EXTENDING HOLOMORPHIC FORMS FROM THE REGULAR LOCUS OF A
COMPLEX SPACE TO A RESOLUTION OF SINGULARITIES

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ABSTRACT. We investigate under what conditions holomorphic forms defined on the regular locus of a complex space extend to a resolution of singularities. Our main result, proved using Hodge modules and the Decomposition Theorem, is that on any reduced and irreducible complex space, the extension problem for forms of a given degree also controls what happens for all forms of smaller degrees. This implies the existence of a functorial pull-back for reflexive differentials on spaces with rational singularities. We also prove a variant of the main result for forms with logarithmic poles, and use our methods to settle the "local vanishing conjecture" proposed by Mustață, Olano, and Popa.

1. OVERVIEW OF THE PAPER

1.1. Extension of holomorphic forms. This paper is about the following "extension problem" for holomorphic differential forms on complex spaces. Let $X$ be a reduced complex space, and let $r: \tilde{X} \to X$ be any resolution of singularities. Under what conditions on the singularities of $X$ does every holomorphic $p$-form defined on the regular locus $X_{\text{reg}}$ extend to a holomorphic $p$-form on the complex manifold $\tilde{X}$?

The best existing result concerning this problem is due to Greb, Kebekus, Kovács, and Peternell [GKKP11, Thm. 1.4]. They show that if $X$ underlies a normal algebraic variety with Kawamata log terminal (=klt) singularities, then all $p$-forms on $X_{\text{reg}}$ extend to $\tilde{X}$, for every $0 \leq p \leq \dim X$. In this paper, we use the Decomposition Theorem and Saito’s theory of mixed Hodge modules to solve the extension problem in much greater generality, without assuming that $X$ is algebraic, normal, or $\mathbb{Q}$-Gorenstein. Our main result says, roughly speaking, that when $X$ is reduced and irreducible, the extension problem for forms of a given degree also controls what happens for all forms of smaller degrees.

1.1.1. Main result. Since there might not be any global $p$-forms on $X_{\text{reg}}$, the effect of the singularities of $X$ is better captured by the following local version of the problem. Given a resolution of singularities $r: \tilde{X} \to X$ of a reduced complex space $X$, under what conditions on the singularities does every holomorphic $p$-form defined on $U_{\text{reg}}$ extend to a holomorphic $p$-form on $\tilde{X}$, for all open subsets $U \subseteq X$? If $j: X_{\text{reg}} \hookrightarrow X$ denotes the embedding of the regular locus, this is equivalent to asking whether the natural morphism of $\mathcal{O}_X$-modules $r_\ast \Omega^k_{\tilde{X}} \hookrightarrow j_\ast \Omega^k_{X_{\text{reg}}}$ is an isomorphism.

Example 1.1. When $X$ is reduced and irreducible, it is easy to see that $r_\ast \mathcal{O}_{\tilde{X}} \hookrightarrow j_\ast \mathcal{O}_{X_{\text{reg}}}$ is an isomorphism if and only if $\dim X_{\text{sing}} \leq \dim X - 2$. (Use the normalisation of $X$.)

Theorem 1.2 (Extension for $p$-forms). Let $X$ be a reduced and irreducible complex space. Let $r: \tilde{X} \to X$ be any resolution of singularities, and $j: X_{\text{reg}} \hookrightarrow X$ the inclusion of the regular locus. If the morphism $r_\ast \Omega^k_{\tilde{X}} \hookrightarrow j_\ast \Omega^k_{X_{\text{reg}}}$ is an isomorphism for some $0 \leq k \leq \dim X$, then $\dim X_{\text{sing}} \leq \dim X - 2$, and $r_\ast \Omega^p_{\tilde{X}} \hookrightarrow j_\ast \Omega^p_{X_{\text{reg}}}$ is an isomorphism for every $0 \leq p \leq k$.  

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Note. When $X$ is normal, an equivalent formulation of Theorem 1.2 is that, if the coherent $\mathcal{O}^k_X$-module $r_*\Omega^k_X$ is reflexive for some $k \leq \dim X$, then $r_*\Omega^p_X$ is reflexive for every $p \leq k$.

An outline of the proof can be found in Section 2 below. The key idea is to use the Decomposition Theorem [BBD82, Sai88], in order to relate the coherent $\mathcal{O}^k_X$-module $r_*\Omega^k_X$ to the intersection complex of $X$, viewed as a polarisable Hodge module. In Appendix B, we look at the example of cones over smooth projective varieties; it gives a hint that the extension problem for all $p$-forms should be governed by what happens for $n$-forms.

Note. One can easily generalise Theorem 1.2 to arbitrary reduced complex spaces. The precise (but somewhat cumbersome) statement is that if the morphism $r_*\Omega^k_X \hookrightarrow j_*\Omega^k_{\operatorname{reg}}$ is an isomorphism for some $k \geq 0$, and if $Z \subseteq X$ denotes the union of all the irreducible components of $X$ of dimension $\geq k$, then $\dim Z\operatorname{sing} \leq k - 2$, and the restriction to $Z$ of the morphism $r_*\Omega^p_X \hookrightarrow j_*\Omega^p_{\operatorname{reg}}$ is an isomorphism for every $0 \leq p \leq k$. The reason is that the irreducible components of $X$ are separated in any resolution of singularities, and so one can simply apply Theorem 1.2 one component at a time.

1.1.2. Independence of the resolution. Since any two resolutions of $X$ are dominated by a common third, the subsheaf $r_*\Omega^k_X \subseteq j_*\Omega^k_{\operatorname{reg}}$ does not depend on the choice of $r: \tilde{X} \rightarrow X$. Both the assumption and the conclusion of Theorem 1.2 are therefore independent of the resolution. Because of its role in the Grauert-Riemenschneider vanishing theorem, we refer to $r_*\omega_X$ as the Grauert-Riemenschneider sheaf and denote it by the symbol $\omega_X^{\operatorname{GR}}$.

1.1.3. Intrinsic description of extendable forms. The following result, whose proof is also based on the Decomposition Theorem, gives an intrinsic description of those holomorphic forms on the regular locus of a complex space that extend holomorphically to one (and hence any) resolution of singularities. We think that it would be very interesting to have an analytic proof of this result, in terms of $L^2$-Hodge theory for the $\bar{\partial}$-operator.

Theorem 1.3 (Intrinsic description). Let $X$ be a reduced complex space of constant dimension $n$, and $r: \tilde{X} \rightarrow X$ a resolution of singularities. A holomorphic $p$-form $\alpha \in H^0(X_{\operatorname{reg}}, \Omega^p_X)$ extends to a holomorphic $p$-form on $\tilde{X}$ if, and only if, for every open subset $U \subseteq X$, and for every pair of Kähler differentials $\beta \in H^0(U, \Omega^{n-p}_X)$ and $\gamma \in H^0(U, \Omega^{n-p-1}_X)$, the holomorphic $n$-forms $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ on $U_{\operatorname{reg}}$ extend to holomorphic $n$-forms on $r^{-1}(U)$.

For $n = \dim X$, it is well-known that a holomorphic $n$-form $\alpha \in H^0(X_{\operatorname{reg}}, \Omega^n_X)$ extends to a holomorphic $n$-form on $\tilde{X}$ if and only if $\alpha \wedge \bar{\alpha}$ is locally integrable on $X$. Griffiths [Gri76, §IIa] gave a similar criterion for extension of $p$-forms in terms of integrals over $p$-dimensional analytic cycles in $X$, but his condition is not easy to verify in practice.

1.1.4. Rational and weakly rational singularities. For example, Theorem 1.2 applies to normal complex spaces with rational singularities. Recall that $X$ has rational singularities if the following equivalent conditions hold. We refer to [KM98, §5.1] for details.

(1.4.1) $X$ is normal, and if $r: \tilde{X} \rightarrow X$ is any resolution of singularities, then $R^ir_*\mathcal{O}_{\tilde{X}} = 0$ for every $i \geq 1$.

(1.4.2) $X$ is Cohen-Macaulay and $\omega_X^{\operatorname{GR}} = \omega_X$.

(1.4.3) $X$ is Cohen-Macaulay and $\omega_X^{\operatorname{GR}}$ is reflexive.

The following corollary is then immediate.

Corollary 1.5 (Extension in the case of rational singularities). Let $X$ be a normal complex space with rational singularities, and let $r: \tilde{X} \rightarrow X$ be a resolution of singularities. Then every holomorphic form defined on $X_{\operatorname{reg}}$ extends uniquely to a holomorphic form on $\tilde{X}$. □
In view of Condition (1.4.3), we say that a normal space $X$ has weakly rational singularities if the Grauert-Riemenschneider sheaf $\omega_X^\text{GR}$ is reflexive. As we will see in Section 6.1, this turns out to be equivalent to the collection of inequalities
$$\dim \text{Supp} R^i r_* \mathcal{O}_X \leq \dim X - 2 - i \quad \text{for every } i \geq 1.$$ 

One can also describe the class of weakly rational singularities in more analytic terms: a normal complex space $X$ of dimension $n$ has weakly rational singularities if and only if, for every open subset $U \subseteq X$ and every holomorphic $n$-form $\omega \in H^0(U_{\text{reg}}, \Omega^n_{U_{\text{reg}}})$, the $(n, n)$-form $\omega \wedge p$ on $U_{\text{reg}}$ is locally integrable on all of $U$. Appendix A discusses examples and establishes elementary properties of this class of singularities.

1.2. Extension of differential forms with logarithmic poles. We also establish a version of Theorem 1.2 with log poles, by adapting the techniques in the proof to a certain class of graded-polarsizable mixed Hodge modules. Recall that a resolution of singularities $r: \tilde{X} \to X$ of a complex space is called a (strong) log resolution if the $r$-exceptional set is a divisor with (simple) normal crossings on $\tilde{X}$.

Theorem 1.6 (Extension for log $p$-forms). Let $X$ be a reduced and irreducible complex space. Let $r: \tilde{X} \to X$ be a log resolution with exceptional divisor $E \subseteq \tilde{X}$, and $j: X_{\text{reg}} \hookrightarrow X$ the inclusion of the regular locus. If the morphism $r_* \Omega^k_X(\log E) \hookrightarrow j_* \Omega^k_X$ is an isomorphism for some $0 \leq k \leq \dim X$, then $\dim X_{\text{sing}} \leq \dim X - 2$, and $r_* \Omega^p_X(\log E) \hookrightarrow j_* \Omega^p_X$ is an isomorphism for every $0 \leq p \leq k$.

Note. By a result of Kovács, Schwede, and Smith [KSS10, Thm. 1], a complex algebraic variety $X$ that is normal and Cohen-Macaulay has Du Bois singularities if and only if $r_* \omega_X(E)$ is a reflexive $\mathcal{O}_X$-module for some log resolution $r: \tilde{X} \to X$.

One has the following analogue of Theorem 1.3.

Theorem 1.7 (Intrinsic description). Let $X$ be a reduced complex space of constant dimension $n$, and $r: \tilde{X} \to X$ a log resolution of singularities with exceptional divisor $E \subseteq \tilde{X}$. A holomorphic $p$-form $\alpha \in H^0(X_{\text{reg}}, \Omega^p_X)$ extends to a holomorphic section of the bundle $\Omega^p_X(\log E)$ on $\tilde{X}$ if, and only if, for every open subset $U \subseteq X$, and for every pair of Kahler differentials $\beta \in H^0(U, \Omega^{n-p+1}_X)$ and $\gamma \in H^0(U, \Omega^n_X)$, the holomorphic $n$-forms $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ on $U_{\text{reg}}$ extend to holomorphic sections of the bundle $\Omega^n_X(\log E)$ on $r^{-1}(U)$.

The tools we develop for the proof of Theorem 1.6 also lead to a slightly better answer in the case of holomorphic forms of degree $\dim X - 1$.

Theorem 1.8 (Extension for $(n-1)$-forms). Let $X$ be a reduced and irreducible complex space. Let $r: \tilde{X} \to X$ be a log resolution with exceptional divisor $E \subseteq \tilde{X}$, and $j: X_{\text{reg}} \hookrightarrow X$ the inclusion of the regular locus. If the natural morphism $r_* \Omega^{n-1}_X(\log E) \hookrightarrow j_* \Omega^{n-1}_X$ is an isomorphism, where $n = \dim X$, then the two morphisms
$$r_* (\Omega^{n-1}_X(\log E)(-E)) \hookrightarrow r_* \Omega^{n-1}_X \hookrightarrow j_* \Omega^{n-1}_X$$

are also isomorphisms.

1.3. Local vanishing conjecture. The methods developed in this paper also settle the “local vanishing conjecture” proposed by Mustaţă, Olano, and Popa [MOP18, Conj. A]. The original conjecture contained the assumption that $X$ is a normal algebraic variety with rational singularities. In fact, the weaker assumption $E^{\dim X - 1} r_* \mathcal{O}_{\tilde{X}} = 0$ is sufficient.

Theorem 1.9 (Local vanishing). Let $X$ be a reduced and irreducible complex space of dimension $n$. Let $r: \tilde{X} \to X$ be a log resolution, with exceptional divisor $E \subseteq \tilde{X}$. If $R^{n-1} r_* \mathcal{O}_{\tilde{X}} = 0$, then $R^{n-1} r_* \Omega^n_X(\log E) = 0$. 

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As shown in [MOP18], this result has interesting consequences for the Hodge filtration on the complement of a hypersurface with at worst rational singularities.

### 1.4. Functorial pull-back.

One can interpret Theorem 1.2 as saying that any differential form \( \sigma \in H^0(X_{\text{reg}}, \Omega^1_X) = H^0(X, \Omega^{[1]}_X) \) induces a pull-back form \( \tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}) \). More generally, we show that pull-back exists for reflexive differentials and arbitrary morphisms between varieties with rational singularities. The paper [Keb13b] discusses these matters in detail.

**Theorem 1.10** (Functorial pull-back for reflexive differentials). Let \( f: X \to Y \) be any morphism between normal complex spaces with rational singularities. Write \( \Omega^{[p]}_X := (\Omega^p_X)^\vee \), ditto for \( \Omega^p_Y \). Then there exists a pull-back morphism

\[
d_{\text{refl}}: f^* \Omega^{[p]}_Y \to \Omega^{[p]}_X,
\]

uniquely determined by natural universal properties.

We refer to Theorem 14.1 and Section 14 for a precise formulation of the “natural universal properties” mentioned in Theorem 1.10. In essence, it is required that the pull-back morphisms agree with the pull-back of Kähler differentials wherever this makes sense, and that they satisfy the composition law.

**Note.** Theorem 1.10 applies to morphisms \( X \to Y \) whose image is entirely contained in the singular locus of \( Y \). Taking the inclusion of the singular set for a morphism, Theorem 1.10 implies that every differential form on \( Y_{\text{reg}} \) induces a differential form on every stratum on the singularity stratification.

### 1.4.1. h-differentials.

One can also reformulate Theorem 1.10 in terms of \( h \)-differentials; these are obtained as the sheafification of Kähler differentials with respect to the \( h \)-topology on the category of complex spaces, as introduced by Voevodsky. We refer the reader to [HJ14] and to the survey [Hub16] for a gentle introduction to these matters. Using the description of \( h \)-differentials found in [HJ14, Thm. 1], the following is an immediate consequence of Theorem 1.10.

**Corollary 1.11** (h-differentials on spaces with rational singularities). Let \( X \) be a normal complex space with rational singularities. Write \( \Omega^{[p]}_X := (\Omega^p_X)^\vee \). Then, \( h \)-differentials and reflexive differentials agree: \( \Omega^{[p]}_h(X) = \Omega^{[p]}_X(X) \). □

The sheaf \( \Omega^{[p]}_h \) of \( h \)-differentials appears under a different name in the work of Barlet, [Bar17], who describes it in analytic terms (“integral dependence equations for differential forms”) as a subsheaf of \( \Omega^{[p]}_X \) and relates it to the normalised Nash transform.

### 1.5. Sample application.

The extension theorem for klt spaces has had a number of applications, pertaining to integral Hodge classes [HV11], hyperbolicity of moduli [Keb13a], the structure of minimal varieties with trivial canonical class [GKP16, GGK17], the nonabelian Hodge correspondence for singular spaces [GKPT17], and quasi-étale uniformisation [LT14, GKPT15]. Here, we mention only one immediate application of Theorem 1.2.

**Theorem 1.12** (Closedness of forms and Bogomolov-Sommese vanishing). Let \( X \) be a normal complex projective variety. If \( \omega^\text{GR}_X \) is reflexive, then any holomorphic differential form on \( X_{\text{reg}} \) is closed. If \( \mathcal{A} \subseteq \Omega^{[p]}_X \) is a locally free subsheaf, then \( \kappa(\mathcal{A}) \leq p \). □

### 1.6. Earlier results.

As mentioned above, Theorem 1.2 was already known for spaces with Kawamata log terminal (=klt) singularities, where \( r_*\omega_X^\text{GR} \) is reflexive by definition [GKK10, GKKP11]. If one is only interested in differential forms of small degree \( p \ll \dim X \), there are earlier results of Steenbrink-van Straten [vSS85] and Flenner [Fle88]. In the special case where \( p = 1 \), Graf-Kovács relate the extension problem to the notion of...
Du Bois singularities [GK14]. For morphisms between varieties with klt singularities, the existence of a pull-back functor was shown in [Keb13b].

We refer the reader to the paper [GKKP11] or to the survey [Keb13a, §4] for a more detailed introduction, and for remarks on the history of the problem. The book [Kol13, §8.5] puts the results into perspective.

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2. **Techniques and main ideas**

In this section, we sketch some of the ideas that go into the proof of Theorem 1.2. The one-line summary is that it is a consequence of the Decomposition Theorem [BBD82, Sai90]. Appendix B contains a short section on cones over projective manifolds that illustrates the extension problem in a particularly transparent case and explains why one might even expect a result such as Theorem 1.2 to hold true.

