \frac{pr_{P*}\Omega^n_{P\times X/P}(*\mathfrak{X})}{d_{P\times X/P}\Bigl(pr_{P*}\Omega^{n-1}_{P\times X/P}(*\mathfrak{X})\Bigr)}. \tag{5.27}$$

By what we just said, $\mathcal{N}^0$ is filtered by

$$F_s\mathcal{N}^0 = \frac{pr_{P*}\Omega^n_{P\times X/P}\bigl((n+s+1)\mathfrak{X}\bigr)}{d_{P\times X/P}\Bigl(pr_{P*}\Omega^{n-1}_{P\times X/P}\bigl((n+s)\mathfrak{X}\bigr)\Bigr)}, \tag{5.28}$$

and $\bigl(\mathcal{N}^0, F\bigr)$ is a regular holonomic $\mathcal{D}$-module, because it underlies a mixed Hodge module. We also let

$$\mathcal{M}_{van} = j_{!*}\bigl(R^{n-1}\pi^{sm}_* \mathbb{Q}_{van} \otimes_{\mathbb{Q}} \mathscr{O}_{P^{sm}}\bigr)$$

be the minimal extension of the $\mathcal{D}$-module in parentheses from P^{sm} to P; of course, it underlies the mixed Hodge module $j_{!*}V_{van}^{n-1}$.

5.5.2 Conclusions about the $\mathcal{D}$-modules

The analysis in 5.4.4 now gives us two conclusions. On the one hand, it tells us the cohomology sheaves $\mathcal{N}^k$ of the relative de Rham complex for $\mathscr{O}_{P\times X}(*\mathfrak{X})$; on the other hand, it identifies the intermediate extension $\mathcal{M}_{van}$ as a quotient of $\mathcal{N}^0$. Note that for a mixed Hodge structure $H \in \mathrm{MHM}(pt)$, the $\mathcal{D}$-module $H \otimes \mathscr{O}_P$ underlying $H \otimes \mathbb{Q}_P^H[d]$ has as its filtration

$$F_s\big(H \otimes \mathscr{O}_P\big) = \big(F_s H\big) \otimes \mathscr{O}_P = \big(F^{-s}H\big) \otimes \mathscr{O}_P, \tag{5.29}$$

induced from the Hodge filtration on H.

Theorem 5.5.1. *When $k < 0$,*

$$\mathcal{N}^k \simeq \frac{H^{n+k}(X,\mathbb{C})}{H^{n+k-2}(X,\mathbb{C})} \otimes \mathscr{O}_P$$

are isomorphic as filtered $\mathcal{D}$-modules, where the right-hand side has the filtration described in (5.29). *Moreover, there is a short exact sequence*

$$H^n(X,\mathbb{C})_{prim} \otimes \mathscr{O}_P \hookrightarrow \mathcal{N}^0 \twoheadrightarrow \mathcal{M}_{van}(-1) \oplus \mathcal{R}$$

of filtered $\mathcal{D}$-modules, strict with respect to the filtrations. Here $\mathcal{R}$ is the $\mathcal{D}$-module underlying the mixed Hodge module R in (5.23).

Proof. The first assertion is an immediate consequence of the isomorphism in (5.21). Indeed, we have just shown that the filtered $\mathcal{D}$-module underlying the left-hand side of (5.21) is $\big(\mathcal{N}^k, F\big)$, whereas the one underlying the right-hand side is evidently $H^{n+k}(X,\mathbb{C})\big/H^{n+k-2}(X,\mathbb{C}) \otimes \mathscr{O}_P$. The second assertion follows in a similar manner from the exact sequence of mixed Hodge modules in (5.23), noting that morphisms between mixed Hodge modules preserve the filtrations strictly. $\square$

If $\mathscr{L}$ is sufficiently ample, then we have $\mathcal{R} = 0$ (see 5.5.4 for a discussion of this point), and so $\mathcal{M}_{van} = \mathcal{M}$, up to a shift in the filtration (see 5.5.3 for the exact numbers). The final result is therefore the following.

Theorem 5.5.2. *Assume that $\mathscr{L}$ is sufficiently ample. Then the error term $\mathcal{R}$ in Theorem 5.5.1 is zero, and we have $(\mathcal{M}_{van}, F) = (\mathcal{M}, F)(-n)$. Consequently, there is a short exact sequence*

$$H^n(X,\mathbb{C})_{prim} \otimes \mathscr{O}_P \hookrightarrow \mathcal{N}^0 \twoheadrightarrow \mathcal{M}(-n-1)$$

of filtered $\mathcal{D}$-modules, where $F_s\big(\mathcal{M}(-n-1)\big) = F_{s+n+1}\mathcal{M}$. The maps in this sequence are again strict on the filtrations.

5.5.3 Comparison of the two $\mathcal{D}$-modules

We shall now make a short comparison of the two $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{M}_{van}$ that have been introduced. To be able to apply the results of 5.5.2, we shall assume that the line bundle $\mathscr{L}$ is sufficiently ample.

Recall that the filtered $\mathcal{D}$-module $(\mathcal{M}, F)$ was defined by taking residues of rational forms; a section of $F_k\mathcal{M}$ is locally the residue of a rational form on $P \times X$ with a pole of order at most k along $\mathfrak{X}$. On the other hand, we used the theory of mixed

Hodge modules to introduce a second filtered $\mathcal{D}$-module $\big(\mathcal{M}_{van}, F\big)$. It underlies the mixed Hodge module $j_{!*}V_{van}^{n-1}$, and is therefore the minimal extension of the variation of Hodge structure from P^{sm}.

It is immediate that $\mathcal{M}$ and $\mathcal{M}_{van}$ are isomorphic $\mathcal{D}$-modules; the only difference is in the indexing of the filtration. Indeed, $\mathcal{N}^0$ is the right-most cohomology sheaf of the complex in (5.25), and $\mathcal{M}_{van}(-1)$ is a quotient of $\mathcal{N}^0$, because of the exact sequence in Theorem 5.5.1. Thus we have a surjective map

$$\mathrm{Res}\colon pr_{P*}\Omega^n_{P\times X/P}(*\mathfrak{X}) \to \mathcal{M}_{van}(-1)$$

of filtered $\mathcal{D}$-modules on P. Because of the Tate twist, this means that

$$\mathrm{Res}\Big(F_s pr_{P*}\Omega^n_{P\times X/P}(*\mathfrak{X})\Big) = F_s\big(\mathcal{M}_{van}(-1)\big) = F_{s+1}\mathcal{M}_{van}.$$

Looking back at the definition of the filtration on the de Rham complex in (5.26), we find that

$$\mathrm{Res}\Big(pr_{P*}\Omega^n_{P\times X/P}\big((n+s+1)\mathfrak{X}\big)\Big) = F_{s+1}\mathcal{M}_{van}$$

for all s with $n+s\geq 0$. In other words, a section of $F_{s+1}\mathcal{M}_{van}$ is locally the residue of a rational form with a pole of order at most $(n+s+1)$. But this means exactly that $F_s\mathcal{M}_{van} = F_{n+s}\mathcal{M}$, and so

$$(\mathcal{M}, F) = \big(\mathcal{M}_{van}, F_{\bullet-n}\big) = \big(\mathcal{M}_{van}, F[n]\big) = (\mathcal{M}_{van}, F)(n)$$

agree up to a shift by n in the filtration.

Note. If desired, the Hodge filtration F can also be written as a decreasing filtration, using the convention that $F^p = F_{-p}$; we then get

$$\mathrm{Res}\Big(pr_{P*}\Omega^n_{P\times X/P}\big((n-p)\mathfrak{X}\big)\Big) = F^{n-p}\mathcal{M} = F^p\mathcal{M}_{van},$$

with the understanding that $F^n \mathcal{M}_{van} = 0$. On the open set P^{sm}, the sheaf $F^p \mathcal{M}_{van}$ restricts to the Hodge bundle $F^p \mathcal{V}_{van}^{n-1}$.

5.5.4 Ampleness assumptions and the vanishing of R

In several places, we have made the assumption that the line bundle $\mathscr{L}$ is "sufficiently ample," without specifying too clearly what this means. We shall now give a more precise statement. In the present context, $\mathscr{L}$ should of course be very ample (to embed X into the projective space Q). In addition, it should satisfy the following two conditions:

(a) The cohomology groups $H^q\big(X, \Omega_X^p \otimes \mathscr{L}^k\big)$ vanish for all $k > 0$ and $q > 0$.

(b) The Hodge modules $E_{0,j}$ in the decomposition (5.14) are zero for $j > 0$.

The first condition allows us to use rational forms to represent cohomology classes; in particular, it implies that the Hodge filtration F on $\mathcal{N}^0$ is given by the order of pole, as in (5.28). The second condition guarantees the vanishing of the error term R in (5.23), and allows us to conclude that $\mathcal{M} = \mathcal{M}_{van}$.

Brosnan et al. (2007, Remark 5.14 on p.15) outline a proof for why (b) holds when $\mathscr{L}$ is sufficiently ample, crediting it to Fakhruddin. They point out that the locus in $X^\vee$ of hypersurfaces with non-isolated singularities is of higher and higher codimension as $\mathscr{L}$ becomes more ample (see also Lemma 6.5.2). Eventually, the codimension is greater than the dimension $n-1$ of the hypersurfaces, and then $E_{0,j} = 0$ for $j > 0$ by support considerations.

Although (a) seems to have nothing whatsoever to do with (b), it turns out that in most cases, it does imply the other condition. The reason for this surprising circumstance is the following. Assuming only (a), we can prove that $H^1\big(P, F_s\mathcal{N}^0\big) = 0$ for all s; this is the content of Theorem 6.1.2. We can then compute the characteristic variety of the $\mathcal{D}$-module $\mathcal{N}^0$, as in Lemma 5.3.2, and find that it is the union of two irreducible components:

1. The zero section of Ω_P, with multiplicity equal to the generic rank of $\mathcal{N}^0$.

2. The cone over the set $\mathfrak{Z}$, with multiplicity one.

The same is then true (with a different multiplicity for the first component) for the holonomic $\mathcal{D}$-module $\mathcal{M}_{van} \oplus \mathcal{R}$, since it differs from $\mathcal{N}^0$ only by the vector bundle $H^n(X, \mathbb{C})_{prim} \otimes \mathscr{O}_P$, according to Theorem 5.5.1. But multiplicity is an additive function on holonomic $\mathcal{D}$-modules, and so one of $\mathcal{M}_{van}$ and $\mathcal{R}$ has to have multiplicity zero along the cone over $\mathfrak{Z}$. Since $\mathcal{R}$ is already supported inside $X^\vee$, we conclude that if $\mathcal{R} \neq 0$, the characteristic variety of $\mathcal{M}_{van}$ has to consist of just the zero section, which means that $\mathcal{M}_{van}$ has to be a locally free sheaf.

But this can only happen when $\mathcal{M}_{van} = 0$, because the monodromy action on the vanishing cohomology is irreducible (Voisin, 2002, Corollaire 15.28 on p. 355). Indeed, if the $\mathcal{D}$-module $\mathcal{M}_{van}$ was locally free, it would be a flat vector bundle, and therefore trivial (because P is simply connected). In particular, the local system $R^{n-1}\pi^{sm}_* \mathbb{C}_{van}$ would be trivial. But since it is known that $\Gamma\big(P^{sm}, R^{n-1}\pi^{sm}_*\mathbb{C}_{van}\big) = 0$, this is not possible unless the vanishing cohomology of the hypersurfaces is trivial.

We can draw from this the following conclusion: Assuming (a), and that $\mathcal{M}_{van} \neq 0$ (which is essentially always the case), we get $\mathcal{R} = 0$, and hence (b).

Note. It is illustrative to compare this discussion with the example of plane conics, given in Brosnan et al. (2007, Example 5.16 on p. 16). This is the special case when $X = \mathbb{P}^2$ and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^2}(2)$; as the authors remark, $E_{0,0} = 0$, and $E_{0,j} = 0$ for all $j \geq 2$, but $E_{0,1} \neq 0$. In other words, we have $\mathcal{M}_{van} = 0$, but $\mathcal{R} \neq 0$; note that only one of the summands of $\mathcal{R}$ is nonzero, as required by the multiplicity calculation above.
In general, when X is a projective space, condition (a) always obtains because of Bott's Vanishing Theorem. Thus $\mathcal{R}$, and in particular all $E_{0,j}$ for $j > 0$ in the decomposition, are necessarily zero as soon as the hypersurfaces have nontrivial vanishing cohomology.

5.5.5 Hypercohomology of the de Rham complex

Note. In this section, we assume that $\mathscr{L}$ is sufficiently ample, so that $\mathcal{M}_{van} = \mathcal{M}$.

As we have seen, the $\mathcal{D}$-module $\mathcal{M}$ underlies the mixed Hodge module $j_{!*}V_{van}^{n-1}$ on P. On the other hand, the corresponding perverse sheaf is

$$\operatorname{rat} j_{!*}V_{van}^{n-1} = j_{!*} \operatorname{rat} V_{van}^{n-1} = j_{!*} R^{n-1}\pi_*^{sm}\mathbb{Q}_{van}[d];$$

after tensoring with $\mathbb{C}$, it has to become isomorphic to the de Rham complex for $\mathcal{M}$, by the definition of mixed Hodge modules. Therefore,

$$\mathrm{DR}_P\big(\mathcal{M}\big) \simeq j_{!*} \operatorname{rat} V_{van}^{n-1} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{5.30}$$

The purpose of this section is to prove that the hypercohomology group $\mathbb{H}^{-d+1}\big(\mathrm{DR}_P\,\mathcal{M}\big)$

of the de Rham complex is isomorphic to the primitive cohomology of X in middle dimension.

Lemma 5.5.3. *Assume that $\mathscr{L}$ is sufficiently ample. Then the Leray spectral sequence gives rise to a (canonical) isomorphism*

$$H^n(X,\mathbb{C})_{prim} \simeq \mathbb{H}^{-d+1}\big(\mathrm{DR}_P(\mathcal{M})\big).$$

Proof. According to the results in 5.4.4, we have

$$H^q\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq \begin{cases} H^{n+q-1}(X)\otimes\mathbb{Q}^H_P[d] & \text{for } q<0, \\ H^{n+q+1}(X)(1)\otimes\mathbb{Q}^H_P[d] & \text{for } q>0, \\ j_{!*}V^{n-1}_{van}\oplus H^{n-1}(X)\otimes\mathbb{Q}^H_P[d] & \text{for } q=0. \end{cases} \tag{5.31}$$

The $\mathcal{D}$-module component of $j_{!*}V^{n-1}_{van}$ is precisely $\mathcal{M}$, and so the hypercohomology of $\mathrm{DR}_P(\mathcal{M})$ computes the complex vector spaces underlying the cohomology modules of $j_{!*}V^{n-1}_{van}$. We calculate that

$$\begin{aligned} H^{-d+1}a_{P*}H^0\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] &\simeq H^{-d+1}a_{P*}j_{!*}V^{n-1}_{van}\oplus H^{n-1}(X)\otimes H^1a_{P*}\mathbb{Q}^H_P \\ &= H^{-d+1}a_{P*}j_{!*}V^{n-1}_{van}, \end{aligned}$$

because $H^1a_{P*}\mathbb{Q}^H_P = H^1(P) = 0$. It follows that $\mathbb{H}^{-d+1}\big(\mathrm{DR}_P(\mathcal{M})\big)$ is the complex vector space of the mixed Hodge structure $H^{-d+1}a_{P*}H^0\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}]$.

Now we bring in the (perverse) Leray spectral sequence,

$$E_2^{p,q} = H^pa_{P*}H^q\pi_*\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \Longrightarrow H^{p+q}a_{\mathfrak{X}*}\mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] = H^{p+q+d_{\mathfrak{X}}}(\mathfrak{X}),$$

which is a spectral sequence of mixed Hodge modules. Note that $E_2^{p,q} = 0$ whenever $p < -d = -\dim P$. The term we are really interested in is

$$E_2^{-d+1,0} \simeq H^{-d+1} a_{P*} j_{!*} V_{van}^{n-1};$$

it sits in degree $-d+1$, and is thus a graded quotient of $H^{-d+1+d_{\mathfrak{X}}}(\mathfrak{X}) = H^n(\mathfrak{X})$. The Decomposition Theorem implies that the spectral sequence degenerates at the E_2-page (in fact, it even implies that $H^k a_{\mathfrak{X}*} \mathbb{Q}_{\mathfrak{X}}^H[d_{\mathfrak{X}}]$ is isomorphic to the direct sum of all the $E_2^{p,q}$ with $p + q = k$, albeit non-canonically). Let us write $L^\bullet$ for the induced filtration on the cohomology of $\mathfrak{X}$. We then have a short exact sequence of mixed Hodge structures

$$0 \longrightarrow L^1 H^n(\mathfrak{X}) \longrightarrow H^n(\mathfrak{X}) \longrightarrow E_2^{-d,1} \longrightarrow 0,$$

and $E_2^{-d,1} \simeq H^{n+2}(X)(1) \otimes H^0(P)$ by virtue of (5.31).

Consider now the pullback map $\phi^*\colon H^n(X) \to H^n(\mathfrak{X})$. As the primitive cohomology is the kernel of $H^n(X) \to H^{n+2}(X)(1)$, we get an induced map from $H^n(X)_{prim}$ to $L^1 H^n(\mathfrak{X})$. The next step of the Leray filtration gives another short exact sequence

$$0 \longrightarrow L^2 H^n(\mathfrak{X}) \longrightarrow L^1 H^n(\mathfrak{X}) \xrightarrow{\rho} E_2^{-d+1,0} \longrightarrow 0,$$

and by composition, we finally obtain a (canonical) map of mixed Hodge structures

$$\rho \circ \phi^*\colon H^n(X)_{prim} \to E_2^{-d+1,0} \simeq H^{-d+1} a_{P*} j_{!*} V_{van}^{n-1}. \tag{5.32}$$

That this map is an isomorphism follows easily from the fact that $\phi\colon \mathfrak{X} \to X$ is a projective space bundle of rank $d-1$. Indeed, we naturally have

$$H^n(\mathfrak{X}) \simeq \bigoplus_{i \geq 0} H^i(P) \otimes H^{n-i}(X) = H^n(X) \oplus \bigoplus_{i \geq 2} H^i(P) \otimes H^{n-i}(X).$$

The terms in the direct sum are precisely the graded quotients of $L^2 H^n(\mathfrak{X})$, because the isomorphisms in (5.31) imply that

$$E_2^{-d+i,-i+1} = H^{-d+i} a_{P*} H^{-i-1} \pi_* \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \simeq H^i(P) \otimes H^{n-i}(X)$$

whenever $i \geq 2$. Therefore, the map from $H^n(X)$ to $H^n(\mathfrak{X})/L^2 H^n(\mathfrak{X})$ is an isomorphism for dimension reasons; this implies that (5.32) is also an isomorphism. By passing to the underlying complex vector spaces, we get the result. $\square$

We are assuming that $\mathscr{L}$ is sufficiently ample to make the error term R in the short exact sequence (5.23) vanish. From the connecting homomorphism for that sequence, we then get another map

$$H^{-d+1} a_{P*} j_{!*} V^{n-1}_{van} \to H^2(P)(1) \otimes H^n(X)_{prim}; \tag{5.33}$$

for later use, we prove the following compatibility result.

Lemma 5.5.4. *The composition of the two maps in* (5.32) *and* (5.33),

$$H^n(X)_{prim} \to H^{-d+1} a_{P*} j_{!*} V^{n-1}_{van} \to H^2(P)(1) \otimes H^n(X)_{prim},$$

is multiplication by $2\pi i \cdot c_1\big(\mathscr{O}_P(1)\big)$.

Proof. By construction, the morphism $\rho \circ \phi^*$ in (5.32) factors through the restriction map $H^n(P \times X) \to H^n(\mathfrak{X})$. Using the distinguished triangle in (5.19), we get a map

$$H^n(\mathfrak{X}) = H^{-d+1} a_{\mathfrak{X}*} \mathbb{Q}^H_{\mathfrak{X}}[d_{\mathfrak{X}}] \to H^{-d+2} pr_{P*} \mathbb{Q}^H_{P\times X}(1)[d_{P\times X}] = H^{n+2}(X)(1). \tag{5.34}$$

It is obvious that the composition

$$H^n(P \times X) \to H^n(\mathfrak{X}) \to H^{n+2}(P \times X)(1)$$

is given by cup product with the class $2\pi i \cdot c_1\bigl(\mathscr{O}_{P\times X}(\mathfrak{X})\bigr)$. To prove the lemma, it is therefore sufficient to show that the map in (5.34) and the map in (5.33) are compatible. This is more or less clear, because both have ultimately been derived from the triangle in (5.19). □

5.5.6 Cohomology sheaves of the de Rham complex

We are now going to compute the cohomology sheaves of the de Rham complex for the $\mathcal{D}$-module $\mathcal{M}$, at least when $\mathscr{L}$ is sufficiently ample. Recall that $\mathrm{DR}_P(\mathcal{M})$ is a perverse complex, because $\mathcal{M}$ underlies a mixed Hodge module; all of its cohomology sheaves

$$\mathcal{H}^k = \mathcal{H}^k \operatorname{DR}_P\bigl(\mathcal{M}_{van}\bigr) \simeq \mathcal{H}^k \operatorname{DR}_P\bigl(\mathcal{M}\bigr)$$

are therefore constructible sheaves (in the Zariski topology). The following lemma describes them very precisely.

Lemma 5.5.5. *Let $\mathscr{L}$ be sufficiently ample. Then*

$$R^{n-1+(d+k)}\pi_*\mathbb{C} \simeq \mathcal{H}^k \operatorname{DR}_P\bigl(\mathcal{M}\bigr) \oplus H^{n-1-(d+k)}(X,\mathbb{C}) \otimes \mathbb{C}_P$$

for all $k \geq -d$.

Proof. The proof mirrors that of Corollary 5.7 in Brosnan et al. (2007). Let $p \in P$ be an arbitrary point. Since the map $\pi\colon \mathfrak{X} \to P$ is proper, we have

$$\bigl(R^{n-1+d+k}\pi_*\mathbb{C}\bigr)_p = H^{n-1+d+k}\bigl(\mathfrak{X}_p,\mathbb{C}\bigr) = \operatorname{rat} H^{n-1+d+k}\bigl(\mathfrak{X}_p\bigr) \otimes_{\mathbb{Q}} \mathbb{C}$$

for the stalk of the higher direct image sheaf at p, by the Proper Base Change Theorem from topology. Because of the decomposition in (5.14), we also have

$$H^{n-1+d+k}\big(\mathfrak{X}_p\big) = H^{d_{\mathfrak{X}}+k}\big(\mathfrak{X}_p\big) = H_p^k\big(\pi_* \mathbb{Q}_{\mathfrak{X}}^H[d_{\mathfrak{X}}]\big) = \bigoplus_{ij} H_p^{k-i}\big(E_{i,j}\big).$$

Since we are assuming that $\mathscr{L}$ is sufficiently ample, we have $E_{i,j} = 0$ for all $j \neq 0$, by the work in 5.4.4. Therefore,

$$H^{n-1+d+k}\big(\mathfrak{X}_p\big) = \bigoplus_{i} H_p^{k-i}\big(E_{i,0}\big).$$

The remaining terms are now easily computed. On the one hand,

$$H_p^k\big(E_{0,0}\big) \simeq H_p^k\big(j_{!*}E^{n-1}\big) \simeq H_p^k\big(j_{!*}V_{van}^{n-1}\big) \oplus \begin{cases} H^{n-1}(X) & \text{if } k = -d, \\ 0 & \text{otherwise,} \end{cases}$$

from Lemma 5.4.6. On the other hand, we have

$$E_{i,0} = \begin{cases} H^{n+i-1}(X) \otimes \mathbb{Q}_P^H[d] & \text{for } i < 0, \\ H^{n-i-1}(X)(-i) \otimes \mathbb{Q}_P^H[d] & \text{for } i > 0, \end{cases}$$

again using the results in 5.4.4. Since $H_p^{k-i}\big(\mathbb{Q}_P^H[d]\big) = 0$ for $k - i \neq -d$, it then follows that

$$H_p^{k-i}\big(E_{i,0}\big) \simeq \begin{cases} H^{n-1+(d+k)}(X) & \text{if } i = d+k < 0, \\ H^{n-1-(d+k)}(X)\big(-(d+k)\big) & \text{if } i = d+k > 0, \\ 0 & \text{if } i \neq d+k \text{ and } i \neq 0. \end{cases}$$

In conclusion, we have for $k \geq -d$ an isomorphism

$$H^{n-1+(d+k)}\big(\mathfrak{X}_p\big) \simeq H_p^k\big(j_{!*}V_{van}^{n-1}\big) \oplus H^{n-1-(d+k)}(X)\big(-(d+k)\big). \tag{5.35}$$

Now apply the functor rat to this, and note that

$$\operatorname{rat} H_p^k\big(j_{!*}V_{van}^{n-1}\big) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathcal{H}_p^k$$

by what was said just before the statement of the lemma. On stalks, we therefore have

$$\big(R^{n-1+(d+k)}\pi_*\mathbb{C}\big)_p \simeq \mathcal{H}_p^k \oplus H^{n-1-(d+k)}(X, \mathbb{C}),$$

from which the asserted identity follows immediately. □

Note. It should be noted that the second part of the isomorphism in (5.35) can be described explicitly. Let $L\colon H^k(X) \to H^{k+2}(X)(1)$ be the Lefschetz operator associated with the very ample line bundle $\mathscr{L}$. For each hypersurface $\mathfrak{X}_p \subseteq X$, smooth or not, there is then naturally a map

$$H^{n-1-(d+k)}(X)\big(-(d+k)\big) \xrightarrow{L^{d+k}} H^{n-1+(d+k)}(X) \longrightarrow H^{n-1+(d+k)}\big(\mathfrak{X}_p\big).$$

So (5.35) tells us in particular that this map is always injective.

CHAPTER 6

PROPERTIES OF THE SHEAVES IN THE FILTRATION

In this chapter, we collect several results about the coherent sheaves $F_k\mathcal{M}$ and $Gr_k^F\mathcal{M}$. Among other things, we prove that their higher cohomology groups vanish; and that, in the range $1 \leq k \leq n$ where $F_k\mathcal{M}$ extends the Hodge bundle $F^{n-k}\mathcal{V}_{van}^{n-1}$, each sheaf $F_k\mathcal{M}$ satisfies Serre's condition S_p for large p. All of these results depend on the assumption that the line bundle $\mathscr{L}$ is sufficiently ample.

6.1 Cohomological properties

We begin by deriving several properties from the fact that the $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{N}^0$ are quotients of the direct image of the relative de Rham complex. The notation throughout is the same as in 5.5.1.

6.1.1 Essentially a locally free resolution

The push-forward of the relative de Rham complex, in (5.25), provides us with a complex of sheaves on P that is essentially a locally free resolution for each of the sheaves $F_k\mathcal{M} = F_{k-n}\mathcal{M}_{van}$.

During the discussion, we let s be a fixed integer. We consider the subcomplex $F_s pr_{P*}\,\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big)$ of the direct image of the relative de Rham complex, as in (5.26). To simplify the notation, let us denote that complex by $\mathcal{B}_s^\bullet$ in this section. It is a complex of $\mathcal{D}$-modules on P, with $\mathscr{O}_P$-linear differentials, and supported in degrees $-n, \dots, 0$. The individual sheaves in the complex $\mathcal{B}_s^\bullet$ are easily computed; they are just direct sums of positive line bundles, because

$$\begin{aligned} \mathcal{B}_s^{p-n} &= pr_{P*}\Big(\Omega^p_{P\times X/P} \otimes F_{s+p}\mathscr{O}_{P\times X}(*\mathfrak{X})\Big) \\ &= \begin{cases} H^0\big(X, \Omega_X^p \otimes \mathscr{L}^{s+p+1}\big) \otimes \mathscr{O}_P(s+p+1) & \text{if } s+p \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.1}$$

By our discussion in 5.5.1 (which involved using Saito's result on strictness of the Hodge filtration on direct images), the cohomology sheaves of the complex are precisely the coherent sheaves $F_s\mathcal{N}^p$. When $p < 0$, we have a concrete description of these sheaves in Theorem 5.5.1, as

$$F_s\mathcal{N}^p \simeq F_s \frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)} \otimes \mathscr{O}_P \simeq \frac{F^{-s}H^{n+p}(X)}{F^{-s-1}H^{n+p-2}(X)} \otimes \mathscr{O}_P. \tag{6.2}$$

But if $\mathscr{L}$ is sufficiently ample, we also have the short exact sequence in Theorem 5.5.2, which shows that

$$0 \longrightarrow F^{-s}H^n(X,\mathbb{C})_{prim} \otimes \mathscr{O}_P \longrightarrow F_s\mathcal{N}^0 \longrightarrow F_{s+n+1}\mathcal{M} \longrightarrow 0 \tag{6.3}$$

is exact. This leads us to consider the augmented complex $\mathcal{B}_s^\bullet \twoheadrightarrow F_{s+n+1}\mathcal{M}$. Its cohomology sheaf at $\mathcal{B}_s^p$ (now including the case when $p = 0$) is still given by the expression in (6.2). By choosing $s = k - n - 1$, we then get the following result.