2.1. **First proof of Theorem 1.2.** We actually give two proofs for Theorem 1.2. The first proof (in Section 1.1) relies on Theorem 1.3, which characterises those holomorphic forms on the regular locus of a complex space that extend holomorphically to any resolution of singularities. This proof is very short and, shows clearly why the extension problem for $k$-forms also controls the extension problem for $(k-1)$-forms (and hence for all forms of smaller degrees).

2.2. **Second proof of Theorem 1.2.** To illustrate the main ideas and techniques used in this paper, we are now going to describe a second, more systematic proof for Theorem 1.2. It is longer, and covers only the case where $k = n$, but it has the advantage of producing a stronger result that has other applications (such as the proof of the local vanishing conjecture). We hope that the description below will make it clear why the Decomposition Theorem is useful in studying the extension problem for holomorphic forms.

**Setup.** We fix a reduced and irreducible complex space $X$ of dimension $n$, and a resolution of singularities $r: \tilde{X} \rightarrow X$. We denote by $j: X_{\text{reg}} \hookrightarrow X$ the embedding of the set of regular points, and assume that the natural morphism $r_! \Omega^n_\tilde{X} \hookrightarrow j_* \Omega^n_{X_{\text{reg}}}$ is an isomorphism. This means concretely that, locally on $X$, holomorphic $n$-forms extend from the regular locus to the resolution. Rather than using the given resolution $\tilde{X}$ to show that $p$-forms extend, we are going to prove directly that the natural morphism $r_! \Omega^p_\tilde{X} \hookrightarrow j_* \Omega^p_{X_{\text{reg}}}$ is an isomorphism for every $p \in \{0, 1, \ldots, n\}$. This is a statement about $X$ itself, because the subsheaf $r_! \Omega^p_\tilde{X}$ does not depend on the choice of resolution, as we have seen in Section 1.1.2.

**Note.** Using independence of the resolution, we may assume without loss of generality that the resolution $r: \tilde{X} \rightarrow X$ is projective, and an isomorphism over $X_{\text{reg}}$. Such resolutions exist for every reduced complex space by [BM97, Thm. 10.7].
Criteria for extension. The first idea in the proof of Theorem 1.2 is to use duality.\footnote{For the sake of exposition, we work directly on $X$ in this section. In the actual proof, we only use duality for coherent sheaves on complex manifolds, after locally embedding $X$ into a complex manifold.} Let $\omega^*_X \in D^b_{\text{coh}}(\mathcal{O}_X)$ denote the dualizing complex of $X$; on the $n$-dimensional complex manifold $\widetilde{X}$, one has $\omega^*_X \cong \omega_X[n]$. The dualizing complex gives rise to a simple numerical criterion for whether sections of a coherent $\mathcal{O}_X$-module extend uniquely over $X_{\text{sing}}$. Indeed, Proposition 6.1 – or rather its generalisation to singular spaces – says that sections of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ extend uniquely over $X_{\text{sing}}$ if and only if
\[
\dim(X_{\text{sing}} \cap \text{Supp } R^k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega^*_X)) \leq -(k + 2) \quad \text{for every } k \in \mathbb{Z}.
\]
When the support of $\mathcal{F}$ has constant dimension $n$, as is the case for the $\mathcal{O}_X$-module $r_\ast \Omega^p_{\widetilde{X}}$ that we are interested in, this amounts to the following two conditions:
\[
\begin{align*}
(2.0.1) \quad & \dim X_{\text{sing}} \leq n - 2 \\
(2.0.2) \quad & \dim \text{Supp } R^k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega^*_X) \leq -(k + 2) \quad \text{for every } k \geq -n + 1
\end{align*}
\]
Unfortunately, there is no good way to compute the dual complex of $r_\ast \Omega^p_{\widetilde{X}}$. But if we work instead with the entire complex $Rr_\ast \Omega^p_{\widetilde{X}}$, things get better: Grothendieck duality \cite{RRV71}, applied to the proper holomorphic mapping $r: \widetilde{X} \to X$, yields
\[
R^k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega^*_X) \cong Rr_* R^k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega^*_X) \cong R^k (\Omega^p_{\widetilde{X}}(\omega^*_X[n]) \Rightarrow R^k \Omega^p_{\widetilde{X}}(\omega^*_X[n]).
\]
In Proposition 6.4, we prove the following variant of the criterion for section extension: if $K \in D^b_{\text{coh}}(\mathcal{O}_X)$ is a complex with $\mathcal{H}^j K = 0$ for $j < 0$, and if
\[
(2.0.4) \quad \dim(X_{\text{sing}} \cap \text{Supp } R^k \mathcal{H}om_{\mathcal{O}_X}(K, \omega^*_X)) \leq -(k + 2) \quad \text{for every } k \in \mathbb{Z},
\]
then sections of the coherent $\mathcal{O}_X$-module $\mathcal{H}^0 K$ extend uniquely over $X_{\text{sing}}$. This observation transforms the problem of showing that sections of $r_\ast \Omega^p_{\widetilde{X}}$ extend uniquely over $X_{\text{sing}}$ into the problem of showing that
\[
\dim \text{Supp } R^k r_\ast \Omega^{n-p}_{\widetilde{X}} \leq n - 2 - k \quad \text{for every } k \geq 1.
\]
In summary, we see that a good upper bound for the dimension of the support of $R^k r_\ast \Omega^{n-p}_{\widetilde{X}}$ would be enough to conclude that $p$-forms extend. Or, to put it more simply, "vanishing implies extension".

Hodge modules and the Decomposition Theorem. The problem with the approach outlined above is that the complex $Rr_\ast \Omega^{n-p}_{\widetilde{X}}$ has too many potentially nonzero cohomology sheaves, which makes it hard to prove the required vanishing. For example, if the preimage of a singular point $x \in X_{\text{sing}}$ is a divisor in the resolution $\widetilde{X}$, then $R^{n-1}r_\ast \Omega^{n-p}_{\widetilde{X}}$ might be supported at $x$, violating the inequality in (2.0.4). Since we are not assuming that the singularities of $X$ are klt, we also do not have enough information about the fibres of $r: \widetilde{X} \to X$ to prove vanishing by restricting to fibres as in \cite[§18]{GKKP11}.

The second idea in the proof, which completely circumvents this problem, is to relate the $\mathcal{O}_X$-module $r_\ast \Omega^p_{\widetilde{X}}$ to the intersection complex of $X$, viewed as a polarisable Hodge module.\footnote{Since the intersection complex is intrinsic to $X$, this also serves to explain once again why the $\mathcal{O}_X$-module $r_\ast \Omega^p_{\widetilde{X}}$ does not depend on the choice of resolution.} In the process, we make use of the Decomposition Theorem. Roughly speaking, the Decomposition Theorem decomposes the push-forward of the constant sheaf into a "generic part" (that only depends on $X$) and a "special part" (that is affected by the positive-dimensional fibres of $r$). The upshot is that the generic part carries all the relevant information, and that the positive-dimensional fibres of $r$ are completely irrelevant for the extension problem. To be more precise, the Decomposition Theorem for the projective
morphism \( r \), together with Saito’s formalism of Hodge modules, leads to a (non-canonical) decomposition

\[
R_r \Omega^p_X = K_p \oplus R_p
\]

into two complexes \( K_p, R_p \in D^b_{coh}(\mathcal{O}_X) \) with the following properties:

(2.0.6) The support of \( R_p \) is contained in the singular locus \( X_{\text{sing}} \).

(2.0.7) One has \( \mathcal{H}^k K_p = 0 \) for \( k \geq n - p + 1 \).

(2.0.8) The complexes \( K_p \) and \( K_{n-p} \) are related by Grothendieck duality in the same way that the complexes \( R_r \Omega^p_X \) and \( R_r \Omega^{n-p}_X \) are related in (2.0.3). More precisely, one has \( R\mathcal{H}om_{\mathcal{O}_X}(K_p, \omega^n_X) \cong K_{n-p}[n] \).

An improved criterion. As an immediate consequence of the decomposition in (2.0.5), we obtain a decomposition of the 0-th cohomology sheaves

\[
r_r \Omega^p_X \cong \mathcal{H}^0 K_p \oplus \mathcal{H}^0 R_p.
\]

Because \( \mathcal{H}^0 R_p \) is supported inside \( X_{\text{sing}} \), whereas \( \Omega^p_X \) is torsion free, we deduce that \( \mathcal{H}^0 R_p = 0 \), and hence that \( r_r \Omega^p_X \cong \mathcal{H}^0 K_p \). According to the criterion for section extension in Proposition 6.4, now applied to the complex \( K_p \), all we therefore need for sections of \( r_r \Omega^p_X \) to extend uniquely over \( X_{\text{sing}} \) is to establish the collection of inequalities

\[
\dim(\text{Sing} \cap \text{Supp } \mathcal{H}^k K_{n-p}) \leq n - 2 - k \quad \text{for all } k \in \mathbb{Z}.
\]

Property (2.0.7) makes this a much more manageable task, compared to the analogous problem for the original complex \( R_r \Omega^{n-p}_X \). We stress that, except in the case \( p = n \), these inequalities are stronger than asking that sections of \( r_r \Omega^p_X \) extend uniquely over \( X_{\text{sing}} \).

The case of isolated singularities. We conclude this outline with a brief sketch how (2.0.9) is proved in the case of isolated singularities. In Section 6.2, we more or less reduce the general case to this special case by locally cutting with hypersurfaces; note that this works because we are proving a stronger statement than just extension of \( p \)-forms.

Because of Property (2.0.7), we have \( \mathcal{H}^k K_{n-p} = 0 \) for \( k \geq p+1 \). Since \( \dim X_{\text{sing}} = 0 \), the inequality in (2.0.9) is therefore true by default as long as \( p \leq n - 2 \). In this way, we recover the result of Steenbrink and van Straten [vSS85, Thm. 1.3] mentioned in the introduction: on an \( n \)-dimensional complex space with isolated singularities, \( p \)-forms extend for every \( p \leq n - 2 \). This only leaves two cases, namely \( p = n - 1 \) and \( p = n \).

The case \( p = n \) is covered by the assumption that \( n \)-forms extend. We have \( K_n \cong \mathcal{H}^0 K_n \cong r_r \Omega^n_X \), and sections of \( r_r \Omega^n_X \) extend uniquely over \( X_{\text{sing}} \). Because of the isomorphism \( R\mathcal{H}om_{\mathcal{O}_X}(K_n, \omega^n_X) \cong K_0[n] \). Proposition 6.1 gives us the desired inequalities

\[
\dim(\text{Sing} \cap \text{Supp } \mathcal{H}^k K_n) = \dim(\text{Sing} \cap \text{Supp } R^{k-n} \mathcal{H}om_{\mathcal{O}_X}(K_n, \omega^n_X)) \leq n - k - 2
\]

for every \( k \in \mathbb{Z} \).

In the other case \( p = n - 1 \), the inequalities in (2.0.9) are easily seen to be equivalent to the single vanishing \( \mathcal{H}^{n-1} K_1 = 0 \). Using the fact that \( \mathcal{H}^k K_0 = 0 \) for \( k \geq n - 1 \), one shows that the \( \mathcal{O}_X \)-module \( \mathcal{H}^{n-1} K_1 \) is a quotient of the (constructible) 0-th cohomology sheaf of the intersection complex of \( X \). But the intersection complex is known to be concentrated in strictly negative degrees, and therefore \( \mathcal{H}^{n-1} K_1 = 0 \).

3. Conventions

3.1. Global conventions. Throughout this paper, all complex spaces are assumed to be countable at infinity. All schemes and algebraic varieties are assumed to be defined over the field of complex numbers. We follow the notation used in the standard reference books [Har77, GR84]. In particular, varieties are assumed to be irreducible, and the support of
a coherent sheaf $\mathcal{F}$ on $X$ is a closed subset of $X$, with the induced reduced structure. For clarity, we will always say explicitly when a complex space needs to be reduced, irreducible, or of constant dimension.

3.2. $\mathcal{D}$-modules. Unless otherwise noted, we use left $\mathcal{D}$-modules throughout this paper. This choice agrees with the notation of the paper [Sch16], which we will frequently cite. It is, however, incompatible with the conventions of the reference papers [Sai88, Sai90] and of the survey [Sch14] that use right $\mathcal{D}$-modules throughout. We refer the reader to [Sch16, §A.5], where the conversion rules for left and right $\mathcal{D}$-modules are recalled.

3.3. Complexes. Let $K$ be a complex of sheaves of Abelian groups on a topological space, for example a complex of sheaves of $\mathcal{O}_X$-modules (or $\mathcal{D}_X$-modules) on a complex manifold $X$. We use the notation $\mathcal{H}^j K$ for the $j$-th cohomology sheaf of the complex. We use the notation $K[n]$ for the shift of $K$. We have $\mathcal{H}^j K[n] = \mathcal{H}^{j+n} K$, and all differentials in the shifted complex are multiplied by $(-1)^n$.

3.4. The dualizing complex. If $X$ is any complex space, we write $\omega^*_X \in D^b_{coh}(\mathcal{O}_X)$ for the dualizing complex as introduced by Ramis and Ruget, see [BS76, VII Thm. 2.6] and the original reference [RR70]. Given a complex of $\mathcal{O}_X$-modules $K \in D^b_{coh}(\mathcal{O}_X)$ with bounded coherent cohomology, we call the complex $R\mathcal{H}om_{\mathcal{O}_X}(K, \omega^*_X) \in D^b_{coh}(\mathcal{O}_X)$ the dual complex of $K$.

Note. When $X$ is a complex manifold of constant dimension, one has $\omega^*_X \cong \omega_X[\dim X]$.

3.5. Reflexive sheaves on normal spaces. Let $X$ be a normal complex space, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Recall that $\mathcal{F}$ is called reflexive if the natural morphism from $\mathcal{F}$ to its double dual $\mathcal{F}^{**} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$ is an isomorphism. The following notation will be used.

Notation 3.1 (Reflexive hull). Given a normal complex space $X$ and a coherent sheaf $\mathcal{F}$ on $X$, write $\Omega^{[\mathcal{F}]}_X := (\Omega^{\mathcal{F}}_X)^{**}$, $\mathcal{F}[m] := (\mathcal{O}^{\mathcal{F}})^{**}$ and $\det \mathcal{F} := (\Lambda^{\mathcal{F}})^{**}$. Given any morphism $f : Y \to X$ of normal complex spaces, write $f^{[\mathcal{F}]} := (f^* \mathcal{F})^{**}$, etc. Ditto for quasi-projective varieties.

4. Mixed Hodge modules

4.1. Mixed Hodge modules. For the convenience of the reader, we briefly recall a number of facts concerning mixed Hodge modules, and lay down the notation that will be used throughout. The standard references for mixed Hodge modules are the original papers by Saito [Sai88, Sai90]. The survey articles [Sai89, Sai94, Sch14] review some aspects of the theory in a smaller number of pages. A good reference for $\mathcal{D}$-modules is the book [HTT08]. We consider the following setting throughout the present section.

Setting 4.1. Assume that a complex manifold $Y$ of constant dimension $d$ and a graded-polarisable mixed Hodge module $M$ on $Y$ are given.

Notation 4.2 (Mixed Hodge modules). In Setting 4.1, we denote by $\text{MHM}(Y)$ the Abelian category of graded-polarisable mixed Hodge modules on $Y$, and by $\text{HM}(Y, w)$ the Abelian category of polarisable Hodge modules of weight $w$. We write $W^w M$ for the weight filtration on $M$. We write $\text{rat} M$ for the underlying perverse sheaf (with coefficients in $\mathbb{Q}$), and $\text{rat} M$ for the underlying filtered $\mathcal{D}_Y$-module. Here, $M$ is a regular holonomic left $\mathcal{D}_Y$-module, and $\mathcal{F}_a M$ is a good filtration by coherent $\mathcal{O}_Y$-modules, sometimes called the $\text{Hodge filtration}$. The support of $M$, denoted by $\text{Supp} M$, is by definition the support of the $\mathcal{D}_Y$-module $M$ (or, equivalently, of the perverse sheaf rat $M$).
Note. In Setting 4.1, we have
\[ \text{gr}_\ell^W M := W_\ell M / W_{\ell - 1} M \in \text{HM}(Y, \ell), \quad \text{for } \ell \in \mathbb{Z}. \]

Conversely, every polarisable Hodge module \( N \in \text{HM}(Y, w) \) may be viewed as a graded-polarisable mixed Hodge module \( N \) with \( W_{w - 1} N = 0 \) and \( W_w N = N \).

4.1.1. Tate twist. Maintain Setting 4.1. Given any integer \( k \in \mathbb{Z} \), define \( Q(k) = (2\pi i)^k \mathbb{Q} \subseteq \mathbb{C} \). The Tate twist \( M(k) \) is the mixed Hodge module whose underlying perverse sheaf is \( Q(k) \otimes \text{rat} M \), whose underlying filtered \( \mathcal{D}_Y \)-module is \( (M, F_{\bullet + k} M) \), and whose weight filtration is given by \( W_\ell M(k) = W_{\ell + 2k} M \). When \( M \) is pure of weight \( w \), it follows that \( M(k) \) is again pure of weight \( w - 2k \).

4.1.2. Decomposition by strict support. In Setting 4.1, one says that the mixed Hodge module \( M \) has strict support if the support of every nontrivial subquotient of \( M \) is equal to \( \text{Supp} M \). Ditto for perverse sheaves and regular holonomic \( \mathcal{D}_Y \)-modules. Note that the strict support property is generally not preserved by restriction to open subsets; for example, \( \text{Supp} M \) may be globally irreducible, but locally reducible. We use the symbol \( \text{HM}_{\mathbb{C}}(Y, w) \) to denote the Abelian category of polarisable Hodge modules on \( Y \) of weight \( w \) with strict support \( X \); this is a full subcategory of \( \text{HM}(Y, w) \).