Lemma 6.1.1. *Fix an integer k, and let $\mathcal{B}^\bullet = F_{k-n-1} pr_{P*} \mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big)$, a complex of locally free sheaves on P with terms*

$$\mathcal{B}^p = \begin{cases} H^0\big(X, \Omega_X^{n+p} \otimes \mathscr{L}^{k+p}\big) \otimes \mathscr{O}_P(k+p) & \text{if } k+p \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the augmented complex

$$\mathcal{B}^{-n} \longrightarrow \mathcal{B}^{-n+1} \longrightarrow \cdots \longrightarrow \mathcal{B}^1 \longrightarrow \mathcal{B}^0 \twoheadrightarrow F_k\mathcal{M}$$

has as its cohomology sheaf at $\mathcal{B}^p$ the locally free sheaf

$$\mathcal{H}^p = F_{k-n-1} \frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)} \otimes \mathscr{O}_P \simeq \frac{F^{n+1-k} H^{n+p}(X)}{F^{n-k} H^{n+p-2}(X)} \otimes \mathscr{O}_P.$$

For all intents and purposes, this is as good as having a locally free resolution for the sheaf $F_k\mathcal{M}$. The main difference is that one gets two spectral sequences when applying a functor to $\mathcal{B}^\bullet$, instead of one as usual.

6.1.2 Vanishing of higher cohomology

We shall now do one calculation with the pseudo-resolution from 6.1.1, to prove the vanishing of higher cohomology for the sheaves $F_s\mathcal{N}^0$ and $F_k\mathcal{M}$.

Theorem 6.1.2. *Assume that $\mathscr{L}$ is sufficiently ample. Then we have*

$$H^i\big(P, F_{s+n+1}\mathcal{M}\big) \simeq H^i\big(P, F_s\mathcal{N}^0\big) = 0$$

for all $i > 0$, and all $s \in \mathbb{Z}$.

Proof. We fix an integer s, and consider the complex of locally free sheaves $\mathcal{B}_s^\bullet$ from Lemma 6.1.1. Each sheaf $\mathcal{B}_s^p$ is a direct sum of positive line bundles on P. One of the two hypercohomology spectral sequences for the complex,

$$'E_1^{p,q} = H^q(P, \mathcal{B}_s^p) \Longrightarrow \mathbb{H}^{p+q}\big(\mathcal{B}_s^\bullet\big),$$

therefore degenerates at the E_1-page, because all but one row is zero. This is illustrated in Figure 6.1, where the only possible nonzero entries are the ones marked with an asterisk.

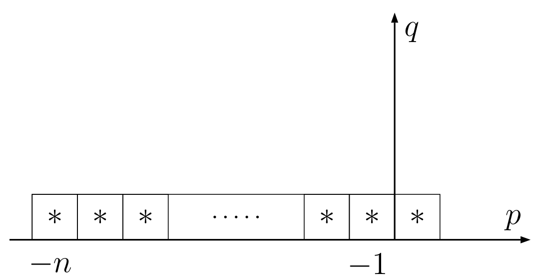

Figure 6.1: The E_1-page of the first spectral sequence

It follows that $\mathbb{H}^i\big(\mathcal{B}_s^\bullet\big)$ is the cohomology in degree i of the complex with terms

$$H^0(P, \mathcal{B}_s^i) = \begin{cases} W_{s+i+n+1}^{i+n} & \text{if } s+i+n \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6.4}$$

in the notation introduced in (5.10). In particular, we see that $\mathbb{H}^i\big(\mathcal{B}_s^\bullet\big) = 0$ for $i > 0$.

We now relate this to the cohomology of $F_s\mathcal{N}^0$ by using the other hypercohomology spectral sequence,

$$''E_2^{p,q} = H^p\big(P, F_s\mathcal{N}^q\big) \Longrightarrow \mathbb{H}^{p+q}\big(\mathcal{B}_s^\bullet\big),$$

remembering that $F_s\mathcal{N}^q$ is the q-th cohomology sheaf of the complex. When $q < 0$, each of these is either a trivial vector bundle, or zero; thus $''E_2^{p,q} = 0$ for $q < 0$ and $p > 0$. This means that only the fields marked with an asterisk in Figure 6.2 can be nonzero.

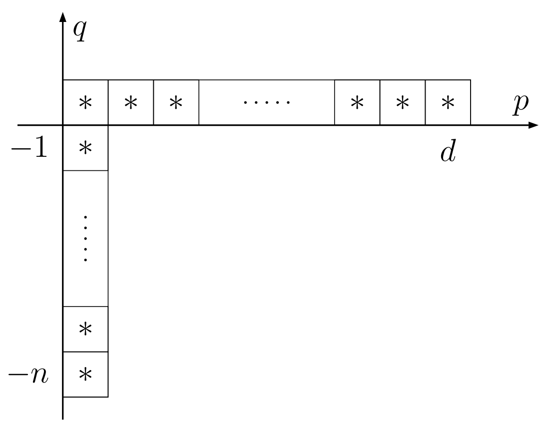

Figure 6.2: The E_2-page of the second spectral sequence

The spectral sequence is thus degenerate as well, and we deduce that

$$H^i\big(P, F_s\mathcal{N}^0\big) \simeq \mathbb{H}^i\big(\mathcal{B}_s^\bullet\big) = 0$$

for all $i > 0$. This proves the vanishing of the higher cohomology groups for the sheaves $F_s\mathcal{N}^0$.

As for the cohomology of the sheaves $F_{s+n+1}\mathcal{M}$, note that we have the exact sequence in (6.3); thus the kernel of $F_s\mathcal{N}^0 \to F_{s+n+1}\mathcal{M}$ is again either zero, or a trivial vector bundle. Since we are on projective space, it follows that

$$0 = H^i\big(P, F_s\mathcal{N}^0\big) \simeq H^i\big(P, F_{s+n+1}\mathcal{M}\big)$$

for all $i > 0$, and this finishes the proof. $\square$

Note. A similar argument can be used to get the vanishing of cohomology groups of the form $H^q\big(P, \Omega_P^p \otimes F_s\mathcal{N}^0\big)$. The precise result is that

$$H^q\big(P, \Omega_P^p \otimes F_s\mathcal{N}^0\big) = 0 \text{ for all } q \geq \min(1, p-1).$$

In particular, all higher cohomology groups of $\Omega_P^1 \otimes F_s\mathcal{N}^0$ are still zero.

6.1.3 The space of global sections

The degeneration of the two spectral sequences used in the proof of Theorem 6.1.2 also gives a way to describe the space of global sections of the sheaf $F_s\mathcal{N}^0$. Indeed, we have

$$H^0\big(P, F_s\mathcal{N}^0\big) = \mathbb{H}^0\big(\mathcal{B}_s^\bullet\big) = \frac{H^0\big(P, \mathcal{B}_s^0\big)}{d_{P\times X/P} H^0\big(P, \mathcal{B}_s^{-1}\big)},$$

and by using the identity in (6.4), we arrive at the following statement.

Lemma 6.1.3. *Still assuming that $\mathscr{L}$ is sufficiently ample, we have*

$$H^0\big(P, F_s\mathcal{N}^0\big) = \frac{W_{s+n+1}^n}{d_{P\times X/P}\big(W_{s+n}^{n-1}\big)},$$

where we write $W_k^p = H^0\big(X, \Omega_X^p \otimes \mathscr{L}^k\big) \otimes H^0\big(P, \mathscr{O}_P(k)\big)$ as in (5.10).

With the help of the vanishing theorems that we proved, we can derive a similar result for the sheaves $F_k\mathcal{M}$ and their graded quotients $Gr_k^F\mathcal{M}$.

Lemma 6.1.4. *For every k, the projection $H^0(P, F_k\mathcal{M}) \to H^0(P, Gr_k^F\mathcal{M})$ is onto. The surjection from $F_{k-n-1}\mathcal{N}^0$ to $F_k\mathcal{M}$ to induces a map*

$$W_k^n \otimes \mathscr{O}_P = H^0\big(P \times X, pr_X^*\Omega_X^n \otimes \mathscr{O}_{P\times X}(k\mathfrak{X})\big) \otimes \mathscr{O}_P \to F_k\mathcal{M},$$

which is also onto. As a consequence, each sheaf $Gr_k^F\mathcal{M}$ is globally generated.

Proof. The surjectivity of $H^0(P, F_k\mathcal{M}) \to H^0(P, Gr_k^F\mathcal{M})$ follows immediately from the vanishing in Theorem 6.1.2. To see why the second assertion holds, note that the exact sequence in (6.3) (for $s = k - n - 1$) proves that the map

$$H^0\big(P, F_{k-n-1}\mathcal{N}^0\big) \to H^0(P, F_k\mathcal{M})$$

is onto, because $H^1(P, \mathscr{O}_P) = 0$. If we combine that fact with the result of Lemma 6.1.3, we get the result. ☐

6.2 The lowest level in the filtration

Kawamata (1981, p. 266) proves the following result:

Theorem 6.2.1 (Kawamata)**.** *Let $f\colon X \to Y$ be an algebraic fiber space (i.e., a surjective map between smooth projective algebraic varieties, having connected fibers). Suppose the following three conditions are satisfied:*

(i) There is a dense Zariski open subset $Y_0 \subseteq Y$, such that $D = Y \setminus Y_0$ is a normal crossing divisor.

(ii) *The restriction* $f_0 = f\big|_{X_0}$ *of* f *to the subset* $X_0 = f^{-1}(Y_0)$ *is smooth over* Y_0.

(iii) *The local system* $R^m f_{0*}\mathbb{C}$ *has unipotent monodromy around the divisor* D*, where* $m = \dim X - \dim Y$.

Then the direct image $f_*\mathscr{O}_X\big(K_{X/Y}\big)$ *of the relative canonical bundle is locally free and nef, and equals* $F^m\overline{\mathscr{V}}$*, where* $\overline{\mathscr{V}}$ *is Deligne's canonical extension of the vector bundle* $R^m f_{0*}\mathbb{C} \otimes \mathscr{O}_{Y_0}$.

The sheaf $F^m\overline{\mathscr{V}}$ is the lowest level in the Hodge filtration on the canonical extension. A similar result is true for the map $\pi\colon \mathfrak{X} \to P$, although $X^\vee$ is not a divisor with normal crossings, and the local monodromies are not unipotent. The next proposition shows that the lowest level in the filtration on $\mathcal{M}$, namely the sheaf $F_1\mathcal{M}$, is also locally free, and related to the relative canonical bundle.

Proposition 6.2.2. *Let* $\mathscr{L}$ *be sufficiently ample. Then* $F_1\mathcal{M}$ *is an ample vector bundle, and forms part of a short exact sequence*

$$0 \longrightarrow F_1\mathcal{M} \longrightarrow \pi_*\mathscr{O}_{\mathfrak{X}}\big(K_{\mathfrak{X}/P}\big) \longrightarrow H^{n-1,0}(X) \otimes \mathscr{O}_P \longrightarrow 0. \tag{6.5}$$

In particular, $\pi_*\mathscr{O}_{\mathfrak{X}}\big(K_{\mathfrak{X}/P}\big)$ *is locally free and nef.*

Proof. Using the pseudo-resolution in Lemma 6.1.1 (for $k = 1$), we find a short exact sequence

$$H^0\big(X, \Omega_X^n\big) \otimes \mathscr{O}_P \hookrightarrow H^0\big(X, \Omega_X^n \otimes \mathscr{L}\big) \otimes \mathscr{O}_P(1) \twoheadrightarrow F_1\mathcal{M}. \tag{6.6}$$

Since the first map is injective at each point (because $H^0\big(X, \Omega_X^n\big) \subseteq H^0\big(X, \Omega_X^n(\mathfrak{X}_p)\big)$ for every point $p \in P$), the quotient $F_1\mathcal{M}$ is locally free, proving the first assertion.

To establish (6.5), we note that the canonical bundle of $\mathfrak{X}$ is

$$\mathscr{O}_{\mathfrak{X}}\big(K_{\mathfrak{X}}\big) \simeq \phi^*\big(\mathscr{O}_X(K_X)\big) \otimes \pi^*\big(\mathscr{O}_P(K_P)\big) \otimes \mathscr{O}_{\mathfrak{X}}(1),$$

because $\mathfrak{X}$ is a smooth hypersurface in the product $P \times X$, with line bundle $\mathscr{O}_{P\times X}(\mathfrak{X}) = \mathscr{O}_{P\times X}(1)$. Thus the relative canonical bundle for the map $\pi\colon \mathfrak{X} \to P$ is given by the formula

$$\mathscr{O}_{\mathfrak{X}}\big(K_{\mathfrak{X}/P}\big) \simeq \phi^*\Omega_X^n \otimes \mathscr{O}_{\mathfrak{X}}(1).$$

By pushing forward the exact sequence

$$0 \longrightarrow pr_X^*\Omega_X^n \longrightarrow pr_X^*\Omega_X^n \otimes \mathscr{O}_{P\times X}(1) \longrightarrow \phi^*\Omega_X^n \otimes \mathscr{O}_{\mathfrak{X}}(1) \longrightarrow 0,$$

and using that $\mathscr{L}$ is sufficiently ample, we then get an exact sequence

$$\begin{tikzcd}
H^0\big(X, \Omega_X^n\big) \otimes \mathscr{O}_P \arrow[r, hook] & H^0\big(X, \Omega_X^n \otimes \mathscr{L}\big) \otimes \mathscr{O}_P(1) \arrow[r] & \pi_*\mathscr{O}_{\mathfrak{X}}\big(K_{\mathfrak{X}/P}\big) \arrow[d] \\
& & H^1\big(X, \Omega_X^n\big) \otimes \mathscr{O}_P
\end{tikzcd}$$

on P. The second assertion follows by upon comparing this with the resolution for $F_1\mathcal{M}$ in (6.6), and noting that $H^1\big(X, \Omega_X^n\big) \simeq H^{n-1,0}(X)$.

Now $F_1\mathcal{M}$ is evidently an ample vector bundle, since it is a quotient of the direct sum $H^0\big(X, \Omega_X^n \otimes \mathscr{L}\big) \otimes \mathscr{O}_P(1)$. Because of the short exact sequence in (6.5), it is then immediate that $\pi_*\mathscr{O}_{\mathfrak{X}}\big(K_{\mathfrak{X}/P}\big)$ is both locally free and nef. $\square$

Note. The result in Proposition 6.2.2 illustrates the principle that $\pi\colon \mathfrak{X} \to P$ already has many of the good properties of a family with unipotent monodromies and normal crossing boundary, provided the line bundle $\mathscr{L}$ is sufficiently ample.

6.3 A duality theorem for the graded quotients

We are now going to consider the individual sheaves $F_k\mathcal{M}$ in the filtration of the $\mathcal{D}$-module $(\mathcal{M}, F)$. Each of them is, of course, a coherent sheaf on the projective space P; of particular interest are the ones for $k = 1, 2, \ldots, n$, because $F_k\mathcal{M}$ naturally extends the Hodge bundle $F^{n-k}\mathcal{V}_{van}^{n-1}$. For the Hodge bundles, we have a duality isomorphism

$$Gr_F^{n-k}\mathcal{V}_{van}^{n-1} \simeq \left(Gr_F^{k-1}\mathcal{V}_{van}^{n-1}\right)^{\vee},$$

induced by the intersection pairing on each nonsingular hypersurface $\mathfrak{X}_p$. In this section, we extend that result to the sheaves $Gr_k^F\mathcal{M}$. Although there is no longer an isomorphism, we do still find a close relationship between $Gr_k^F\mathcal{M}$ and $\left(Gr_{n+1-k}^F\mathcal{M}\right)^{\vee}$. In addition, we compute the sheaves $\mathcal{E}xt^i\left(Gr_k^F\mathcal{M}, \mathscr{O}_P\right)$, for all $i \geq 0$.

The method we will use was suggested by Green; it relies on the fact that the set $\mathfrak{Z} \subseteq \mathfrak{X}$ of singular points in the hypersurfaces is a local complete intersection in $P \times X$, which allows the use of a Koszul complex.[1] Using a spectral sequence, we shall deduce our duality theorem from the duality inherent in the Koszul complex.

6.3.1 Resolutions by Koszul complexes

Whenever we have a section s of a vector bundle $\mathscr{E}$, we get two Koszul complexes. The first is

$$\bigwedge^{\operatorname{rk}\mathscr{E}}\mathscr{E}^{\vee} \longrightarrow \bigwedge^{\operatorname{rk}\mathscr{E}-1}\mathscr{E}^{\vee} \longrightarrow \cdots \longrightarrow \bigwedge^{2}\mathscr{E}^{\vee} \longrightarrow \mathscr{E}^{\vee} \longrightarrow \mathscr{O}_X,$$

[1] An exposition of this idea, when applied to a single smooth hypersurface, can be found in Lecture 4 of "Infinitesimal methods in Hodge theory" (Green et al., 1994, pp. 39–51).

and is obtained by considering the section as a map of vector bundles $\mathscr{E}^\vee \to \mathscr{O}_X$. It resolves the ideal sheaf $\mathscr{O}_Z$ of the zero scheme $Z = Z(s)$ of the section, provided that $Z \subseteq X$ is a local complete intersection in X. The second Koszul complex is

$$\mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \bigwedge^2 \mathscr{E} \longrightarrow \cdots \longrightarrow \bigwedge^{\operatorname{rk}\mathscr{E}-1} \mathscr{E} \longrightarrow \bigwedge^{\operatorname{rk}\mathscr{E}} \mathscr{E},$$

where the differentials are given by taking the wedge product with s. This second complex is of course just the first complex, tensored by the line bundle $\det \mathscr{E}$, because we have the isomorphisms

$$\det \mathscr{E} \otimes \bigwedge^k \mathscr{E}^\vee \simeq \bigwedge^{\operatorname{rk}\mathscr{E}-k} \mathscr{E}.$$

Under the same assumption on Z, the second complex is therefore a resolution of the sheaf $\det \mathscr{E} \otimes \mathscr{O}_Z$.

The following lemma will give us a resolution of the structure sheaf $\mathscr{O}_{\mathfrak{Z}}$ by a Koszul complex of the first type.

Lemma 6.3.1. *Suppose B is an algebraic variety, and*

$$0 \longrightarrow \mathscr{E}' \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}'' \longrightarrow 0$$

is a short exact sequence of vector bundles on B. Let $p\colon \mathbb{P}(\mathscr{E}) \to B$ be the projectivization of $\mathscr{E}$. Then $\mathbb{P}(\mathscr{E}'') \subseteq \mathbb{P}(\mathscr{E})$ is a local complete intersection, and is in fact the zero scheme of a section s of $\left(p^*\mathscr{E}'\right)^\vee \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. As a consequence, the structure sheaf $\mathscr{O}_{\mathbb{P}(\mathscr{E}'')}$ admits a locally free resolution by the Koszul complex

$$\bigwedge^{\operatorname{rk}\mathscr{E}'} p^*\mathscr{E}' \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-\operatorname{rk}\mathscr{E}') \longrightarrow \cdots \longrightarrow p^*\mathscr{E}' \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}.$$

Proof. The map $p^*\mathscr{E}' \to p^*\mathscr{E} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ gives a section s of the bundle $\bigl(p^*\mathscr{E}'\bigr)^\vee \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$, and it is easily seen that $Z(s) = \mathbb{P}\bigl(\mathscr{E}''\bigr)$. Since this is evidently a local complete intersection, the Koszul complex for the map $p^*\mathscr{E}' \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$ is exact except at the end, and resolves the sheaf $\mathscr{O}_{\mathbb{P}(\mathscr{E}'')}$. $\square$

We may apply this lemma to the case of $\mathfrak{Z} \subseteq \mathfrak{X}$, by virtue of the exact sequence in (5.6). In this case, we have a section of the bundle $\phi^*\Omega^1_X \otimes \mathscr{O}_{\mathfrak{X}}(1)$; it can easily be described more concretely. Let $s_{\mathfrak{X}}$ be the section of $\mathscr{O}_{P\times X}(\mathfrak{X})$ that defines $\mathfrak{X}$; then the section of $\phi^*\Omega^1_X$ is a kind of relative differential of $s_{\mathfrak{X}}$, and so we shall write it in the form $d_X s_{\mathfrak{X}}$. Indeed,

$$\mathscr{O}_{\mathfrak{X}}(-1) \xrightarrow{ds_{\mathfrak{X}}} \Omega^1_{P\times X}\big|_{\mathfrak{X}} \simeq \phi^*\Omega^1_X \oplus \pi^*\Omega^1_P$$

is part of the co-normal sequence for $\mathfrak{X} \subseteq P \times X$, and the section in question is obtained by composing with the projection to the first summand.

From the lemma, we then obtain the following resolution for the structure sheaf $\mathscr{O}_{\mathfrak{Z}}$ on $\mathfrak{X}$:

$$\textstyle\bigwedge^n \phi^*T_X \otimes \mathscr{O}_{\mathfrak{X}}(-n) \longrightarrow \cdots \longrightarrow \phi^*T_X \otimes \mathscr{O}_{\mathfrak{X}}(-1) \longrightarrow \mathscr{O}_{\mathfrak{X}}.$$

As before, we are writing $\mathscr{O}_{\mathfrak{X}}(1) = \mathscr{O}_{\mathbb{P}(i^*T_Q)}(1) = \pi^*\mathscr{O}_P(1) \otimes \phi^*\mathscr{L}$ for the universal line bundle on $\mathfrak{X}$. For our purposes, the second type of Koszul complex will be of greater use. If we tensor it by $\mathscr{O}_{\mathfrak{X}}(n-p)$, we arrive at the following statement.

Lemma 6.3.2. *For every integer k, the Koszul-type complex*

$$\mathscr{O}_{\mathfrak{X}}(k-n) \longrightarrow \phi^*\Omega^1_X \otimes \mathscr{O}_{\mathfrak{X}}(k-n+1) \longrightarrow \cdots \longrightarrow \phi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{X}}(k).$$

gives a locally free resolution on $\mathfrak{X}$ for the sheaf $\psi^*\Omega_X^n \otimes \mathscr{O}_{\mathfrak{Z}}(k)$. The differential in this complex is given by the rule $\beta \mapsto d_X s_{\mathfrak{X}} \wedge \beta$, where $s_{\mathfrak{X}}$ is the section of $\mathscr{O}_{P\times X}(1)$ defining $\mathfrak{X}$ in $P \times X$.

6.3.2 First-order differential operators on a line bundle

For the time being, it is actually more convenient to work on the bigger ambient space $P \times X$. In order to get a resolution for $\mathscr{O}_{\mathfrak{Z}}$ there, we have to introduce the sheaf of first-order differential operators on the line bundle $\mathscr{L}$, together with its dual $\mathscr{E}_{\mathscr{L}} = \mathcal{D}_1(\mathscr{L})^\vee$. We need a few basic properties of the two sheaves, and briefly review those here.

The sheaf $\mathcal{D}_1(\mathscr{L})$ is locally free of rank $n+1$, and part of an exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathcal{D}_1(\mathscr{L}) \longrightarrow T_X \longrightarrow 0.$$

The map to the tangent bundle associates to a first-order operator D its symbol $\sigma(D)$. For any (local) section s of $\mathscr{L}$, and any (local) holomorphic function f, we have the identity

$$D(f \cdot s) = f \cdot Ds + [D, f]s = f \cdot Ds + \sigma(D) f \cdot s.$$

Locally, $\mathcal{D}_1(\mathscr{L})$ is isomorphic to $\mathscr{O}_X \oplus T_X$, and based on the formula above, one easily works out the transition functions. Take an open cover of X, and say U_i and U_j are two open sets over which $\mathscr{L}$ is locally trivial. Set $U_{ij} = U_i \cap U_j$, and let $g_{ij} \in \Gamma\big(U_{ij}, \mathscr{O}_X^\times\big)$ be the transition function from U_i to U_j. Then the transition functions for $\mathcal{D}_1(\mathscr{L})$ are given by

$$\mathscr{O}_{U_{ij}} \oplus T_{U_{ij}} \to \mathscr{O}_{U_{ji}} \oplus T_{U_{ji}}, \quad (a, \xi) \mapsto \big(a - g_{ij}^{-1}(\xi g_{ij}), \xi\big)$$

The dual sheaf will be denoted by $\mathscr{E}_{\mathscr{L}} = \mathcal{D}_1(\mathscr{L})^\vee$; it is part of the dual of the exact sequence above,

$$0 \longrightarrow \Omega^1_X \longrightarrow \mathscr{E}_{\mathscr{L}} \longrightarrow \mathscr{O}_X \longrightarrow 0, \tag{6.7}$$

and its transition functions[2] are

$$\Omega^1_{U_{ij}} \oplus \mathscr{O}_{U_{ij}} \to \Omega^1_{U_{ji}} \oplus \mathscr{O}_{U_{ji}}, \quad (\omega, a) \mapsto \bigl(\omega + a \cdot d \log g_{ij}, a\bigr).$$

Based on this data, it is not hard to show that the extension class of the sequence, in $H^1(X, \Omega^1_X)$, is exactly $2\pi i \cdot c_1(\mathscr{L})$.

Since they will occur frequently in the Koszul complexes below, we shall abbreviate the wedge products by writing

$$\mathscr{E}^p_{\mathscr{L}} = \textstyle\bigwedge^p \mathscr{E}_{\mathscr{L}}.$$

The transition functions for these, relative to the given open cover, are again easily worked out; they are

$$\Omega^p_{U_{ij}} \oplus \Omega^{p-1}_{U_{ij}} \to \Omega^p_{U_{ji}} \oplus \Omega^{p-1}_{U_{ji}}, \quad (\Omega, \omega) \mapsto \bigl(\Omega + d \log g_{ij} \wedge \omega, \omega\bigr).$$

We also need to consider the tensor product bundle $\mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}$. It is again locally isomorphic to $\Omega^1_X \oplus \mathscr{O}_X$, with transition functions

$$\Omega^1_{U_{ij}} \oplus \mathscr{O}_{U_{ij}} \to \Omega^1_{U_{ji}} \oplus \mathscr{O}_{U_{ji}}, \quad (\omega, a) \mapsto \bigl(g_{ij}\omega + a \cdot dg_{ij}, g_{ij}a\bigr). \tag{6.8}$$

Now given any global section $s \in H^0(X, \mathscr{L})$, we obtain a global section of the tensor product, as follows. Locally, the section s is represented by a holomorphic function

[2]Unfortunately, there is a misprint in Green et al. (1994, p. 42) at this point.

f_i on each open set U_i, with $f_i = g_{ij} f_j$ for all i and j. Then (df_i, f_i) patch together under the transition maps in (6.8), and thus give a global section $d\tilde{s}$ of $\mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}$. If the line bundle $\mathscr{L} = \mathscr{O}_X(D)$ comes from a divisor on X, then we can consider $\mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}$ as being locally isomorphic to $\Omega^1_X(D) \oplus \mathscr{O}_X(D)$, allowing first-order poles. We may rewrite the transition functions in the form

$$\Omega^1_{U_{ij}}(D) \oplus \mathscr{O}_{U_{ij}}(D) \to \Omega^1_{U_{ji}}(D) \oplus \mathscr{O}_{U_{ji}}(D), \quad (\omega, a) \mapsto \big(\omega + a \cdot d \log g_{ij}, a\big).$$

Take the section s_D to be the image of 1 under the map $\mathscr{O}_X \to \mathscr{O}_X(D)$. We may then think of f_i as being a local defining equation for D on U_i, and the section $d\tilde{s}_D$ is locally represented by the pair $(d \log f_i, 1)$.

Lastly, we need to know what happens when the section s_D defines a smooth hypersurface D. In that case, we have the following commutative diagram.

$$\begin{tikzcd}
 & \Omega^1_X \arrow[r, equal] \arrow[d, hook] & \Omega^1_X \arrow[d, hook] \\
\mathscr{L}^{-1} \arrow[r, hook] \arrow[d, equal] & \mathscr{E}_{\mathscr{L}} \arrow[r, two heads, "\lambda"] \arrow[d, two heads] & \Omega^1_X(\log D) \arrow[d, two heads, "-\operatorname{Res}"'] \\
\mathscr{L}^{-1} \arrow[r, hook] & \mathscr{O}_X \arrow[r, two heads] & \mathscr{O}_D
\end{tikzcd} \tag{6.9}$$

The map $\mathscr{L}^{-1} \to \mathscr{E}_{\mathscr{L}}$ in the second row is induced by the section $d\tilde{s}_D$. As for the map $\lambda\colon \mathscr{E}_{\mathscr{L}} \to \Omega^1_X(\log D)$, it can be described most conveniently using the local trivializations from above. Over an open set U_i, we can represent sections of $\mathscr{E}_{\mathscr{L}}$ by pairs (ω, a). We then have

$$\lambda(\omega, a) = \omega - a \cdot \frac{d \log f_i}{2\pi i},$$

which makes the middle row in (6.9) exact, and the whole diagram commute.