If \( M \) is a polarisable Hodge module, then \( M \) has strict support if and only if the support of every nontrivial subobject (or quotient object) is equal to \( \text{Supp} M \); the reason is that polarisable Hodge modules are semisimple [Sai88, Cor. 5.2.13]. By definition, every polarisable Hodge module admits, on every open subset of \( X \), a decomposition by strict support as a (locally finite) direct sum of polarisable Hodge modules with strict support [Sai88, §5.1.6].

4.1.3. Weight filtration and dual module. In Setting 4.1, we write \( M' = \mathbb{D} M \in \text{MHM}(Y) \) for the dual mixed Hodge module. This is again a graded-polarisable mixed Hodge module [Sai90, Prop. 2.6], with the property that
\[ \mathbb{D}(\text{gr}_\ell^W M) \cong \text{gr}_{-\ell}^W \mathbb{D} M. \]

In particular, if \( M \) is pure of weight \( \ell \), then \( \mathbb{D} M \) is again pure of weight \( -\ell \). The underlying perverse sheaf \( \text{rat} M' \) is isomorphic to the Verdier dual [HTT08, Def. 4.5.2] of \( \text{rat} M \). The regular holonomic left \( \mathcal{D}_Y \)-module \( (M', F_\bullet M') \) underlying \( M' = \mathbb{D} M \) is isomorphic to the holonomic dual [HTT08, Def. 2.6.1]
\[ R^d \mathcal{H}om_{\mathcal{D}_Y}(\omega_Y \otimes_{\mathcal{O}_Y} M, \mathcal{D}_Y) \]
of the regular holonomic left \( \mathcal{D}_Y \)-module \( M \).

4.1.4. The de Rham complex. In Setting 4.1, the complex of sheaves of \( \mathbb{C} \)-vector spaces
\[ \text{DR}(M) = \left[ M \rightarrow \Omega^1_Y \otimes_{\mathcal{O}_Y} M \rightarrow \cdots \rightarrow \Omega^d_Y \otimes_{\mathcal{O}_Y} M \right] [d], \]
concentrated in degrees \(-d, \ldots, 0\) is called the de Rham complex of \( M \). Since \( M \) is a regular holonomic \( \mathcal{D}_Y \)-module, the de Rham complex \( \text{DR}(M) \) has constructible cohomology sheaves, and is in fact a perverse sheaf on \( Y \) by a theorem of Kashiwara [HTT08, Thm. 4.6.6]. In particular, it is always semiperverse, which means concretely that
\[ \dim \text{Supp} \mathcal{H}^j \text{DR}(M) \leq -j \quad \text{for every } j \in \mathbb{Z}. \]

The perverse sheaf \( \text{rat} M \) and the de Rham complex of \( M \) are related through an isomorphism \( \mathbb{C} \otimes_{\mathbb{Q}} \text{rat} M \cong \text{DR}(M) \) that is part of the data of a mixed Hodge module.
4.1.5. Subquotients of the de Rham complex. Assume Setting 4.1. The filtration $F_* M$ induces an increasing filtration on the de Rham complex by
\[(4.3.1) \quad F_p \text{DR}(M) = \left[ F_p M \to \Omega^1_Y \otimes_{\mathcal{O}_Y} F_{p+1} M \to \cdots \to \Omega^d_Y \otimes_{\mathcal{O}_Y} F_{p+d} M \right][d].\]
The $p$-th subquotient of this filtration is the complex of $\mathcal{O}_Y$-modules
\[(4.3.2) \quad \text{gr}_p^F \text{DR}(M) = \left[ \text{gr}_p^F M \to \Omega^1_Y \otimes_{\mathcal{O}_Y} \text{gr}_{p+1}^F M \to \cdots \to \Omega^d_Y \otimes_{\mathcal{O}_Y} \text{gr}_{p+d}^F M \right][d].\]
For a more detailed discussion of these complexes, see for example [Sch16, §7]. The following simple lemma will be useful later.

**Lemma 4.4.** In Setting 4.1, if $\text{gr}_p^F \text{DR}(M)$ is acyclic for every $p \leq m$, then $F_{m+\dim Y} M = 0$.

**Proof.** Since $F_* M$ is a good filtration, there is, at least locally on $Y$, an integer $p_0$ such that $F_{p_0} M = 0$ and, hence, $F_p M = 0$ for every $p \leq p_0$. To show that $F_{m+\dim Y} M = 0$, it is therefore enough to prove that $\text{gr}_p^F M = 0$ for every $p \leq m + d$. Because $\text{gr}_p^F \text{DR}(M)$ is acyclic for $p \leq m$, this follows from (4.3.2) by induction on $p \geq p_0$.

4.2. Duality. Next, we review how the duality functor for mixed Hodge modules affects the subquotients of the de Rham complex. The following nontrivial result by Saito shows that the dual complex of $\text{gr}_p^F \text{DR}(M)$ is nothing but $\text{gr}_p^F \text{DR}(M')$.

**Proposition 4.5** (Duality, mixed case). Assume Setting 4.1. Then,
\[R\mathcal{H}om_{\mathcal{O}_Y} \left( \text{gr}_p^F \text{DR}(M), \omega_Y^* \right) \cong \text{gr}_p^F \text{DR}(M') \quad \text{for every } p \in \mathbb{Z},\]
where $(M', F_* M')$ is the filtered $\mathcal{O}_Y$-module underlying $M' = \mathcal{D} M$.

**Proof.** This is proved in [Sai88, §2.4.3]; see also [Sch16, Lem. 7.4]. The crucial point in the proof is that $\text{gr}_p^F M$ is a Cohen-Macaulay module over $\text{gr}_F^* \mathcal{O}_Y$, due to the fact that $(M, F_* M)$ underlies a mixed Hodge module.

In the special case where $M$ is a polarisable Hodge module, the de Rham complex is self-dual, up to a shift in the duality. Duality therefore relates different subquotients of $\text{DR}(M)$, in a way that will be very useful for the proof of Theorem 1.2.

**Corollary 4.6** (Duality, pure case). Let $M \in \text{HM}(Y, w)$ be a polarisable Hodge module of weight $w$ on a complex manifold $Y$. Any polarisation on $M$ induces an isomorphism
\[R\mathcal{H}om_{\mathcal{O}_Y} \left( \text{gr}_p^F \text{DR}(M), \omega_Y^* \right) \cong \text{gr}_{p-w}^F \text{DR}(M) \quad \text{for every } p \in \mathbb{Z}.\]

**Proof.** A polarisation on $M$ induces an isomorphism $\mathcal{D} M \cong M(w)$ [Sai88, §5.2.10], and therefore an isomorphism $(M', F_* M') \cong (M, F_* M)$. Now apply Proposition 4.5.

4.3. Acyclicity. The following proposition contains an acyclicity criterion for subquotients of the de Rham complex, involving both the weight filtration $W_* M$ and the Hodge filtration $F_* M$.

**Proposition 4.7** (Acyclic subquotients). Assume Setting 4.1. If $w, c \in \mathbb{Z}$ are such that $W_{w-c} M = 0$ and $F_{c-1} M = 0$, then $\text{gr}_p^F \text{DR}(M)$ is acyclic unless $c - d \leq p \leq w - c$.

**Proof.** Since $F_{c-1} M = 0$ and $d = \dim Y$, a look at the formula (4.3.2) for the $p$-th subquotient of $\text{DR}(M)$ reveals that $\text{gr}_p^F \text{DR}(M) = 0$ for $p \leq c - 1 - d$. The other inequality is going to follow by duality. Let us first consider the pure case, meaning that $M \in \text{HM}(Y, w')$ is a polarisable Hodge module of weight $w'$. By Corollary 4.6, we have
\[\text{gr}_p^F \text{DR}(M) \cong R\mathcal{H}om_{\mathcal{O}_Y} \left( \text{gr}_{p-w'}^F \text{DR}(M), \omega_Y^* \right)\]
and since the complex on the right-hand side is acyclic for $-p - w' \leq c - 1 - d$, we get the result when $M$ is pure. The general case follows from this by considering the subquotients of the weight filtration $W_* M$. 

\[\square\]
Proposition 4.7 is especially useful when combined with the following general fact, which an easy consequence of the filtration $F_*M$ being exhaustive.

**Proposition 4.8.** Let $Y$ be a complex manifold. Let $(M, F_*M)$ be a coherent $\mathcal{D}^*_Y$-module with a good filtration. If $gr^p F_* DR(M)$ is acyclic for all $p \geq p_0 + 1$, then the inclusion $F_{p_0} DR(M) \hookrightarrow DR(M)$ is a quasi-isomorphism. □

4.3. Direct images and the Decomposition Theorem. Let $f : X \to Y$ be a projective holomorphic mapping between two complex manifolds, and let $M \in \mathcal{M}H\mathcal{M}(X)$ be a graded-polarisable mixed Hodge module on $X$. One of the most important results in Saito’s theory is that, in this setting, one can define a direct image functor, compatible with the direct image functor for perverse sheaves and filtered $\mathcal{D}$-modules, and that the $i$-th higher direct image $H^i f_* M$ is again a graded-polarisable mixed Hodge module on $Y$. In this section, we briefly review this result and its implications for the underlying filtered $\mathcal{D}_X$-module $(M, F_*M)$ and the de Rham complex $DR(M)$.

4.3.1. Filtered $\mathcal{D}$-modules and strictness. Let $X$ be a complex manifold. Following Saito, we denote by $D^b_{coh} F(\mathcal{D}_X)$ the derived category of (certain cohomologically bounded and coherent complexes of) filtered $\mathcal{D}_X$-modules, as defined in [Sai88, §2.1.15]. The category of filtered $\mathcal{D}_X$-modules is only an exact category, but it embeds into the larger Abelian category of graded $R_F \mathcal{D}_X$-modules, where

$$ R_F \mathcal{D}_X = \bigoplus_{p \in \mathbb{Z}} F_p \mathcal{D}_X $$

is the Rees algebra of $\mathcal{D}_X$ with respect to the order filtration. The embedding takes a coherent filtered $\mathcal{D}_X$-module $(M, F_*M)$ to the associated Rees module

$$ R_F M = \bigoplus_{p \in \mathbb{Z}} F_p M, $$

which is coherent over $R_F \mathcal{D}_X$. Let $D^b_{coh} G(R_F \mathcal{D}_X)$ be the derived category of (cohomologically bounded and coherent complexes of) graded $R_F \mathcal{D}_X$-modules. Then the Rees module construction gives an equivalence of categories

$$ D^b_{coh} F(\mathcal{D}_X) \cong D^b_{coh} G(R_F \mathcal{D}_X), $$

according to [Sai88, Prop. 2.1.16]. The cohomology modules of an object in $D^b_{coh} F(\mathcal{D}_X)$ are therefore in general not filtered $\mathcal{D}_X$-modules, but graded $R_F \mathcal{D}_X$-modules.

**Definition 4.9** (Strictness). A graded $R_F \mathcal{D}_X$-module is called strict if it is isomorphic to the Rees module of a coherent filtered $\mathcal{D}_X$-module. A complex $K \in D^b_{coh} G(R_F \mathcal{D}_X)$ is called strict if all of its cohomology modules $H^i K$ are strict.

The functor that takes a coherent filtered $\mathcal{D}_X$-module $(M, F_*M)$ to the underlying $\mathcal{D}_X$-module $M$ extends uniquely to an exact functor

$$ D^b_{coh} G(R_F \mathcal{D}_X) \to D^b_{coh} (\mathcal{D}_X). $$

Indeed, if we denote by $z \in R_F \mathcal{D}_X$ the degree-one element obtained from $1 \in F_1 \mathcal{D}_X$, then the functor is simply the derived tensor product with $R_F \mathcal{D}_X / (1 - z) R_F \mathcal{D}_X$. Similarly, the functor that takes a coherent filtered $\mathcal{D}_X$-module $(M, F_*M)$ to the coherent graded $\mathcal{D}_X$-module $gr^p_* M$ extends uniquely to an exact functor

$$ gr^p_* : D^b_{coh} G(R_F \mathcal{D}_X) \to D^b_{coh} (\mathcal{D}_X). $$

This time, the functor is given by the derived tensor product with $R_F \mathcal{D}_X / z R_F \mathcal{D}_X$. Lastly, for every $p \in \mathbb{Z}$, the functor that takes a coherent filtered $\mathcal{D}_X$-module $(M, F_*M)$ to the complex of coherent $\mathcal{O}_X$-modules $gr^p_* DR(M)$ extends uniquely to an exact functor

$$ gr^p_* DR : D^b_{coh} G(R_F \mathcal{D}_X) \to D^b_{coh} (\mathcal{O}_X). $$
Indeed, by [Sai88, Prop. 2.2.10], the de Rham functor (which Saito denotes by the symbol $\mathcal{D}R$) defines an equivalence of categories between $D^b_{coh}(\mathcal{O}_X)$ and the derived category of filtered differential complexes $D^b_{coh}F^i(\mathcal{O}_X, \text{Diff})$, and $gr^F$ of a filtered differential complex is by construction a (cohomologically bounded and coherent) complex of $\mathcal{O}_X$-modules [Sai88, §2.2.4].

4.3.2. Direct image functor for filtered $\mathcal{D}$-modules. Now suppose that $f: X \rightarrow Y$ is a proper holomorphic mapping between complex manifolds. In this setting, one can construct a direct image functor

$$f_*: D^b_{coh}G(R_F\mathcal{D}_X) \rightarrow D^b_{coh}G(R_F\mathcal{D}_Y);$$

see [Sai88, §2.3.5] for the precise definition. This functor is compatible with the functor $gr^F\mathcal{D}R$ in the following manner [Sai88, §2.3.7].

**Proposition 4.10.** Let $f: X \rightarrow Y$ be a proper holomorphic mapping between complex manifolds. For every $p \in \mathbb{Z}$, one has a natural isomorphism of functors

$$Rf_* \circ gr^F\mathcal{D}R \cong gr^F\mathcal{D}R \circ f_*$$

as functors from $D^b_{coh}G(R_F\mathcal{D}_X)$ to $D^b_{coh}(\mathcal{O}_Y)$.

**Proof.** By [Sai88, Lem. 2.3.6], the de Rham functor exchanges the direct image functor $f_*: D^b_{coh}G(R_F\mathcal{D}_X) \rightarrow D^b_{coh}G(R_F\mathcal{D}_Y)$ and the direct image functor

$$f_*: D^b_{coh}F^i(\mathcal{O}_X, \text{Diff}) \rightarrow D^b_{coh}F^i(\mathcal{O}_Y, \text{Diff})$$

for filtered differential complexes. But the latter commutes with taking $gr^F$, as is clear from the construction in [Sai88, §2.3.7].

**Note.** In the case of a single coherent filtered $\mathcal{D}_X$-module, this says that

$$Rf_* gr^F\mathcal{D}R(M) \cong gr^F\mathcal{D}R(f_*(R_F M)), $$

as objects of the derived category $D^b_{coh}(\mathcal{O}_Y)$.

4.3.3. Direct image theorem, pure case. We now assume that the proper holomorphic mapping $f: X \rightarrow Y$ is actually projective. Then we have the following important “direct image theorem” due to Saito [Sai88, §5.3].

**Theorem 4.11 (Direct image theorem, pure case).** Let $f: X \rightarrow Y$ be a projective morphism between complex manifolds, and let $\xi \in H^2(X, \mathbb{Z}(1))$ be the first Chern class of a relatively ample line bundle. If $M \in \text{HM}(X, w)$ is a polarisable Hodge module $X$, then:

1. The complex $f_*(R_F M)$ is strict, and each $\mathcal{H}^i f_* (R_F M)$ is the filtered $\mathcal{D}_Y$-module underlying a polarisable Hodge module $H^i f_* M \in \text{HM}(Y, w + i)$.

2. For every $i \geq 0$, the Lefschetz morphism

$$\ell^i: H^{i+1} f_* M \rightarrow H^i f_* M(i)$$

is an isomorphism between Hodge modules of weight $w-i$.

3. Any polarisation on $M$ induces a polarisation on $\bigoplus_i H^i f_* M$ in the Hodge-Lefschetz sense (on primitive parts with respect to the action of $\ell$).

One consequence of Theorem 4.11 is a version of the Decomposition Theorem for those filtered $\mathcal{D}$-modules that underlie polarisable Hodge modules.

**Corollary 4.12 (Decomposition Theorem).** Let $f: X \rightarrow Y$ be a projective morphism between complex manifolds. Let $M \in \text{HM}(X, w)$ be a polarisable Hodge module on $X$, and let $M_i = H^i f_* M \in \text{HM}(Y, w + i)$. Write $(M, \mathcal{F}_i M)$ respectively $(M_i, \mathcal{F}_i M_i)$ for the underlying filtered $\mathcal{D}$-modules. Then

$$f_* (R_F M) \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i f_* (R_F M)[-i] \cong \bigoplus_{i \in \mathbb{Z}} R_F M_i [-i].$$
De/f_inition 4.15

4.34. Direct image theorem, mixed case. In the case of mixed Hodge modules, there are some additional results, having to do with the weight filtration. We summarise them in the following theorem [Sai90, Thm. 2.14 and Prop. 2.15].

Theorem 4.13 (Direct image theorem, mixed case). Let \( f : X \to Y \) be a projective morphism between complex manifolds, and let \( M \in \text{MHM}(X) \) be a graded-polarisable mixed Hodge module on \( X \).

(4.13.1) The complex \( f_* (R_f M) \) is strict, and each \( \mathcal{H}^i f_* (R_f M) \) is the filtered \( \mathcal{D}_Y \)-module underlying a graded-polarisable mixed Hodge module \( H^i f_* M \in \text{MHM}(Y) \).

(4.13.2) One has a convergent weight spectral sequence

\[
E_1^{p,q} = H^{p+q} f_* \gr_p^W M \Rightarrow H^{p+q} f_* M,
\]

and each differential \( d_i : E_1^{p,q} \to E_1^{p+1,q} \) is a morphism in \( \text{HM}(Y,q) \).