6.3.3 A second, more convenient Koszul resolution

Using the sheaf $\mathscr{E}_{\mathscr{L}}$, we can now extend the exact sequence in (5.6) to a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r, equal] \arrow[d, hook] & \mathscr{O}_X \arrow[d, hook] & \\
\mathcal{D}_1(\mathscr{L}) \arrow[r, hook] \arrow[d, two heads] & V^\vee \otimes \mathscr{L} \arrow[r, two heads] \arrow[d, two heads] & N_{X\subseteq Q} \arrow[d, equal] \\
T_X \arrow[r, hook] & i^*T_Q \arrow[r, two heads] & N_{X\subseteq Q}
\end{tikzcd}$$

with exact rows and columns; here $V = H^0(X, \mathscr{L})$. Applying Lemma 6.3.1 to the middle row of the diagram, and using that $P \times X = \mathbb{P}\big(V^\vee \otimes \mathscr{L}\big)$, we obtain another Koszul complex

$$\bigwedge^{n+1} pr_X^* \mathcal{D}_1(\mathscr{L}) \otimes \mathscr{O}_{P\times X}(-n-1) \longrightarrow \cdots \longrightarrow pr_X^*\mathcal{D}_1(\mathscr{L}) \otimes \mathscr{O}_{P\times X}(-1) \longrightarrow \mathscr{O}_{P\times X}$$

as a resolution for $\mathscr{O}_{\mathfrak{Z}}$ on $P \times X$. Now note that $\det \mathscr{E}_{\mathscr{L}} \simeq \Omega_X^n$; also note the identity $\bigwedge^r \mathcal{D}_1(\mathscr{L}) \otimes \det \mathscr{E}_{\mathscr{L}} \simeq \mathscr{E}_{\mathscr{L}}^{n+1-r}$. We can thus tensor the complex above by $pr_X^* \det \mathscr{E}_{\mathscr{L}} \otimes \mathscr{O}_{P\times X}(k) = pr_X^*\Omega_X^n \otimes \mathscr{O}_{P\times X}(k)$ to get the following lemma.

Lemma 6.3.3. *The Koszul-type complex*

$$\mathscr{O}_{P\times X}(k-n-1) \longrightarrow pr_X^*\mathscr{E}_{\mathscr{L}} \otimes \mathscr{O}_{P\times X}(k-n) \longrightarrow \cdots \longrightarrow pr_X^*\mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{O}_{P\times X}(k).$$

on $P \times X$ is a locally free resolution of the sheaf $\psi^*\Omega_X^n \otimes \mathscr{O}_{\mathfrak{Z}}(k)$. The differential is given by taking the wedge product with the section $d\tilde{s}_{\mathfrak{X}}$ of $\mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}$, where $s_{\mathfrak{X}}$ is the section of $\mathscr{O}_{P\times X}(1)$ defining $\mathfrak{X}$.

In 5.3.4, we saw that the set $\mathfrak{Z}$ is the projectivized characteristic variety of the filtered holonomic $\mathcal{D}$-module $(\mathcal{M}, F)$. In fact, we showed that the characteristic sheaf $\mathscr{C}_H(\mathcal{M}, F)$, associated to the graded module

$$C_H(\mathcal{M}, F) = \bigoplus_{k \in \mathbb{Z}} Gr_k^F \mathcal{M},$$

is equal to $\Omega^n_{\mathfrak{X}/P}$. It can be proved[3] that this is isomorphic to $\psi^*\Omega^n_X$; therefore,

$$\delta_*\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big) \simeq Gr_k^F \mathcal{M}$$

for all sufficiently large values of k. The resolution in Lemma 6.3.3 now allows us to derive much finer results, especially in the interesting range $1 \leq k \leq n$.

6.3.4 A spectral sequence from the Koszul complex

We obtain results on the graded quotients $Gr_k^F \mathcal{M}$ of the $\mathcal{D}$-module $\mathcal{M}$ by comparing two complexes: the push-forward of the Koszul complex in Lemma 6.3.3, and the push-forward of the relative de Rham complex for $\mathscr{O}_{P\times X}(*\mathfrak{X})$. To begin with, we apply the functor pr_{P*} to the Koszul resolution get a spectral sequence[4]

$$E_1^{p,q} \Longrightarrow R^{p+q}\delta_*\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big); \tag{6.10}$$

the individual terms are given by the formula

$$\begin{aligned} E_1^{p,q} &= R^q pr_{P*}\big(pr_X^*\mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{O}_{P\times X}(k+p)\big) \\ &= H^q\big(X, \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{L}^{k+p}\big) \otimes \mathscr{O}_P(k+p). \end{aligned}$$

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[3]See 194 below.

[4]Recall that $\delta\colon \mathfrak{Z} \to P$ is the restriction of pr_P to the subvariety $\mathfrak{Z}$.

The indexing in the spectral sequence may seem unusual, but the purpose is to make the terms in degree $p+q=i$ compute the i-th higher direct image sheaf.

Provided now that $\mathscr{L}$ is sufficiently ample (in the sense of condition (a) in 5.5.4), we have

$$E_1^{p,q} = 0 \qquad \text{if } p < -k \text{ and } q < n, \text{ or if } p > -k \text{ and } q > 0.$$

Thus the E_1-page of the spectral sequence appears as in Figure 6.3, with only the marked entries being nonzero.

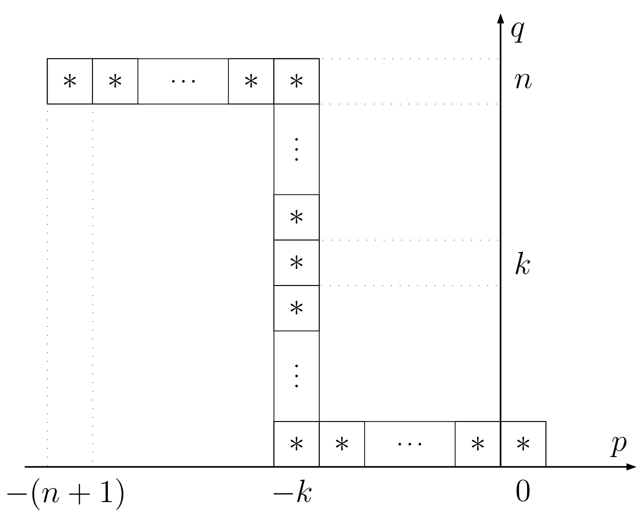

Figure 6.3: The E_1-page of the spectral sequence

We analyze the spectral sequence in three steps: In 6.3.5, we calculate the middle column; in 6.3.6, we study the limit of the lower half (for $q < k$); and, finally, in 6.3.10, we analyze the upper half (for $q \geq k$) of the spectral sequence.

6.3.5 The middle column of the spectral sequence

The first step in our analysis has to be the computation of the (nonzero) middle column, for $p = -k$; of course, it is only there when the value of k is between 0 and $n+1$. It consists of the locally free sheaves

$$E_1^{-k,q} = H^q\big(X, \mathscr{E}_{\mathscr{L}}^{n+1-k}\big) \otimes \mathscr{O}_P;$$

the next lemma expresses these in terms of the cohomology of X.

Lemma 6.3.4. *Let us write $H^{p,q}$ for the cohomology group $H^{p,q}(X) = H^q\big(X, \Omega_X^p\big)$. Then*

$$E_1^{-k,q} = \begin{cases} H^{n+1-k,q}\big/ H^{n-k,q-1} \otimes \mathscr{O}_P & \text{if } q < k, \\ H_{prim}^{n-k,q} \otimes \mathscr{O}_P & \text{if } q \geq k. \end{cases}$$

Here $H_{prim}^{n-k,q} = \ker\big(L\colon H^{n-k,q} \to H^{n-k+1,q+1}\big)$, and L is the Lefschetz operator given by cup product with the class $2\pi i \cdot c_1(\mathscr{L})$.

Proof. We use the exact sequence in (6.7), whose extension class is $2\pi i \cdot c_1(\mathscr{L})$. By taking the $(n+1-k)$-th wedge product, we get

$$0 \longrightarrow \Omega_X^{n+1-k} \longrightarrow \mathscr{E}_{\mathscr{L}}^{n+1-k} \longrightarrow \Omega_X^{n-k} \longrightarrow 0,$$

and thus in cohomology, we have the following exact sequence, with $H^{a,b} = H^{a,b}(X)$ for the sake of brevity:

$$H^{n-k,q-1} \xrightarrow{L} H^{n+1-k,q} \longrightarrow H^q\big(X, \mathscr{E}_{\mathscr{L}}^{n+1-k}\big) \longrightarrow H^{n-k,q} \xrightarrow{L} H^{n+1-k,q+1}$$

If $q < k$, then $n+1-k+q \leq n$, and so the first and last map are injective by the Hard Lefschetz Theorem; this gives the first half of the statement. On the other hand, if $q \geq k$, then $n - k + q \geq n$, so the first map is surjective, while the kernel of the last map is exactly the primitive cohomology. The second half of the statement follows. $\square$

6.3.6 The lower half of the spectral sequence

We can now pass to the second step of the analysis, by considering the "lower" half of the spectral sequence (the part where $q < k$). Note that we are going to have $E_\infty^{p,q} = 0$ for all $p + q < 0$, since the limit of the spectral sequence lives in degrees $p + q \geq 0$.

The bottom row of Figure 6.3 contains the following complex

$$E_1^{-k+1,0} \longrightarrow E_1^{-k+2,0} \longrightarrow \cdots \longrightarrow E_1^{-1,0} \longrightarrow E_1^{0,0},$$

which, more concretely, is the complex of vector bundles

$$H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+2-k} \otimes \mathscr{L}\big) \otimes \mathscr{O}_P(1) \longrightarrow \cdots \longrightarrow H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k\big) \otimes \mathscr{O}_P(k) \tag{6.11}$$

on P. Since this complex will be used a lot, we shall abbreviate it by the symbol $\mathscr{C}_k^\bullet = E_1^{\bullet,0}$; thus

$$\mathscr{C}_k^\bullet = \Big[\mathscr{C}_k^{-k+1} \longrightarrow \mathscr{C}_k^{-k} \longrightarrow \cdots \longrightarrow \mathscr{C}_k^{-1} \longrightarrow \mathscr{C}_k^0\Big], \tag{6.12}$$

where the degree of each sheaf is indicated by the raised index.

We let $\mathscr{F}_k$ be the cohomology sheaf at the end of the complex (for $p = 0$, that is); because $E_\infty^{p,q} = 0$ for $p + q < 0$, the other cohomology sheaves are isomorphic to the sheaves $E_1^{-k,q}$ in the middle column that were computed in Lemma 6.3.4. In addition, the map

$$d_k^{-k,k-1} \colon E_1^{-k,k-1} \simeq H^{n+1-k,k-1}(X)_{prim} \otimes \mathscr{O}_P \to \mathscr{F}_k$$

has to be injective for the same reason. Note that the cokernel of this map is the term $E_{k+1}^{0,0}$. We can reorganize this information slightly, by considering the augmented complex

$$\mathscr{C}_k^{-k+1} \longrightarrow \mathscr{C}_k^{-k+2} \longrightarrow \cdots \longrightarrow \mathscr{C}_k^{0} \longrightarrow\!\!\!\!\!\rightarrow E_{k+1}^{0,0}.$$

Writing $\mathscr{H}_k^p$ for the cohomology sheaf at $\mathscr{C}_k^p$, we then find that

$$\begin{aligned} \mathscr{H}_k^p = E_1^{-k,p+k-1} &\simeq \frac{H^{n+1-k,p+k-1}(X)}{H^{n-k,p+k-2}(X)} \otimes \mathscr{O}_P \\ &\simeq Gr_F^{n+1-k} \frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)} \otimes \mathscr{O}_P = Gr_{k-n-1}^F \frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)} \otimes \mathscr{O}_P. \end{aligned}$$

We shall prove in a minute (see (6.17) below) that $E_{k+1}^{0,0}$ is precisely the sheaf $Gr_k^F \mathcal{M}$. In summary, we can then say that, once we reach the E_{k+1}-page, the lower part of the spectral sequence looks as in Figure 6.4.

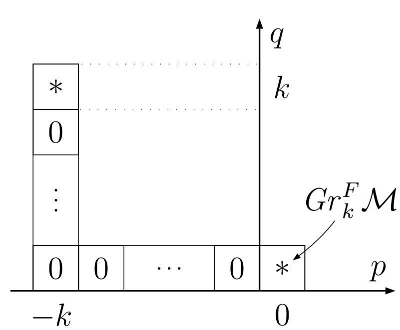

Figure 6.4: The lower half of the E_{k+1}-page

6.3.7 Connections with the de Rham complex

Now let $\mathscr{L} = \mathscr{O}_X(D)$ be the line bundle associated to a divisor $D \subseteq X$, and write s_D for the canonical section of $\mathscr{O}_X(D)$. As in 6.3.2, let $d\tilde{s}_D$ be the corresponding section of $\mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}$. For any values of p and k, we then have a map from $\mathscr{E}^p_{\mathscr{L}} \otimes \mathscr{L}^k$ to $\mathscr{E}^{p+1}_{\mathscr{L}} \otimes \mathscr{L}^{k+1}$, by taking the wedge product with $d\tilde{s}_D$. The purpose of this section is to compare that map, and the differential in the de Rham complex $\Omega^\bullet_X(*D)$.

We consider the following diagram, where d means the usual differential of the de Rham complex, and $d^p_k(x) = d\tilde{s}_D \wedge x$ is the differential in the Koszul complex.

$$\begin{tikzcd}
\Omega^p_X(kD) \arrow[r, "i^p_k"] & \mathscr{E}^p_{\mathscr{L}} \otimes \mathscr{L}^k \arrow[r, "q^p_k"] \arrow[d, "d^p_k"] & \Omega^{p-1}_X(kD) \arrow[d, "d"] \\
& \mathscr{E}^{p+1}_{\mathscr{L}} \otimes \mathscr{L}^{k+1} \arrow[r, "q^{p+1}_{k+1}"] & \Omega^p_X\big((k+1)D\big)
\end{tikzcd}$$

The reader should note that the square is *not* commutative; the next lemma explains to what extent commutativity fails.

Lemma 6.3.5. *In the notation of the above diagram, let*

$$c^p_k = k \cdot q^{p+1}_{k+1} \circ d^p_k - d \circ q^p_k.$$

Then the image of c^p_k is contained in the subsheaf $\Omega^p_X(kD) \subseteq \Omega^{p+1}_X\big((k+1)D\big)$. Moreover, we have $c^p_k \circ i^p_k = k \cdot \mathrm{id}$.

Proof. The proof consists in a local computation, on one of the open sets U_i in the trivialization for $\mathscr{L}$. We use the same notation as in 6.3.2. As explained there, we let f_i be a local defining equation for D on U_i, and then $d\tilde{s}_D$ is represented by the pair (df_i, f_i).

Now take an arbitrary section of $\mathscr{E}^p_{\mathscr{L}} \otimes \mathscr{L}^k$ on U_i; it can be represented by a pair (β, α), where β is a p-form on U_i, and α a $(p-1)$-form. The projection q^p_k takes the pair (β, α) to the rational form α / f_i^k, and so

$$d\big(q^p_k(\beta, \alpha)\big) = d\left(\frac{\alpha}{f_i^k}\right) = \frac{d\alpha}{f_i^k} - k\frac{df_i}{f_i^{k+1}} \wedge \alpha.$$

On the other hand, noting that $(df_i, 0) \wedge (0, \alpha) = (0, -df_i \wedge \alpha)$, we compute that

$$d^p_k(\beta, \alpha) = (df_i, f_i) \wedge (\beta, \alpha) = (df_i \wedge \beta, f_i\beta - df_i \wedge \alpha),$$

and under the map q^{p+1}_{k+1}, this goes to the section

$$q^{p+1}_{k+1}\big(d^p_k(\beta, \alpha)\big) = \frac{f_i\beta - df_i \wedge \alpha}{f_i^{k+1}} = \frac{\beta}{f_i^k} - \frac{df_i}{f_i^{k+1}} \wedge \alpha$$

of $\Omega^p_X\big((k+1)D\big)$. By combining both expressions, we arrive at

$$c^p_k(\beta, \alpha) = k \cdot q^{p+1}_{k+1}\big(d^p_k(\beta, \alpha)\big) - d\big(q^p_k(\beta, \alpha)\big) = k \cdot \frac{\beta}{f_i^k} - \frac{d\alpha}{f_i^k}. \tag{6.13}$$

This is a p-form with a pole of order at most k, and is therefore contained in the subsheaf $\Omega^p_X(kD)$, as asserted.

The same calculation proves that $c^p_k \circ i^p_k = k \cdot \mathrm{id}$. For suppose we start from a section β / f_i^k of $\Omega^p_X(kD)$ on the open set U_i. Then the corresponding pair is $(\beta, 0)$, and the formula in (6.13) shows that we get k times the original section back upon application of c^p_k. ☐

6.3.8 Relation to the $\mathcal{D}$-module

To proceed, and to justify writing $Gr^F_k \mathcal{M}$ in place of $E^{0,0}_{k+1}$ we need to relate the complex $\mathscr{C}^\bullet_k$ to the direct image of the relative de Rham complex in (5.26),

$$F_s pr_{P*} \operatorname{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big),$$

whose cohomology sheaves $F_s\mathscr{N}^q$ we also computed before (in Theorem 5.5.1). Of course, the relationship comes from the fact that $\mathscr{E}_{\mathscr{L}}$ is an extension of $\mathscr{O}_X$ by Ω^1_X; the following lemma gives a precise statement.

Lemma 6.3.6. *The complex $\mathscr{C}_k^\bullet$ is isomorphic to the mapping cone complex of*

$$F_{k-n-2}pr_{P*}\,\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big) \longrightarrow F_{k-n-1}pr_{P*}\,\mathrm{DR}_{P\times X/P}\big(\mathscr{O}_{P\times X}(*\mathfrak{X})\big).$$

Proof. The isomorphism is essentially a formal consequence of Lemma 6.3.5; to make this clearer, we shall use similar notation. Let us write the displayed mapping of complexes in the form $\mathscr{A}^\bullet \to \mathscr{B}^\bullet$; In other words, for each value of p, we let

$$\mathscr{A}^p = \begin{cases} H^0\big(X, \Omega_X^{n+p} \otimes \mathscr{L}^{k+p-1}\big) \otimes \mathscr{O}_P(k+p-1) & \text{if } k+p-1 \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

which occurs in the direct image of the relative de Rham complex for $s = k-n-1$; and, similarly, we let

$$\mathscr{B}^p = \begin{cases} H^0\big(X, \Omega_X^{n+p} \otimes \mathscr{L}^{k+p}\big) \otimes \mathscr{O}_P(k+p) & \text{if } k+p \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

which occurs when $s = k-n-2$. We shall also abbreviate by writing $\mathscr{C}^p = \mathscr{C}_k^p$; then

$$\mathscr{C}^p = \begin{cases} H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{L}^{k+p}\big) \otimes \mathscr{O}_P(k+p) & \text{if } k+p \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each p, we thus have a short exact sequence

$$0 \longrightarrow \mathscr{A}^{p+1} \longrightarrow \mathscr{C}^p \longrightarrow \mathscr{B}^p \longrightarrow 0. \tag{6.14}$$

We now arrive at the following diagram, where d stands for the differential induced from the relative de Rham complex, and $d^p(\beta) = d\tilde{s}_{\mathfrak{X}} \wedge \beta$ is the differential in the Koszul complex.

$$\begin{tikzcd} \mathscr{A}^{p+1} \arrow[r, hook, "i^p"] & \mathscr{C}^p \arrow[r, two heads, "q^p"] \arrow[d, "d^p"] & \mathscr{B}^p \arrow[d, "d"] \\ & \mathscr{C}^{p+1} \arrow[r, two heads, "q^{p+1}"] & \mathscr{B}^{p+1} \end{tikzcd}$$

The first row in the diagram is exact, but—just as in 6.3.7—the square is *not* commutative. For each $p \in P$, however, we do have the result of Lemma 6.3.5, taking the divisor as $D = \mathfrak{X}_p$. By considering the diagram fiber-wise, we therefore see that

$$c^p = (k+p) \cdot q^{p+1} \circ d^p - d \circ q^p$$

takes its image in $\mathscr{A}^{p+1}$, and that $c^p \circ i^p = (k+p)\,\mathrm{id}$.

Said differently, the map $c^p \colon \mathscr{C}^p \to \mathscr{A}^{p+1}$ essentially provides a splitting of the short exact sequence in (6.14). Note that all three terms in the sequence are zero for $k+p \leq 0$. It is then a purely formal exercise to check that

$$\mathscr{C}^p \to \mathrm{Cone}^p(\mathscr{A}^\bullet \to \mathscr{B}^\bullet) = \mathscr{A}^{p+1} \oplus \mathscr{B}^p, \qquad x \mapsto (k+p-1)! \cdot \big(c^p(x), q^p(x)\big)$$

defines an isomorphism of complexes[5] between $\mathscr{C}^\bullet$ and $\mathrm{Cone}(\mathscr{A}^\bullet \to \mathscr{B}^\bullet)$. Going back to our special situation, we get the assertion of the lemma. $\square$

Now consider the map of complexes in the lemma. By strictness of the Hodge filtration on direct images under the projective morphism pr_P (Saito, 1993, Remark 4.6 on

[5]To make things consistent, the differential $d_{\mathrm{Cone}}(a,b) = \big(-d_A(a), a + d_B(b)\big)$ is used in the cone complex. In our case, both d_A and d_B are induced by the relative differential $d_{P\times X/P}$.

p. 72), the cohomology sheaves of the first inject into those of the other; the lemma now lets us conclude that

$$\mathscr{H}_k^p \simeq Gr_{k-n-1}^F \mathcal{N}^p \simeq Gr_{k-n-1}^F \frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)} \otimes \mathscr{O}_P \tag{6.15}$$

for $p < 0$, which agrees with our findings in Theorem 5.5.1. More importantly, we obtain the isomorphism

$$\mathscr{F}_k \simeq Gr_{k-n-1}^F \mathcal{N}^0. \tag{6.16}$$

From the short exact sequence in Theorem 5.5.2, we get, by taking one graded piece,

$$0 \longrightarrow H^{n+1-k,k-1}(X)_{prim} \otimes \mathscr{O}_P \longrightarrow Gr_{k-n-1}^F \mathcal{N}^0 \longrightarrow Gr_k^F \mathcal{M} \longrightarrow 0.$$

Since the first term in this sequence is exactly $\mathscr{H}_k^0$, it now follows that

$$E_{k+1}^{0,0} = \mathscr{F}_k \big/ \mathscr{H}_k^0 \simeq Gr_k^F \mathcal{M} \tag{6.17}$$

This completes our analysis of the lower half of the spectral sequence. We summarize the results obtained so far in the following lemma.

Lemma 6.3.7. *Let $\mathscr{C}_k^\bullet$ be the complex of locally free sheaves on P, concentrated in degrees $-k+1 \leq p \leq 0$, with terms*

$$\mathscr{C}_k^p = H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{L}^{k+p}\big) \otimes \mathscr{O}_P(k+p).$$

Then the augmented complex $\mathscr{C}_k^\bullet \longrightarrow Gr_k^F \mathcal{M}$ is exact at the end; moreover, its cohomology sheaf in degree $p \leq 0$ is precisely

$$\mathscr{H}_k^p \simeq Gr_{k-n-1}^F \frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)} \otimes \mathscr{O}_P = \frac{H^{n+1-k,p+k-1}(X)}{H^{n-k,p+k-2}(X)} \otimes \mathscr{O}_P.$$

6.3.9 The duality relating upper and lower half

The remainder of the argument uses the natural duality between the lower and upper halves of the spectral sequence; it is essentially the self-duality inherent in the Koszul complex. The following lemma gives an abstract statement.

Lemma 6.3.8. *Let $C_m \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow G$ be a complex of coherent sheaves on a variety P, such that*

1. *each sheaf C_i is locally free;*
2. *the map $C_0 \longrightarrow G$ is surjective;*
3. *the cohomology sheaf H_i at each C_i is also locally free.*

If we let H^i be the cohomology sheaf of the dual complex, $C_0^\vee \longrightarrow C_1^\vee \longrightarrow \cdots \longrightarrow C_m^\vee$, at the term $C_i^\vee$, then the sequence

$$0 \longrightarrow G^\vee \longrightarrow H^0 \longrightarrow H_0^\vee \longrightarrow \mathcal{E}xt^1(G, \mathcal{O}_P) \longrightarrow H^1 \longrightarrow H_1^\vee \longrightarrow \cdots$$

is exact.

Proof. Let $C_{-1} = G$. We take an injective resolution $I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$ of the structure sheaf $\mathcal{O}_P$, and consider the double complex with terms $\mathcal{H}om(C_p, I^q)$. Note that this includes the sheaves $\mathcal{H}om(G, I^q)$, for $p = -1$. As usual, we get two spectral sequences for the double complex. The first, corresponding to the filtration by rows, degenerates at the E_2-page; indeed, each sheaf H_p is locally free, and so $\mathcal{E}xt^q(H_p, \mathcal{O}_P) = 0$ whenever $q > 0$. It follows that the cohomology of the total complex is $H_{p+q}^\vee$ in degree $p + q$.

The second spectral sequence (for the filtration by columns) also has few nonzero entries; we have

$$E_1^{-1,q} = \mathcal{E}xt^q(G, \mathscr{O}_P) \qquad \text{and} \qquad E_1^{p,0} = C_p^\vee \quad \text{for } p \geq 0,$$

while all other entries on the E_1-page are zero. It is then easy to derive the long exact sequence in the statement of the lemma. $\square$

6.3.10 The upper half of the spectral sequence

Finally, we consider the remaining "upper" half of the spectral sequence, consisting of those terms $E_1^{p,q}$ with $q \geq k$. This is the place where the self-dual nature of the Koszul complex becomes apparent, since the terms in the upper half of the spectral sequence for a certain value of k are naturally dual to those in the lower half for the value $n+1-k$. This manifests itself in the following isomorphisms. Firstly, we have for $q \geq k$,

$$E_1^{-k,q} = H^{n-k,q}(X)_{prim} \otimes \mathscr{O}_P \simeq \left(\frac{H^{k,n-q}(X)}{H^{k-1,n-q-1}(X)} \otimes \mathscr{O}_P \right)^\vee = \left(\mathscr{H}_{n+1-k}^{k-q} \right)^\vee.$$

Secondly, we can invoke Serre duality on X to compute the entries in the top row. Indeed, $\Omega_X^n \otimes \big(\mathscr{E}_{\mathscr{L}}^{n+1+p} \big)^\vee \simeq \det \mathscr{E}_{\mathscr{L}} \otimes \big(\mathscr{E}_{\mathscr{L}}^{n+1+p} \big)^\vee \simeq \mathscr{E}_{\mathscr{L}}^{-p}$, and so

$$H^n\big(X, \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{L}^{k+p}\big) \simeq \Big(H^0\big(X, \mathscr{E}_{\mathscr{L}}^{-p} \otimes \mathscr{L}^{-k-p}\big) \Big)^\vee.$$

This means that

$$\begin{aligned} E_1^{p,n} &= H^n\big(X, \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{L}^{k+p}\big) \otimes \mathscr{O}_P(k+p) \\ &\simeq \Big(H^0\big(X, \mathscr{E}_{\mathscr{L}}^{-p} \otimes \mathscr{L}^{-k-p}\big) \otimes \mathscr{O}_P(-k-p) \Big)^\vee = \Big(\mathscr{C}_{n+1-k}^{-n-1-p} \Big)^\vee. \end{aligned}$$

The top row of the E_1-page of our spectral sequence is thus precisely the dual of the complex $\mathscr{C}^\bullet_{n+1-k}$, while the upper half of the middle column consists of the duals of the cohomology sheaves $\mathscr{H}^p_{n+1-k}$.

If we abstract a little, we arrive at the statement of Lemma 6.3.8; the situation at hand has $m = n - k$, and $C_i = \mathscr{C}^{-i}_{n+1-k}$, and $G = Gr^F_{n+1-k}\mathcal{M}$. In the notation of the lemma, the upper half of the spectral sequence thus has the shape shown in Figure 6.5.