(4.13.3) The weight spectral sequence degenerates at \( E_2 \), and one has

\[
\gr_q^W H^{p+q} f_* M \cong E_2^{p,q} \quad \text{for every } p, q \in \mathbb{Z}.
\]

One can use this result to control the range in which the Hodge filtration on the direct image of a graded-polarisable mixed Hodge module is nontrivial.

Proposition 4.14. Let \( f : X \to Y \) be a projective morphism between complex manifolds, and let \( M \in \text{MHM}(X) \) be a graded-polarisable mixed Hodge module on \( X \). Suppose that the underlying \( \mathcal{D}_X \)-module \( (M,F,\mathcal{M}) \) satisfies \( F_{m-1}M = 0 \). Then one has

\[
F_{m+c-1} \mathcal{H}^i f_* (R_f M) = 0
\]

for every \( i \in \mathbb{Z} \), where \( c = \dim Y - \dim X \).

Proof. One can deduce this from the construction of the direct image functor in [Sai88, §2.3]. Here we outline another proof based on Theorem 4.11 and Theorem 4.13.

We first deal with the case where \( M \in \text{HM}(X,W) \) is a polarisable pure Hodge module. By Proposition 4.10 and Corollary 4.12, we have for every \( p \in \mathbb{Z} \) an isomorphism

\[
R_f \gr_p^F \text{DR}(M) \cong \gr_p^F \text{DR}(f_*(R_f M)) \cong \bigoplus_{i \in \mathbb{Z}} \gr_p^F \text{DR}(\mathcal{M}_i)[-i],
\]

where \( (\mathcal{M}_i, F, \mathcal{M}_i) \) is the filtered \( \mathcal{D}_Y \)-module underlying \( H^i f_* M \in \text{HM}(Y,w+i) \). Since \( F_{m-1}M = 0 \), we get \( \gr_p^F \text{DR}(\mathcal{M}_i) = 0 \) for all \( p \leq m - 1 - \dim X \), and \( \gr_p^F \text{DR}(\mathcal{M}_i) \) is therefore acyclic as long as \( p \leq m - 1 - \dim X \). According to Lemma 4.4, this is enough to conclude that \( F_{m+c-1}M_i = F_{m-1-\dim X} M_i = 0 \) for every \( i \in \mathbb{Z} \).

Now suppose that \( M \in \text{MHM}(X) \) is a graded-polarisable mixed Hodge module. The underlying \( \mathcal{D}_X \)-module of the Hodge module \( \gr_p^F M \in \text{HM}(X,w) \) is \( \gr_p^W M \), with the induced Hodge filtration; because \( F_{m-1}M = 0 \), we have \( F_{m-1} \gr_p^W M = 0 \). Since we already have the result in the pure case, the assertion now follows by looking at the spectral sequence in (4.13.2).

4.4. Non-characteristic restriction to hypersurfaces. We briefly review the non-characteristic restriction of a mixed Hodge module to a hypersurface. For a more general discussion of non-characteristic restriction, see [Sai88, §3.5] or [Sch16, §8].

Definition 4.15 (Non-characteristic hypersurfaces). Let \( X \) be a complex manifold, and let \( D \subseteq X \) be a smooth hypersurface. The inclusion \( i_D : D \hookrightarrow X \) gives rise to the following
morphisms between cotangent bundles:

\[(T^*X) \times_X D \longrightarrow T^*D\]

(4.15.1)

Given a regular holonomic left \(\mathcal{D}_X\)-module \(M\) on \(X\), let \(\text{Ch}(M) \subseteq T^*X\) denote its characteristic variety. We say that \(D \subseteq X\) is non-characteristic for \(M\) if \(p_1^{-1}\text{Ch}(M)\) is finite over its image in \(T^*D\).

Note. As explained for example in [Sch16, §8], \(D \subseteq X\) is non-characteristic for \(M\) if and only if \(D\) is transverse to every stratum in a Whitney stratification of \(X\) that is adapted to the perverse sheaf \(\text{DR}(M)\). In particular, generic hyperplane sections (in \(\mathbb{P}^n\) or \(\mathbb{C}^n\)) are always non-characteristic.

The following result of Saito [Sai90, Lem. 2.25] describes what happens to mixed Hodge modules under non-characteristic restriction to smooth hypersurfaces.

**Theorem 4.16** (Restriction to non-characteristic hypersurfaces). Let \(X\) be a complex manifold, and let \(M \in \text{MHM}(X)\) be a graded-polarised mixed Hodge module on \(X\), with underlying filtered \(\mathcal{D}_X\)-module \((M, F, \mathcal{I})\). Suppose that \(i_D: D \hookrightarrow X\) is a smooth hypersurface that is non-characteristic for \(M\). Then there is a graded-polarised mixed Hodge module \(H^{-1}i_D^!\mathcal{I}M \in \text{MHM}(D)\), whose underlying filtered \(\mathcal{D}_D\)-module is isomorphic to

\[(\mathcal{O}_D \otimes_{i_D^!\mathcal{I}} F_X^j \mathcal{I}D^{-i}M, \mathcal{O}_D \otimes_{i_D^!\mathcal{I}} F_X^j \mathcal{I}D^{-i}F_i M),\]

and whose de Rham complex is quasi-isomorphic to

\[i_D^! \text{DR}(M)[-1].\]

Moreover, if \(M\) is pure of weight \(w\), then \(H^{-1}i_D^!\mathcal{I}M\) is again pure of weight \(w - 1\).

As the discussion in Saito’s paper is rather brief, we include a sketch of the proof of **Theorem 4.16** for the convenience of the reader. It relies on the following result of Saito [Sai88, Lemma 3.5.6] whose proof we reproduce here.

**Lemma 4.17** (Existence of \(V\)-filtration). In the setting of **Definition 4.15**, suppose that the smooth hypersurface \(D \subseteq X\) is non-characteristic for \(M\). Then the rational \(V\)-filtration of \(M\) relative to \(D\) exists and is given by

\[
V^\alpha M = \begin{cases} 
\mathcal{F}D^\alpha M & \text{for } \alpha \leq 0, \\
\mathcal{F}D^{\alpha - 1}M & \text{for } \alpha > 0,
\end{cases}
\]

where \(\mathcal{F}D \subseteq \mathcal{O}_X\) denotes the coherent ideal sheaf of \(D\).

**Proof.** The problem is local, and after shrinking \(X\), we may assume that \(D = t^{-1}(0)\), where \(t: X \to \mathbb{C}\) is holomorphic and submersive. We may also assume that we have a global holomorphic vector field \(\partial_t\) with the property that \([\partial_t, t] = 1\). In this situation, the rational \(V\)-filtration is the unique exhaustive decreasing filtration \(V^\bullet M\), indexed discretely and left-continuously by the set of rational numbers, with the following properties:

(4.17.1) Each \(V^\alpha M\) is coherent over \(V^0\mathcal{D}_X\), the \(\mathcal{O}_X\)-subalgebra of \(\mathcal{D}_X\) preserving \(\mathcal{F}D\).

(4.17.2) One has \(t \cdot V^\alpha M \subseteq V^{\alpha + 1}M\) and \(\partial_t \cdot V^\alpha M \subseteq V^{\alpha - 1}M\) for every \(\alpha \in \mathbb{Q}\).

(4.17.3) For \(\alpha > -1\), multiplication by \(t\) induces an isomorphism \(V^\alpha M \cong V^{\alpha + 1}M\).

(4.17.4) The operator \(t\partial_t - \alpha\) acts nilpotently on \(\text{gr}^V_\alpha M = V^\alpha M/V^{\alpha + 1}M\).

If we define the filtration \(V^\bullet M\) as in the statement of the lemma, then the last three properties are immediate; the only thing we need to check is that \(M\) itself is coherent over \(V^0\mathcal{D}_X\). After choosing a good filtration \(F_i M\), it is enough to show that \(\text{gr}^F_\alpha M\) is coherent over \(\text{gr}^F V^0\mathcal{D}_X\). Note that \(\text{gr}^F_\alpha M\) is always coherent over \(\text{gr}^F \mathcal{D}_X \cong \text{Sym} \mathcal{D}_X\).
Now it is easy to see that the category of (mixed) Hodge modules in [Sai88, Cauchy-Kovalevskaya theorem [HTT08, Cor. 4.3.4], which says that non-characteristic varieties, the support of the coherent sheaf on $T^*X$ corresponding to $gr^t \mathcal{M}$ is precisely $\text{Ch}(\mathcal{M})$. Because push forward by finite holomorphic mappings preserves coherence, it follows that $gr^t \mathcal{M}$ is coherent over the subalgebra $\text{Sym} \mathcal{E}/\mathcal{C} \subseteq \text{Sym} \mathcal{E}$. Now it is easy to see that

$$gr^t \mathcal{V}^0 \mathcal{D}_X \cong \text{Sym} \mathcal{E}/\mathcal{C} [t \partial_t],$$

and so $gr^t \mathcal{M}$ is coherent over this larger $\partial_X$-algebra as well. □

We use the above description of the rational $\mathcal{V}$-filtration to prove Theorem 4.16.

**Proof of Theorem 4.16.** Since all the assertions are local on $X$, we may assume that $D = t^{-1}(0)$, where $t : X \to \mathbb{C}$ is submersive. We keep the notation introduced during the proof of Lemma 4.17. Since $(\mathcal{M}, \mathcal{F}_\mathcal{M})$ underlies a mixed Hodge module, multiplication by $t$ induces an isomorphism between $F_p \mathcal{V}^a \mathcal{M}$ and $F_p \mathcal{V}^{a+1} \mathcal{M}$ for every $a > -1$; see [Sai88, §3.2.1], but keep in mind that we are talking about filtered $\mathcal{D}$-modules. Specialising to $a = 0$, we conclude that

$$F_p \mathcal{M} \cap t \mathcal{M} = t F_p \mathcal{M}.$$  

It follows that $t : gr^t \mathcal{M} \to gr^t \mathcal{M}$ is injective, and hence that $\mathcal{O}_D \otimes_{\mathcal{O}_X} t^{-1} \mathcal{F}_\mathcal{M}$ defines a good filtration of $\mathcal{O}_D \otimes_{\mathcal{O}_X} t^{-1} \mathcal{M}$ by coherent $\mathcal{O}_D$-submodules. In particular, $i_D : D \hookrightarrow X$ is strictly non-characteristic for $(\mathcal{M}, \mathcal{F}_\mathcal{M})$, in the terminology of [Sch16, §8].

According to Lemma 4.17, we have

$$gr^0 \mathcal{M} \cong M/t \mathcal{M} \cong \mathcal{O}_D \otimes_{\mathcal{O}_X} t^{-1} \mathcal{M},$$

and the action of the (nilpotent) operator $N = t \partial_t$ is trivial. Consequently, the relative weight filtration of $N$ is equal to the filtration $W_\mathcal{M}/t W_\mathcal{M} \cong \mathcal{O}_D \otimes_{\mathcal{O}_X} t^{-1} W_\mathcal{M}$ induced by the weight filtration of $\mathcal{M}$ itself [Sai90, §2.3]. Now Saito’s inductive definition of the category of (mixed) Hodge modules in [Sai88, §5.1] and [Sai90, (2.4)] implies the first and third assertion. The second assertion is a special case of Kashiwara’s version of the Cauchy-Kovalevskaya theorem [HTT08, Cor. 4.3.4], which says that non-characteristic restriction is compatible with passage to the de Rham complex. □

We end this section by describing the relation between the de Rham complexes of the two mixed Hodge modules $\mathcal{M}$ and $H^{-1} t^{-1} \mathcal{M}$; see [Sch16, (13.3)] for the proof.

**Proposition 4.18** (Comparison of de Rham complexes). In the setting of Theorem 4.16, denote by $(\mathcal{M}_D, \mathcal{F}_\mathcal{M}_D)$ the filtered $\mathcal{D}$-module underlying the mixed Hodge module $\mathcal{M}_D = H^{-1} t^{-1} \mathcal{M}$. Given any $p \in \mathbb{Z}$, one has a short exact sequence of complexes

$$0 \to N^*_D|X \otimes_{\mathcal{O}_D} gr^p \mathcal{D}(\mathcal{M}_D) \to \mathcal{O}_D \otimes_{\mathcal{O}_X} gr^p \mathcal{D}(\mathcal{M}) \to gr^p \mathcal{D}(\mathcal{M}_D)[1] \to 0,$$

where $N^*_D|X$ means the conormal bundle for the inclusion $D \subseteq X$. □

5. A vanishing theorem for intersection complexes

We briefly discuss a vanishing theorem for certain perverse sheaves that applies in particular to intersection complexes. Recall that a perverse sheaf $\mathcal{K}$ on a complex manifold $Y$ is, by definition, always semiperverse, meaning that

$$\dim \text{Supp} \mathcal{K} \leq -j, \quad \text{for every } j \in \mathbb{Z}. \quad (5.1.1)$$
These inequalities can be improved, provided that $K$ does not admit any nontrivial morphisms to perverse sheaves whose support is properly contained in $\text{Supp} \, K$. This applies for example to the intersection complex on any irreducible complex space, and more generally to the de Rham complex of any polarisable Hodge module with strict support.

**Proposition 5.2.** Let $K$ be a perverse sheaf on a complex manifold $Y$, and assume that $\text{Supp} \, K$ has constant dimension $n$. Then the following two conditions are equivalent:

(P.2.1) If $L$ is a perverse sheaf on $Y$ with $\dim \text{Supp} \, L \leq n - 1$, then $\text{Hom}(K, L) = 0$.

(P.2.2) For every $j \geq -n + 1$, one has $\dim \text{Supp} \, \mathcal{H}^j K \leq -(j + 1)$.

**Proof.** Let us show that (P.2.1) implies (P.2.2). Since $K$ is a perverse sheaf, one has $\mathcal{H}^j K = 0$ for $j \leq -n - 1$, and the inequalities in (5.1.1) imply that $\mathcal{H}^{-n} K$ is supported on all of $X$, whereas $\dim \text{Supp} \, \mathcal{H}^j K \leq -j$ for every $j \geq -n + 1$. If we truncate $K$ with respect to the standard t-structure on $D^b_c(X)$, the resulting constructible complex $K' := r_{-n+1} K$ is still semiperverse, and supported in a complex subspace that is properly contained in $X$. By (P.2.1), the natural composed morphism

$$K \to K' \to p.\mathcal{H}^0 K'$$

of the identity morphism. By construction, $\dim \text{Supp} \, \mathcal{H}^j K'' \leq -(j + 1)$, and therefore also $\dim \text{Supp} \, \mathcal{H}^j K \leq -(j + 1)$ for every $j \geq -n + 1$, proving (P.2.2).

It remains to show that, conversely, (P.2.2) implies (P.2.1). Suppose we are given a morphism of perverse sheaves $\varphi : K \to L$ with $\dim \text{Supp} \, L \leq n - 1$. After replacing $L$ by $\text{im} \varphi$, we can assume that $\varphi$ is surjective. As before, we have $\mathcal{H}^j L = 0$ for $j \leq -n$. Now fix some $j \geq -n + 1$, and consider the short exact sequence

$$\mathcal{H}^j K \to \mathcal{H}^j L \to \mathcal{H}^{j+1}(\ker \varphi).$$

We have $\dim \text{Supp} \, \mathcal{H}^j K \leq -(j + 1)$ by (P.2.2), and $\dim \text{Supp} \, \mathcal{H}^{j+1}(\ker \varphi) \leq -(j + 1)$ by (5.1.1). Consequently, $\dim \text{Supp} \, \mathcal{H}^j L \leq -(j + 1)$ for every $j \in \mathbb{Z}$, and since $L$ is a perverse sheaf, the properties of the perverse t-structure imply that $L = 0$.

The following vanishing theorem for the de Rham complex plays a crucial role in the proof of our main theorem, and so we state it as a corollary.

**Corollary 5.3.** Let $Y$ be a complex manifold, and let $M \in \mathcal{H}^0 X(Y, w)$ be a polarisable Hodge module of weight $w$ with strict support an irreducible complex subspace $X \subseteq Y$. If $F_{c-1} M = 0$ for some $c \in \mathbb{Z}$, one has $\mathcal{H}^0 F_{\dim Y - (w + c)} \text{DR}(M) = 0$.

**Proof.** According to Proposition 4.7, the complex $\mathcal{H}^p F_{c} \text{DR}(M)$ is acyclic for $p \geq \dim Y - (w + c) + 1$. By Proposition 4.8, this implies that the inclusion of the subcomplex $F_{c} \text{DR}(M)$ into $\text{DR}(M)$ is a quasi-isomorphism for $p_0 = \dim Y - (w + c)$. In particular, the inclusion induces an isomorphism $\mathcal{H}^0 F_{\dim Y - (w + c)} \text{DR}(M) \cong \mathcal{H}^0 \text{DR}(M)$. But now $M$ has strict support $X$, and so the perverse sheaf $\text{DR}(M)$ does not have nontrivial quotient objects whose support is properly contained in $X$. We conclude that $\mathcal{H}^0 \text{DR}(M) = 0$, by Proposition 5.2.

6. Coherent Sheaves and Mixed Hodge modules

The present section forms the technical core of the present paper. Its main results, Theorem 6.6 and Theorem 6.11, as well as Corollary 6.7 and Corollary 6.12 are criteria to guarantee that sections of certain coherent sheaves derived from the de Rham complex of certain (mixed) Hodge modules on $X$ extend across the singular locus $X_{\text{sing}}$. 

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6.1. Extending sections of coherent sheaves. In this paragraph, we give a homological formulation of the property that sections of a coherent sheaf extend uniquely over a given complex subspace. The material covered here will be known to experts.

Proposition and Definition 6.1 (Extension across subsets). Let $Y$ be a complex manifold.

Let $A \subseteq Y$ be a complex subspace, and let $j: Y \setminus A \hookrightarrow Y$ be the open embedding. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_Y$-modules, then the following conditions are equivalent:

(6.1.1) The natural morphism $\mathcal{F} \to j_* j^* \mathcal{F}$ is an isomorphism.