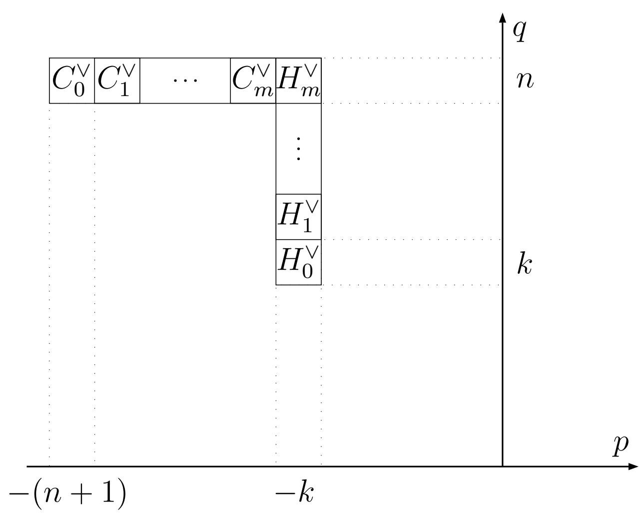

Figure 6.5: The upper half of the E_1-page

Of course, starting from the E_{n+1-k}-page, there will not be any differentials between entries in the upper half of the spectral sequence. From the long exact sequence in Lemma 6.3.8, we conclude that the upper part converges (for $p+q \geq 0$) to $\mathcal{E}xt^{p+q+1}(G, \mathscr{O}_P) = \mathcal{E}xt^{p+q+1}\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big)$. Moreover, we are going to have

$$E^{-n-1,n}_{n+1-k} = G^\vee = \big(Gr^F_{n+1-k}\mathcal{M}\big)^\vee.$$

With this, we have completely analyzed the upper half of the spectral sequence.

6.3.11 Conclusion of the argument

We can now put the results of 6.3.4–6.3.10 together and draw a useful conclusion. As a result of our analysis, the E_{n+1}-page of the spectral sequence appears as in Figure 6.6.

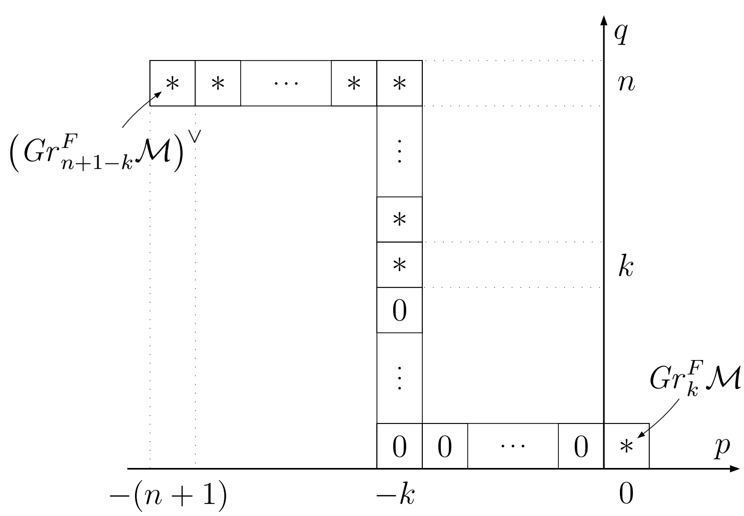

Figure 6.6: The E_{n+1}-page of the spectral sequence

The entries for $p+q \geq 0$ and $q \geq k$ have stabilized, and the upper half of the spectral sequence has as its limit the sheaves $\mathcal{E}xt^{p+q+1}\bigl(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\bigr)$. But, since we already know that the spectral sequence converges to $R^{p+q}\delta_*\bigl(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\bigr)$, we obtain

$$\mathcal{E}xt^{i+1}\bigl(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\bigr) \simeq R^i\delta_*\bigl(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\bigr) \qquad \text{for all } i \geq 1. \tag{6.18}$$

Moreover, there is only one nonzero differential, namely

$$d_{n+1}^{-n-1,n} \colon E_{n+1}^{-n-1,n} \simeq \bigl(Gr^F_{n+1-k}\mathcal{M}\bigr)^\vee \to E^{0,0}_{n+1} \simeq Gr^F_k\mathcal{M}.$$

This map has to be injective (because the spectral sequence converges to zero for $p+q<0$), and its cokernel, $E^{0,0}_\infty$ is part of the filtration for the limit $\delta_*\bigl(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\bigr)$ in degree $p+q=0$. The other part is given by the limit $\mathcal{E}xt^1\bigl(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\bigr)$ that we found in our analysis of the upper half. In other words, we also have a four-term exact sequence

$$\begin{tikzcd}
\bigl(Gr^F_{n+1-k}\mathcal{M}\bigr)^\vee \arrow[r, hook] & Gr^F_k\mathcal{M} \arrow[r] & \delta_*\bigl(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\bigr) \arrow[d] \\
 & & \mathcal{E}xt^1\bigl(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\bigr)
\end{tikzcd} \tag{6.19}$$

What (6.19) shows is that not every section of $Gr^F_k\mathcal{M}$ extends to a linear functional on $Gr^F_{n+1-k}\mathcal{M}$; there is a sort of growth condition near points of $X^\vee$, which is measured by the sheaf $\delta_*\bigl(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\bigr)$.

6.4 More about the duality theorem

In this section, we first verify that the left-most map in (6.19) is the expected one, coming from the intersection pairing. We then reformulate the duality result using the language of the derived category, and deduce several global consequences.

6.4.1 Relating the duality to the intersection pairing

As we have seen before, $Gr_k^F\mathcal{M}$ is a natural extension of the Hodge bundle $Gr_F^{n-k}\mathcal{V}_{van}^{n-1}$ on P^{sm} to a coherent sheaf on P; similarly, $Gr_{n+1-k}^F\mathcal{M}$ is an extension of $Gr_F^{k-1}\mathcal{V}_{van}^{n-1}$. The two Hodge bundles are evidently dual to each other; the duality is induced by the intersection pairing on each smooth hypersurface $\mathfrak{X}_p$. From this point of view, it is not surprising that $Gr_{n+k-1}^F\mathcal{M}$ and $Gr_k^F\mathcal{M}$ should be related through duality. Unfortunately, it is not clear from our analysis that the map

$$\Delta_k \colon \bigl(Gr_{n+1-k}^F\mathcal{M}\bigr)^\vee \to Gr_k^F\mathcal{M}$$

in (6.19) is given by the intersection pairing. This is certainly very plausible; but Δ_k was obtained as a differential of the spectral sequence in (6.10), and so the map might be something else. The purpose of this section is to prove that Δ_k is indeed the expected map. Here is the precise statement.

Lemma 6.4.1. *Let U be an open set in P, and φ any section of $\bigl(Gr_{n+1-k}^F\mathcal{M}\bigr)^\vee$ over U. Then we have*

$$\varphi_p(\beta_p) = \frac{\pm 1}{(2\pi i)^{n-1}} \int_{\mathfrak{X}_p} \Delta_{k,p}(\varphi_p) \cup \beta_p$$

for any $\beta \in \Gamma\bigl(U, Gr_{n+1-k}^F\mathcal{M}\bigr)$, and any point $p \in P^{sm}$.

Note. There are actually two ways of identifying $H^n\bigl(X, \Omega_X^n\bigr)$ with $\mathbb{C}$. One is via the trace map from duality theory; the other comes about by representing elements of $H^n\bigl(X, \Omega_X^n\bigr)$ as smooth (n,n)-forms, using the Dolbeault isomorphism, and then integrating over X. The relationship between the two isomorphisms is worked out, for instance, in Sastry and Tong (2003); the result is that

$$\frac{\pm 1}{(2\pi i)^{\dim X}} \int_X (\_)\colon H^n\bigl(X, \Omega_X^n\bigr) \to \mathbb{C}$$

is the trace map in Serre duality. The value of the sign (a function of $\dim X$ only) depends on the set of sign conventions used. Since this is irrelevant for our purposes, we shall content ourselves by giving all results in this section only up to a sign.

The proof of Lemma 6.4.1 occupies the remainder of this section. A first observation is that it suffices to deal with one smooth hypersurface $\mathfrak{X}_p$ at a time. Indeed, the spectral sequence in (6.10) is clearly compatible with restricting to smooth $\mathfrak{X}_p$; the same is therefore true for the differential $d_{n+1}^{-n-1,n}$ that gave rise to the map Δ_k. For the remainder of this section, we may thus fix a point $p \in P^{sm}$, and consider the hypersurface $D = \mathfrak{X}_p$. Let $i\colon D \to X$ be the inclusion map.

In this setting, we naturally have two Koszul complexes, both exact (because D does not meet the singular locus $\mathfrak{Z}$). The first one comes from the complex in Lemma 6.3.3, by restricting to the slice $\{p\} \times X$; it has the form

$$\mathscr{L}^{k-n-1} \longrightarrow \mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}^{k-n} \longrightarrow \cdots \longrightarrow \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^{k}. \tag{6.20}$$

Evidently, this is a complex of locally free sheaves on X; the differential is given by taking the wedge product with the section $d\tilde{s}_D$ of $\mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}$. (As usual, s_D is the section of $\mathscr{L}$ defining the divisor D.)

The second one is obtained by restricting the Koszul complex in Lemma 6.3.2 to $D = \pi^{-1}(p)$; in other words, it is the complex

$$i^*\mathscr{L}^{k-n} \longrightarrow i^*\bigl(\Omega_X^1 \otimes \mathscr{L}^{k-n+1}\bigr) \longrightarrow \cdots \longrightarrow i^*\bigl(\Omega_X^n \otimes \mathscr{L}^{k}\bigr) \tag{6.21}$$

of locally free sheaves on D itself. Here the section ds_D of $i^*\big(\Omega_X^1 \otimes \mathscr{L}\big)$ defines the differential.

Our proof depends on comparing the two complexes. Since both are exact, they are certainly quasi-isomorphic. An explicit quasi-isomorphism between them is given by the next lemma.

Lemma 6.4.2. *For each integer p, let r_p be the composition*

$$\mathscr{E}_{\mathscr{L}}^{p+1} \otimes \mathscr{L}^{k-n+p} \to \Omega_X^p \otimes \mathscr{L}^{k-n+p} \to i^*\big(\Omega_X^p \otimes \mathscr{L}^{k-n+p}\big).$$

Then the maps $(-1)^p r_p$ give a quasi-isomorphism between the complexes in (6.20) *and* (6.21)*.*

Proof. It suffices to prove that the maps $(-1)^p r_p$ form a map of complexes; this amounts to showing that the following diagram is commutative ($m = k - n + p$).

$$\begin{tikzcd}
\mathscr{E}_{\mathscr{L}}^{p+1} \otimes \mathscr{L}^m \arrow[r, "d\tilde{s}_D"] \arrow[d, "r_p"'] & \mathscr{E}_{\mathscr{L}}^{p+2} \otimes \mathscr{L}^{m+1} \arrow[d, "-r_{p+1}"] \\
i^*\big(\Omega_X^p \otimes \mathscr{L}^m\big) \arrow[r, "ds_D"] & i^*\big(\Omega_X^{p+1} \otimes \mathscr{L}^{m+1}\big)
\end{tikzcd}$$

We use local trivializations as in Lemma 6.3.5, letting f_i be a local defining equation for D on a suitable open set U_i. Then a section of $\mathscr{E}_{\mathscr{L}}^{p+1} \otimes \mathscr{L}^m$ is represented by a pair (β, α), where β is a $(p+1)$-form on U_i, and α a p-form. We now have

$$-r_{p+1}\big(d\tilde{s}_D \wedge (\beta, \alpha)\big) = -r_{p+1}\big(df_i \wedge \beta, f_i\beta - df_i \wedge \alpha\big) = df_i \wedge \alpha,$$

because multiples of f_i are in the kernel of r_{p+1}. On the other hand, $ds_D = df_i$ on U_i, and so

$$ds_D \wedge r_p(\beta, \alpha) = df_i \wedge \alpha,$$

proving commutativity of the diagram. $\square$

To abbreviate, let us agree to write

$$M_k = \big(Gr_k^F \mathcal{M}\big)_p \quad \text{and} \quad M_{n+1-k}^\vee = \big(Gr_{n+1-k}^F \mathcal{M}\big)_p^\vee$$

for the fibers of the two sheaves that we are trying to compare; at p, both are locally free, and so M_k and $M_{n+1-k}^\vee$ are complex vector spaces. In fact, since D is nonsingular, we have

$$M_k \simeq H^{n-k,k-1}(D) \simeq H^{k-1}\big(D, \Omega_D^{n-k}\big),$$

and a similar description for the other space. To prove Lemma 6.4.1, we have to show that the map

$$\Delta_k = \Delta_{k,p} \colon M_{n+1-k}^\vee \to M_k$$

is given by integration over D. For the time being, we can say that Δ_k is certainly an isomorphism. (This follows from the four-term sequence in (6.19), because $\operatorname{coker} \Delta_k$ is supported over $X^\vee$, whereas $p \in P^{sm}$.)

The relationship between M_k and the complex in (6.21) is easily worked out. The complex is based on the co-normal sequence

$$0 \longrightarrow i^*\mathscr{L}^{-1} \xrightarrow{d} i^*\Omega_X^1 \longrightarrow \Omega_D^1 \longrightarrow 0; \tag{6.22}$$

it can therefore be broken down into several short exact sequences

$$\Omega_D^{p-1} \otimes i^*\mathscr{L}^{k-n+p-1} \hookrightarrow i^*\big(\Omega_X^p \otimes \mathscr{L}^{k-n+p}\big) \twoheadrightarrow \Omega_D^p \otimes i^*\mathscr{L}^{k-n+p}$$

The associated long exact sequences in cohomology now give us connecting homomorphisms

$$\partial_{n-1-p}\colon H^{n-1-p}\big(D, \Omega_D^p \otimes i^*\mathscr{L}^{k-n+p}\big) \to H^{n-p}\big(D, \Omega_D^{p-1} \otimes i^*\mathscr{L}^{k-n+p-1}\big).$$

Each connecting homomorphism is given by cup product with the extension class ε_D of (6.22); this is an element

$$\varepsilon_D \in \operatorname{Ext}_D^1\big(\Omega_D^1, i^*\mathscr{L}^{-1}\big) \simeq H^1\big(D, T_D \otimes i^*\mathscr{L}^{-1}\big).$$

Using that $H^0\big(D, i^*\big(\Omega_X^n \otimes \mathscr{L}^k\big)\big) \simeq H^0\big(D, \Omega_D^{n-1} \otimes i^*\mathscr{L}^{k-1}\big)$ (by adjunction, i.e., the short exact sequence for $p = n$), and applying all n connecting homomorphisms $\partial_0, \ldots, \partial_{n-1}$, we thus get a map

$$S_k\colon H^0\big(D, i^*\big(\Omega_X^n \otimes \mathscr{L}^k\big)\big) \to H^{n-1}\big(D, i^*\mathscr{L}^{k-n}\big)$$

On the other hand, if we only apply $\partial_0, \ldots, \partial_{k-1}$, then we get

$$R_k\colon H^0\big(D, i^*\big(\Omega_X^n \otimes \mathscr{L}^k\big)\big) \to H^{k-1}\big(D, \Omega_D^{n-k}\big) \simeq M_k.$$

Lemma 6.4.3. *Let $\tilde{\alpha}$ be a section of $i^*\big(\Omega_X^n \otimes \mathscr{L}^k\big) \simeq \Omega_D^{n-1} \otimes i^*\mathscr{L}^{k-1}$, and let $\tilde{\beta}$ be a section of $\Omega_D^{n-1} \otimes i^*\mathscr{L}^{n-k}$. Then we have*

$$\int_D S_k(\tilde{\alpha}) \cup \tilde{\beta} = \int_D R_k(\tilde{\alpha}) \cup R_{n-k}(\tilde{\beta}).$$

Proof. Since $S_k(\tilde{\alpha}) \cup \tilde{\beta}$ is an element of $H^{n-1}\big(D, \Omega_D^{n-1}\big)$, the integration on the left-hand side makes sense. The equality itself is easily proved; just note that $S_k(\tilde{\alpha})$ is obtained by from $\tilde{\alpha}$ by taking the cup product with ε_D exactly n times. Thus

$$S_k(\tilde{\alpha}) \cup \tilde{\beta} = \tilde{\alpha} \cup \varepsilon_D^n \cup \tilde{\beta} = \big(\tilde{\alpha} \cup \varepsilon_D^k\big) \cup \big(\tilde{\beta} \cup \varepsilon_D^{n-k}\big) = R_k(\tilde{\alpha}) \cup R_{n-k}(\tilde{\beta}),$$

which leads to the assertion. ∎

Lemma 6.4.4. *The composition $R_k \circ r_n$, which is a map from $H^0\big(X, \Omega^n_X(kD)\big) \simeq H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k\big)$ to M_k, is the usual residue map.*

Proof. This is explained on pp. 44–5 of Green's lectures in Green et al. (1994). $\square$

Thus M_k is a quotient of both $H^0\big(D, i^*\big(\Omega^n_X \otimes \mathscr{L}^k\big)\big)$ and $H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k\big)$, and the two quotient maps are compatible. By Serre Duality, the space $M^\vee_{n+1-k}$ is then a subspace of both $H^{n-1}\big(D, i^*\mathscr{L}^{n-k}\big)$ and $H^n\big(X, \mathscr{L}^{k-n-1}\big)$, again in a way that is compatible with the obvious map between the two spaces.

The spectral sequence for the first complex (6.20) is similarly based on the short exact sequence

$$0 \longrightarrow \mathscr{L}^{-1} \longrightarrow \mathscr{E}_{\mathscr{L}} \longrightarrow \Omega^1_X(\log D) \longrightarrow 0$$

in (6.9). Arguing as before, we get a natural map

$$S'_k \colon H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k\big) \to H^n\big(X, \mathscr{L}^{k-n-1}\big).$$

The next lemma asserts the compatibility of the three maps S_k, S'_k, and Δ_k.

Lemma 6.4.5. *The following diagram is commutative (up to a sign).*

$$\begin{tikzcd}
H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k\big) \arrow[r, "S'_k"] \arrow[d, "r_n"'] & H^n\big(X, \mathscr{L}^{k-n-1}\big) \\
H^0\big(D, i^*\big(\Omega^n_X \otimes \mathscr{L}^k\big)\big) \arrow[r, "S_k"] \arrow[d, "R_k"'] & H^{n-1}\big(D, i^*\mathscr{L}^{k-n}\big) \arrow[u, hook] \\
M_k \arrow[r, "\Delta_k^{-1}"] & M^\vee_{n+1-k}. \arrow[u, hook]
\end{tikzcd} \tag{6.23}$$

Note that $\Delta_k \colon M^\vee_{n+1-k} \to M_k$ is an isomorphism, because $p \in P^{sm}$.

Proof. The commutativity (up to a sign) of the top square follows almost directly from Lemma 6.4.2, because both maps are defined in terms of connecting homomorphisms. There is one small point, namely what happens at the left end of both complexes. Here, we use the following diagram

$$\begin{tikzcd}
 & \Omega_X^1 \otimes \mathscr{L}^{k-n} \arrow[r, equal] \arrow[d, hook] & \Omega_X^1 \otimes \mathscr{L}^{k-n} \arrow[d, hook] \\
\mathscr{L}^{k-n-1} \arrow[r, hook] \arrow[d, equal] & \mathscr{E}_{\mathscr{L}} \otimes \mathscr{L}^{k-n} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\square"] & \Omega_X^1(\log D) \otimes \mathscr{L}^{k-n} \arrow[d] \\
\mathscr{L}^{k-n-1} \arrow[r, hook] & \mathscr{L}^{k-n} \arrow[r, two heads] & i^*\mathscr{L}^{k-n},
\end{tikzcd}$$

derived from (6.9) (the marked square is only anti-commutative). The $(n+1)$-st connecting map used in the definition of S_k' is the one for the middle row; because the diagram commutes (up to a sign), it is compatible with the map from $H^{n-1}\big(D, i^*\mathscr{L}^{k-n}\big)$ to $H^n\big(X, \mathscr{L}^{n-k-1}\big)$ derived from the bottom row. Thus the top square of (6.23) commutes up to a sign.

For the outer square of (6.23), note that $R_k \circ r_n$ is the residue map by Lemma 6.4.4; thus it is the map from $H^0\big(X, \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k\big)$ to M_k that occurs in the spectral sequence for the complex (6.20). The map Δ_k is induced by the differential $d_{n+1}^{-n-1,n}$. Now the differentials in a spectral sequence for an exact complex are given, quite generally, by the inverses (on suitable sub-quotients) of the various connecting homomorphisms. Since the ∂_p were also used to define the map S_k, the outer square has to commute. Finally, note that the left-hand arrows r_n and R_k in (6.23) are surjective, and therefore the bottom square is automatically commutative (up to a sign) as well. $\square$

It is now easy to complete the proof of Lemma 6.4.1. Recall that we only need to prove the statement for one smooth hypersurface $D = \mathfrak{X}_p$ at a time, where we may write $\Delta_k = \Delta_{k,p}$. Let $\varphi \in M_{n+1-k}^{\vee}$ be an arbitrary element, and put $\alpha = \Delta_k(\varphi)$. We may lift α to a section $\tilde{\alpha}$ of $i^*\bigl(\Omega_X^n \otimes \mathscr{L}^k\bigr)$; then $R_k(\tilde{\alpha}) = \alpha$. Because the diagram in Lemma 6.4.5 is commutative (up to a sign), we then have

$$S_k(\tilde{\alpha}) = \pm\Delta_k^{-1}(\alpha) = \pm\varphi.$$

Now let $\beta \in M_{n+1-k}$ be an arbitrary vector; let $\tilde{\beta}$ be such that $R_{n-k}(\tilde{\beta}) = \beta$. Using the identity in Lemma 6.4.3, and the fact that Serre Duality is given (up to a factor) by integration over D, we compute that

$$\begin{aligned}\varphi(\beta) &= \frac{\pm 1}{(2\pi i)^{n-1}} \int_D S_k(\tilde{\alpha}) \cup \tilde{\beta} = \frac{\pm 1}{(2\pi i)^{n-1}} r \int_D R_k(\tilde{\alpha}) \cup R_{n-k}(\tilde{\beta}) \\ &= \frac{\pm 1}{(2\pi i)^{n-1}} \int_D \alpha \cup \beta = \frac{\pm 1}{(2\pi i)^{n-1}} \int_D \Delta_k(\phi) \cup \beta\end{aligned}$$

This finishes the proof of Lemma 6.4.1. ∎

We also note the following consequence of the proof. The factor of $\pm(2\pi i)$ is there because $\dim X - \dim D = 1$, and because we have not been keeping track of the signs carefully.

Lemma 6.4.6. *Let α' be a section of $\mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{L}^k \simeq \Omega_X^n \otimes \mathscr{L}^k$, and let β' be a section of $\Omega_X^n \otimes \mathscr{L}^{n+1-k}$. Then we have*

$$\int_X S_k'(\alpha') \cup \beta' = \pm(2\pi i) \int_D \operatorname{Res}_D(\alpha') \cup \operatorname{Res}_D(\beta').$$

Proof. As before, the term $S_k'(\alpha') \cup \beta'$ is an element of $H^n\bigl(X, \Omega_X^n\bigr)$, and as such can be integrated over X. Noting that $R_k\bigl(r_n(\alpha')\bigr) = \operatorname{Res}_D(\alpha')$ by Lemma 6.4.4, the identity follows from the preceding computations. ∎

Note. The map S'_k gives rise to a pairing between the spaces $H^0\big(X, \Omega_X^n \otimes \mathscr{L}^k\big)$ and $H^0\big(X, \Omega_X^n \otimes \mathscr{L}^{n+1-k}\big)$; as we have seen, the pairing is perfect on the quotients M_k and M_{n+1-k}. Of course, this is an instance of the Generalized Macaulay's Theorem (Green, 1985, Theorem 2.15 on p. 145), which is derived in a similar fashion. Green informs me that the statement in Lemma 6.4.6 has been established by Carlson, although the proof does not seem to have appeared in print. For the convenience of the reader, a proof has therefore been included here.

6.4.2 Other results obtainable by the same method

We gave a fairly detailed analysis of the spectral sequence in 6.3.4–6.3.11, to illustrate the general method. We will now state another result that can be derived in the same way, this time omitting the proof.

To begin with, we need a formula for the canonical bundle on $\mathfrak{Z}$. Since the latter is a projective bundle over X, this is easily found. In general, we have

$$\omega_{\mathbb{P}(\mathscr{E})} \simeq p^*\big(\omega_X \otimes \det \mathscr{E}\big) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-\operatorname{rk} \mathscr{E})$$

if $p \colon \mathbb{P}(\mathscr{E}) \to X$ is a projective bundle. In our case, $\mathfrak{Z} = \mathbb{P}(N_{X \subseteq Q})$, and so

$$\omega_{\mathfrak{Z}} \simeq \psi^* \omega_X^{\otimes 2} \otimes \delta^* \omega_P \otimes \mathscr{O}_{\mathfrak{Z}}(n+1).$$

By going through a similar analysis of the spectral sequence coming from the Koszul complex resolving $\mathscr{O}_{\mathfrak{Z}}$, we get the following result: for all $i \geq 0$,

$$R^i \delta_* \mathscr{O}_{\mathfrak{Z}} \simeq \mathscr{E}xt^{i+1}\big(\delta_* \omega_{\mathfrak{Z}/P}, \mathscr{O}_P\big). \tag{6.24}$$

As before, the $\mathcal{E}xt$-sheaf is to be taken in the category of $\mathscr{O}_P$-modules. Note also that the higher direct image sheaves $R^i\delta_*\omega_{\mathfrak{Z}/P}$ are zero; this is a special case of Kollár's Vanishing Theorem (Lazarsfeld, 2004a, Remark 4.3.8 on p. 257), because $\mathfrak{Z}$ and $X^\vee$ have the same dimension.

Note. Of course, it is not necessary to use a spectral sequence for deriving (6.24), since the isomorphism is a simple consequence of the Duality Theorem for the morphism $\delta\colon \mathfrak{Z} \to P$.

6.4.3 The Duality Theorem for a projective morphism

We are now going to recast the results of 6.3.11 in the language of the derived category. In the process, we are going to use the Duality Theorem for projective morphisms (Hartshorne, 1966, Theorem 11.1 on p. 210). We recall the statement, in the special form that shall be needed below.

Theorem 6.4.7. *Let $f\colon X \to Y$ be a projective morphism between algebraic varieties. Let $\mathbb{F}$ be an element of $D^b(X)$. Then the duality morphism*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X\bigl(\mathbb{F}, f^!\mathscr{O}_Y\bigr) \to \mathbf{R}\mathcal{H}om_Y\bigl(\mathbf{R}f_*\mathbb{F}, \mathscr{O}_Y\bigr)$$

is an isomorphism in $D^b(Y)$.

As for notation, we are essentially following the conventions in "Residues and Duality." Thus $D^b(Y)$ is the bounded derived category of coherent sheaves on Y (the Duality Theorem holds much more generally, but we do not need this generality here). The right-derived functors of a functor are indicated by the symbol $\mathbf{R}$; for instance,

$\mathbf{R}f_*\colon D^b(X) \to D^b(Y)$ is the derived functor of the push-forward. Shifting a complex $\mathbb{F}$ to the left is indicated by the symbol $\mathbb{F}[1]$; thus $\bigl(\mathbb{F}[1]\bigr)^p = \mathbb{F}^{p+1}$. We also use the natural t-structure on the triangulated category $D^b(P)$ (Beĭlinson et al., 1982, Exemples 1.3.2 on p. 29). We write $D^b_{\leq k}(P)$ for the subcategory of complexes that are exact in degrees greater than k, and denote the corresponding truncation functors by $\tau_{\leq k}$. Similarly, we have $D^b_{\geq k}(P)$ and $\tau_{\geq k}$.

To make the Duality Theorem useful, we need to compute the relative dualizing complexes $f^!\mathscr{O}_Y$. These computations are the content of the next lemma; of course, one can get the same result by viewing $\mathfrak{X} \to X$ and $\mathfrak{Z} \to X$ as projective bundles, and using the (relative) Euler sequence.

Lemma 6.4.8. *We have $\pi^!\mathscr{O}_P \simeq \phi^*\omega_X \otimes \mathscr{O}_{\mathfrak{X}}(1)[n-1]$, as objects of $D^b(\mathfrak{X})$. Similarly, we have $\delta^!\mathscr{O}_P \simeq \psi^*\omega_X^{\otimes 2} \otimes \mathscr{O}_{\mathfrak{Z}}(n+1)[-1]$ in $D^b(\mathfrak{Z})$.*

Proof. We factor the map π as shown in the diagram.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "i"] \arrow[dr, "\pi"'] & P \times X \arrow[d, "pr_P"] \\ & P \end{tikzcd}$$

Since the projection $pr_P\colon P \times X \to P$ is a smooth morphism, we have

$$pr_P^!\mathscr{O}_P = \omega_{P\times X/P}[n] \simeq pr_X^*\omega_X[n],$$

the shift by $n = \dim X$ being due to the difference in dimension. Secondly, the inclusion $i\colon \mathfrak{X} \to P \times X$ is a local complete intersection morphism, because $\mathfrak{X}$ is a hypersurface, and so we have

$$i^!\mathscr{E} = i^*\mathscr{E} \otimes \det N_{\mathfrak{X}\subseteq P\times X}[-1] \simeq i^*\mathscr{E} \otimes \mathscr{O}_{\mathfrak{X}}(1)[-1]$$

for any locally free sheaf $\mathscr{E}$ on $P \times X$. Combining both isomorphisms now gives the desired result.