(6.1.2) For every $k \in \mathbb{Z}$, one has $\dim(A \cap \text{Supp} R^k \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^*)) \leq -(k+2)$.

If these conditions are satisfied, we say that sections of $\mathcal{F}$ extend uniquely across $A$.

We will often apply Proposition 6.1 in the following form.

Corollary 6.2. Let $Y$ be a complex manifold, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Y$-modules.

If $\text{Supp} \mathcal{F}$ has constant dimension $n$, then the following conditions are equivalent:

(6.2.1) Sections of $\mathcal{F}$ extend uniquely across any $A \subseteq Y$ with $\dim A \leq n-2$.

(6.2.2) For every $k \geq -n+1$, one has $\dim \text{Supp} R^k \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^*) \leq -(k+2)$.

Proof. According to [Sta18, Tag 0A7U], one has $R^k \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^*) = 0$ for every $k \leq -n$. If $A \subseteq Y$ is a complex subspace with $\dim A \leq n-2$, then of course

$$\dim(A \cap \text{Supp} R^k \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^*)) \leq n-2,$$

and so the condition in (6.2.2) is equivalent to the condition in (6.1.2). The assertion now follows from Proposition 6.1. $\square$

Before giving the proof of Proposition 6.1, we briefly review some facts about singular sets of coherent sheaves. Let $Y$ be a complex manifold, and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_Y$-modules. Recall that the singular sets of $\mathcal{F}$ are defined as

$$S_m(\mathcal{F}) := \{ y \in Y \mid \text{depth}_y \mathcal{F} \leq m \}.$$

The singular sets $S_m(\mathcal{F})$ are closed complex subspaces of $Y$; we refer the reader to [BS76, Chapt. II.2] for a detailed discussion. The following homological fact about regular local rings [Sta18, Tag 0A7U] relates the singular sets to the dualizing complex.

Proposition 6.3 (Singular sets and duality). If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_Y$-modules on a complex manifold $Y$, then the singular sets of $\mathcal{F}$ are described as

$$S_m(\mathcal{F}) = \bigcup_{k \geq 0} \text{Supp} R^{k-m} \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y^*),$$

where $\omega_Y^*$ is the dualizing complex. $\square$

Proof of Proposition 6.1. We consider the standard exact sequence for sheaves of local cohomology with supports, see for example [BS76, II Cor. 1.10].

$$0 \to \mathcal{H}^0_A \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \to \mathcal{H}^1_A \to 0$$

Because of this sequence, (6.1.1) is equivalent to the condition that $\mathcal{H}^0_A \mathcal{F} = \mathcal{H}^1_A \mathcal{F} = 0$. The vanishing theorem for local cohomology of Scheja-Trautmann [BS76, II Thm. 3.6] relates this to the singular sets of $\mathcal{F}$: it asserts that $\mathcal{H}^0_A \mathcal{F} = \mathcal{H}^1_A \mathcal{F} = 0$ is equivalent to the collection of inequalities

$$\dim(A \cap S_m(\mathcal{F})) \leq m-2 \quad \text{for all } m \in \mathbb{Z}.$$

But Proposition 6.3 shows that this last line is in turn equivalent to (6.1.2). $\square$

We will later need the following variant of Proposition 6.1 that works for complexes of $\mathcal{O}_Y$-modules rather than single sheaves. We stress that, in the case of a complex with two or more nonzero cohomology sheaves, the condition below is stronger than asking that sections of $\mathcal{H}^m K$ extend uniquely across $A$.  

Proposition 6.4. Let $Y$ be a complex manifold, let $A \subseteq Y$ be a complex subspace, and let $K \in D^b_{coh}(\mathcal{O}_Y)$ be a complex with $\mathcal{H}^j K = 0$ for $j < 0$. If
\[
dim(A \cap \text{Supp} R^k \mathcal{H}om_{\mathcal{O}_Y}(K, \omega^*_Y)) \leq -(k + 2) \quad \text{for every } k \in \mathbb{Z},
\]then sections in $\mathcal{H}^0 K$ extend uniquely across $A$.

Proof. Let $\tau_{\geq 1} K$ denote the truncation of the complex $K$ in cohomological degree $\geq 1$. In the derived category $D^b_{coh}(\mathcal{O}_Y)$, one has a distinguished triangle
\[
\mathcal{H}^0 K \to K \to \tau_{\geq 1} K \to (\mathcal{H}^0 K)[1].
\]
After applying the functor $R\mathcal{H}om_{\mathcal{O}_Y}(-, \omega^*_Y)$ and taking cohomology, we obtain the following exact sequence:
\[
R^k \mathcal{H}om_{\mathcal{O}_Y}(K, \omega^*_Y) \to R^k \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}^0 K, \omega^*_Y) \to R^{k+1} \mathcal{H}om_{\mathcal{O}_Y}(\tau_{\geq 1} K, \omega^*_Y)
\]
Thus $A \cap \text{Supp} R^k \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}^0 K, \omega^*_Y)$ is contained in the union of the two sets
\[
A \cap \text{Supp} R^k \mathcal{H}om_{\mathcal{O}_Y}(K, \omega^*_Y) \quad \text{and} \quad \text{Supp} R^{k+1} \mathcal{H}om_{\mathcal{O}_Y}(\tau_{\geq 1} K, \omega^*_Y)
\]
By assumption, the dimension of the first set is at most $-(k + 2)$ for every $k \in \mathbb{Z}$. As $\tau_{\geq 1} K \in D^b_{coh}(\mathcal{O}_Y)$, the same is true for the second set; this follows from [Sta18, Tag 0A7U] by considering the spectral sequence
\[
E^{p,q}_2 = R^p \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}^{-q} \tau_{\geq 1} K, \omega^*_Y) \Rightarrow R^{p+q} \mathcal{H}om_{\mathcal{O}_Y}(\tau_{\geq 1} K, \omega^*_Y).
\]
We conclude the proof by applying Proposition 6.1 to the coherent $\mathcal{O}_Y$-module $\mathcal{H}^0 K$. □

6.2. The case of Hodge modules. In this section, we apply the criteria from Section 6.1 to certain coherent sheaves derived from the de Rham complex of certain Hodge modules. We specify the precise setting first.

Setting 6.5. Let $Y$ be a complex manifold, and let $X \subseteq Y$ be a reduced and irreducible complex subspace of dimension $n$. Let $c$ be the codimension of the closed embedding $i_X : X \hookrightarrow Y$, so that $\dim Y = n + c$. Suppose that $M \in \text{HM}_X(Y, n)$ is a polarisable Hodge module of weight $n$ with strict support equal to $X$. We denote the underlying filtered left $\mathcal{D}_Y$-module by $(\mathcal{M}, F_\cdot \mathcal{M})$, and make the following assumptions about $M$.

(6.5.1) One has $F_{-1} \mathcal{M} = 0$.

(6.5.2) One has $\dim \text{Supp} \mathcal{H}^j \mathcal{O}_Y \text{gr}^p_0 \text{DR}(\mathcal{M}) \leq -(j + 2)$ for every $j \geq -n + 1$.

Note. By [Sai90, Thm. 3.21], there is a dense Zariski-open subset of $X$ on which $M$ is a polarisable variation of Hodge structure of weight 0. The condition $F_{-1} \mathcal{M} = 0$ is equivalent to asking that the variation of Hodge structure is entirely of type $(0,0)$; being polarisable, it must therefore be a unitary flat bundle. Now $F_c \mathcal{M}$ is a certain extension of this unitary flat bundle to a coherent $\mathcal{O}_Y$-module, and (6.5.2) is equivalent to asking that sections of $F_c \mathcal{M}$ extend uniquely over any complex subspace of $X$ of dimension at most $n - 2$.

Theorem 6.6 (Inequalities for Hodge modules). Assume Setting 6.5 and let $p \in \mathbb{Z}$ be any integer. Then one has
\[
\dim \text{Supp} \mathcal{H}^j \mathcal{O}_Y \text{gr}^p_0 \text{DR}(\mathcal{M}) \leq -(p + j + 2) \quad \text{for every } j \text{ with } p + j \geq -n + 1.
\]

A proof of Theorem 6.6 is given in Section 6.2.1 and Section 6.2.2 below. First, however, we note that the dimension estimates in Theorem 6.6 imply the promised extension property for certain coherent sheaves derived from the de Rham complex.

Corollary 6.7 (Extending sections). Assume Setting 6.5. Then for any $p \in \mathbb{Z}$, sections of $\mathcal{H}^{-n-p} \text{gr}^p_0 \text{DR}(\mathcal{M})$ extend uniquely across any complex subspace of dimension $\leq n - 2$. 

Proof. Recall from Proposition 4.7 that \( \text{gr}^F_p \text{DR}(M) \) is acyclic, unless \( 0 \leq p \leq n \). Assuming that \( p \) is in this range, we aim to apply Proposition 6.4 to the complex

\[
K_p := \text{gr}^F_p \text{DR}(M)[p - n],
\]

which requires first of all that \( K_p \) is contained in \( D^{>0}_\text{coh} (\mathcal{O}_X) \). To this end recall from Assumption (6.5.1) that \( F_{-1} M = 0 \). An application of Formula (4.3.2) for the subquotients of the de Rham complex then shows that

\[
\mathcal{H}^j K_p = \mathcal{H}^{j+p-n} \text{gr}^F_{-p} \text{DR}(M) \quad (4.3.2),
\]

for every \( j \leq -1 \).

So \( K \in D^{>0}_\text{coh} (\mathcal{O}_X) \), as desired. Next, choose a polarisation on the Hodge module \( M \), in order to obtain an isomorphism as follows,

\[
R \text{Hom}_{\mathcal{O}_Y} \left( K_p, \omega^* \right) \overset{\text{Cor. 4.6}}{\cong} \text{gr}^F_{-(n-p)} \text{DR}(M)[n - p].
\]

The Inequalities (6.6.1) of Theorem 6.6 therefore take the form

\[
\dim \text{Supp} R^j \text{Hom}_{\mathcal{O}_X} \left( K_p, \omega^* \right) = \dim \text{Supp} \mathcal{H}^{j+n+p} \text{gr}^F_{-(n-p)} \text{DR}(M) \leq -(j + 2)
\]

for every \( j \geq -n+1 \). We conclude from Proposition 6.4 that sections of the coherent \( \mathcal{O}_Y \)-module \( \mathcal{H}^0 K_p = \mathcal{H}^{-(n-p)} \text{gr}^F_{-p} \text{DR}(M) \) extend uniquely across any complex subspace \( A \subseteq Y \) with \( \dim A \leq n - 2 \).

6.2.1. Preparation for proof of Theorem 6.6. In cases where \( p + j \geq \max(-n+1, -1) \), the inequality (6.6.1) in Theorem 6.6 is claiming that \( \mathcal{H}^j \text{gr}^F_p \text{DR}(M) = 0 \). As it turns out, the proof of this special case is the core of the argument; the other cases follow quickly from the following lemma by induction, taking repeated hyperplane sections.

Lemma 6.8. Assume Setting 6.5. If \( p + j \geq \max(-n+1, -1) \), then \( \mathcal{H}^j \text{gr}^F_p \text{DR}(M) = 0 \).

Proof. The complex \( \text{gr}^F_p \text{DR}(M) \) is concentrated in non-positive degrees, and acyclic for \( p \geq 1 \) by Proposition 4.7 and by Assumption (6.5.1). This means that \( \mathcal{H}^j \text{gr}^F_p \text{DR}(M) = 0 \) whenever \( j \geq 1 \) or \( p \geq 1 \). Assumption (6.5.2) implies the claim when \( p = 0 \). This leaves only one case to consider, namely \( p = -1 \) and \( j = 0 \). We shall argue that \( \mathcal{H}^0 \text{gr}^F_{-1} \text{DR}(M) = 0 \), too.

Recall that \( M \) has strict support \( X \). Assumption (6.5.1) therefore allows us to apply Corollary 5.3. We obtain \( \mathcal{H}^0 F_0 \text{DR}(M) = 0 \). Now consider the short exact sequence of complexes (of sheaves of \( \mathbb{C} \)-vector spaces)

\[
0 \to F_{-1} \text{DR}(M) \to F_0 \text{DR}(M) \to \text{gr}^F_0 \text{DR}(M) \to 0.
\]

Since \( \mathcal{H}^j \text{gr}^F_0 \text{DR}(M) = 0 \) for \( j \geq -1 \), we get

\[
(6.8.1) \quad \mathcal{H}^0 F_{-1} \text{DR}(M) \cong \mathcal{H}^0 F_0 \text{DR}(M) \overset{\text{Cor. 5.3}}{=} 0
\]

from the long exact sequence in cohomology. By the same logic, the short exact sequence of complexes (of sheaves of \( \mathbb{C} \)-vector spaces)

\[
0 \to F_{-2} \text{DR}(M) \to F_{-1} \text{DR}(M) \to \text{gr}^F_{-1} \text{DR}(M) \to 0
\]

gives us an exact sequence

\[
\cdots \to \mathcal{H}^0 F_{-1} \text{DR}(M) \to \mathcal{H}^0 \text{gr}^F_{-1} \text{DR}(M) \to \mathcal{H}^0 F_{-2} \text{DR}(M) \to \cdots.
\]

As a consequence, we obtain the desired vanishing \( \mathcal{H}^0 \text{gr}^F_{-1} \text{DR}(M) = 0 \). \( \square \)

6.2.2. Proof of Theorem 6.6. We prove Theorem 6.6 by induction on \( n = \dim X \). If \( n = 1 \) or \( n = 2 \), then the desired statement follows from Lemma 6.8 above, and we are done. We will therefore assume for the remainder of the proof that \( n \geq 3 \), and that Theorem 6.6 is already known for all strictly smaller values of \( n \).
Cutting down. The statement we are trying to prove is local on \( Y \), and so we can assume for the remainder of this proof that \( Y \) is an open ball in \( \mathbb{C}^{n+c} \). (If the restriction of \( M \) no longer has strict support, for example because \( X \) was locally reducible, then we simply replace \( M \) by any of the summands in the decomposition by strict support, and \( X \) by the support of that summand.) Let \( H \subseteq Y \) be the intersection of \( Y \) with a generic hyperplane in \( \mathbb{C}^{n+c} \). The intersection \( H \cap X \) is then reduced and irreducible of dimension \( n-1 \geq 2 \).

The inclusion mapping \( i_H: H \hookrightarrow Y \) is non-characteristic for \( M \), and the inverse image \( M_H = H^{-1}i_H^*M \) is a polarisable Hodge module of weight \((n-1)\) with strict support \( H \cap X \); see Section 4.4 for a discussion of non-characteristic restriction to smooth hypersurfaces. Denoting the underlying filtered \( \mathcal{D}_H \)-module by \((M_H, F^\bullet M_H)\), we have moreover

\[
(M_1) \quad M_H \cong \mathcal{O}_H \otimes \mathcal{O}_Y \text{ and } F^\bullet M_H \cong \mathcal{O}_H \otimes \mathcal{O}_Y \text{ for } j \geq 0.
\]

This is explained in Theorem 4.16.

Properties of \( M_H \). The isomorphisms in \((M_1)\) imply that \( F_{-1}M_H = 0 \), and so \( M_H \) also satisfies Assumption \((6.5.1)\). We claim that \( M_H \) also satisfies Assumption \((6.5.2)\). To this end, recall from Proposition 4.18 that there exists a short exact sequence of complexes,

\[
\begin{align*}
&0 \to N^*_H \otimes O_Y \text{ and } F^1_M \to \mathcal{O}_H \otimes O_Y \text{ and } F^1_M \to \mathcal{O}_H \otimes O_Y \to \mathcal{O}_H \to \mathcal{O}_H \to 0, \\
&\text{where } N^*_H \text{ is the conormal bundle for the inclusion } H \subseteq Y. \text{ As } F_{-1}M_H = 0, \text{ one shows as before that the complex } gr^F \text{ is acyclic for every } p \geq 1. \text{ This gives}
\end{align*}
\]

\[
\mathcal{O}_H \otimes O_Y \cong gr^F \mathcal{O}_H \text{ and } F^\bullet M_H \cong gr^F \mathcal{O}_H \text{ for } j \geq 0.
\]

and because Assumption \((6.5.2)\) holds for \( M \), we obtain that

\[
dim \text{Supp } H^j gr^F \mathcal{O}_H \leq (j+1)
\]

for every \( j \geq -\dim(H \cap X) + 1 \). But this is exactly \((6.5.2)\) for \( M_H \).

Conclusion. We have established that \( M_H \in HM_{H \cap X}(H, n-1) \) again satisfies the two assumptions in \((6.5.1)\) and \((6.5.2)\). Since \( \dim(H \cap X) = n-1 \), we can therefore conclude by induction that

\[
\dim \text{Supp } H^j gr^F \mathcal{O}_H \leq -(p+j+2), \quad \text{whenever } p + j \geq -(n-1) + 1.
\]

Taking cohomology, \((6.9.2)\) gives us an exact sequence of \( \mathcal{O}_H \)-modules,

\[
N^*_H \otimes H^j gr^F \mathcal{O}_H \to \mathcal{O}_H \otimes O_Y \to H^j gr^F \mathcal{O}_H \to H^{j+1} gr^F \mathcal{O}_H \text{ and therefore the inequality}
\]

\[
\dim \text{Supp}(H^j gr^F \mathcal{O}_H) \leq -(p+j+3), \quad \text{whenever } p + j \geq -(n-1).
\]

Since \( H \subseteq Y \) was a generic hyperplane section of \( Y \), this inequality clearly implies that

\[
\dim \text{Supp } H^j gr^F \mathcal{O}_H \mathcal{O}_H \leq -(p+j+2,0) \text{ whenever } p + j \geq -(n+1).
\]

This is enough for our purposes, because we have already shown in Lemma 6.8 that \( H^j gr^F \mathcal{O}_H \mathcal{O}_H = 0 \) whenever \( p + j \geq -1 \). The proof of Theorem 6.6 is thus complete.

6.3. The case of mixed Hodge modules. In this section, we generalise Theorem 6.6 and Corollary 6.7 to a certain class of mixed Hodge modules. While the main line of argument follows Section 6.2, there are some noteworthy differences. To keep the text readable, we chose to include full arguments, at the cost of introducing some repetition.

Setting 6.10. Let \( Y \) be a complex manifold of constant dimension \( n + c \), and let \( X \subseteq Y \) be a complex subspace of constant dimension \( n \). As before, \( c \) is equal to the codimension of the closed embedding \( \iota_X: X \hookrightarrow Y \). Suppose that \( M \in MHM(Y) \) is a graded-polarisable mixed Hodge module with support equal to \( X \). We denote the underlying filtered left \( \mathcal{D}_Y \)-module by \((M, F^\bullet M)\), and make the following assumptions about \( M \):

\[
(6.10.1) \quad \text{One has } \dim \text{Supp } H^j gr^F \mathcal{O}_H \mathcal{O}_H \leq -(j+1) \text{ for every } j \geq -(n+1).
\]
The complex of $\mathcal{O}_Y$-modules $\text{gr}_p^F \text{DR}(M)$ is acyclic for every $p \geq 1$.

One has $\dim \text{Supp} \mathcal{H}^j \text{gr}_p^F \text{DR}(M) \leq -(j + 2)$ for every $j \geq -n + 1$.

These are the natural generalisations of (6.5.1) and (6.5.2) to the mixed case, formulated in a way that is convenient for a proof by induction on the dimension. As before, write $M^* := DM \in \text{MHM}(Y)$ to denote the dual mixed Hodge module, which is again graded-polarisable, and write $(M', \mathcal{F}_s M')$ for its underlying filtered left $\mathcal{O}_Y$-module. Recall that the support does not change when taking duals, so $\text{Supp} M' = \text{Supp} M = X$.

**Note.** The cohomology sheaves of the de Rham complex $\text{DR}(M)$ are constructible sheaves on $Y$. Since $\text{DR}(M)$ is a perverse sheaf, the dimension of the support of $\mathcal{H}^j \text{DR}(M)$ is always at most $-j$ for every $j \in \mathbb{Z}$. In light of **Proposition 5.2**, the condition in (6.10.1) is saying that $\text{DR}(M)$ does not admit nontrivial quotients whose support has dimension $\leq n - 1$.

**Theorem 6.11** (Inequalities for mixed Hodge modules). Assume **Setting 6.10** and let $p \in \mathbb{Z}$ be any integer. Then one has

$$\dim \text{Supp} \mathcal{H}^j \text{gr}_p^F \text{DR}(M) \leq -(p + j + 2)$$

for every $j$ with $p + j \geq -n + 1$.

The proof of **Theorem 6.11** is given in **Section 6.3.1** and **Section 6.3.2** below. As before, **Theorem 6.11** leads to extension theorems for certain coherent sheaves derived from the de Rham complex.

**Corollary 6.12** (Extending sections). Assume **Setting 6.10**. Then for any $p \in \mathbb{Z}$, sections of $\mathcal{H}^p \text{gr}_p^F \text{DR}(M')$ extend uniquely across any complex subspace of dimension $\leq n - 2$.

**Proof.** Write $K_p := \text{gr}_p^F \text{DR}(M')[−p]$. As in the proof of **Corollary 6.7**, we begin by showing that $K_p \in D^{<0}_\text{coh}(\mathcal{O}_X)$. To this end, **Proposition 4.5**, implies that

$$\text{gr}_p^F \text{DR}(M') \cong R\mathcal{H}om_{\mathcal{O}_Y} \left(\text{gr}_p^F \text{DR}(M), \mathcal{O}_Y^\bullet\right)$$

for every $p \in \mathbb{Z}$.

By (6.10.2), this complex is acyclic for all $\ell \leq -1$. In particular, it follows from **Lemma 4.4** that $F_{−1}M' = 0$. The description (4.3.2) of the graded pieces in the de Rham complex then implies that $\mathcal{H}^j \text{gr}_p^F \text{DR}(M') = 0$ for $j < −p$. In other words, we obtain that $K_p \in D^{<0}_\text{coh}(\mathcal{O}_X)$ as desired.

As before, **Proposition 4.5** gives isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_Y}(K_p, \mathcal{O}_Y^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y} \left(\text{gr}_{−p}^F \text{DR}(M')[−p], \mathcal{O}_Y^\bullet\right) \cong \text{gr}_p^F \text{DR}(M)[p]$$

With these identifications, the inequalities (6.11.1) in **Theorem 6.11** take the form

$$\dim \text{Supp} R\mathcal{H}om_{\mathcal{O}_Y}(K_p, \omega_Y^\bullet) = \dim \text{Supp} \mathcal{H}^{j+p} \text{gr}_p^F \text{DR}(M) \leq -(j + 2)$$

for every $j \geq -n + 1$. As before, we conclude from **Proposition 6.4** that sections of the coherent $\mathcal{O}_Y$-module $\mathcal{H}^p K_p = \mathcal{H}^p \text{gr}_p^F \text{DR}(M')$ extend uniquely across any complex subspace $A \subseteq Y$ with $\dim A \leq n - 2$.

**6.3.1. Preparation for proof of Theorem 6.11.** In cases where $p + j \geq \max(-n + 1, -1)$, the inequality (6.11.1) in **Theorem 6.11** is claiming that $\mathcal{H}^j \text{gr}_p^F \text{DR}(M) = 0$. We begin by proving that this is indeed the case.

**Lemma 6.13.** Assume **Setting 6.10**. If $p + j \geq \max(-n + 1, -1)$, then $\mathcal{H}^j \text{gr}_p^F \text{DR}(M) = 0$.

**Proof.** The complex $\text{gr}_p^F \text{DR}(M)$ is concentrated in non-positive degrees, and is acyclic for $p \geq 1$ by **Assumption 6.10.2**. This means that $\mathcal{H}^j \text{gr}_p^F \text{DR}(M) = 0$ whenever $j \geq 1$ or $p \geq 1$. **Assumption 6.10.3** implies the claim when $p = 0$. This leaves only one case to consider, namely $p = 1$ and $j = 0$. We show that $\mathcal{H}^0 \text{gr}_1^F \text{DR}(M) = 0$ too.

The inclusion $F_0 \text{DR}(M) \subseteq \text{DR}(M)$ is a quasi-isomorphism; this follows from **Assumption 6.10.2** and **Proposition 4.8**. In particular, the inclusion induces an isomorphism
The inequality in (6.10.1) shows that $\mathcal{H}^0$ DR$(M) = 0$, and therefore $\mathcal{H}^0 F_0\text{DR}(M) = 0$. Now consider the short exact sequence of complexes (of sheaves of $\mathbb{C}$-vector spaces)

$$0 \to F_{-1} \text{DR}(M) \to F_0 \text{DR}(M) \to \text{gr}_0^F \text{DR}(M) \to 0.$$ 

Since $\mathcal{H}^j \text{gr}_0^F \text{DR}(M) = 0$ for $j \geq -1$, we obtain

$$\mathcal{H}^0 F_{-1} \text{DR}(M) \cong \mathcal{H}^0 F_0\text{DR}(M) = 0$$

from the long exact sequence in cohomology. The rest of the proof now proceeds exactly as in Lemma 6.8.

6.3.2. Proof of Theorem 6.11. We prove Theorem 6.11 by induction on $n = \dim X$. If $n = 1$ or $n = 2$, then the desired statement follows from Lemma 6.13 above, and we are done. We will therefore assume for the remainder of the proof that $n \geq 3$, and that Theorem 6.11 is already known for smaller values of $n$.

Cutting down. The statement we are trying to prove is local on $Y$, and so we can assume for the remainder of the argument that $Y$ is an open ball in $\mathbb{C}^{\pi+c}$, and that $X \subseteq Y$ is connected. Let $H \subseteq Y$ be the intersection of $Y$ with a generic hyperplane in $\mathbb{C}^{\pi+c}$. The intersection $H \cap X$ is then a connected complex subspace of constant dimension $n - 2$. The inclusion mapping $i_H: H \hookrightarrow Y$ is non-characteristic for $M$, and the inverse image $M_H = H^{-1}i_H^* M$ is again a graded-polarisable mixed Hodge module with support $H \cap X$; see Theorem 4.16 for the details. Note that the support of $M_H \in \text{MHM}(H)$ still has codimension $c$ in the ambient complex manifold $H$. Denoting the underlying filtered $\mathcal{D}_H$-module by $(M_H, F_\bullet M_H)$, Theorem 4.16 give

$$M_H \cong \mathcal{O}_H \otimes_{i_H^{[-1]}H} i_H^* M \quad \text{and} \quad F_\bullet M_H \cong \mathcal{O}_H \otimes_{i_H^{[-1]}H} i_H^{[-1]} F_\bullet M,$$

as well as an isomorphism of perverse sheaves

$$(6.13.1) \quad \text{DR}(M_H) \cong i_H^{[-1]} \text{DR}(M)[-1].$$

Properties of $M_H$. As before, we claim that $M_H \in \text{MHM}(H)$ satisfies all assumptions made in Setting 6.10. We consider the assumptions one by one. Because $M$ satisfies Assumption (6.10.1) and because of the choice of $H$ as a generic hyperplane section, (6.13.1) yields

$$\dim \text{Supp} \mathcal{H}^j \text{DR}(M_H) = \dim \left(H \cap \text{Supp} \mathcal{H}^{j-1} \text{DR}(M)\right) \leq -(j + 1),$$

for every $j \geq -\dim(H \cap X) + 1$. In other words, $M_H$ satisfies (6.10.1) as well.

According Proposition 4.18, one has a short exact sequence of complexes

$$(6.13.2) \quad 0 \to N^*_{H/Y} \otimes_{\mathcal{O}_H} \text{gr}_p^F \text{DR}(M_H) \to \mathcal{O}_H \otimes_{\mathcal{O}_H} \text{gr}_p^F \text{DR}(M) \to \text{gr}_p^F \text{DR}(M_H)[1] \to 0,$$

where $N^*_{H/Y}$ is the conormal bundle for the inclusion $H \subseteq Y$. Since $\text{gr}_p^F \text{DR}(M_H)$ is acyclic for $p \gg 0$, and since Assumption (6.10.2) holds for $M$, we can use descending induction on $p$ to show that $\text{gr}_p^F \text{DR}(M_H)$ is acyclic for every $p \geq 1$, and hence that $M_H$ satisfies (6.10.2). It also follows that

$$\mathcal{O}_H \otimes_{\mathcal{O}_Y} \mathcal{H}^{j-1} \text{gr}_p^F \text{DR}(M) \cong \mathcal{H}^j \text{gr}_0^F \text{DR}(M_H),$$

and because of Assumption (6.10.3), we get

$$\dim \text{Supp} \mathcal{H}^j \text{gr}_0^F \text{DR}(M_H) = -1 + \dim \text{Supp} \mathcal{H}^{j-1} \text{gr}_0^F \text{DR}(M) \leq -(j + 2)$$

for every $j \geq -\dim(H \cap X) + 1$. But this is exactly (6.10.3) for $M_H$. 
Conclusion. In summary, we have established that $MH \in \text{MHM}(H)$ also has the three properties in (6.10.1) to (6.10.3), but with $\dim \text{Supp} M_H = \dim(H \cap X) = n - 1$. We can therefore conclude by induction on the dimension of the support that

$$\dim \text{Supp} \mathcal{H}^j_{\mathcal{O}_H} \mathcal{O}^p_{\mathcal{D}R}(M_H) \leq -(p + j + 2)$$ whenever $p + j \geq -\dim(H \cap X) + 1$.

Taking cohomology in the short exact in (6.13.2), we obtain an exact sequence of coherent $\mathcal{O}_H$-modules

$$N^*_{H/Y} \otimes \mathcal{H}^j_{\mathcal{O}_H} \mathcal{O}^p_{\mathcal{D}R}(M_H) \rightarrow \mathcal{O}_H \otimes \mathcal{O}_Y \mathcal{H}^j \mathcal{O}^p_{\mathcal{D}R}(M) \rightarrow \mathcal{H}^{j+1}_{\mathcal{O}_H} \mathcal{O}^p_{\mathcal{D}R}(M_H),$$

and therefore the inequality

$$\dim \text{Supp} \left( \mathcal{O}_H \otimes \mathcal{O}_Y \mathcal{H}^j \mathcal{O}^p_{\mathcal{D}R}(M) \right) \leq -(p + j + 3)$$ whenever $p + j \geq -\dim X + 1$.

Since $H \subseteq Y$ was a generic hyperplane section of $Y$, this inequality clearly implies that

$$\dim \text{Supp} \mathcal{H}^j_{\mathcal{O}_H} \mathcal{O}^p_{\mathcal{D}R}(M) \leq -\min(p + j + 2, 0)$$ for $p + j \geq -\dim X + 1$.

This is enough for our purposes, because we have already shown that $\mathcal{H}^j \mathcal{O}^p_{\mathcal{D}R}(M) = 0$ whenever $p + q \geq -1$. The proof of Theorem 6.11 is thus complete. \qed

7. Setup for the proof

We will prove the main results of the present paper in the following sections. Since we want to work locally, and since an irreducible complex space is not necessarily locally irreducible, we relax the assumptions a little bit and allow any reduced complex space of constant dimension. Except for Theorem 1.10, the proofs all work in essentially the same setup. We will therefore fix the setup here and introduce notation that will be consistently used throughout the following sections.

Setup 7.1. Consider a reduced complex space $X$ of constant dimension $n$, together with an embedding $i_X : X \hookrightarrow Y$ into an open ball. Choose a strong log resolution $r : \tilde{X} \rightarrow X$ that is projective as a morphism of complex spaces.

Notation 7.2. We denote dimensions and codimensions by

$$n := \dim X \quad \text{and} \quad c := \text{codim}_Y X,$$

which means that $Y$ is an open ball in $\mathbb{C}^{n+c}$. The assumption that $r$ is a strong log resolution implies that $\text{X}_{\text{reg}}$ is isomorphic to its preimage $r^{-1}(\text{X}_{\text{reg}})$. Finally, let $E := r^{-1}(\text{X}_{\text{sing}})$ be the reduced $r$-exceptional set. The assumption that $r$ is a strong log resolution implies that $E \subset \tilde{X}$ is a divisor with simple normal crossings; we write its irreducible components as $E = \bigcup_{i \in I} E_i$. The following diagram summarises the relevant morphisms in our setting.

```
X_{\text{reg}} \xrightarrow{j, \text{open embedding}} \tilde{X} \xrightarrow{r, \text{strong log resolution}} X \xrightarrow{i_X, \text{closed embedding}} Y
```

8. Pure Hodge modules and differentials on the resolution

Maintaining the assumptions and notation of Setting 7.1, we explain in this section how the (higher) direct images of $\Omega^p_X$ are related to the intersection complex on $X$. We begin with a discussion of the constant Hodge module on the complex manifold $\tilde{X}$. 
8.1. The constant Hodge module on the resolution. On the complex manifold $\tilde{X}$, consider the locally constant sheaf $\mathbb{Q}_X$, viewed as a polarised variation of Hodge structure of type $(0,0)$. Following Saito [Sai88, Thm. 5.4.3], we denote by $\mathbb{Q}_X[n] \in \text{HM}(\tilde{X}, n)$ the corresponding polarised Hodge module of weight $n$; see also [Pop16, Ex. 2.4]. Its underlying regular holonomic left $\mathcal{D}_{\tilde{X}}$-module is $\mathcal{O}_X$, with the usual action by differential operators, and the Hodge filtration $F^p_{\mathcal{O}_X}$ is given by

$$F^p_{\mathcal{O}_X} = \begin{cases} 0 & \text{if } p \leq -1 \\ \mathcal{O}_X & \text{otherwise.} \end{cases}$$

The de Rham complex $\text{DR}(\mathcal{O}_X)$, which is quasi-isomorphic to $\mathbb{C}_X[n]$, is

$$\text{DR}(\mathcal{O}_X) = \left[ \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \right][n].$$

It is filtered in the usual way, by degree, and the $(-p)$-th graded piece is then

$$(8.0.1) \quad \text{gr}^{-p}_{\mathcal{O}_X} \text{DR}(\mathcal{O}_X) \cong \Omega_X^n[n-p].$$

Following the discussion in Section 4.3, we consider the direct image $f_*(-R_F\mathcal{O}_X)$ of the filtered $\mathcal{D}_{\tilde{X}}$-module $(\mathcal{O}_{\tilde{X}}, F, \mathcal{O}_{\tilde{X}})$, as an object of the bounded derived category of coherent graded $R_F\mathcal{D}_Y$-modules. The direct image functor commutes with taking the associated graded of the de Rham complex by Proposition 4.10, which allows us to identify the graded pieces of the de Rham complex for $f_*(-R_F\mathcal{O}_X)$ as

$$(8.0.2) \quad \text{gr}^{-p}_{\mathcal{O}_X} \text{DR}(f_* (R_F\mathcal{O}_X)) \cong Rf_* \text{gr}^{-p}_{\mathcal{O}_X} \text{DR}(\mathcal{O}_X) \cong Rf_* \Omega_X^n[n-p].$$

8.2. The intersection complex of $X$. Consider the constant variation of Hodge structure of type $(0,0)$ on $X_{\text{reg}}$. By Saito’s fundamental theorem [Sai90, Thm. 3.21], applied to each irreducible component of the complex space $X$, it determines a polarised Hodge module $M_X \in \text{HM}(Y, n)$ of weight $n = \dim X$ on the complex manifold $Y$, with support equal to $X$. Its underlying perverse sheaf is the intersection complex of $X$. Denoting the filtered regular holonomic $\mathcal{D}_Y$-module underlying $M_X$ by $(\mathcal{M}_X, F, \mathcal{M}_X)$, we have $F_{-c}M_X = 0$ by construction. The de Rham complex $\text{DR}(\mathcal{M}_X)$ is again filtered, and its subquotients are

$$\text{gr}^{-p}_{\mathcal{M}_X} \text{DR}(\mathcal{M}_X) = \left[ \text{gr}^{-p}_{\mathcal{M}_X} M_X \rightarrow \Omega_X^n \otimes \text{gr}^{-p+1}_{\mathcal{M}_X} M_X \rightarrow \cdots \rightarrow \Omega_X^n \otimes \text{gr}^{-p+n+c}_{\mathcal{M}_X} M_X \right][n+c].$$

Note that this complex is concentrated in degrees $-(n+c), \ldots, 0$.