For the analogous computation of $\delta^! \mathscr{O}_P$, we use the factorization

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "j"] \arrow[dr, "\delta"'] & P \times X \arrow[d, "pr_P"] \\ & P. \end{tikzcd}$$

Noting that, because of the diagram in 6.3.3, the normal bundle to $\mathfrak{Z}$ in $P \times X$ is $\psi^* \mathscr{E}_{\mathscr{L}} \otimes \mathscr{O}_{\mathfrak{Z}}(1)$, we have

$$j^! \mathscr{E} = j^* \mathscr{E} \otimes \det N_{\mathfrak{Z} \subseteq P \times X} \simeq j^* \mathscr{E} \otimes \psi^* \omega_X \otimes \mathscr{O}_{\mathfrak{Z}}(n+1)[-n-1].$$

We then conclude as before. $\square$

6.4.4 A formulation using a distinguished triangle

We are now going to show that the exact sequences in 6.3.11 actually come from a distinguished triangle in $D^b(P)$, by reworking the previous proof in the derived category. Just as before, the starting point is the Koszul-type complex

$$\mathscr{O}_{P \times X}(k-n-1) \longrightarrow pr_X^* \mathscr{E}_{\mathscr{L}} \otimes \mathscr{O}_{P \times X}(k-n) \longrightarrow \cdots \longrightarrow pr_X^* \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{O}_{P \times X}(k).$$

from Lemma 6.3.3. We shall write $\mathbb{E}_k$ for the entire complex, supported in degrees $-(n+1), -n, \ldots, 0$, viewing it as an object of $D^b(P \times X)$. Since it is a resolution, we have

$$\mathbb{E}_k \simeq \psi^* \Omega_X^n \otimes \mathscr{O}_{\mathfrak{Z}}(k).$$

To mimic the analysis of the spectral sequence, we let $\mathbb{E}_k^+$ be the complex

$$pr_X^* \mathscr{E}_{\mathscr{L}}^{n-k+2} \otimes \mathscr{O}_{P\times X}(1) \longrightarrow pr_X^* \mathscr{E}_{\mathscr{L}}^{n-k+3} \otimes \mathscr{O}_{P\times X}(2) \longrightarrow \cdots \longrightarrow pr_X^* \mathscr{E}_{\mathscr{L}}^{n+1} \otimes \mathscr{O}_{P\times X}(k)$$

supported in degrees $-k+1, \dots, 0$; this will sometimes be referred to as the "right" end of the Koszul complex. We also let $\mathbb{E}_k^-$ be the complex

$$\mathscr{O}_{P\times X}(k-n-1) \longrightarrow pr_X^* \mathscr{E}_{\mathscr{L}} \otimes \mathscr{O}_{P\times X}(k-n) \longrightarrow \cdots \longrightarrow pr_X^* \mathscr{E}_{\mathscr{L}}^{n-k} \otimes \mathscr{O}_{P\times X}(-1)$$

supported in degrees $-n, \dots, -k$. What might be called the "left" end of the Koszul complex is then $\mathbb{E}_k^-[1]$. The full complex $\mathbb{E}_k$ is consisting of three pieces: the right end $\mathbb{E}_k^+$, the left end $\mathbb{E}_k^-[1]$, and the single sheaf $pr_X^* \mathscr{E}_{\mathscr{L}}^{n+1-k}$ in the middle, in degree $-k$. We write $\mathbb{M}_k = pr_X^* \mathscr{E}_{\mathscr{L}}^{n+1-k}[k-1]$.

The following lemma explains the seeming asymmetry in the indexing.

Lemma 6.4.9. *We have* $\mathbf{R}\mathcal{H}om_{P\times X}\big(\mathbb{E}_k^+, pr_P^! \mathscr{O}_P\big) \simeq \mathbb{E}_{n+1-k}^-$*, for all values of* k*.*

Proof. Clearly, $pr_P^! \mathscr{O}_P = pr_X^* \omega_X[n]$, since $pr_P \colon P \times X \to P$ is just the projection to the first factor. Now each sheaf in the complex $\mathbb{E}_k^+$ is locally free, and so

$$\mathbf{R}\mathcal{H}om_{P\times X}\big(\mathbb{E}_k^+, pr_P^! \mathscr{O}_P\big) \simeq \mathcal{H}om_{P\times X}\big(\mathbb{E}_k^+, pr_X^* \omega_X\big)[n]$$

is obtained by applying the functor term-wise. The term in degree p of $\mathbb{E}_k^+$ is the sheaf $pr_X^* \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{O}_{P\times X}(k+p)$; on the other hand, the complex $\mathbb{E}_{n+1-k}^-$ has the sheaf $pr_X^* \mathscr{E}_{\mathscr{L}}^{n+q} \otimes \mathscr{O}_{P\times X}(n-k+q)$ in degree q.

Recall that $\omega_X \simeq \det \mathscr{E}_{\mathscr{L}}$. In the dualized complex, we therefore have

$$\begin{aligned} &\mathcal{H}om_{P\times X}\big(pr_X^* \mathscr{E}_{\mathscr{L}}^{n+1+p} \otimes \mathscr{O}_{P\times X}(k+p), pr_X^* \omega_X\big) \\ &\qquad\qquad \simeq pr_X^* \mathscr{E}_{\mathscr{L}}^{-p} \otimes \mathscr{O}_P(-k-p) = pr_X^* \mathscr{E}_{\mathscr{L}}^{n+q} \otimes \mathscr{O}_P(n-k+q) \end{aligned}$$

in degree $q = -n-p$. But this is exactly the degree q part of $\mathbb{E}^-_{n+1-k}$. The differentials also agree, because of the self-duality of the Koszul complex, and so the lemma is proved. $\square$

The duality between left and right end gives one reason for splitting up the complex $\mathbb{E}_k$ into three pieces instead of two. The second reason is the special nature of the sheaf in the middle.

Lemma 6.4.10. *The object* $\mathbf{R}pr_{P*}\mathbb{M}_k$ *of* $D^b(P)$ *is* split*; in other words, it is isomorphic to a complex with zero differentials.*

Proof. By the Base Change Theorem (Hartshorne, 1966, Proposition 5.12 on p. 111), we have

$$\mathbf{R}pr_{P*}\mathbb{M}_k = \mathbf{R}pr_{P*}\big(pr_X^*\mathscr{E}_{\mathscr{L}}^{n+1-k}\big)[k-1] \simeq pr_P^*\big(\mathbf{R}pr_{X*}pr_X^*\mathscr{E}_{\mathscr{L}}^{n+1-k}\big)[k-1].$$

The term in parentheses is an element of $D^b(pt)$, meaning a complex of vector spaces. It is therefore isomorphic to the complex with terms $H^p\big(X, \mathscr{E}_{\mathscr{L}}^{n+1-k}\big)$ and zero differentials. The same is then true for its pullback under the map pr_P. $\square$

Note. The lemma implies that the canonical distinguished triangle

$$\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k \xrightarrow{[1]} \cdots$$

is also split, meaning that $\mathbf{R}pr_{P*}\mathbb{M}_k \simeq \tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k \oplus \tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k$.

From the Koszul complex $\mathbb{E}_k$, we get a map of complexes $\mathbb{M}_k \longrightarrow \mathbb{E}_k^+$; after pushing forward to P, it induces a map $\mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \mathbf{R}pr_{P*}\mathbb{E}_k^+$. By composing with the canonical arrow $\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \mathbf{R}pr_{P*}\mathbb{M}_k$, coming from the t-structure, we then get a map

$$\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \mathbf{R}pr_{P*}\mathbb{E}_k^+. \tag{6.25}$$

Lemma 6.4.11. *The mapping cone of* (6.25) *is the sheaf $Gr_k^F\mathcal{M}$; in other words, for each value of k, we naturally have a distinguished triangle*

$$\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \mathbf{R}pr_{P*}\mathbb{E}_k^+ \longrightarrow Gr_k^F\mathcal{M} \xrightarrow{[1]} \cdots$$

in the category $D^b(P)$.

Proof. This is essentially the same result as in Lemma 6.3.7; indeed, since each sheaf in the complex $\mathbb{E}_k^+$ is acyclic for the functor pr_{P*}, the complex $\mathscr{C}_k^\bullet$ in that lemma represents $\mathbf{R}pr_{P*}\mathbb{E}_k^+$. We therefore have an augmentation map

$$\mathbf{R}pr_{P*}\mathbb{E}_k^+ \longrightarrow Gr_k^F\mathcal{M},$$

and the cohomology sheaves of the augmented complex are nonzero only in degrees $p \leq 0$, and there equal to

$$\mathscr{H}_k^p = \frac{H^{n+1-k,p+k-1}(X)}{H^{n-k,p+k-2}(X)} \otimes \mathscr{O}_P.$$

But, because of Lemma 6.3.4, we also have

$$\mathcal{H}^p\big(\mathbf{R}pr_{P*}\mathbb{M}_k\big) = H^{p+k-1}\big(X, \mathscr{E}_{\mathscr{L}}^{n+1-k}\big) \otimes \mathscr{O}_P = \frac{H^{n+1-k,p+k-1}(X)}{H^{n-k,p+k-2}(X)} \otimes \mathscr{O}_P$$

for every $p \leq 0$. Therefore $\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k$ is quasi-isomorphic to the augmented complex, and so we get the result. $\square$

A similar result holds for the left end of the Koszul complex. Here we have a map $\mathbb{E}_k^-[-1] \longrightarrow \mathbb{M}_k$, and then by push-forward, $\mathbf{R}pr_{P*}\mathbb{E}_k^-[-1] \longrightarrow \mathbf{R}pr_{P*}\mathbb{M}_k$. By composition with the canonical arrow from $\mathbf{R}pr_{P*}\mathbb{M}_k$ to $\tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k$, we obtain

$$\mathbf{R}pr_{P*}\mathbb{E}_k^-[-1] \longrightarrow \tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k. \tag{6.26}$$

Lemma 6.4.12. *The mapping in* (6.26) *is part of another distinguished triangle*

$$\tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k \longrightarrow \mathbf{R}\mathcal{H}om_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \longrightarrow \mathbf{R}pr_{P*}\mathbb{E}_k^- \xrightarrow{[1]} \cdots$$

dual to the one in Lemma 6.4.11 (for the parameter value $n+1-k$).

Proof. To compute the direct image of $\mathbb{E}_k^-$, we use the Duality Theorem 6.4.7, together with Lemma 6.4.9. We find that

$$\begin{aligned}
\mathbf{R}pr_{P*}\mathbb{E}_k^- &\simeq \mathbf{R}pr_{P*}\mathbf{R}\mathcal{H}om_{P\times X}\big(\mathbb{E}^+_{n+1-k}, pr_P^!\mathscr{O}_P\big) \\
&\simeq \mathbf{R}\mathcal{H}om_P\big(\mathbf{R}pr_{P*}\mathbb{E}^+_{n+1-k}, \mathscr{O}_P\big).
\end{aligned}$$

If we dualize the triangle in Lemma 6.4.11 for $n+1-k$, and use the isomorphism we have just obtained, we arrive at the following distinguished triangle.

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \arrow[rr] & & \mathbf{R}pr_{P*}\mathbb{E}_k^- \arrow[dl] \\
& \mathbf{R}\mathcal{H}om_P\big(\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_{n+1-k}, \mathscr{O}_P\big) \arrow[ul, "{[1]}"] &
\end{tikzcd} \tag{6.27}$$

To simplify that triangle, we are now going to show that there is an isomorphism

$$\mathbf{R}\mathcal{H}om_P\big(\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_{n+1-k}, \mathscr{O}_P\big)[-1] \simeq \tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k.$$

Since both sides are split (by Lemma 6.4.10), is suffices to prove that they have the same cohomology. Note that both complexes are in $D^b_{\geq 1}(P)$. By using the calculations in Lemma 6.3.4, we immediately find that

$$\mathcal{H}^p\Big(\tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k\Big) = H^{p+k-1}\big(X, \mathscr{E}_{\mathscr{L}}^{n+1-k}\big) \otimes \mathscr{O}_P \simeq H^{n-k,p+k-1}(X)_{prim} \otimes \mathscr{O}_P,$$

for all $p \geq 1$. As for the other complex, it is split, and so its cohomology sheaves are just the duals (in the appropriate degrees) of those of $\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_{n+1-k}$. The result is that

$$\begin{aligned}\mathcal{H}^p\Big(\mathbf{R}\mathcal{H}om_P\big(\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_{n+1-k}, \mathscr{O}_P\big)[-1]\Big) &\simeq \Big(\mathcal{H}^{1-p}\big(\tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_{n+1-k}\big)\Big)^{\vee} \\ &\simeq \Big(H^{1-p+n-k}\big(X, \mathscr{E}_{\mathscr{L}}^k\big) \otimes \mathscr{O}_P\Big)^{\vee}\end{aligned}$$

With the help of Lemma 6.3.4, the latter evaluates to

$$\left(\frac{H^{k,n-(p+k-1)}(X)}{H^{k-1,n-(p+k)}(X)} \otimes \mathscr{O}_P\right)^{\vee} \simeq H^{n-k,p+k-1}(X)_{prim} \otimes \mathscr{O}_P,$$

as desired. We can thus rewrite the triangle in (6.27) in the form

$$\mathbf{R}\mathcal{H}om_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \longrightarrow \mathbf{R}pr_{P*}\mathbb{E}_k^- \longrightarrow \big(\tau_{\geq 1}\mathbf{R}pr_{P*}\mathbb{M}_k\big)[1] \xrightarrow{[1]} \cdots$$

The assertion of the lemma follows upon rotating this triangle one step to the right.

$\square$

After these preliminaries, we are now ready to interpret the result of 6.3.11 as coming from a distinguished triangle in $D^b(P)$.

Proposition 6.4.13. *Let k be an arbitrary integer. In $D^b(P)$, we have the following distinguished triangle.*

$$\mathbf{R}\mathcal{H}om_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \longrightarrow Gr^F_k\mathcal{M} \longrightarrow \mathbf{R}\delta_*\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big) \xrightarrow{[1]} \cdots$$

The resulting maps on cohomology are the same as in 6.3.11.

Proof. The existence of this triangle is (essentially) a formal consequence of the previous results. We begin by breaking the Koszul complex up into two triangles. First, we introduce an auxiliary complex $\mathbb{T}_k$, as the mapping cone of $\mathbb{E}_k^-[-1] \longrightarrow \mathbb{M}_k$. We then have two distinguished triangles in $D^b(P \times X)$,

$$\mathbb{E}_k^-[-1] \longrightarrow \mathbb{M}_k \longrightarrow \mathbb{T}_k \longrightarrow \mathbb{E}_k^- \tag{6.28}$$

and

$$\mathbb{T}_k \longrightarrow \mathbb{E}_k^+ \longrightarrow \psi^*\Omega_X^n \otimes \mathscr{O}_3(k) \longrightarrow \mathbb{T}_k[1] \tag{6.29}$$

because the full Koszul complex $\mathbb{E}_k$ (which is the mapping cone of $\mathbb{T}_k \longrightarrow \mathbb{E}_k^+$) is a resolution of $\psi^*\Omega_X^n \otimes \mathscr{O}_3(k)$.

We are now going to apply Lemma 6.4.14, which is a general statement about triangulated categories. In the notation of the lemma, let

$$A = \mathbf{R}\mathcal{H}om_P\big(Gr_{n+1-k}^F\mathcal{M}, \mathscr{O}_P\big), \quad B = Gr_k^F\mathcal{M}, \quad \text{and} \quad C = \mathbf{R}\delta_*\big(\psi^*\Omega_X^n \otimes \mathscr{O}_3(k)\big).$$

Also define three complexes

$$M = \mathbf{R}pr_{P*}\mathbb{M}_k, \quad M' = \tau_{\leq 0}M, \quad \text{and} \quad M'' = \tau_{\geq 1}M,$$

and denote the various parts of the Koszul complex by

$$E^+ = \mathbf{R}pr_{P*}\mathbb{E}_k^+, \quad E^- = \mathbf{R}pr_{P*}\mathbb{E}_k^-, \quad \text{and} \quad T = \mathbf{R}pr_{P*}\mathbb{T}_k.$$

By virtue of (6.28), (6.29), Lemma 6.4.11, and Lemma 6.4.12, we have five distinguished triangles, as required. We shall also denote the various arrows by the same letters that are used in the statement of Lemma 6.4.14.

It remains to verify the two relations that are part of the assumptions of the lemma. We note that the composition $e \circ d$ is just the map from $M = \mathbf{R}pr_{P*}\mathbb{M}_k$ to $E^+ = \mathbf{R}pr_{P*}\mathbb{E}_k^+$ coming from the Koszul complex. In (6.25), we had defined the map g from $M' = \tau_{\leq 0}\mathbf{R}pr_{P*}\mathbb{M}_k$ to E^+ as the composition of a and $e \circ d$, and so the first relation holds. The second relation, $b \circ c = f[-1]$, is dual to the first, because the maps involved were constructed in Lemma 6.4.12 by duality.

Lemma 6.4.14 therefore applies to our situation, and provides us with the desired triangle involving A, B, and C. That the maps are the same as in 6.3.11 is clear; after all, the whole argument is patterned after the analysis of the spectral sequence that we had used there. ∎

Lemma 6.4.14. *Let A, B, C, M, M', M'', E^+, E^-, and T be objects in a triangulated category. Suppose that we are given five distinguished triangles*

$$M' \xrightarrow{a} M \xrightarrow{b} M'' \longrightarrow M'[1] \tag{6.30}$$

$$E^-[-1] \xrightarrow{c} M \xrightarrow{d} T \longrightarrow E^- \tag{6.31}$$

$$T \xrightarrow{e} E^+ \longrightarrow C \longrightarrow T[1] \tag{6.32}$$

$$M'' \longrightarrow A \longrightarrow E^- \xrightarrow{f} M''[1] \tag{6.33}$$

$$M' \xrightarrow{g} E^+ \longrightarrow B \longrightarrow M'[1] \tag{6.34}$$

subject to the condition that $e \circ d \circ a = g$ and $b \circ c = f[-1]$. Then there is a sixth distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

Proof. The proof consists in two applications of the Octahedral Axiom. Step one is to consider the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & M' \arrow[r, "\mathrm{id}"] \arrow[d, "a"] & M' \arrow[r] \arrow[d, "d \circ a"] & 0 \arrow[d] \\
E^-[-1] \arrow[r, "c"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\square"] & M \arrow[r, "d"] \arrow[d, "b"] & T \arrow[r] \arrow[d, dashed] & E^- \arrow[d, "\mathrm{id}"] \\
E^-[-1] \arrow[r, "{f[-1]}"] \arrow[d] & M'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, dashed] & E^- \arrow[d] \\
0 \arrow[r] & M'[1] \arrow[r, "\mathrm{id}"] & M'[1] \arrow[r] & 0
\end{tikzcd}
$$

The second and third row are derived from (6.31) and (6.33), respectively. The second column is the triangle from (6.30), and the square marked $\square$ commutes by hypothesis. According to the Octahedral Axiom, the two broken arrows can be filled in to make the third column a distinguished triangle, and all squares commute (or anti-commute). We thus obtain an auxiliary triangle

$$
M' \xrightarrow{d \circ a} T \longrightarrow A \longrightarrow M'[1], \tag{6.35}
$$

which is again distinguished.

Step two is to incorporate the new triangle into another 3×3-diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C[-1] \arrow[r, "\mathrm{id}"] \arrow[d] & C[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
M' \arrow[r, "d \circ a"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\square"] & T \arrow[r] \arrow[d, "e"'] & A \arrow[r] \arrow[d, dashed] & M'[1] \arrow[d, "\mathrm{id}"'] \\
M' \arrow[r, "g"] \arrow[d] & E^{+} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & M'[1] \arrow[d] \\
0 \arrow[r] & C \arrow[r, "\mathrm{id}"] & C \arrow[r] & 0
\end{tikzcd}$$

The second row is (6.35), the third row is (6.34), and the second column is derived from (6.32). Again, commutativity of the marked square is assumed, and so we get the conclusion by another application of the Octahedral Axiom. $\square$

6.4.5 Several applications of the distinguished triangle

From Proposition 6.4.13, we immediately recover the results of 6.3.11 by taking cohomology. Indeed, both the four-term exact sequence in (6.19), and the isomorphisms in (6.18), follow immediately upon noting that $\mathcal{H}^i Gr^F_k \mathcal{M} = 0$ for $i \neq 0$.

The advantage of having the duality between $Gr^F_{n+1-k}\mathcal{M}$ and $Gr^F_k\mathcal{M}$ expressed as a triangle in $D^b(P)$, is the greater flexibility this offers. As an example, we shall now derive a global version of the duality without any additional effort.

We use the map $a_P \colon P \to pt$ to get a global statement. Recall that $\mathbf{R}a_{P*} \circ \mathbf{R}\delta_* = \mathbf{R}a_{\mathfrak{Z}*}$ (the Leray spectral sequence, in the context of the derived category), and similarly

$\mathbf{R}a_{P*} \circ \mathbf{R}\mathcal{H}om_P = \mathbf{R}\operatorname{Hom}_P$. If we apply the functor $\mathbf{R}a_{P*}$ to the triangle in Proposition 6.4.13, we get the distinguished triangle

$$\mathbf{R}\operatorname{Hom}_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \longrightarrow \mathbf{R}a_{P*}Gr^F_k\mathcal{M} \longrightarrow \mathbf{R}a_{\mathfrak{Z}*}\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big) \xrightarrow{[1]} \cdots$$

The higher cohomology of the sheaves $Gr^F_k\mathcal{M}$ vanishes by Theorem 6.1.2. If we now take cohomology, we obtain the following result.

Proposition 6.4.15. *For each value of k, we have a four-term[6] exact sequence*

$$\begin{tikzcd} \operatorname{Hom}_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \arrow[r, hook] & H^0\big(P, Gr^F_k\mathcal{M}\big) \arrow[r] & H^0\big(\mathfrak{Z}, \psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big) \arrow[d] \\ & & \operatorname{Ext}^1_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big). \end{tikzcd}$$

Moreover, we have $\operatorname{Ext}^{i+1}_P\big(Gr^F_{n+1-k}\mathcal{M}, \mathscr{O}_P\big) \simeq H^i\big(\mathfrak{Z}, \psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big)$ for all $i \geq 1$, and all values of k.

In particular, we can choose the value of k such that $k \geq n+1$; then $Gr^F_{n+1-k}\mathcal{M} = 0$, and so we get an isomorphism

$$H^0\big(P, Gr^F_k\mathcal{M}\big) \simeq H^0\big(\mathfrak{Z}, \psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big).$$

As a byproduct, we obtain another calculation of the characteristic module

$$C_H(\mathcal{M}, F) = \bigoplus_{k\in\mathbb{Z}} H^0\big(P, Gr^F_k\mathcal{M}\big),$$

this time in all degrees $k \geq n+1$. In particular, we find that the characteristic sheaf is $\mathscr{C}_H(\mathcal{M}, F) \simeq \psi^*\Omega^n_X$. When computing the characteristic variety of $(\mathcal{M}, F)$ in Lemma 5.3.3, we had found the answer $\Omega^n_{\mathfrak{X}/P}$; we conclude that $\Omega^n_{\mathfrak{X}/P} \simeq \psi^*\Omega^n_X$.

[6]In Theorem 6.4.17 below, we will show that the first term is actually zero.

6.4.6 Global duality results

To make the result in Proposition 6.4.15 useful, we need to do one further computation.

Lemma 6.4.16. *Let i and s be two integers. Then if $s+i<2d-n$, we have*

$$\operatorname{Ext}^i_P\bigl(F_s\mathcal{N}^0,\mathscr{O}_P\bigr)\simeq\begin{cases}0 & \text{for } i\leq 1,\\ \left(F_s\dfrac{H^{n-i+1}(X)}{H^{n-i-1}(X)(-1)}\right)^{\vee} & \text{for } i\geq 2.\end{cases}$$

Proof. Once again, we consider the complex $\mathscr{B}_s^{\bullet}$ from 6.1.1; according to (6.1), each sheaf $\mathscr{B}_s^p$ is a direct sum of line bundles $\mathscr{O}_P(n+p+s+1)$, if $n+p+s\geq 0$, or zero. From (6.2), we also know that the cohomology sheaf in degree p is $F_s\mathcal{N}^p$; and that, for $p<0$, this equals

$$F_s\mathcal{N}^p\simeq F_s\frac{H^{n+p}(X)}{H^{n+p-2}(X)(-1)}\otimes\mathscr{O}_P. \tag{6.36}$$

Just as before, we get two spectral sequences when applying the functor $\operatorname{Hom}_P(\_\_,\mathscr{O}_P)$ to the complex, both converging to the same limit. The first has terms

$$'E_1^{p,q}=\operatorname{Ext}^q_P\bigl(\mathscr{B}_s^{-p},\mathscr{O}_P\bigr)\simeq H^{d-q}\bigl(P,\mathscr{B}_s^{-p}\otimes\mathscr{O}_P(-d-1)\bigr),$$

by Serre Duality. Since each $\mathscr{B}_s^{-p}$ is a sum of positive line bundles, $'E_1^{p,q}$ can only be nonzero if $q=d$, and also $(-d-1)+(n+p+s+1)\geq 0$. Thus $p\geq d-n-s$, and so the limit of the spectral sequence is zero whenever $p+q<2d-n-s$.

We conclude that the second spectral sequence, with terms

$$''E_2^{p,q}=\operatorname{Ext}^p_P\bigl(F_s\mathcal{N}^{-q},\mathscr{O}_P\bigr)$$

converges to zero in degrees $p + q < 2d - n - s$. Because the cohomology sheaves $F_s\mathcal{N}^{-q}$ are trivial bundles for $q > 0$, the E_2-page of this spectral sequence presents the appearance shown in Figure 6.7.

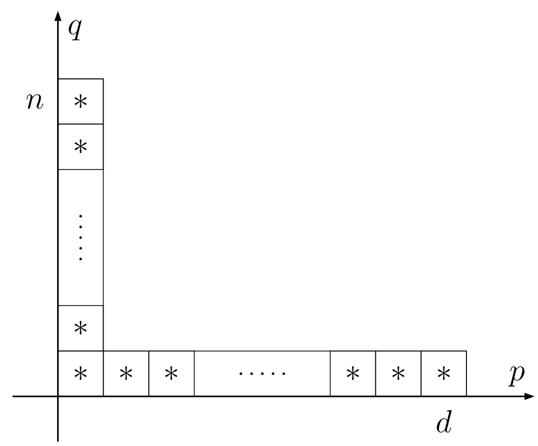

Figure 6.7: The E_2-page of the second spectral sequence

When $i = 0$ or $i = 1$, we therefore have $\operatorname{Ext}^i_P\bigl(F_s\mathcal{N}^0, \mathscr{O}_P\bigr) = 0$. For values of i in the range $2 \leq i < 2d - n - s$, we have

$$\operatorname{Ext}^i_P\bigl(F_s, \mathcal{N}^0, \mathscr{O}_P\bigr) = {}''E_2^{i,0} \simeq {}''E_2^{0,i-1} = \operatorname{Hom}_P\bigl(F_s\mathcal{N}^{i-1}, \mathscr{O}_P\bigr)$$

and the asserted formula follows from this and (6.36). $\square$

We now use the short exact sequence

$$H^n(X, \mathbb{C})_{prim} \otimes \mathscr{O}_P \hookrightarrow \mathcal{N}^0 \twoheadrightarrow \mathcal{M}(-n-1) \tag{6.37}$$

from Theorem 5.5.2 to get results for the Ext-groups of the sheaves $F_k\mathcal{M}$.

Theorem 6.4.17. *Let k and i be two integers satisfying $k + i \leq 2d$. Then we have*

$$\operatorname{Ext}_p^i\big(F_k\mathcal{M}, \mathscr{O}_P\big) \simeq \left(F^{n+1-k}\frac{H^{n-i+1}(X)}{H^{n-i-1}(X)(-1)}\right)^\vee \simeq \frac{H^{n+i-1}(X)_{prim}}{F^k H^{n+i-1}(X)_{prim}}.$$

In particular, $\operatorname{Hom}_P\big(F_k\mathcal{M}, \mathscr{O}_P\big) = 0$, *and* $\operatorname{Ext}_P^1\big(F_k\mathcal{M}, \mathscr{O}_P\big) \simeq \big(F^{n+1-k}H^n(X)_{prim}\big)^\vee$.