8.3. Decomposition. As discussed in Section 4.3, the fact that the holomorphic mapping $f: \tilde{X} \rightarrow Y$ is projective implies that each $H^\ell f_* \mathcal{Q}^H[n]$ is again a polarisable Hodge module of weight $n + \ell$ on $Y$. Using the decomposition by strict support, we obtain moreover

$$H^\ell f_* M_{\tilde{X}} \cong \begin{cases} M_X \oplus M_0 & \text{if } \ell = 0 \\ M_\ell & \text{if } \ell \neq 0, \end{cases}$$

where $M_X \in \text{HM}(Y, n)$ is as above, and where the other summands $M_\ell \in \text{HM}(Y, n+\ell)$ are polarisable Hodge modules on $Y$ whose support is contained inside $X_{\text{sing}}$. Denoting the associated $\mathcal{D}_Y$-modules by $M_\ell$, the properties of the direct image functor imply that $F_\ell M_X = 0$, as a special case of Proposition 4.14.

Note. For dimension reasons, one has $M_\ell = 0$ once $|\ell|$ is greater than the “defect of semismallness” of $r: \tilde{X} \rightarrow X$; in particular, this holds for $|\ell| \geq n-1$. 

8.4. Relation with differential forms. Saito’s version of the Decomposition Theorem, Corollary 4.12, together with the isomorphism in (8.0.2), allows us to identify, for every $p \in \mathbb{Z}$, the derived push forward of the sheaf of $p$-forms on $\tilde{X}$ as

$$(8.0.3) \quad Rf_* \Omega^p_{\tilde{X}}[n-p] \cong \text{gr}^F_p \text{DR}(M_X) \oplus \bigoplus_{\ell \geq 1} \text{gr}^F_{p-\ell} \text{DR}(M_\ell)[-\ell].$$

In the situation at hand, the relation between $f_* \Omega^p_{\tilde{X}}$ and the intersection complex of $X$ is an almost direct consequence of the isomorphism in (8.0.3) above.

**Proposition 8.1.** Maintaining Setting 7.1 and using the notation introduced above, we have

$$f_* \Omega^p_{\tilde{X}} \cong \mathcal{H}^{-(n-p)} \text{gr}^F_p \text{DR}(M_X) \quad \text{for every } p \in \mathbb{Z}.$$

**Proof.** Recall from (8.0.3) that we have a decomposition

$$Rf_* \Omega^p_{\tilde{X}}[n-p] \cong \text{gr}^F_p \text{DR}(M_X) \oplus \text{Rest}_p,$$

in which the support of the complex $\text{Rest}_p \in D^b_c(O_Y)$ is contained inside $X_{\text{sing}}$. Taking cohomology in degree $-(n-p)$, we get

$$f_* \Omega^p_{\tilde{X}} \cong \underbrace{\text{gr}^F_p \text{DR}(M_X)}_{=A} \oplus \underbrace{\mathcal{H}^{-(n-p)} \text{Rest}_p}_{=B},$$

and therefore $f_* \Omega^p_{\tilde{X}}$ is the direct sum of $A$ and a coherent $O_X$-module $B$ supported on $X_{\text{sing}}$.

The claim follows because $\Omega^p_{\tilde{X}}$ is torsion free: the functor $f^*$ is a left adjoint for $f_*$, and the adjoint morphism $f^*B \to \Omega^p_{\tilde{X}}$ vanishes because $f^*B$ is supported on $f^{-1}(X_{\text{sing}})$. □

**Note.** The proof shows once again that $F_c M_\ell = 0$ for every $\ell \in \mathbb{Z}$. (Use Lemma 4.4.) This fact is also proved in much greater generality in [Sai91, Prop. 2.6].

The two values $p = n$ and $p = 0$ are special, because there is no contribution from the Hodge modules $M_\ell$ in those cases.

**Proposition 8.2.** Maintaining Setting 7.1 and using the notation introduced above, we have

$$Rf_* \Omega^p_{\tilde{X}} \cong \text{gr}^F_n \text{DR}(M_X) \quad \text{and} \quad Rf_* \partial_{\tilde{X}}[n] \cong \text{gr}^F_0 \text{DR}(M_X).$$

**Proof.** By Proposition 4.14, we have $F_c M_\ell = 0$ for every $\ell \in \mathbb{Z}$, and so $\text{gr}^F_n \text{DR}(M_\ell) = 0$. Together with (8.0.3), this implies the first isomorphism. The second isomorphism follows by duality, using Corollary 4.6 and the fact that $M_X \in H^0(M)(Y, n)$. □

The higher direct images of $\Omega^p_{\tilde{X}}$ can of course also be computed from (8.0.3), but they generally involve some of the other terms $M_\ell$. We give one example, in the special case $p = 1$, that will serve to illustrate the general technique.

**Proposition 8.3.** Maintaining Setting 7.1 and using the notation introduced above, we have

$$R^{n-1}f_* \Omega^1_{\tilde{X}} \cong \mathcal{H}^0 \text{gr}^F_{-1} \text{DR}(M_X) \oplus \mathcal{H}^0 \text{gr}^F_{-1} \text{DR}(M_0).$$

**Proof.** Formula (8.0.3) identifies the left side of the desired equality as

$$R^{n-1}f_* \Omega^1_{\tilde{X}} \cong \mathcal{H}^0 \text{gr}^F_{-1} \text{DR}(M_X) \oplus \bigoplus_{\ell \geq 0} \mathcal{H}^{-\ell} \text{gr}^F_{-1} \text{DR}(M_\ell).$$

To prove Proposition 8.3, it is therefore enough to show that $\text{gr}^F_{-1} \text{DR}(M_\ell)$ is acyclic for every $\ell \geq 1$. But using the fact that the Hodge modules $M_\ell \in H^0(M)(Y, n + \ell)$ are polarisable of weight $n + \ell$, Corollary 4.6 yields

$$\text{gr}^F_{-1} \text{DR}(M_\ell) \cong R\text{Hom}_{D^b}(\text{gr}^F_{-1-n-\ell} \text{DR}(M_\ell), \omega^*_Y).$$
Now a look back at the description of the filtration on the de Rham complex, in (4.3.1), reveals that the complex $\text{gr}^F_{1-(n+\ell)} \text{DR}(M_\ell)$ only involves the $\mathcal{O}_Y$-modules $\text{gr}^F_0 M_\ell$ with $k \leq c + 1 - \ell$. As $F_c M_\ell = 0$, it follows that $\text{gr}^F_{1-(n+\ell)} \text{DR}(M_\ell) = 0$ for every $\ell \geq 1$. □

8.5. Application to the extension problem. We conclude this section with a brief discussion of the effect that extendability of $n$-forms has on $\text{DR}(M_X)$ and its subquotients. The following result, together with Corollary 6.7, can be used to prove that if $n$-forms extend, then all forms extend. As explained in Section 2.2, this gives another proof for Theorem 1.2 in the (most important) case $k = n$.

**Proposition 8.4** (Extension of $n$-forms and $M_X$). Maintaining Setting 7.1 and using the notation introduced above, assume that $r.\Omega^n_X \hookrightarrow j.\Omega^n_{X_{\text{reg}}}$ is an isomorphism. Then one has

$$\dim \text{Supp} H^j \text{gr}^F_p \text{DR}(M_X) \leq -(j + p + 2)$$

for all integers $p, j \in \mathbb{Z}$ with $p + j \geq -n + 1$.

**Proof.** After replacing the Hodge module $M_X \in \text{HM}(Y, n)$ by any of the summands in its decomposition by strict support, and $X$ by the support of that summand, we may assume without loss of generality that $X$ is reduced, irreducible, and $n$-dimensional, and that $M_X$ has strict support $X$; in symbols, $M_X \in \text{HM}_X(Y, w)$. We aim to apply Theorem 6.6. Recalling from Section 8.2 that $F_{c-1} M_X = 0$, where $c = \dim Y - \dim X$, all the conditions in Theorem 6.6 hold in our context, provided we manage to prove the inequalities

$$\dim \text{Supp} H^\ell \text{gr}^F_p \text{DR}(M_X) \leq -(\ell + 2)$$

for every number $\ell \geq -n + 1$. But we have

$$-(\ell + 2) \geq \dim \text{Supp} R^\ell \mathcal{H}om_{\mathcal{O}_Y}(f_* \Omega^n_X, \omega^*_Y)$$

by Corollary 6.2

$$= \dim \text{Supp} R^\ell \mathcal{H}om_{\mathcal{O}_Y}(\text{gr}^F_n \text{DR}(M_X), \omega^*_Y)$$

by Proposition 8.2

$$= \dim \text{Supp} H^\ell \text{gr}^F_n \text{DR}(M_X)$$

by Corollary 4.6

This completes the proof. □

9. Mixed Hodge modules and log differentials on the resolution

We maintain the assumptions and notation of Setting 7.1. While the direct images of $\Omega^n_X$ are described in terms of the pure Hodge modules discussed in the previous Section 8, the study of logarithmic differentials requires us to look at certain mixed Hodge modules.

9.1. The mixed Hodge module on the complement of the exceptional divisor. Recall that $X$ is a reduced complex space of constant dimension $n$, and that $r: X \to X$ is a log resolution with exceptional divisor $E$. We denote by $j: X \setminus E \hookrightarrow X$ the open embedding of the complement of the normal crossing divisor $E$. By analogy with the argument in Section 8.1, we consider the constant Hodge module $\mathcal{O}^H_{\tilde{X} \setminus E}[n]$ on the complement of $E$, and its extension to a mixed Hodge module

$$j_* \mathcal{O}^H_{\tilde{X} \setminus E}[n] \in \text{MHM}(\tilde{X})$$

on $\tilde{X}$, as discussed in [Sai90, Thm. 3.27]. For the reader’s convenience, we summarise its main properties, properly translated to our convention of using left $\mathcal{D}$-modules.
9.1.1. Perverse sheaf and filtered $\mathcal{D}$-module. The underlying perverse sheaf of the mixed Hodge module $j_* Q^H_{\mathcal{X} \setminus E}[n]$ is, by construction, $Rj_* Q^H_{\mathcal{X} \setminus E}[n]$. The underlying regular holonomic $\mathcal{D}_{\mathcal{X}}$-module is $\mathcal{O}_{\mathcal{X}}(\ast E)$, the sheaf of meromorphic functions on the complex manifold $\mathcal{X}$ that are holomorphic outside the normal crossing divisor $E$. The Hodge filtration is given by
\[
F_p \mathcal{O}_{\mathcal{X}}(\ast E) = \begin{cases} 
0 & \text{if } p \leq -1, \\
F_p \mathcal{D}_{\mathcal{X}} \cdot \mathcal{O}_{\mathcal{X}}(E) & \text{if } p \geq 0.
\end{cases}
\]
The de Rham complex of $\mathcal{O}_{\mathcal{X}}(\ast E)$ is the complex of meromorphic differential forms
\[
DR(\mathcal{O}_{\mathcal{X}}(\ast E)) = \left[ \mathcal{O}_{\mathcal{X}}(\ast E) \xrightarrow{d} \Omega^1_{\mathcal{X}}(\ast E) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathcal{X}}(\ast E) \right][n],
\]
placed in degrees $-n, \ldots, 0$ as always. Saito [Sai90, Prop. 3.11] has shown that this complex, with the filtration induced by $F_p \mathcal{O}_{\mathcal{X}}(\ast E)$, is filtered quasi-isomorphic to the log de Rham complex $\Omega^*_X(\log E)[n]$, with the usual filtration by degree; in fact, the Hodge filtration on $\mathcal{O}_{\mathcal{X}}(\ast E)$ is defined so as to make this true.

**Proposition 9.1.** Maintaining Setting 7.1 and using the notation introduced above, the natural inclusion $\Omega^*_X(\log E)[n] \hookrightarrow DR(\mathcal{O}_{\mathcal{X}}(\ast E))$ is a filtered quasi-isomorphism. In particular, we have canonical isomorphisms
\[
\Omega^p_X(\log E)[n - p] \cong gr^p_F \Omega^*_X(\log E)[n] \cong gr^p_F DR(\mathcal{O}_{\mathcal{X}}(\ast E)). \quad \square
\]

9.1.2. Weight filtration. The weight filtration on the mixed Hodge module $j_* Q^H_{\mathcal{X} \setminus E}[n]$ is governed by how the components of the normal crossing divisor $E$ intersect. Since this fact is not explicitly mentioned in [Sai90, Thm. 3.27], we include a precise statement and a proof.

**Proposition 9.2** (Description of weight filtration). Maintaining Setting 7.1 and using the notation introduced above, the first pieces of the weight filtration on the mixed Hodge module $j_* Q^H_{\mathcal{X} \setminus E}[n]$ of the filtrations are given by
\[
W_{n-1} j_* Q^H_{\mathcal{X} \setminus E}[n] = 0 \quad \text{and} \quad W_n j_* Q^H_{\mathcal{X} \setminus E}[n] \equiv Q^H_{\mathcal{X}}[n].
\]
Likewise, for $\ell \geq 1$, the Hodge module $gr^W_{n+\ell} j_* Q^H_{\mathcal{X} \setminus E}[n] \in HM(\mathcal{X}, n+\ell)$ is isomorphic to the direct sum, over all subsets $J \subseteq I$ of size $\ell$, of the Hodge modules
\[
Q^H_{E_J}(\ast E)[n-\ell] \in HM(E_J, n+\ell),
\]
pushed forward from the complex submanifold $E_J := \cap_{i \in J} E_i$ into $\mathcal{X}$.

**Proof.** One possibility is to factor $j_*$ as a composition of open embeddings over the irreducible components of the simple normal crossing divisor $E$, as in [Sai90, Thm. 3.27]. Here, we explain a different argument, based on Saito’s computation of the nearby cycles functor in the normal crossing case [Sai90, Thm. 3.3].

To begin with, we observe that the weight filtration on a graded-polarisable mixed Hodge module is, even locally, unique: the reason is that there are no nontrivial morphisms between polarisable Hodge modules of different weights. This reduces the problem to the case where $\mathcal{X}$ is a polydisk, say with coordinates $x_1, \ldots, x_n$, and where $E$ is the divisor $g = x_1 \cdots x_r = 0$. Moreover, it is enough to prove the statement for the underlying $\mathcal{D}$-modules. Indeed, by [Sai88, Thm. 3.21], every polarisable Hodge module on $\mathcal{X}$, whose underlying $\mathcal{D}$-module is the direct image of $\mathcal{O}_{E_J}$, comes from a polarisable variation of Hodge structure on $E_J$, hence must be isomorphic to the push forward of $Q^H_{E_J}(k)$ for some $k \in \mathbb{Z}$. The Tate twist is then determined by the weight, because $n+\ell = \dim E_J + k$. 


After embedding $\tilde{X}$ into $\tilde{X} \times \mathbb{C}$, via the graph of $g = x_1 \cdots x_r$, we have, according to [Sai90, (2.11.10)], that

$$\text{gr}_{a+t}^W j_* Q^H_{X,E}[n] \cong \begin{cases} 0 & \text{if } \ell < 0, \\ Q^H_{X,E}[n] & \text{if } \ell = 0, \\ P_N \text{gr}_{a+t-2}^W \psi_{g^1} Q^H_{X,E}[n](-1) & \text{if } \ell > 0, \end{cases}$$

where $\psi_{g^1}$ denotes the nearby cycles functor (with respect to the coordinate function $t$ on $\tilde{X} \times \mathbb{C}$). In our normal crossing setting, the nearby cycles functor is computed explicitly in [Sai90, Thm. 3.3]. In the notation introduced in [Sai90, §3.4], the right $\mathcal{D}_{X,E}$-module associated to $\mathcal{O}_X$ is isomorphic to $M(\mu, \varnothing)$, where $\mu = (-1, \ldots, -1) \in \mathbb{Z}^n$. By [Sai90, (3.5.4)], the right $\mathcal{D}_{X,E}$-module underlying $\text{gr}_{a+t}^W j_* Q^H_{X,E}[n]$ is therefore isomorphic to the direct sum of $M(\mu, J)$, where $J \subseteq \{1, \ldots, r\}$ runs over all subsets of size $\ell$. But $M(\mu, J)$ is exactly the right $\mathcal{D}_{X,E}$-module associated to the push forward of $\mathcal{O}_{E_j}$, and so we get the desired result.

\[\square\]

9.2. Push forward to $Y$. Recall that $f : \tilde{X} \to Y$ is the projective holomorphic mapping obtained by composing our resolution of singularities $r : \tilde{X} \to X$ with the closed embedding $i_X : X \hookrightarrow Y$. We now define a family of mixed Hodge modules $N_\ell \in \text{MHM}(Y)$, indexed by $\ell \in \mathbb{Z}$, by setting

$$N_\ell := H^f f_* (j_* Q^H_{X,E}[n]).$$

Note that each $N_\ell$ is again a graded-polarsizable mixed Hodge module on $Y$, due to the fact that $f$ is a projective morphism (see Theorem 4.11). Clearly, $\text{Supp } N_0 = X$, and $\text{Supp } N_\ell \subseteq X_{\text{sing}}$ for $\ell \neq 0$.