Proof. The second isomorphism follows in a straightforward way from the duality between $F^pH^i(X)$ and $H^{2n-i}(X)\big/F^{n-p+1}H^{2n-i}(X)$. It remains to verify the first one. From the short exact sequence in (6.37), we get

$$F^{n+1-k}H^n(X)_{prim} \otimes \mathscr{O}_P \hookrightarrow F_{k-n-1}\mathcal{N}^0 \twoheadrightarrow F_k\mathcal{M} \tag{6.38}$$

upon taking F_{k-n-1}. From the long exact sequence for the functor $\operatorname{Hom}_P(\_\!\_, \mathscr{O}_P)$, and the vanishing of $\operatorname{Ext}_P^i\big(F_{k-n-1}\mathcal{N}^0, \mathscr{O}_P\big)$ for $i = 0$ and $i = 1$, we see that

$$\operatorname{Hom}_P\big(F_k\mathcal{M}, \mathscr{O}_P\big) = 0 \qquad \text{and} \qquad \operatorname{Ext}_P^1\big(F_k\mathcal{M}, \mathscr{O}_P\big) \simeq \big(F^{n+1-k}H^n(X)_{prim}\big)^\vee;$$

both isomorphisms conform to what is asserted in the theorem. For values of i in the range $2 \leq i < 2d - n - s = 2d - n - (k - n - 1) = 2d - k + 1$, we can simply apply the result of Lemma 6.4.16 (for $s = k - n - 1$), noting that $\operatorname{Ext}_P^i(\mathscr{O}_P, \mathscr{O}_P) = 0$ on projective space. $\square$

The proof shows that the isomorphism between the two groups $\operatorname{Ext}_P^1\big(F_k\mathcal{M}, \mathscr{O}_P\big)$ and $\big(F^{n+1-k}H^n(X, \mathbb{C})_{prim}\big)^\vee$ is essentially given by the extension class of the sequence (6.38). This class is an element ε_k of

$$\operatorname{Ext}_P^1\big(F_k\mathcal{M}, F^{n+1-k}H^n(X)_{prim} \otimes \mathscr{O}_P\big).$$

Any map $f\colon F^{n+1-k}H^n(X)_{prim} \to \mathbb{C}$ defines a homomorphism

$$f_*\colon \operatorname{Ext}_P^1\big(F_k\mathcal{M}, F^{n+1-k}H^n(X)_{prim} \otimes \mathscr{O}_P\big) \to \operatorname{Ext}_P^1\big(F_k\mathcal{M}, \mathscr{O}_P\big),$$

and $f_*(\varepsilon_k)$ is the element corresponding to f under the isomorphism in the theorem. It follows that there are many nontrivial extensions of $F_k\mathcal{M}$ by $\mathscr{O}_P$; in a sense, the obstruction to a "global" splitting (on P) of such sequences is the existence of primitive classes of a certain Hodge type.

Note. Under the same assumptions as in Theorem 6.4.17, we also have an isomorphism $\operatorname{Ext}^1_P\bigl(Gr^F_k\mathcal{M}, \mathscr{O}_P\bigr) \simeq H^{n+1-k,k-1}(X)_{prim}$, derived again from (6.37).

6.4.7 A curious vanishing theorem

We digress to point out a curious application of the computations for the groups $\operatorname{Ext}^i_P\bigl(Gr^F_k\mathcal{M}, \mathscr{O}_P\bigr)$ when $i > 0$. We have obtained two different expressions for these groups, one in Proposition 6.4.15, the other in Theorem 6.4.17. By comparing the two, we get the following statement.

Proposition 6.4.18. *Assume that $\mathscr{L}$ is sufficiently ample. For all $k \geq 0$ and all $q > 0$, we have isomorphisms*

$$H^q\bigl(X, \Omega^n_X \otimes \operatorname{Sym}^k N_{X\subseteq Q}\bigr) \simeq H^q\bigl(\mathfrak{Z}, \psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\bigr) \simeq H^{n-k,q+k}(X)_{prim},$$

where $N_{X\subseteq Q}$ is the normal bundle for the embedding of X into the projective space $Q = \mathbb{P}\bigl(H^0(X, \mathscr{L})\bigr)$.

Proof. Recall that $\psi \colon \mathfrak{Z} \to X$ is the projective bundle $\mathbb{P}\bigl(N_{X\subseteq Q}\bigr)$. For $k \geq 0$, we thus have $\psi_*\mathscr{O}_{\mathfrak{Z}}(k) \simeq \operatorname{Sym}^k N_{X\subseteq Q}$. The first asserted isomorphism is then obtained by push-forward along the map ψ.

As for the second isomorphism, it follows immediately by combining the result of Proposition 6.4.15 and Theorem 6.4.17 (for $i = q+1$ and the sheaf $F_{n+1-k}\mathcal{M}$). Note that the numerical condition there is vacuous if $k \geq 0$, at least when $\mathscr{L}$ is sufficiently ample, and consequently d very large. $\square$

In particular, one gets a vanishing theorem for the ample vector bundle $N_{X \subseteq Q}$. For example, the condition $q + k \geq n + 1$ is sufficient to make the cohomology group $H^q\big(X, \Omega_X^n \otimes \operatorname{Sym}^k N_{X \subseteq Q}\big)$ be zero. This does not seem to follow from the standard vanishing theorems for ample vector bundle, such as Griffiths' Theorem (Lazarsfeld, 2004b, Theorem 7.3.1 on p. 90), because the factor of $\det N_{X \subseteq Q}$ is missing. At the same time, whenever X has nontrivial primitive cohomology in degree $(n - k, q + k)$, one gets an example where the group in question is *not* zero. This shows that there are certain restrictions on the kind of vanishing theorem one can get for ample vector bundles (not present in the case of ample line bundles).

6.5 Bounding the singularities of hypersurfaces

The purpose of this part is to prove the following general statement:

> If $\mathscr{L}$ is sufficiently ample, then the set of hypersurface $\mathfrak{X}_p$ with "many" singularities has high codimension in P.

This is supposed to mean that, except on a subset of $X^\vee$ of high codimension, the hypersurfaces have only a small number of isolated singular points; moreover, each singularity is not too complicated.[7]

[7] The complexity of an isolated singularity will be measured by its Milnor number, see 6.5.4.

Our goal is to get a lower bound on the codimension of the set of hypersurfaces with many singular points. To achieve this, we basically use a deformation-theoretic argument, as follows: We first stratify the dual variety $X^\vee$, so that the fibers over each connected component of a stratum are equi-singular. Let S be any such component; then we find an lower bound for the codimension of the tangent space to S at any point $p \in S$, by analyzing local deformations. If the hypersurface $\mathfrak{X}_p$ has many singularities, this lower bound is large, giving the result.

6.5.1 Reduction of the problem through stratifications

We now show why it suffices to estimate the codimension of certain tangent spaces. This reduction of the problem is made possible by using stratifications. As explained for instance in de Cataldo and Migliorini (2005, Theorem 3.2.3 on p. 710), there exist Whitney stratifications for $\mathfrak{X}$ and $X^\vee$, so that $\pi\colon \mathfrak{X} \to X^\vee$ becomes a stratified morphism. Moreover, we can assume that the set $\mathfrak{Z}$ of all singular points is itself a union of strata of $\mathfrak{X}$.

Let S be a connected component of some stratum; we shall refer to such an S as a *piece* of the stratification. S is a smooth locally closed algebraic subset of P. Thom's Isotopy Lemmas imply that $\pi^{-1}(S)$ is a topological fiber bundle over S; locally, in the Euclidean topology, it is homeomorphic to a product. Thus any point $p \in S$ has a small open neighborhood U in S, such that

$$\pi^{-1}(U) \simeq U \times \mathfrak{X}_p$$

are homeomorphic. In other words, the topological type of the hypersurfaces is unchanged as we move along S.

Moreover, π is a stratified morphism, so $\pi^{-1}(S)$ is a union of strata of $\mathfrak{X}$. Let T be one component of such a stratum; then T is smooth, and maps submersively to S. This implies a strong equi-singularity property for the family $\pi^{-1}(S) \to S$, because $\mathfrak{Z}$ is a union of strata. Indeed, each component T is either outside of $\mathfrak{Z}$, in which case it consists of smooth points of the fibers. Or it lies inside of $\mathfrak{Z}$, and then it consists entirely of singular points of the fibers. If, for instance, the singular locus of $\mathfrak{X}_p$ has a component Z of positive dimension, then the component T containing Z has to lie in $\mathfrak{Z}$, hence $T \cap \mathfrak{X}_p = Z$. But since T maps submersively to S, all nearby hypersurfaces $\mathfrak{X}_{p'}$ have a similar component, namely $T \cap \mathfrak{X}_{p'}$, in their singular locus.

With that in mind, our line of argument is as follows. We need to have a lower bound on the codimension of two loci; one is, where the hypersurfaces have non-isolated singularities; the other, where the singularities are isolated, but the sum of all Milnor numbers is large. We show that either of these loci W is a union of pieces S of the stratification. We then find a (uniform) lower bound for the codimension of each such S by estimating the codimension of its tangent space $T_{S,p}$ at any point. Then W is a union of strata in $X^\vee$, of high codimension in P, which is the desired conclusion.

6.5.2 The tangent space to a stratum

At a point $p \in P$, corresponding to a line $[s]$ in $H^0(X, \mathscr{L})$, the tangent space to the projective space P is

$$T_{P,p} \simeq \frac{H^0(X, \mathscr{L})}{\mathbb{C} \cdot s}.$$

Thus we may describe a subspace of $T_{P,p}$ by giving the subspace of $H^0(X, \mathscr{L})$ that maps onto it. The following lemma provides a general statement about the tangent space to (a component of) a stratum in $X^\vee$.

Lemma 6.5.1. *Let $S \subseteq X^\vee$ be a piece of the stratification, and let $p \in S$ be any point. Let T be an arbitrary component of a stratum in $\pi^{-1}(S) \cap \mathfrak{Z}$, and set $Z = T \cap \mathfrak{X}_p$. Also let $V(Z) \subseteq H^0(X, \mathscr{L})$ consist of all sections of $\mathscr{L}$ that vanish along Z. Then*

$$T_{S,p} \subseteq \frac{V(Z)}{\mathbb{C} \cdot s}.$$

Proof. It suffices to show that $T_{S,p}$ is contained in the image of the subspace $V(x)$, for every point $x \in Z$. Furthermore, S being smooth at the point p, we can find a one-parameter family of hypersurfaces in the direction of any given vector in $T_{S,p}$. By choosing local coordinates near $x \in X$ and $p \in S$, we can therefore reduce the proof to the following local problem: We are given a holomorphic function $f(z,t)$ on $\Delta^n \times \Delta$, satisfying $f(0,0) = 0$; moreover, there is a section $\xi \colon \Delta \to \Delta^n$, with $\xi(0) = 0$, such that each $\bigl(\xi(t), t\bigr)$ is a singular point on the hypersurface defined by $f(\_\_, t)$. Such a section exists in the case at hand, because $T \to S$ is submersive, and all of T is contained in the singular locus.

The tangent vector corresponding to the given family is $\frac{\partial f}{\partial t}(0,0)$; thus we need to prove the vanishing of this partial derivative. By differentiating the relation $f\bigl(\xi(t), t\bigr) = 0$ once, and setting $t = 0$, we find that

$$0 = \sum_{i=1}^n \frac{\partial f}{\partial z_i}(0,0) \cdot \xi_i'(0) + \frac{\partial f}{\partial t}(0,0) = \frac{\partial f}{\partial t}(0,0),$$

because $(0,0)$ is a singular point. The assertion of the lemma follows. $\square$

6.5.3 The locus of non-isolated singularities

As a first step, we shall estimate the codimension of the subset of $X^\vee$ where the hypersurfaces $\mathfrak{X}_p$ have a singular set of positive dimension. Let H be a very ample divisor on X, and let

$$V_m = H^0\big(X, \mathscr{O}_X(mH)\big)$$

be the space of sections of $\mathscr{O}_X(mH)$.

Lemma 6.5.2. *For a closed subvariety $Z \subseteq X$ of positive dimension $k > 0$, write $V_m(Z) \subseteq V_m$ for the subspace of sections that vanish along Z. Then*

$$\operatorname{codim}\big(V_m(Z), V_m\big) \geq \binom{m+k}{k}.$$

Proof. Since H is very ample, we may find $(k+1)$ points $P_0, P_1, \ldots, P_k$ on Z, together with $(k+1)$ sections $s_0, s_1, \ldots, s_k \in V_1$, such that each s_i vanishes at all points P_j with $j \neq i$, but does not vanish at P_i. Then all the sections

$$s_0^{\otimes i_0} \otimes s_1^{\otimes i_1} \otimes \cdots \otimes s_k^{\otimes i_k} \in V_m,$$

for $i_0 + i_1 + \cdots + i_k = m$, are easily seen to be linearly independent on Z. The lower bound on the codimension follows immediately. $\square$

We can now take $\mathscr{L} = \mathscr{O}_X(mH)$, for a sufficiently large value of m. Then $P = \mathbb{P}(V_m)$, and we set

$$X^\vee_{pos} = \big\{\, p \in X^\vee \;\big|\; \text{the singular locus of } \mathfrak{X}_p \text{ has positive dimension} \,\big\}.$$

That this set is a union of components of strata is clear; if $\mathfrak{X}_p$ has a singular locus of positive dimension for one $p \in S$, then the same has to be true at all points of the component S.

We now pick an arbitrary such component S, and any point $p \in S$. Let $Z \subseteq \mathfrak{X}_p$ be a positive-dimensional component of the singular locus. As outlined above, we may find a component T of a stratum in $\mathfrak{Z}$, with $T \cap \mathfrak{X}_p = Z$, and T submersive over S. By Lemma 6.5.1, the tangent space $T_{S,p}$ has to be contained in the subspace $V_m(Z)$ of sections of $\mathscr{L}$ that vanish at every point of Z. Lemma 6.5.2 now assures us that

$$\operatorname{codim}(S, P) \geq \operatorname{codim}\bigl(V_m(Z), V_m\bigr) \geq \binom{m+1}{1} \geq m+1,$$

because $\dim Z \geq 1$. It follows that the codimension of $X^\vee_{pos}$ in P is at least $m+1$.

6.5.4 Milnor number and Tyurina number

From now on, we consider hypersurfaces with only isolated singularities, and try to bound the number and type of singularities that can occur. To measure how singular such a hypersurface is, we recall two definitions.

Let $f\colon U \to \mathbb{C}$ be a holomorphic function on a neighborhood U of the origin in $\mathbb{C}^n$. We shall use coordinates $z = (z_1, \ldots, z_n)$ on U, and write $\mathscr{O}_{U,0}$ for the local ring of U at the origin. Now suppose that f has an isolated singularity at $z = 0$. Then the *Milnor number* of f is the integer

$$\mu = \mu(f, 0) = \dim_{\mathbb{C}} \mathscr{O}_{U,0} \Big/ \Bigl(\tfrac{\partial f}{\partial z_1}, \ldots, \tfrac{\partial f}{\partial z_n}\Bigr);$$

it measures the topological multiplicity of the singularity. A related quantity is the *Tyurina number* of f,

$$\tau = \tau(f,0) = \dim_{\mathbb{C}} \mathscr{O}_{U,0} \Big/ \left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right).$$

The Tyurina number τ is the dimension of the miniversal deformation space of the singularity. This means the following: The quotient ring $\mathscr{O}_{U,0} \Big/ \left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$ is a finite-dimensional complex vector space. It naturally parametrizes infinitesimal deformations of the hypersurface germ defined by f. Inside of it, there is a small open neighborhood of the origin, parametrizing actual (analytic) deformations of the germ.

Both numbers can be defined for an isolated singular point x on a hypersurface D in a smooth ambient variety X; in local coordinates $z_1, \ldots, z_n$ on X, centered at x, the hypersurface is given by a single holomorphic equation $f = 0$. We can then let $\mu(D,x) = \mu(f,0)$, and similarly, $\tau(D,x) = \tau(f,0)$. If the hypersurface has only isolated singularities, we can also define its *total Milnor number* $\mu(D)$ as the sum of the Milnor numbers at all singular points; in a similar manner, we have the *total Tyurina number* $\tau(D)$.

As we said above, the Tyurina number at an isolated singular point $x \in D$ is the dimension of the space of deformations of the germ (D,x). In general, the deformation theory of the singularities of D is governed by the coherent sheaf $\mathscr{E}xt^1(\Omega^1_D, \mathscr{O}_D)$. It is supported on the singular locus of D, and naturally a quotient of the line bundle $\mathscr{O}_X(D)$. Indeed, consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-D)\big|_D \longrightarrow \Omega^1_X\big|_D \longrightarrow \Omega^1_D \longrightarrow 0$$

on D. By dualizing it, and using that Ω^1_X is locally free, we get a presentation

$$T_X\big|_D \longrightarrow \mathscr{O}_X(D)\big|_D \twoheadrightarrow \mathcal{E}\chi t^1(\Omega^1_D, \mathscr{O}_D).$$

This presentation allows the computation of $\mathcal{E}\chi t^1(\Omega^1_D, \mathscr{O}_D)$ at an isolated singular point x, using local coordinates. The result is that

$$\mathcal{E}\chi t^1(\Omega^1_D, \mathscr{O}_D)_x \simeq \mathscr{O}_X(D) \otimes \mathscr{O}_{X,x} \Big/ \Big(f, \tfrac{\partial f}{\partial z_1}, \dots, \tfrac{\partial f}{\partial z_n}\Big),$$

where $f \in \mathscr{O}_{X,x}$ is a local defining equation for D. This is a sheaf of length τ, supported at the point x.

We are now going to discuss briefly the meaning of the two quantities μ and τ, when the hypersurface in question is $\mathfrak{X}_p$, for $p \in X^\vee$. To begin with, a result by Dimca (1986) states that the total Milnor number of $\mathfrak{X}_p$ equals the multiplicity of the dual variety at the point p. A more detailed description is the following: If $\mathfrak{X}_p$ has exactly r singular points $x_1, \dots, x_r$, then the tangent cone to the dual variety at p is a union of r components, with the multiplicity of the i-th component equal to the Milnor number $\mu\big(\mathfrak{X}_p, x_i\big)$.

The Tyurina numbers, on the other hand, are related to the geometry of the set $\mathfrak{Z}$ of singular points, and thus to the $\mathcal{D}$-module $(\mathcal{M}, F)$. We had seen in Lemma 5.3.3 that the characteristic sheaf of that $\mathcal{D}$-module is $\Omega^n_{\mathfrak{X}/P}$; at an isolated singular point $x \in \mathfrak{X}_p$, its length is precisely the quantity τ. Indeed, let $(z_1, \dots, z_n)$ be local coordinates at x in X, and let f be a local defining equation for the hypersurface $\mathfrak{X}_p$. Then $\Omega^1_{\mathfrak{X}/P}\big|_{\mathfrak{X}_p} = \Omega^1_{\mathfrak{X}_p}$; clearly

$$\Omega^1_{\mathfrak{X}_p} \otimes \mathscr{O}_{X,x} \simeq \Big(\mathscr{O}_{X,x} \cdot dz_1 \oplus \dots \oplus \mathscr{O}_{X,x} \cdot dz_n\Big) \Big/ \Big(f, \tfrac{\partial f}{\partial z_1} dz_1 + \dots + \tfrac{\partial f}{\partial z_n} dz_n\Big),$$

and so the stalk of $\Omega^n_{\mathfrak{X}/P}$ at the point x is precisely the module

$$\Omega^n_X\big|_{\mathfrak{X}_p} \simeq \Omega^n_{\mathfrak{X}_p} \otimes \mathscr{O}_{X,x} \simeq \mathscr{O}_{X,x}\Big/\Big(f, \tfrac{\partial f}{\partial z_1}, \dots, \tfrac{\partial f}{\partial z_n}\Big), \tag{6.39}$$

with generator $dz_1 \wedge \cdots \wedge dz_n$.

6.5.5 Local analysis at an isolated singularity

We now analyze the situation locally, near one isolated singular point. We take a holomorphic function f on a neighborhood U of the origin in $\mathbb{C}^n$, with an isolated singularity at $z = 0$. Let $R = \mathscr{O}_{U,0}$ be the local ring at the origin, and let $j(f)$ be the ideal in R generated by the partial derivatives $\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}$ of f. We write $A = R/j(f)$ for the quotient ring, which is an Artinian $\mathbb{C}$-algebra of complex dimension $\mu = \mu(f, 0)$.

We also let $\tau = \tau(f, 0)$ be the Tyurina number of the singularity; it is the dimension of the quotient A/Af. Then the base of the miniversal deformation space is an open set $B \subseteq A/Af$, and so A/Af is naturally identified with the tangent space $T_{B,0}$. To begin with, we have the following relationship between the Milnor number and the Tyurina number.

Lemma 6.5.3. *Let f be a holomorphic function on a neighborhood U of the origin in $\mathbb{C}^n$, with an isolated singularity at $z = 0$. Then*

$$\tau \leq \mu \leq n\tau,$$

where $\tau = \tau(f, 0)$, and $\mu = \mu(f, 0)$.

Proof. The first inequality is obvious from the definition of μ and τ. To prove the second one, we recall that the function f is contained in the integral closure of the ideal $j(f)$ generated by its partial derivatives (Lazarsfeld, 2004b, p. 223), because the singularity is isolated. By the Briançon-Skoda Theorem (Lazarsfeld, 2004b, Theorem 9.6.26 on p. 223), we then have $f^n \in j(f)$. The Milnor number μ is the vector space dimension of the ring $A = R/j(f)$. By considering the chain

$$A \supseteq Af \supseteq Af^2 \supseteq \cdots \supseteq Af^n = 0,$$

we see that $\mu = \dim_{\mathbb{C}} A \leq n \cdot \dim_{\mathbb{C}} A/Af = n\tau$, which is the desired inequality. □

In order to bound the codimension of the set of hypersurfaces $\mathfrak{X}_p$ with large total Milnor number, we need an estimate for the tangent space to the stratum where μ remains constant. The following lemma provides a weak estimate, with a rather crude proof. (Much better statements must be known to the experts.)

Lemma 6.5.4. *In the notation from above, let B be the miniversal deformation space of the singularity. Let $B^\mu \subseteq B$ be the locus where the Milnor number of the singularity remains equal to μ. Then*

$$T_{B^\mu,0} \subseteq \overline{j(f)}/\big(f + j(f)\big)$$

inside $T_{B,0} \simeq A/Af$. Moreover, we have a lower bound

$$\operatorname{codim}(B^\mu, B) \geq \operatorname{codim}\big(T_{B^\mu,0}, T_{B,0}\big) \geq \tfrac{1}{n}\mu^{1/n}$$

for the codimension of B^μ.

Proof. As before, we let $j(f)$ be the ideal in $R = \mathscr{O}_{U,0}$ generated by the n partial derivatives of the function f, and define the ring $A = R/j(f)$; the dimension of A as a vector space is then equal to μ. The tangent space to B at the origin is naturally identified with the τ-dimensional vector space A/Af, and we clearly have

$$\operatorname{codim}(B^\mu, B) \geq \operatorname{codim}\bigl(T_{B^\mu,0}, T_{B,0}\bigr) = \operatorname{codim}\bigl(T_{B^\mu,0}, A/Af\bigr).$$

The tangent space to the locus of constant Milnor number is controlled by a result of Lê, K. Saito, and Teissier; one possible reference is Greuel (1986). The theorem on p. 161 there gives a criterion for a one-parameter family $F(z,t)$ of deformations of the germ over the disk Δ to have constant Milnor number.[8] An immediate consequence of the criterion is that, for any such family, $\partial F/\partial t\big|_{t=0} \in \overline{j(f)}$. This implies that the tangent space to B^μ at 0 is contained in the subspace of A/Af given by the image of $\overline{j(f)}$. As a consequence,

$$\operatorname{codim}(B^\mu, B) \geq \operatorname{codim}\bigl(\overline{j(f)}, R\bigr) = \dim_{\mathbb{C}} R/\overline{j(f)}.$$

To complete the proof, we simply apply Lemma 6.5.6 below, which gives a lower bound on $\dim_{\mathbb{C}} R/\overline{j(f)}$ in terms of the Milnor number μ. $\square$

The next two lemmas complete the proof of Lemma 6.5.4. We state them for an arbitrary field k; of course, we only need the case $k = \mathbb{C}$.

Lemma 6.5.5. *Let $(R, \mathfrak{m})$ be a local k-algebra with residue field k. Let $I \subseteq R$ be any ideal whose codimension $d = \dim_k R/I$ as a linear subspace is finite.*

[8] Here $f(z) = F(z,0)$.

(a) *We have* $\mathfrak{m}^d \subseteq I$.

(b) *If* $\dim_k R/\big(I + \mathfrak{m}^{b+1}\big) \leq b$, *then even* $\mathfrak{m}^b \subseteq I$.

Proof. The proof of (a) is a simple application of the Pigeonhole Principle. By passing to the quotient ring R/I we reduce to the case $I = 0$; thus we need to show that if $(R, \mathfrak{m})$ is an Artinian k-algebra with $\dim_k R = d$, then $\mathfrak{m}^d = 0$. Consider the chain of subspaces

$$R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \supseteq \mathfrak{m}^d \supseteq \mathfrak{m}^{d+1}.$$

Since there are $d+1$ sub-quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, at least one of them has to be the trivial vector space. By Nakayama's Lemma, this implies $\mathfrak{m}^i = 0$ for some $i \leq d$.
A similar line of reasoning proves (b). This time, we look at the chain

$$R \supseteq I + \mathfrak{m} \supseteq \cdots \supseteq I + \mathfrak{m}^b \supseteq I + \mathfrak{m}^{b+1}.$$

If the dimension of the quotient space $R/\big(I + \mathfrak{m}^{b+1}\big)$ is less than $b+1$, one of the sub-quotients has to be trivial. Seeing that

$$\frac{I + \mathfrak{m}^i}{I + \mathfrak{m}^{i+1}} \simeq \frac{\mathfrak{m}^i}{I \cap \mathfrak{m}^i + \mathfrak{m}^{i+1}},$$

this implies that $\mathfrak{m}^i \subseteq I$ for some $i \leq b$. ☐

Lemma 6.5.6. *Let R be a n-dimensional local k-algebra with residue field k. Assume that R is regular, and let $z_1, \ldots, z_n$ be a system of parameters. Let $I \subseteq R$ be an ideal of finite codimension $\mu = \dim_k R/I$. Then the codimension of the integral closure $\overline{I}$ satisfies the inequalities*

$$\tfrac{1}{n}\mu^{1/n} \leq \dim_k R/\overline{I} \leq \mu.$$

Proof. Let us write $d = \dim_k R/\overline{I}$ for the codimension of $\overline{I}$ in R. Since $I \subseteq \overline{I}$, the inequality $d \leq \mu$ is obvious. To prove the other inequality, we observe that $\mathfrak{m}^d \subseteq \overline{I}$ because of Lemma 6.5.5. By the Briançon-Skoda Theorem, we have $(\overline{I})^n \subseteq I$; thus I contains the elements $z_1^{dn}, \ldots, z_n^{dn}$. But then we get

$$\mu = \dim_k R/I \leq \dim_k R/R\bigl(z_1^{dn}, \ldots, z_n^{dn}\bigr) = (dn)^n,$$

from which the remaining half of the inequality follows immediately. □

6.5.6 The locus of large total Milnor number

Having established the local bound in Lemma 6.5.4, we are now ready to estimate the codimension of the subset of points $p \in X^\vee$ where $\mathfrak{X}_p$ has a large total Milnor number. We begin with a technical lemma about separation of jets.

Lemma 6.5.7. *Fix a positive integer N. If $\mathscr{L}$ is sufficiently ample, then for any set of N points $x_1, \ldots, x_N \in X$, the map*

$$H^0(X, \mathscr{L}) \to \bigoplus_{i=1}^{N} \mathscr{L} \otimes \frac{\mathscr{O}_{X,x_i}}{\mathfrak{m}_{x_i}^{N+1}} \tag{6.40}$$

is surjective. More generally, for any 0-dimensional subscheme Z in X of length at most N, the map

$$H^0(X, \mathscr{L}) \otimes \mathscr{O}_X \to \mathscr{L} \otimes \mathscr{O}_Z$$

is surjective.

Proof. The first assertion is a straightforward generalization of Theorem 5.1.17 on p. 273 of Lazarsfeld (2004a). The second assertion is easily reduced to the first one, by observing that the support of Z can consist of no more than N points. At each such point x, the ideal of Z in $\mathscr{O}_{X,x}$ has codimension at most N, and therefore contains $\mathfrak{m}_x^N$ by Lemma 6.5.5. Thus $\mathscr{L} \otimes \mathscr{O}_Z$ is naturally isomorphic to a subsheaf of the right-hand side of (6.40). $\square$

Lemma 6.5.8. *Fix a positive integer N. If the line bundle $\mathscr{L}$ is sufficiently ample, then the codimension of the locus*

$$X^\vee_{\mu \geq N} = \left\{\, p \in X^\vee \;\middle|\; \mathfrak{X}_p \text{ has isolated singularities, and } \mu(\mathfrak{X}_p) \geq N \,\right\}$$

in P is at least $\gamma = \frac{1}{n} N^{1/n}$.

Proof. We already know from Lemma 6.5.2 that the locus of $p \in X^\vee$, where $\mathfrak{X}_p$ has non-isolated singularities, is of high codimension for sufficiently ample $\mathscr{L}$. Thus we may concentrate on those $\mathfrak{X}_p$ with only isolated singularities.

Following our general strategy, we note that $X^\vee_{\mu \geq N}$ is again a union of connected components of strata of $X^\vee$. Indeed, along each stratum S, the homeomorphism type of the hypersurface remains unchanged. Since the Milnor number is a topological invariant (being determined by the homology of the link), it follows that all hypersurfaces over S have the same number of isolated singularities, with the same Milnor numbers. In particular, all of S is contained in $X^\vee_{\mu \geq N}$. Thus the lemma will be proved if we manage to show that the codimension of each S in P is at least equal to γ. We shall assume that $\mathscr{L}$ has been chosen sufficiently ample, for the conclusion of Lemma 6.5.7 to hold for the given value of N.

Now let $p \in S$ be an arbitrary point; thus $\mathfrak{X}_p$ has isolated singularities, and $\mu(\mathfrak{X}_p) \geq N$. Let $x \in \mathfrak{X}_p$ be any of the singular points; write $R = \mathscr{O}_{X,x}$ for the local ring at x, and $\mathfrak{m} = \mathfrak{m}_x$ for its maximal ideal. Take a local equation $f \in R$ for the hypersurface, and define $j(f) \subseteq R$, and $A = R/j(f)$ as above.

From our previous discussion, we know that there is a restriction map

$$\rho\colon H^0(X, \mathscr{L}) \to \mathcal{E}xt^1(\Omega^1_{\mathfrak{X}_p}, \mathscr{O}_{\mathfrak{X}_p})_x \simeq \mathscr{L} \otimes R/\bigl(Rf + j(f)\bigr).$$

By Lemma 6.5.4, the tangent space to the stratum of constant Milnor number is contained in the subspace determined by $\overline{j(f)}$. Since the Milnor number remains constant along S, the tangent space to S at p has to lie in the preimage of this subspace under ρ. Now the map from $H^0(X, \mathscr{L})$ to $\mathscr{L} \otimes R/\mathfrak{m}^{N+1}$ is onto, and so the codimension is at least equal to

$$\operatorname{codim}(S, P) \geq \dim_{\mathbb{C}} R/\bigl(\overline{j(f)} + \mathfrak{m}^{N+1}\bigr).$$

If this is greater than N, then we already have our lower bound, because $N \geq \gamma$. If not, then we conclude, with the help of Lemma 6.5.5, that

$$\mathfrak{m}^N \subseteq \overline{j(f)}. \tag{6.41}$$

We will assume from now on that this inclusion holds at each singularity of $\mathfrak{X}_p$.

Now let $x_1, \dots, x_r$ be the set of singular points of $\mathfrak{X}_p$; write $\mu_i = \mu(\mathfrak{X}_p, x_i)$. We distinguish two cases. The *first case* is that $r \leq N$. Using Lemma 6.5.7, and the inclusion in (6.41), we conclude that the codimension of the tangent space to S is at least

$$\sum_{i=1}^r \frac{1}{n}\mu_i^{1/n} \geq \frac{1}{n}\left(\sum_{i=1}^r \mu_i\right)^{1/n} = \frac{1}{n}\bigl(\mu(\mathfrak{X}_p)\bigr)^{1/n} \geq \frac{1}{n}N^{1/n} = \gamma.$$

The *second case* is that $r \geq N$. We only consider the first N singular points $x_1, \dots, x_N$. By Lemma 6.5.1, the tangent space to S at p is contained in the image of the subspace

$$\left\{\, s \in H^0(X, \mathscr{L}) \;\middle|\; s(x_i) = 0 \text{ for all } i = 1, \dots, N \,\right\}.$$

Lemma 6.5.7 now implies that the points impose N independent conditions; therefore, the codimension of that subspace in $H^0(X, \mathscr{L})$ is at least N. Since the section defining $\mathfrak{X}_p$ is contained in the subspace, it follows that the codimension of $T_{S,p}$ in $T_{P,p}$ is at least N, and so again $\operatorname{codim}(S, P) \geq \gamma$. In either case, we have a satisfactory lower bound, and the proof is therefore complete. $\square$

6.6 The Serre conditions for the graded quotients

In this section, we shall use the duality results obtained above to study some properties of the sheaves $F_k\mathcal{M}$. The general philosophy is that, as $\mathscr{L}$ becomes more ample, they should become more well-behaved, because the complexity of the dual variety $X^\vee$ appears in successively higher codimension. Ideally, each $F_k\mathcal{M}$ (being a natural extension of the Hodge bundle $F^{n-k}\mathcal{V}_{van}^{n-1}$ on P^{sm}) would be locally free; but, as we shall see below, this is in general impossible.[9] On the other hand, it is reasonable to expect that $F_k\mathcal{M}$ is at least a reflexive sheaf.

[9]This impossibility is related to the question of whether normal functions have singularities.

The geometry of the universal hypersurface $\mathfrak{X}$ now allows us to deduce such properties of the sheaves $F_k\mathcal{M}$. Using the computation of the local $\mathcal{E}xt$-sheaves

$$\mathcal{E}xt^i\bigl(Gr_k^F\mathcal{M}, \mathscr{O}_P\bigr),$$

we will prove that each $F_k\mathcal{M}$ in the range $1 \leq k \leq n$ satisfies Serre's condition S_p, for large values of p. How large one can choose p depends on the ampleness of $\mathscr{L}$; this seems to be a general rule.

6.6.1 The Serre conditions

A coherent sheaf $\mathscr{F}$ on a nonsingular algebraic variety P is said to satisfy Serre's condition S_k, if the inequality

$$\operatorname{depth} \mathscr{F}_p \geq \min\bigl(k, \dim \mathscr{O}_{P,p}\bigr)$$

holds at every point $p \in P$. A locally free sheaf satisfies S_k for all values of p; on the other hand, condition S_1 is equivalent to being torsion-free, and S_2 to being reflexive. In that sense, Serre's conditions measure to what extent a sheaf fails to be locally free.

The following lemma gives a useful criterion[10] for verifying condition S_k. On a nonsingular variety, a coherent sheaf $\mathscr{F}$ is locally free if, and only if, all $\mathcal{E}xt^i(\mathscr{F}, \mathscr{O}_P)$ vanish for $i > 0$. One consequence of the lemma is that an S_k-sheaf is locally free on the complement of a set of codimension $k + 1$.

[10] I thank Mihnea Popa for pointing this out to me.

Lemma 6.6.1. *Let P be a nonsingular algebraic variety, and $\mathscr{F}$ a coherent sheaf on P. The following three conditions are equivalent:*

1. *The sheaf $\mathscr{F}$ satisfies condition S_k.*
2. *For all $i > 0$, the codimension of the support of $\mathcal{E}xt^i(\mathscr{F}, \mathscr{O}_P)$ is at least $i + k$.*
3. *$\mathscr{F}$ is a k-th syzygy sheaf; in other words, there is an exact sequence*

$$\mathscr{F} \hookrightarrow \mathscr{E}_{k-1} \longrightarrow \mathscr{E}_{k-2} \longrightarrow \cdots \longrightarrow \mathscr{E}_0$$

with each $\mathscr{E}_i$ locally free.

6.6.2 Good behavior of the sheaves in the interesting range

The following result shows that the sheaves $F_k\mathcal{M}$ in the interval $1 \leq k \leq n$ are very well-behaved, once $\mathscr{L}$ becomes sufficiently ample.

Theorem 6.6.2. *Fix a positive integer p. If the line bundle $\mathscr{L}$ is sufficiently ample, then each sheaf $Gr_k^F\mathcal{M}$ (and consequently, each $F_k\mathcal{M}$) in the interval $1 \leq k \leq n$ satisfies S_p.*

Proof. We are going to show that the sheaf $\mathcal{E}xt^i\big(Gr_k^F\mathcal{M}, \mathscr{O}_P\big)$ is supported in codimension at least $i + p$, for all $i > 0$; because of Lemma 6.6.1, this is equivalent to condition S_p.

First, we treat the case when $i \geq 2$, where we have

$$\mathcal{E}xt^i\big(Gr_k^F\mathcal{M}, \mathscr{O}_P\big) \simeq R^{i-1}\delta_*\big(\psi^*\Omega_X^n \otimes \mathscr{O}_{\mathfrak{Z}}(n+1-k)\big)$$

by (6.18). The support of this sheaf is therefore contained in the locus $X^\vee_{pos}$ where the singular set of the fibers has positive dimension. The result of 6.5.3 is that the codimension of this locus can be made arbitrarily large, in particular larger than $i+p$, by choosing $\mathscr{L}$ sufficiently ample.

Next, we have to deal with the case $i = 1$; here, we only have the four-term exact sequence in (6.19), which gives a presentation

$$Gr^F_{n+1-k}\mathcal{M} \longrightarrow \delta_*\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(n+1-k)\big) \twoheadrightarrow \mathcal{E}xt^1\left(Gr^F_k\mathcal{M}, \mathscr{O}_P\right)$$

for the $\mathcal{E}xt$-sheaf under consideration. To complete the proof, it remains to show that the first arrow is surjective except on a set of codimension at least $i+p$. Since $1 \leq k \leq n$, we can consider without loss of generality the map

$$Gr^F_k\mathcal{M} \to \delta_*\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{X}}(k)\big) \tag{6.42}$$

instead. To prove surjectivity, we now look at the (large) space of global sections of the left-hand sheaf.

In Lemma 6.1.4, we had already shown that the map

$$H^0\big(P \times X, pr_X^*\Omega^n_X \otimes \mathscr{O}_{P\times X}(k\mathfrak{X})\big) \otimes \mathscr{O}_P \to Gr^F_k\mathcal{M}$$

is surjective for every value of k. Now we may restrict our attention to the open subset of $X^\vee$ where the projection $\delta\colon \mathfrak{Z} \to X^\vee$ is finite (its complement has very high codimension, so this is permissible in our context). Take any point p in that set. Because the map δ is affine over a neighborhood of p, the sheaf on the right-hand side of (6.42) has fibers

$$\delta_*\big(\psi^*\Omega^n_X \otimes \mathscr{O}_{\mathfrak{Z}}(k)\big) \otimes \mathbb{C}(p) \simeq H^0\Big(\mathfrak{Z}_p, i_p^*\big(\Omega^n_X \otimes \mathscr{L}^k\big)\Big),$$

where $i_p \colon \mathfrak{Z}_p \to X$ is the inclusion map from the singular locus $\mathfrak{Z}_p$ of $\mathfrak{X}_p$ into X. To prove surjectivity, it therefore suffices to show that the map

$$H^0\big(X, \Omega_X^n \otimes \mathscr{L}^k\big) \to H^0\Big(\mathfrak{Z}_p, i_p^*\big(\Omega_X^n \otimes \mathscr{L}^k\big)\Big) \tag{6.43}$$

is surjective, except when p lies in a subset of $X^\vee$ of high codimension.

But this follows in a straightforward way from the results in Section 6.5. Indeed, we proved in Lemma 6.5.8 that the set of hypersurfaces with total Milnor number at least N has high codimension, provided $\mathscr{L}$ is sufficiently ample. By taking N large enough, and then $\mathscr{L}$ sufficiently ample, we can make this codimension be larger that $i+p$; we may thus assume, for the remainder of the proof, that the hypersurface $\mathfrak{X}_p$ in question has total Milnor number at most N. Because the singularities are isolated, and using (6.39), the sheaf

$$i_p^*\big(\Omega_X^n \otimes \mathscr{L}^k\big) \simeq \Omega_{\mathfrak{X}_p}^n \otimes i_p^*\mathscr{L}^k$$

has 0-dimensional support, and length equal to the total Tyurina number $\tau(\mathfrak{X}_p)$. By Lemma 6.5.3, this quantity is at most $\mu(\mathfrak{X}_p) \leq N$. After making $\mathscr{L}$ even more ample, if necessary, we can apply Lemma 6.5.7 to conclude that the map in (6.43) is surjective.

In conclusion, we have surjectivity in (6.42) except on a set of codimension at least $i + p$ in P. We thus get the required bound on the codimension of the support of $\mathcal{E}\mathcal{X}t^1\big(Gr_k^F \mathcal{M}, \mathscr{O}_P\big)$, implying that $Gr_k^F \mathcal{M}$ satisfies condition S_p for the chosen value of p. $\square$

CHAPTER 7

APPLICATIONS TO THE STUDY OF NORMAL FUNCTIONS

In this chapter, we apply the preceding results to the study of normal functions.[1] We first give two ways of representing primitive cohomology classes in $H^n(X, \mathbb{C})$ by data on the entire projective space P. Both of them can in particular be used for a primitive Hodge class, and give a new interpretation for the singularities of the associated normal function.

7.1 Primitive cohomology classes as $\mathcal{M}$-valued one-forms

Let X be a smooth projective variety of dimension $n \geq 2$. We shall now see how the filtered $\mathcal{D}$-module $(\mathcal{M}, F)$ can be used to represent primitive cohomology classes on X by holomorphic one-forms on P with coefficients in a certain part of the filtration. We first given an elementary construction, before explaining its meaning by using mixed Hodge modules.

[1] Some basic facts about normal functions are reviewed in 7.3.1.

Let $\zeta \in F^pH^n(X)_{prim}$ be a primitive cohomology class on X. If $\mathscr{L}$ is sufficiently ample, Nori's Connectedness Theorem implies that the map

$$H^n(X,\mathbb{C})_{prim} \to H^1\big(P^{sm}, R^{n-1}\pi^{sm}\mathbb{C}_{van}\big)$$

is injective (this is proved in Proposition 4.3.1, for $\dim X \geq 2$). Thus no information about ζ is lost by working with its image in the right-hand cohomology group.

Let $\mathcal{V}^{n-1}_{van} = R^{n-1}\pi^{sm}_*\mathbb{C}_{van} \otimes_{\mathbb{C}} \mathscr{O}_{P^{sm}}$ be the vector bundle associated to the local system, and ∇ the Gauss-Manin connection. By the holomorphic Poincaré Lemma, the de Rham complex

$$\mathcal{V}^{n-1}_{van} \xrightarrow{\nabla} \Omega^1_{P^{sm}} \otimes \mathcal{V}^{n-1}_{van} \xrightarrow{\nabla} \Omega^2_{P^{sm}} \otimes \mathcal{V}^{n-1}_{van} \longrightarrow \cdots \longrightarrow \Omega^d_{P^{sm}} \otimes \mathcal{V}^{n-1}_{van}$$

is a resolution of the local system; since P^{sm} is affine, we thus obtain

$$\begin{aligned} H^1\big(P^{sm}, R^{2m-1}\pi^{sm}_*\mathbb{Q}_{van}\big) &\simeq \mathbb{H}^1\big(\Omega^\bullet_{P^{sm}} \otimes \mathcal{V}^{n-1}_{van}\big) \\ &\simeq H^1\Big(\Gamma\big(P^{sm}, \Omega^\bullet_{P^{sm}} \otimes \mathcal{V}^{n-1}_{van}\big)\Big). \end{aligned}$$

The class ζ can thus be represented by a holomorphic one-form ω^{sm}_ζ with coefficients in the bundle $\mathcal{V}^{n-1}_{van}$, defined on P^{sm}, and closed under the differential ∇ of the de Rham complex; of course, it is only unique up to $\nabla\Gamma\big(P^{sm}, \mathcal{V}^{n-1}_{van}\big)$. Since ζ is a class in $F^pH^n(X)_{prim}$, it is actually possible to chose ω^{sm}_ζ to have its coefficients in $F^{p-1}\mathcal{V}^{n-1}_{van}$.

7.1.1 An elementary construction of such forms

Seeing that ω^{sm}_ζ is only defined on P^{sm}, we have to ask ourselves whether the construction above can somehow be extended to all of P. Using residues—more precisely, the $\mathcal{D}$-module $\mathcal{M}$—it is indeed possible to do this, as we shall now see.

Note. To understand why $F_{n-p+1}\mathcal{M}$ should appear in the statement, note that the one-form has to take its values in the extension of $F^{p-1}\mathcal{V}^{n-1}_{van}$, which is $F^{p-1}\mathcal{M}_{van}$. Because the filtrations of $(\mathcal{M}_{van}, F)$ and $(\mathcal{M}, F)$ are offset by n steps, we then get

$$F^{p-1}\mathcal{M}_{van} = F_{-p+1}\mathcal{M}_{van} = F_{n-p+1}\mathcal{M},$$

as explained in 5.5.3.

The following elementary construction of a one-form ω_ζ on all of P, using residues, is based on ideas of Clemens.

Lemma 7.1.1. *Let $\mathscr{L}$ be sufficiently ample. Associated to a primitive cohomology class $\zeta \in F^pH^n(X)_{prim}$, there is a global section ω_ζ of $\Omega^1_P \otimes F_{n-p+1}\mathcal{M}$, closed under the differential of the de Rham complex for $\mathcal{M}$.*

Proof. We first give the very simple construction of the form ω_ζ. Take the pullback $pr_X^*\zeta$ to the product $P \times X$, and restrict it to the complement of the universal hypersurface. Since the resulting cohomology class is in $F^pH^n(P \times X \setminus \mathfrak{X})$, it can be represented by a closed rational n-form on $P \times X$, with a pole of order at most $n-p+1$ (by Proposition 5.1.2). Let us denote this rational form by α for the time being.

To define our one-form, we use the Euler sequence

$$0 \longrightarrow \mathscr{O}_P \longrightarrow V \otimes \mathscr{O}_P(1) \longrightarrow T_P \longrightarrow 0.$$

on the projective space P. Since

$$pr_{P*}\mathscr{O}_{P\times X}(\mathfrak{X}) \simeq H^0(X, \mathscr{L}) \otimes \mathscr{O}_P(1) = V \otimes \mathscr{O}_P(1),$$

any section of the bundle $V \otimes \mathscr{O}_P(1)$ over some open set $U \subseteq P$ can be interpreted as a rational function on $U \times X$ with a first-order pole along $\mathfrak{X}$. Of course, the sections of the subbundle $\mathscr{O}_P$ correspond to rational functions that are constant along each fiber of $U \times X \to U$. We thus get a morphism of sheaves

$$V \otimes \mathscr{O}_P(1) \to F_{n-p+2}\mathcal{M},$$

by multiplying a rational function f with α, and then taking fiber-wise residue. The result, $\operatorname{Res}_{\mathfrak{X}/P}(f\alpha)$, takes its image in $F_{n-p+2}\mathcal{M}$ because α has a pole of order $n-p+1$, and multiplication by f can increase the order at most by one. Since the fiber-wise residue of α itself is zero (on each fiber, α represents the cohomology class ζ by construction), the subbundle $\mathscr{O}_P$ is in the kernel, and so we get a morphism

$$\omega_\zeta \colon T_P \to F_{n-p+2}\mathcal{M},$$

in other words, a section of $\Omega_P^1 \otimes F_{n-p+2}\mathcal{M}$.

We can, however, use the fact that $d\alpha = 0$ to prove a sort of "horizontality" statement, namely that the image of ω_ζ is actually contained in $F_{n-p+1}\mathcal{M}$.

This is a local question, and so we choose a basis $s_0, s_1, \dots, s_d$ of sections in V, and work on the affine subspace of P where $s_0 \neq 0$. Then $(t_1, \dots, t_d) \mapsto [s_0 + t_1 s_1 + \dots + t_d s_d]$ defines a coordinate system, and we can write α in the form

$$\alpha = \frac{\alpha(t)}{\bigl(s_0 + \sum t_i s_i\bigr)^{n-p+1}},$$

with $\alpha(t)$ a holomorphic section of $H^0\bigl(X, \Omega_X^n \otimes \mathscr{L}^{n-p+1}\bigr)$. The vector field $\partial_j = \partial/\partial t_j$ corresponds to the rational function $f_j = s_j/\bigl(s_0 + \sum t_i s_i\bigr)$, and so the product with α is

$$f_j\alpha = \frac{s_j\alpha(t)}{\bigl(s_0 + \sum t_i s_i\bigr)^{n-p+2}}.$$

Now note that we have

$$0 = \partial_j \lrcorner\, d\alpha = -(n-p+1) \cdot \frac{s_j \alpha(t)}{\bigl(s_0 + \sum t_i s_i\bigr)^{n-p+2}} + \frac{\partial_j \alpha(t)}{\bigl(s_0 + \sum t_i s_i\bigr)^{n-p+1}}$$
$$= -(n-p+1) \cdot f_j \alpha + \frac{\partial_j \alpha(t)}{\bigl(s_0 + \sum t_i s_i\bigr)^{n-p+1}},$$

because α is closed. Appearance to the contrary, the product $f_j \alpha$ therefore has a pole of order at most $n-p+1$, and the statement about the image of ω_ζ follows.

It remains to see that ω_ζ is closed under the differential $\nabla_{\mathcal{M}}$ of the de Rham complex

$$\mathcal{M} \longrightarrow \Omega_P^1 \otimes \mathcal{M} \longrightarrow \Omega_P^2 \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega_P^d \otimes \mathcal{M}.$$

To prove that $\nabla_{\mathcal{M}} \omega_\zeta = 0$, we need to show that for any two vector fields ξ and η,

$$\bigl(\nabla_{\mathcal{M}} \omega_\zeta\bigr)(\xi, \eta) = \xi \cdot \omega_\zeta(\eta) - \eta \cdot \omega_\zeta(\xi) - \omega_\zeta\bigl([\xi, \eta]\bigr) = 0.$$

We do this in the same coordinate system as above, for $\xi = \partial_j$ and $\eta = \partial_k$. Because of the way in which the $\mathcal{D}$-module structure on $\mathcal{M}$ is defined, the element $\partial_j \cdot \omega_\zeta(\partial_k) - \partial_k \cdot \omega_\zeta(\partial_j)$ is represented by the residue of $\partial_j \lrcorner\, d(f_k \alpha) - \partial_k \lrcorner\, d(f_j \alpha)$. But $d(f_k \alpha) = (df_k) \wedge \alpha$, and so

$$\partial_j \lrcorner\, d(f_k \alpha) - \partial_k \lrcorner\, d(f_j \alpha) = \frac{s_j s_k}{\bigl(s_0 + \sum t_i s_i\bigr)^2} \cdot \alpha - \frac{s_k s_j}{\bigl(s_0 + \sum t_i s_i\bigr)^2} \cdot \alpha = 0,$$

as asserted. $\square$

Here is another interpretation of the form ω_ζ. Consider again the exact sequence in Theorem 5.5.2. If we apply F_{-p} to it, and then tensor by Ω_P^1, we obtain the short exact sequence

$$0 \longrightarrow \Omega_P^1 \otimes F^p H^n(X)_{prim} \longrightarrow \Omega_P^1 \otimes F_{-p} \mathcal{N}^0 \longrightarrow \Omega_P^1 \otimes F_{n-p+1} \mathcal{M} \longrightarrow 0.$$

Now take cohomology; by arguments similar to those used in the proof of Theorem 6.1.2, it can be shown that $H^1\big(P, \Omega_P^1 \otimes F_{-p}\mathcal{N}^0\big) = 0$ (see the comments on p. 147). Thus we get a surjective map

$$H^0\big(P, \Omega_P^1 \otimes F_{n-p+1}\mathcal{M}\big) \to H^1\big(P, \Omega_P^1\big) \otimes F^p H^n(X), \tag{7.1}$$

and ζ can be recovered from the image of ω_ζ, by virtue of the following lemma.

Lemma 7.1.2. *Under the map in* (7.1), *the element* ω_ζ *is sent to* $2\pi i \cdot c_1\big(\mathscr{O}_P(1)\big) \otimes \zeta$.

Proof. To prove the assertion, we need to compute the connecting homomorphism for the exact sequence. This is most easily done using affine open sets $U \subseteq P$ of the kind considered during the proof of Lemma 7.1.1. On each such U, we have coordinates $(t_1, \dots, t_d) \mapsto [s_0 + t_1 s_1 + \dots + t_d s_d]$, and ω_ζ is given as a residue by the formula

$$\omega_\zeta\big|_U = \sum_{j=1}^d dt_j \otimes \operatorname{Res}_{\mathfrak{X}/P}\left(\frac{s_j \alpha}{s_0 + \sum t_i s_i}\right),$$

in the notation used before. Thus a natural lifting to a section of $\Omega_P^1 \otimes F_{-p}\mathcal{N}^0$ over each U is given by

$$\omega_U = \sum_{j=1}^d dt_j \otimes \frac{s_j \alpha}{s_0 + \sum t_i s_i}.$$

Now consider a second open set U', for a different basis $s_0', s_1', \dots, s_d'$ of V. The corresponding coordinates $t_1', \dots, t_d'$ are related to $t_1, \dots, t_d$ by an identity of the form

$$s_0' + \sum t_i' s_i' = g_{UU'}\big(s_0 + \sum t_i s_i\big),$$

where $g_{UU'}$ is a non-vanishing function on $U \cap U'$. An easy computation shows that

$$\omega_{U'} = d \log g_{UU'} \otimes \alpha + \omega_U,$$

and so the class in $H^1\big(P, \Omega_P^1\big) \otimes F^p H^n(X)_{prim}$ is represented by the Čech co-cycle

$$\big(U, U'\big) \mapsto d \log g_{UU'} \otimes \alpha = d \log g_{UU'} \otimes \zeta,$$

remembering that α represents ζ on each fiber. But clearly the class of the co-cycle

$$\big(U, U'\big) \mapsto \frac{1}{2\pi i} \cdot d \log g_{UU'}$$

is the first Chern class of $\mathscr{O}_P(1)$, since $s_0 + \sum t_i s_i$ is a section of $\mathscr{O}_P(1)$ over U. This observation completes the proof. $\square$

Note. The factor of $2\pi i$ is consistent with the Tate twist occurring in the sequence of mixed Hodge modules in (5.23).

7.1.2 Dependence of the choice of rational form

The elementary construction in Lemma 7.1.1 depends on the choice of a rational n-form α on $P \times X$ to represent the pullback of the class ζ. This means that the one-form ω_ζ is not uniquely determined by the primitive cohomology class. It should be the case that changing α by an exact form only changes ω_ζ by an element in the image of $\nabla_{\mathcal{M}}$; however, my attempts at proving this directly have not been successful.

Note. When $\zeta \in H^{n,0}(X)$, there is no ambiguity in the construction; we can uniquely represent ζ by a holomorphic n-form on X, and simply let α be the pullback of that form to $P \times X$.

We shall take a different approach to the problem of ambiguity, using the results about the de Rham complex that were derived in 5.5.5. We begin by observing that the one-form ω_ζ defines a class in the hypercohomology of the de Rham complex, since it is closed under the differential $\nabla_{\mathcal{M}}$. Taking into account that the complex is supported in degrees $-d, \dots, 0$, we thus have

$$[\omega_\zeta] \in \mathbb{H}^{-d+1}\big(\mathrm{DR}_P(\mathcal{M})\big).$$

By Lemma 5.5.3, vector space on the right is isomorphic to $H^n(X, \mathbb{C})_{prim}$. Thus the question is whether ζ and ω_ζ correspond to each other under the isomorphism. That they do follows from Lemma 5.5.4. Indeed, after passage to the underlying complex vector spaces, the lemma shows that the composition

$$H^n(X, \mathbb{C})_{prim} \xrightarrow{\simeq} \mathbb{H}^{-d+1}\big(\mathrm{DR}_P(\mathcal{M})\big) \to H^2(P, \mathbb{C}) \otimes H^n(X, \mathbb{C})_{prim}$$

is multiplication by $2\pi i \cdot c_1\big(\mathscr{O}_P(1)\big)$. But we already know from Lemma 7.1.2 that $[\omega_\zeta]$ goes to $2\pi i \cdot c_1\big(\mathscr{O}_P(1)\big) \otimes \zeta$ under the second map; the only way this can happen is that the first map takes ζ to $[\omega_\zeta]$, which is what we needed to show.

It follows that the cohomology class ζ does determine the form ω_ζ up to elements in the image of $\nabla_{\mathcal{M}}$, at least when $\mathscr{L}$ is sufficiently ample.[2]

Note. In particular, we see that the elementary construction of ω_ζ via residues provides a natural interpretation for the isomorphism in Lemma 5.5.3. Herb Clemens has also shown me a direct proof that the map $\zeta \mapsto [\omega_\zeta]$ is the one given by the Leray spectral sequence.

[2] I do not know whether this last condition is really necessary.

7.1.3 The local obstruction to integrating the one-form

In contrast to the situation over P^{sm}, the de Rham complex

$$\mathcal{M} \longrightarrow \Omega_P^1 \otimes \mathcal{M} \longrightarrow \Omega_P^2 \otimes \mathcal{M} \longrightarrow \cdots$$

for the $\mathcal{D}$-module $\mathcal{M}$ is typically not locally exact. Indeed, since $\mathcal{M} = \mathcal{M}_{van}$ underlies the mixed Hodge module $j_{!*}V_{van}^{n-1}$, the de Rham complex $\mathrm{DR}_P(\mathcal{M})$ is isomorphic to the perverse sheaf $\operatorname{rat} j_{!*}V_{van}^{n-1} \otimes_{\mathbb{Q}} \mathbb{C} = j_{!*}R^{2m-1}\pi_*^{sm}\mathbb{C}_{van}[d]$. Its cohomology sheaves, which we denote by $\mathcal{H}^k \mathrm{DR}_P(\mathcal{M})$, have already been computed in Lemma 5.5.5; the two that are of interest to us are the ones in degree $-d$ and $-d+1$. According to the lemma, the former satisfies

$$R^{n-1}\pi_*\mathbb{C} \simeq \mathcal{H}^{-d} \mathrm{DR}_P(\mathcal{M}) \oplus H^{n-1}(X, \mathbb{C}) \otimes \mathbb{C}_P,$$

while the latter is described by

$$R^n\pi_*\mathbb{C} \simeq \mathcal{H}^{-d+1} \mathrm{DR}_P(\mathcal{M}) \oplus H^{n-2}(X, \mathbb{C}) \otimes \mathbb{C}_P.$$

The $\mathcal{M}$-valued form ω_ζ gives a global section of the cohomology sheaf $\mathcal{H}^{-d+1} \mathrm{DR}_P(\mathcal{M})$, and thus also an element σ_ζ of $H^0\big(P, R^n\pi_*\mathbb{C}\big)$. This element is evidently the obstruction to locally integrating ω_ζ; that is to say, ω_ζ is locally in the image of $\nabla_{\mathcal{M}}$ at exactly those points where $\sigma_\zeta = 0$. The following lemma describes the obstruction in terms of the original cohomology class $\zeta \in F^k H^n(X)_{prim}$.

Theorem 7.1.3. *The section σ_ζ of $R^n\pi_*\mathbb{C}$ is the image of ζ under the map*

$$H^n(X, \mathbb{C}) \xrightarrow{\phi^*} H^n(\mathfrak{X}, \mathbb{C}) \to H^0\big(P, R^n\pi_*\mathbb{C}\big).$$

Thus ω_ζ can be locally integrated near a point $p \in P$ if, and only if, $\zeta\big|_{\mathfrak{X}_p} = 0$.

Proof. This is again a consequence of the results in Chapter 5. Fix a point $p \in P$. We then have a commutative diagram

$$\begin{tikzcd} H^n(X, \mathbb{C})_{prim} \arrow[r, "\simeq"] \arrow[d] & \mathbb{H}^{-d+1}\big(\mathrm{DR}_P(\mathcal{M})\big) \arrow[d] \\ H^n\big(\mathfrak{X}_p, \mathbb{C}\big) \arrow[r, hook] & \big(\mathcal{H}^{-d+1}\,\mathrm{DR}_P(\mathcal{M})\big)_p \end{tikzcd}$$

in which the isomorphism in the top row is defined using the Leray spectral sequence (see Lemma 5.5.3), and the map in the bottom row is given by Lemma 5.5.5. We proved in 7.1.2 that the image of ζ under the isomorphism is exactly the class of ω_ζ. The diagram now shows that ω_ζ goes to zero in the group $\big(\mathcal{H}^{-d+1}\,\mathrm{DR}_P(\mathcal{M})\big)_p$ (and thus ω_ζ is locally integrable near p) if, and only if, the image of ζ in $H^n\big(\mathfrak{X}_p, \mathbb{C}\big)$ is zero. □

7.2 Primitive cohomology as extensions of coherent sheaves

A second way of representing a primitive cohomology class $\zeta \in F^k H^n(X)_{prim}$ by data on the projective space P is given in this section. We show that ζ determines a certain nontrivial extension of coherent sheaves on P; in analogy with Theorem 7.1.3, it should be the case that the local obstruction to splitting the extension at a point $p \in P$ is the image of ζ in $H^n\big(\mathfrak{X}_p, \mathbb{C}\big)$. I have not, however, been able to prove this yet.

7.2.1 The construction of such extensions

In Theorem 6.4.17, we proved that

$$\operatorname{Ext}^1_P\bigl(F_{k+1}\mathcal{M}, \mathscr{O}_P\bigr) \simeq \bigl(F^{n-k}H^n(X)_{prim}\bigr)^\vee \simeq \frac{H^n(X)_{prim}}{F^{k+1}H^n(X)_{prim}}. \tag{7.2}$$

By exactly the same method, one can also show that

$$\operatorname{Ext}^1_P\bigl(Gr^F_{k+1}\mathcal{M}, \mathscr{O}_P\bigr) \simeq \bigl(H^{n-k,k}(X)_{prim}\bigr)^\vee \simeq H^{k,n-k}(X)_{prim}. \tag{7.3}$$

Both isomorphisms can be described very concretely, as follows from the proof. Namely, we have the short exact sequence of filtered $\mathcal{D}$-modules in (6.37); after passing to the graded quotient of the filtration (in degree $k-n$), we obtain a short exact sequence of coherent sheaves

$$H^{n-k,k}(X)_{prim} \otimes \mathscr{O}_P \hookrightarrow Gr^F_{k-n}\mathcal{N}^0 \twoheadrightarrow Gr^F_{k+1}\mathcal{M} \tag{7.4}$$

Now let $\zeta \in H^{k,n-k}(X)_{prim}$ be any primitive class; we may view it as a linear functional on $H^{n-k,k}(X)_{prim}$, using the intersection pairing on X. We can use to form a new exact sequence, by push-out from (7.4):

$$\begin{tikzcd}
H^{n-k,k}(X)_{prim} \otimes \mathscr{O}_P \arrow[r, hook] \arrow[d, "\zeta"'] & Gr^F_{k-n}\mathcal{N}^0 \arrow[r, two heads] \arrow[d] & Gr^F_{k+1}\mathcal{M} \arrow[d, equal] \\
\mathscr{O}_P \arrow[r, hook] & \mathscr{F}_\zeta \arrow[r, two heads] & Gr^F_{k+1}\mathcal{M}
\end{tikzcd} \tag{7.5}$$

Here $\mathscr{F}_\zeta$ is a certain coherent sheaf on P. The extension class of the bottom row in (7.5) is exactly the image of ζ under the isomorphism (7.3).

7.2.2 The local obstruction to splitting the extension

On the entire projective space P, the extension

$$\mathscr{O}_P \hookrightarrow \mathscr{F}_\zeta \twoheadrightarrow Gr^F_{k+1}\mathcal{M} \tag{7.6}$$

constructed above does not split for nonzero $\zeta \in H^{n-k,k}(X)_{prim}$. However, there may be local splittings; these certainly exist at each place where $Gr^F_{n+1-k}\mathcal{M}$ is locally free (so, for instance, on all of P^{sm}). But since the sheaf $Gr^F_{n+1-k}\mathcal{M}$ is usually not everywhere locally free, this is a subtle question.

It seems likely that the existence of local splitting of the extension in (7.6) would be related to the behavior of the cohomology class ζ. A result like the following therefore appears plausible:

Lemma 7.2.1 (Unproved). *Let $\zeta \in H^{k,n-k}(X)_{prim}$ be a primitive cohomology class. Provided that the line bundle $\mathscr{L}$ is sufficiently ample, the following two things should be equivalent for a point $p \in P$:*

(i) The exact sequence in (7.6) *splits in a neighborhood of p.*

(ii) The class ζ restricts to zero in $H^n\big(\mathfrak{X}_p, \mathbb{C}\big)$.

In this form, I am currently not able to prove the statement; however, I will give some motivation for thinking that it should be true, and also prove a weakened version of Lemma 7.2.1. The basic idea is the following: Let $D \subseteq X$ be a single (possibly singular) hypersurface. We have an exact sequence

$$\cdots \longrightarrow \big(H^n(D,\mathbb{C})\big)^\vee \xrightarrow{r} H^n(X,\mathbb{C}) \longrightarrow H^n(U,\mathbb{C}) \longrightarrow \big(H^{n-1}(D,\mathbb{C})\big)^\vee \longrightarrow \cdots,$$

dual the one for cohomology with compact support; here r is dual to the restriction map $H^n(X,\mathbb{C}) \to H^n(D,\mathbb{C})$.

Now suppose we have a class $\zeta \in H^n(X,\mathbb{C})$, which (using the pairing) we may view as a map $H^n(X,\mathbb{C}) \to \mathbb{C}$. Having a splitting for the exact sequence in (7.6) essentially means that this map factors through $H^n(U,\mathbb{C})$, as shown in the following diagram.

$$\begin{tikzcd} \cdots \arrow[r] & \big(H^n(D,\mathbb{C})\big)^\vee \arrow[r, "r"] & H^n(X,\mathbb{C}) \arrow[r] \arrow[d, "\zeta"'] & H^n(U,\mathbb{C}) \arrow[r] \arrow[dl, dashed] & \big(H^{n-1}(D,\mathbb{C})\big)^\vee \arrow[r] & \cdots \\ & & \mathbb{C} & & & \end{tikzcd}$$

This means that $\zeta \circ r = 0$. Then we find for any $\phi \in \big(H^n(D,\mathbb{C})\big)^\vee$ that

$$0 = \int_X \zeta \cup r(\phi) = \phi\big(\zeta\big|_D\big),$$

which is equivalent to having $\zeta\big|_D = 0$.

7.2.3 Proof of a weaker statement

Let $\zeta \in H^{k,n-k}(X)_{prim}$ be a primitive class; it is naturally a linear functional on $F^{n-k}H^n(X)_{prim}$. According to (7.2), we have

$$\operatorname{Ext}^1_P\big(F_{k+1}\mathcal{M}, \mathscr{O}_P\big) \simeq \big(F^{n-k}H^n(X)_{prim}\big)^\vee,$$

and the extension corresponding to ζ is derived from Theorem 5.5.2 by push-out. Thus ζ determines a global section of the sheaf $\mathcal{E}xt^1\big(F_{k+1}\mathcal{M}, \mathscr{O}_P\big)$ on P. While Lemma 7.2.1 is currently unproved, the following weaker form of it does hold.

Lemma 7.2.2. *If the restriction of ζ to a hypersurface $\mathfrak{X}_p$ is zero, then the section of $\mathcal{E}xt^1\big(F_{k+1}\mathcal{M}, \mathscr{O}_P\big)$ vanishes at p.*

Proof. We use the commutative diagram of filtered $\mathcal{D}$-modules

$$\begin{tikzcd} 0 \arrow[r] & H^n(X,\mathbb{C})_{prim} \otimes \mathscr{O}_P \arrow[r] \arrow[d, "\zeta"'] & \mathcal{N}^0 \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{O}_P \arrow[r] & \mathcal{N}_\zeta \arrow[r] & \mathcal{M} \arrow[r] & 0, \end{tikzcd} \tag{7.7}$$

in which the top row is the exact sequence from Theorem 5.5.2, and the bottom row is obtained by pushing out along the map given by ζ.

From Theorem 7.3.3, we know that the form ω_ζ is locally integrable at p; let $U \subseteq P$ be a small neighborhood of the point p where ω_ζ is exact. We shall now see that the bottom row of (7.7) is split on U as a sequence of $\mathcal{D}$-modules (without preserving the filtration).

To that end, we consider the dual diagram, obtained by applying the duality functor $\mathbb{D}_P$. Both $\mathcal{M}$ and $H^n(X)_{prim} \otimes \mathscr{O}_P$ are self-dual, and so we get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d, equal] & \mathbb{D}_P\mathcal{N}_\zeta \arrow[r] \arrow[d] & \mathscr{O}_P \arrow[r] \arrow[d, "\zeta"'] & 0 \\ 0 \arrow[r] & \mathcal{M} \arrow[r] & \mathbb{D}_P\mathcal{N}^0 \arrow[r] & H^n(X,\mathbb{C})_{prim} \otimes \mathscr{O}_P \arrow[r] & 0. \end{tikzcd} \tag{7.8}$$

The bottom row now gives us a map

$$H^n(X,\mathbb{C})_{prim} \simeq H^n(X,\mathbb{C})_{prim} \otimes \mathbb{H}^{-d}\big(\mathrm{DR}_P(\mathscr{O}_P)\big) \to \mathbb{H}^{-d+1}\big(\mathrm{DR}_P(\mathcal{M})\big),$$

and we know from Lemma 5.5.3 that it is an isomorphism; and from 7.1.2 that it sends ζ to the class of ω_ζ.

Now let us prove that the top row in (7.8) is locally split. Taking the filtrations into account, we get an exact sequence of coherent sheaves

$$0 \longrightarrow F_k\mathcal{M} \longrightarrow F_k\mathbb{D}_P\mathcal{N}_\zeta \longrightarrow \mathscr{O}_P \longrightarrow 0,$$

and since $H^1\big(P, F_{k+1}\mathcal{M}\big) = 0$, we can find a global section s_0 of $F_k\mathbb{D}_P\mathcal{N}_\zeta$ mapping to the constant section 1 of $\mathscr{O}_P$. To split the sequence as $\mathcal{D}$-modules, we need to find a section $s \in \Gamma(U, \mathcal{M})$, such that

$$\nabla\big(s_0 - s\big) = 0,$$

because $s_0 - s$ is then a flat lifting of 1. Note that ∇s_0 is actually a section of $\Omega^1_P \otimes F_{k+1}\mathcal{M}$; since we know that ζ maps to the class of ω_ζ for the bottom row, we get $\nabla s_0 = \omega_\zeta$ (up to exact forms). Thus ∇s_0 can be integrated on U, and so we can find $s \in \Gamma(U, \mathcal{M})$ with the property $\nabla s_0 = \nabla s$. The top row of (7.8) is therefore split on U.

By applying $\mathbb{D}_P$ again, we go back to the original diagram; it follows that the bottom row of (7.7) is also locally split as $\mathcal{D}$-modules. Thus there is, in the neighborhood U of the point, a map of $\mathcal{D}$-modules $\sigma\colon \mathcal{N}_\zeta \to \mathscr{O}_P$ splitting the sequence. Passing to the $(k-n)$-th level of the filtrations, we get

$$0 \longrightarrow \mathscr{O}_P \longrightarrow F_{k-n}\mathcal{N}_\zeta \longrightarrow F_{k+1}\mathcal{M} \longrightarrow 0,$$

and the restriction of σ to $F_{k-n}\mathcal{N}_\zeta$ provides a local splitting of the sequence as coherent sheaves. It follows that the section of $\mathcal{E}xt^1\big(F_{k+1}\mathcal{M}, \mathscr{O}_P\big)$ vanishes on U, as asserted in the lemma. $\square$

Note. To conclude that the sequence

$$0 \longrightarrow \mathscr{O}_P \longrightarrow Gr^F_{k-n}\mathcal{N}_\zeta \longrightarrow Gr_{k+1}\mathcal{M} \longrightarrow 0$$

is also split on U, one would need to know that the map $\sigma\colon \mathcal{N}_\zeta \to \mathscr{O}_P$ is trivial on

$F_{k-n-1}\mathcal{N}_\zeta$. It is not clear to me how to find a $\mathcal{D}$-module splitting with this additional property.

7.3 An interpretation for the singularity of a normal function

We now interpret the construction above in the context of normal functions. In this section, we let X be a smooth complex projective variety of even dimension $n = 2m$. Let $\zeta \in H^{2m}(X, \mathbb{Z})_{prim} \cap H^{m,m}(X)$ be a primitive Hodge class on X, and let ν_ζ be the corresponding normal function on P^{sm}.

7.3.1 Background on normal functions

For a general introduction to normal functions, see (Voisin, 2002, Chapitre 19). Here we just recall the basic definitions in the case of hypersurfaces, and explain what a *singularity* of a normal function is, following (Green and Griffiths, 2007; Brosnan et al., 2007).

Intermediate Jacobians Given any integral Hodge structure H of odd weight $2m-1$, say with underlying complex vector space $H_\mathbb{C}$, Hodge filtration $F^\bullet H_\mathbb{C}$, and integral lattice $H_\mathbb{Z}$, one can define a complex torus

$$J(H) = \frac{H_\mathbb{C}}{F^m H_\mathbb{C} + H_\mathbb{Z}}.$$

This torus is called Griffiths' intermediate Jacobian for the Hodge structure.

Now let X be a smooth complex projective variety of dimension $2m$, and let $\pi\colon \mathfrak{X} \to P$ be the universal hypersurface associated to some very ample line bundle $\mathscr{L}$ on

X. As usual, let P^{sm} be the open set parametrizing smooth hypersurfaces, and let $\mathcal{H}_{\mathbb{Z}}^{2m-1} = R^{2m-1}\pi_*^{sm}\mathbb{Z}_{van}$ be the local system of their vanishing cohomology. At each point $p \in P^{sm}$, the cohomology group $H^{2m-1}\big(\mathfrak{X}_p, \mathbb{Z}\big)_{van}$ carries a Hodge structure of weight $2m-1$, and so we can consider its intermediate Jacobian

$$\mathfrak{J}_p = J\big(H^{2m-1}(\mathfrak{X}_p)_{van}\big).$$

Since $\mathcal{H}_{\mathbb{Z}}^{2m-1}$ underlies a variation of Hodge structure, we get a fiber bundle $\mathfrak{J} \to P^{sm}$, whose fibers are the individual tori $\mathfrak{J}_p$. Let $\mathscr{H}_{\mathbb{C}}^{2m-1} = \mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes_{\mathbb{Z}} \mathscr{O}_{P^{sm}}$ be the vector bundle associated to the local system, and let $F^\bullet \mathscr{H}_{\mathbb{C}}^{2m-1}$ be the Hodge filtration. It satisfies Griffiths' transversality axiom,

$$\nabla F^p \mathscr{H}_{\mathbb{C}}^{2m-1} \subseteq \Omega_{P^{sm}}^1 \otimes F^{p-1} \mathscr{H}_{\mathbb{C}}^{2m-1}, \tag{7.9}$$

where ∇ is the Gauss-Manin connection.

Denote by $\mathscr{J}$ the sheaf of holomorphic sections of the torus bundle; by construction, it is part of an exact sequence

$$0 \longrightarrow \mathcal{H}_{\mathbb{Z}}^{2m-1} \oplus F^m \mathscr{H}_{\mathbb{C}}^{2m-1} \longrightarrow \mathscr{H}_{\mathbb{C}}^{2m-1} \longrightarrow \mathscr{J} \longrightarrow 0. \tag{7.10}$$

Normal functions A *normal function* is a holomorphic section $\nu \colon P^{sm} \to \mathfrak{J}$ of the bundle of intermediate Jacobians, satisfying the following additional *horizontality* condition: Locally, ν can be lifted to a section σ of $\mathscr{H}_{\mathbb{C}}^{2m-1}$; then $\nabla\sigma$ should be a section of $\Omega_{P^{sm}}^1 \otimes F^{m-1}\mathscr{H}_{\mathbb{C}}^{2m-1}$. This requirement makes sense, because (7.9) implies that it holds for sections of $F^m\mathscr{H}_{\mathbb{C}}^{2m-1} + \mathcal{H}_{\mathbb{Z}}^{2m-1}$, which are the ambiguity in the choice of σ.

In particular, every normal function ν is an element of $H^0\big(P^{sm}, \mathscr{J}\big)$; under the connecting homomorphism for the short exact sequence in (7.10), it goes to a certain class $\delta(\nu) \in H^1\big(P^{sm}, \mathcal{H}_{\mathbb{Z}}^{2m-1}\big)$, called the *cohomology class* of the normal function.

Hodge classes and admissible normal functions Given a primitive Hodge class $\zeta \in H^{2m}(X,\mathbb{Z})_{prim} \cap H^{m,m}(X)$, there is an associated normal function ν_ζ for every very ample line bundle $\mathscr{L}$. Such "geometric" normal functions satisfy an additional condition called *admissibility* (Saito, 1996, Definition 1.4 on p. 241), related to their behavior near points of $X^\vee$.

The cohomology group $H^1\big(P^{sm}, \mathcal{H}_{\mathbb{Z}}^{2m-1}\big)$ actually carries a mixed Hodge structure, with weights $2m$ and above. For an admissible normal function ν, it is a theorem, due Zucker and Saito (Saito, 1996), that $\delta(\nu)$ is always a Hodge class, meaning of weight $2m$ and type (m,m). A more precise statement is the following: The intersection cohomology group $IH^1\big(P, \mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big)$ has a pure Hodge structure of weight $2m$, there is a natural map

$$IH^1\big(P, \mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big) \to H^1\big(P^{sm}, \mathcal{H}_{\mathbb{Z}}^{2m-1}\big) \otimes \mathbb{Q}$$

to the ordinary cohomology of the local system. Then $\delta(\nu)$ is the image of a Hodge class of type (m,m) in the intersection cohomology group.

When $\mathscr{L}$ is sufficiently ample, Proposition 4.3.1 implies that

$$IH^1\big(P, \mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big) \simeq H^1\big(P^{sm}, \mathcal{H}_{\mathbb{Z}}^{2m-1}\big) \otimes \mathbb{Q} \simeq H^{2m}(X,\mathbb{Q})_{prim}; \tag{7.11}$$

as one might expect, $\delta\big(\nu_\zeta\big)$ is then exactly the original Hodge class ζ. Thus ζ and ν_ζ determine each other in this case.

7.3.2 Singularities of normal functions

The concept of *singularities* of a normal function was first introduced by Green and Griffiths (2007, 2006). The general idea is the following: The bundle of intermediate Jacobians $\mathfrak{J} \to P^{sm}$ should have an extension to an object (no longer an analytic space) over P, whose fiber over any boundary point $p \in X^\vee$ is an extension of a finitely generated Abelian group by a complex semi-torus.[3] Any admissible normal function should be a section of this *Néron model*. In this way of thinking, the normal function has a singularity at a point $p \in X^\vee$ if the corresponding section does not pass through the identity component of the fiber. Equivalently, a singularity means that the image of ν in the Abelian group parametrizing the components is nonzero. The construction of the Néron model is still work in progress (by Green, Griffiths, Kerr, and Young on the one hand, and Brosnan, Pearlstein, and Saito on the other). The group is question has been defined, however, at least with rational coefficients; here is the construction.[4] Let $p \in X^\vee$ be any boundary point. The limit

$$IH_p^1\big(\mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big) = \lim_{U \ni p} IH^1\big(U, \mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big) \subseteq \lim_{U \ni p} H^1\big(U, \mathcal{H}_{\mathbb{Z}}^{2m-1}\big) \otimes \mathbb{Q}$$

is a finite-dimensional $\mathbb{Q}$-vector space. By restricting $\delta(\nu) \in IH^1\big(P, \mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big)$ to this space, we get an element

$$\sigma_p(\nu) \in IH_p^1\big(\mathcal{H}_{\mathbb{Z}}^{2m-1} \otimes \mathbb{Q}\big),$$

[3]Here, semi-torus means the quotient of a complex vector space by a discrete subgroup; the quotient may not be compact.

[4]This appears in Brosnan et al. (2007); also, Green and Griffiths (2007) have another definition which applies when the complement of the open set is a divisor with normal crossings.

which is the singularity of the normal function at p (with rational coefficients).

The main result of Brosnan et al. (2007) is the following theorem; it gives a precise form to the idea by Green and Griffiths that the Hodge conjecture is equivalent to the existence of singularities for normal functions of the form ν_ζ.

Theorem 7.3.1 (Brosnan, Fang, Nie, and Pearlstein). *The Hodge conjecture is equivalent to the following statement: For every smooth projective variety X of even dimension $2m$, and every Hodge class $\zeta \in H^{2m}(X, \mathbb{Q})_{prim} \cap H^{m,m}(X)$, there is a sufficiently ample line bundle $\mathscr{L}$, and a point $p \in X^\vee$ where $\sigma_p\big(\nu_\zeta\big) \neq 0$.*

The proof makes use of the following statement, interesting in its own right.

Proposition 7.3.2 (Brosnan, Fang, Nie, and Pearlstein). *Let ζ be a primitive Hodge class of type (m, m) on a smooth projective variety X of dimension $2m$. If the line bundle $\mathscr{L}$ is sufficiently ample, then the normal function ν_ζ has a singularity at a point $p \in X^\vee$ if, and only if, the restriction of ζ to $\mathfrak{X}_p$ is nonzero.*

7.3.3 An interpretations of the singularity

First of all, we can apply the construction in 7.1.1 to the Hodge class $\zeta \in H^{2m}(X, \mathbb{Z})_{prim} \cap H^{m,m}(X)$. According to Lemma 7.1.1, we then get a holomorphic one-form ω_ζ with coefficients in $F_{m+1}\mathcal{M}$, defined on all of P, and closed under the differential $\nabla_\mathcal{M}$ of the de Rham complex

$$\mathcal{M} \longrightarrow \Omega^1_P \otimes \mathcal{M} \longrightarrow \Omega^2_P \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega^d_P \otimes \mathcal{M}$$

for the $\mathcal{D}$-module $\mathcal{M}$. Over P^{sm}, this one-form represents the element in the space $H^1\big(P^{sm}, R^{2m-1}\pi_*^{sm}\mathbb{Q}_{van}\big)$ corresponding to ζ under the isomorphism in (7.11).

On P^{sm}, the de Rham complex is exact, because $\mathcal{M}$ is locally free there. At points of $X^\vee$, however, $\mathcal{M}$ is typically not locally free, and so ω_ζ is not necessarily in the image of $\nabla_{\mathcal{M}}$. In other words, the form ω_ζ cannot always be integrated in a neighborhood of a boundary point. The following theorem shows that this is one possible interpretation of the singularity of ν_ζ.

Theorem 7.3.3. *Let $\zeta \in H^{2m}(X, \mathbb{Z})_{prim} \cap H^{m,m}(X)$ be a primitive Hodge class on the smooth projective m-dimensional variety X. Assume that the line bundle $\mathscr{L}$ is sufficiently ample, and let $p \in X^\vee$ be a boundary point. Then the following two things are equivalent:*

(i) The section ω_ζ of $\Omega_P^1 \otimes F_{m+1}\mathcal{M}$ can be locally integrated near p, i.e., is locally in the image of $\nabla_{\mathcal{M}}$.

(ii) The singularity $\sigma_p\big(\nu_\zeta\big)$ of the associated normal function ν_ζ is nonzero.

Proof. In Theorem 7.1.3, we showed that a necessary and sufficient condition for local integrability of ω_ζ at a point p is that $\omega_\zeta\big|_{\mathfrak{X}_p} = 0$. By Proposition 7.3.2, this is equivalent to ν_ζ having a singularity at the point p, because the line bundle $\mathscr{L}$ is assumed to be sufficiently ample. ☐

A second interpretation for the singularity $\sigma_p\big(\nu_\zeta\big)$ may come from the construction in 7.2.1. Under the isomorphism

$$\mathrm{Ext}_P^1\big(Gr_{m+1}^F\mathcal{M}, \mathscr{O}_P\big) \simeq H^{m,m}(X)_{prim}$$

given there, the Hodge class ζ corresponds to the extension class of the short exact sequence

$$H^{m,m}(X)_{prim} \otimes \mathscr{O}_P \hookrightarrow Gr^F_{-m}\mathcal{N}^0 \twoheadrightarrow Gr^F_{m+1}\mathcal{M}. \tag{7.12}$$

This sequence is derived from the one in Theorem 5.5.2, by taking graded quotients in degree $-m$.

Over P^{sm}, the sheaf $Gr^F_{m+1}\mathcal{M}$ restricts to the Hodge bundle with fibers $H^{m-1,m}\big(\mathfrak{X}_p\big)_{van}$, and is thus locally free; in particular, the short exact sequence splits over P^{sm}. Once again, this does not remain true at points of of $X^\vee$,[5] but it stands to reason that the existence of local splittings is related to the presence of singularities in the normal function. Therefore, I suspect that something like the following result should be true; however, I am currently not able to prove more than what is given in 7.2.3.

Theorem 7.3.4 (Unproved)**.** *Under the same assumptions as in Theorem 7.3.3, the following two conditions on a point $p \in P$ should be equivalent:*

(i) The short exact sequence in (7.12) *is split in a neighborhood of the point p.*

(ii) The singularity $\sigma_p\big(\nu_\zeta\big)$ of the associated normal function ν_ζ is nonzero.

Indeed, if Lemma 7.2.1 were available, it would show that the short exact sequence splits at p if, and only if, ζ restricts to zero in $H^n\big(\mathfrak{X}_p, \mathbb{C}\big)$. Just as in Theorem 7.3.3, this latter condition is equivalent to ν_ζ having a singularity at p.

[5] We certainly cannot expect $Gr^F_{m+1}\mathcal{M}$ to be locally free on all of P.

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