**Lemma 9.3.** Maintaining Setting 7.1 and using the notation introduced above, we have $N_\ell = 0$ for $\ell \leq -1$. The mixed Hodge module $N_0$ has no nontrivial subobjects whose support is contained in $X_{\text{sing}}$.

**Proof.** It suffices to prove this for the underlying perverse sheaves $R\mathcal{F}_!$. By construction, $R\mathcal{F}_!$ is the $\ell$-th perverse cohomology sheaf of the constructible complex

$$R\mathcal{F}_! (j_* Q^H_{X,E}[n]) \cong Rj_* Q_{X_{\text{reg}}}[n].$$

Now, if $K \in \text{D}^b_c(Q_X)$ is any constructible complex, then

$$\text{Hom}_{\text{D}^b_c(Q_X)}(K, Rj_* Q_{X_{\text{reg}}}[n]) \cong \text{Hom}_{\text{D}^b_c(Q_{X_{\text{reg}}})}(j^{-1} K, Q_{X_{\text{reg}}}[n]),$$

and the right-hand side vanishes if $\text{Supp } K \subseteq X_{\text{sing}}$. The first assertion of Lemma 9.3 thus follows by taking $K = \text{rat } N_\ell[-\ell]$ for $\ell \leq -1$. Once it is known that $N_\ell = 0$ for $\ell \leq -1$, the second assertion follows by taking $K$ to be any subobject of $\text{rat } N_0$.

Each mixed Hodge module $N_\ell$ has weight $\geq n + \ell$, in the following sense.

**Lemma 9.4.** Maintaining Setting 7.1 and using the notation introduced above, we have $W_{n+\ell-1} N_\ell = 0$. The module $W_{n+\ell} N_\ell$ is a quotient of $H^f f_* Q^H_{X,E}[n]$.

**Proof.** This is proved in [Sai90, Prop. 2.26]. For the convenience of the reader, we explain how to deduce it from the degeneration of the weight spectral sequence in Theorem 4.13. Since $f$ is a projective morphism, the weight spectral sequence

$$E_1^{p,q} = H^{p+q} f_* \text{gr}_{-p}^W j_* Q^H_{X,E}[n] \Rightarrow N_{p+q}$$

degenerates at $E_2$, and the induced filtration on $N_\ell$ is the weight filtration $W_\bullet N_\ell$. More precisely, $E_1^{p,q}$ and $E_2^{p,q}$ are Hodge modules of weight $q$, and

$$\text{gr}_q N_{p+q} \cong E_2^{p,q}.$$
As \(j_*\Omega^H_X|E \nabla E\) has weight \(\geq n\), we have \(E^{p,q}_i = 0\) for \(p \geq -n+1\), whence \(\text{gr}^W_w N_\ell = 0\) for \(w \leq n + \ell - 1\). This also shows that \(W_{n+\ell} N_\ell\) is a quotient of \(E^{-n,n+\ell}_i\).

## 9.3. Relation with logarithmic differentials on the resolution

Now we can relate the coherent \(\mathcal{D}_Y\)-module \(f_*\Omega^P_X(\log E)\) to the de Rham complex of the mixed Hodge module \(N_0\). In line with the notation used before, write \((N_\ell, F_\ell N_\ell)\) for the filtered regular holonomic \(\mathcal{D}_Y\)-module underlying the mixed Hodge module \(N_\ell\).

**Proposition 9.5.** Maintaining Setting 7.1 and using the notation introduced above, we have \(f_*\Omega^P_X(\log E) \cong \mathcal{H}^{-n} \text{gr}^F_p \text{DR}(N_0)\) for every \(p \in \mathbb{Z}\).

**Proof.** Fix an integer \(p \in \mathbb{Z}\). Proposition 9.1, together with Proposition 4.10 about the compatibility of the de Rham complex with direct images, implies that

\[
\text{R} f_* \Omega^P_X(\log E)[n-p] \cong \text{gr}^F_p \text{DR}\left(f_*(\mathcal{D}_E(\log E))\right).
\]

Because the complex computing the direct image is strict by Theorem 4.13, we have a convergent spectral sequence

\[
E^{a,b}_2 = \mathcal{H}^a \text{gr}^F_p \text{DR}(N_0) \Rightarrow \mathcal{H}^{a+b} \text{gr}^F_p \text{DR}\left(f_*(\mathcal{D}_E(\log E))\right),
\]

and we are interested in the terms with \(a + b = p - n\). Proposition 4.14 guarantees that \(F_{-\ell} N_\ell = 0\) for every \(\ell \in \mathbb{Z}\), whence \(E^{a,b}_2 = 0\) for \(a \leq p - n - 1\). Also, \(N_\ell = 0\) for \(\ell \leq -1\) by Lemma 9.3, and so \(E^{a,b}_2 = 0\) for \(b \leq -1\). The spectral sequence therefore gives us the desired isomorphism.

The analysis of the higher direct images quickly gets complicated. For that reason, we shall only consider what happens in the case of 1-forms with log poles. Here, one has the following simple relation between \(R f_* \Omega^1_X(\log E)\) and the complex \(\text{gr}^{\geq 1}_F \text{DR}(N_0)\).

**Proposition 9.6.** Maintaining Setting 7.1 and using the notation introduced above, we have a canonical isomorphism

\[
R f_* \Omega^1_X(\log E)[n-1] \cong \text{gr}^F_{-1} \text{DR}(N_0).
\]

In particular, \(R^{n-1} f_* \Omega^1_X(\log E) \cong \mathcal{H}^0 \text{gr}^F \text{DR}(N_0)\).

The proof of Proposition 9.6 relies on the following lemma, which we discuss first.

**Lemma 9.7.** Maintaining Setting 7.1 and using the notation introduced above, the complex \(\text{gr}^F_{-1} \text{DR}(N_\ell)\) is acyclic for every \(\ell \neq 0\).

**Proof.** Recall from Lemma 9.4 that \(N_\ell \in \text{MHM}(Y)\) has weight \(\geq n + \ell\), which means that \(\text{gr}^W_w N_\ell = 0\) for \(w \leq n + \ell - 1\). Proposition 4.14 guarantees that \(F_{-\ell} N_\ell = 0\) for every \(\ell \geq 0\). This implies that \(F_{-1} \text{gr}^W_w N_\ell = 0\) for every \(w \in \mathbb{Z}\). According to Corollary 4.6, we have

\[
\text{gr}^F_{-1} \text{DR}(\text{gr}^W_w N_\ell) \cong R \mathcal{H}om_{\mathcal{O}_Y}\left(\text{gr}^F_{1-w} \text{DR}(\text{gr}^W_w N_\ell), \omega_Y^\bullet\right),
\]

and the complex \(\text{gr}^{\geq 1}_F \text{DR}(\text{gr}^W_w N_\ell)\) only uses the \(\mathcal{D}_Y\)-modules \(\text{gr}^F_p \text{gr}^W_w N_\ell\) in the range

\[
1 - w \leq p \leq 1 - w + \dim Y = c + 1 - \ell - (w - (n + \ell)).
\]

As \(F_{-1} \text{gr}^W_w N_\ell = 0\) and \(\ell \geq 1\), we see that \(\text{gr}^F_{1-w} \text{DR}(\text{gr}^W_w N_\ell) = 0\), except maybe in the special case \(w = n + \ell\). But by the \(E_2\)-degeneration of the weight spectral sequence, \(\text{gr}^{n+\ell}_w N_\ell\) is a quotient of \(M_\ell = H^0 F_n \Omega^H_X[n]\), and since we already know that \(F_{-\ell} M_\ell = 0\), we also have \(F_{-1} \text{gr}^W_{n+\ell} N_\ell = 0\) for \(\ell \geq 1\). This proves that \(\text{gr}^F_{-1} \text{DR}(\text{gr}^W_w N_\ell)\) is acyclic for every \(\ell \geq 1\) and every \(w \in \mathbb{Z}\). Since the functor \(\text{gr}^{\geq 1}_F \text{DR}\) is exact on mixed Hodge modules, it follows that the complex \(\text{gr}^F_{-1} \text{DR}(N_\ell)\) is also acyclic. \(\square\)
Proof of Proposition 9.6. Because $N_j = 0$ for $j \leq -1$, and because the complex computing the direct image is strict by Theorem 4.13, we have a canonical morphism

$$(N_0, f_* N_0) \to f_*(R_F \mathcal{O}_X(\ast E))$$

in the derived category $D^b_{\text{coh}} G(R_F \mathcal{D}_Y)$. As a first step, we are going to show that the induced morphism

$$(9.7.1) \quad \text{gr}^F_{E-1} \text{DR}(N_0) \to \text{gr}^F_{E-1} \text{DR} \left( f_*(R_F \mathcal{O}_X(\ast E)) \right)$$

between complexes of $\mathcal{O}_Y$-modules is a quasi-isomorphism. Lemma 9.7 implies that the spectral sequence

$$(9.8.2) \quad \text{E}^1_{\ast, b} = \mathcal{H}^a \text{gr}^F_{E-1} \text{DR}(N_0) \Rightarrow \mathcal{H}^{a+b} \text{gr}^F_{E-1} \text{DR} \left( f_*(R_F \mathcal{O}_X(\ast E)) \right),$$

degenerates at $E_2$, and so we have a collection of isomorphisms

$$(9.8.3) \quad \mathcal{H}^a \text{gr}^F_{E-1} \text{DR}(N_0) \cong \mathcal{H}^a \text{gr}^F_{E-1} \text{DR} \left( f_*(R_F \mathcal{O}_X(\ast E)) \right).$$

These isomorphisms are induced by the morphism in $(9.7.1)$, which is therefore a quasi-isomorphism. Now the compatibility of the de Rham complex with direct images, together with Proposition 9.1, implies that

$$\text{gr}^F_{E-1} \text{DR}(N_0) \cong f_* \text{gr}^F_{E-1} \text{DR} \left( \mathcal{O}_X(\ast E) \right) \cong f_* \Omega_X^1(\log E)[n-1],$$

as asserted by the proposition. \qed

9.4. The weight filtration on $N_0$. We describe how the weight filtration interacts with the complex $\text{gr}^F_{E-1} \text{DR}(N_0)$.

Proposition 9.8 (The complex $\text{gr}^F_{E-1} \text{DR}(N_0)$). Maintaining Setting 7.1 and using the notation introduced above, the complex $\text{gr}^F_{E-1} \text{DR}(\text{gr}^W_{m} N_0)$ is acyclic for $w \notin \{n, n+1\}$ and

$$(9.8.1) \quad \text{gr}^F_{E-1} \text{DR}(\text{gr}^W_{m} N_0) \equiv \text{gr}^F_{E-1} \text{DR}(M_X)$$

and

$$(9.8.2) \quad \text{gr}^F_{E-1} \text{DR}(\text{gr}^W_{n+1} N_0) \equiv \bigoplus_{i+j \in I} \text{R}^j f_* \mathcal{O}_{E_i}[n-1].$$

Proof. Consider again the weight spectral sequence

$$(9.8.3) \quad E^{p, q}_{1} = H^{p+q} f_* \text{gr}^W_{m} j_* Q_{X_{1 \backslash E}}^H[n] \Rightarrow N_{p+q}.$$ 

Because $f$ is projective, the spectral sequence degenerates at $E_2$, and the induced filtration on $N_f$ is the weight filtration $W_n N_f$, see Theorem 4.13. More precisely, what happens is that $E^{p, q}_{1}$ and $E^{p, q}_{2}$ are polarisable Hodge modules of weight $q$, and

$$\text{gr}^W_{q} N_{p+q} \equiv E^{p, q}_{2}.$$ 

Now $j_* Q_{X_{1 \backslash E}}^H[n]$ has weight $\geq n$, and so $E^{p, q}_{1} = 0$ for $p \geq -n+1$, and $W_{n-1} N_0 = 0$. Moreover, $W_n N_0$ is the cokernel of the morphism $d_1 : E^{-n-1, n} \to E_{n-1,n}$. Using the description of the weight filtration in Proposition 9.2, we compute that

$$(9.9.1) \quad E_{n-1,n} = H^0 f_* \text{gr}^W_{n+1} j_* Q_{X_{1 \backslash E}}^H[n] \equiv H^0 f_* Q_{X_{1 \backslash E}}[n] \equiv M_X \oplus M_0$$

and that the support of $E^{-n+1, n}$ is contained inside $X_{\text{sing}}$. Because $N_0$ has no subobjects that are supported inside $X_{\text{sing}}$ (by Lemma 9.3), and $M_X$ has neither subobjects nor quotient objects that are supported inside $X_{\text{sing}}$ (by construction), we conclude that $W_n N_0 \cong M_X$. This already proves $(9.8.1)$.

Likewise, $\text{gr}^W_{n+1} N_0$ is the cohomology of the complex of Hodge modules of weight $n + 1$

$$(9.8.3) \quad E^{n-2, n+1} \xrightarrow{d_1} E^{n-1, n+1} \xrightarrow{d_1} E^{-n+1, n+1}.$$ 

By a similar computation as above, we have $E_{1}^{-n,n+1} \cong M_{1}$ and
\[
E_{1}^{-n-1,n+1} \cong \bigoplus_{i \in I} H^{0} f_{*} \mathcal{Q}^{H}_{E_{i}}(-1)[n-1]
\]
\[
E_{1}^{-n-2,n+1} \cong \bigoplus_{i,j \in I} H^{-1} f_{*} \mathcal{Q}^{H}_{E_{i} \cap E_{j}}(-2)[n-2].
\]

We showed during the proof of Proposition 8.3 that $\text{gr}^{F}_{E_{1}} \text{DR} (M_{1})$ is acyclic. At the same time, using the compatibility of the de Rham complex with direct images, we have
\[
\text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{E}_{1}^{-n-1,n+1}) \cong \bigoplus_{i \in I} R f_{*} \text{gr}^{F}_{E_{i}} \text{DR} (\mathcal{O}_{E_{i}}) \cong \bigoplus_{i \in I} R f_{*} \mathcal{O}_{E_{i}}[n-1].
\]

By a similar calculation and the Decomposition Theorem, the complex $\text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{E}_{1}^{-n-2,n+1})$ is isomorphic to a direct summand in
\[
\bigoplus_{i \in I} R f_{*} \text{gr}^{F}_{E_{i}} \text{DR} (\mathcal{O}_{E_{i} \cap E_{j}})
\]
and therefore acyclic. Since morphisms between mixed Hodge modules strictly preserve the Hodge filtration, it now follows from (9.8.3) that
\[
\text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{W}_{n+1,N_{0}}) \cong \bigoplus_{i \in I} R f_{*} \mathcal{O}_{E_{i}}[n-1],
\]
proving (9.8.2).

Since $W_{n+1,N_{0}} = 0$, the complex $\text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{W}_{n,N_{0}})$ is certainly acyclic for $w \leq n - 1$. It remains to show that it is also acyclic for $w \geq n + 2$. The proof of this fact is the same as that of Lemma 9.7, and so we omit it $\square$.

**Corollary 9.9.** Maintaining Setting 7.1 and using the notation introduced above, we obtain a long exact sequence
\[
\cdots \to \mathcal{H}^{j} \text{gr}^{F}_{E_{1}} \text{DR} (M_{X}) \to \mathcal{H}^{j} \text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{N}_{0}) \to \bigoplus_{i \in I} R^{n+1-j} f_{*} \mathcal{O}_{E_{i}} \to \cdots
\]

**Proof.** Proposition 9.8 implies that the complex $\text{gr}^{F}_{E_{1}} \text{DR} (W_{n,N_{0}})$ is acyclic, and that the natural morphism
\[
\text{gr}^{F}_{E_{1}} \text{DR} (W_{n+1,N_{0}}) \to \text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{N}_{0})
\]
is a quasi-isomorphism. We therefore get a distinguished triangle
\[
\text{gr}^{F}_{E_{1}} \text{DR} (M_{X}) \to \text{gr}^{F}_{E_{1}} \text{DR} (\mathcal{N}_{0}) \to \bigoplus_{i \in I} R f_{*} \mathcal{O}_{E_{i}}[n-1] \to \text{gr}^{F}_{E_{1}} \text{DR} (M_{X})[1]
\]
in the derived category $D_{\text{coh}}^{b} (\mathcal{O}_{Y})$. The claim follows by passing to cohomology $\square$.

**9.5. Application to the extension problem.** In analogy with Section 8.5, we conclude with a brief discussion of the effect that extendability of log $n$-forms has on $\text{DR} (\mathcal{N}_{0})$. Once again, Corollary 6.12 and the result below can be used to show if $n$-forms extend with log poles, then all forms extend with log poles. This gives another proof for Theorem 1.6 in the (most important) case $k = n$. Since we are now working with mixed Hodge modules, the reader may find it instructive to compare the proof below with that of the analogous result for pure Hodge modules in Section 8.5.

**Proposition 9.10 (Extension of log $n$-forms and $N_{Y}$).** Maintaining Setting 7.1 and using the notation introduced above, assume that the morphism $r, \Omega^{\mathcal{E}}_{X} (\log E) \leftrightarrow j, \Omega^{\mathcal{E}}_{X_{\text{reg}}} \log is an isomorphism. Then one has
\[
\dim \text{Supp} \mathcal{H}^{j} \text{gr}^{F}_{p} \text{DR} (\mathcal{N}_{Y}) \leq -(j + p + 2)
\]
for all integers $j, p \in \mathbb{Z}$ with $p + j \geq -n + 1$. 

Proof. This time, we aim to apply Theorem 6.11. Recall that $X$ is reduced of constant dimension $n$; that the mixed Hodge module $N_0 \in MHM(Y)$ has support equal to $X$; and that we defined $N_Y := \mathbb{D}(N_0)(-n) \in MHM(Y)$ by taking the $(-n)$-th Tate twist of the dual mixed Hodge module. Taking into account the Tate twist, the formula for the de Rham complex of the dual mixed Hodge module in Proposition 4.5 becomes

\begin{equation}
\text{gr}_p^F \text{DR}(N_Y) \cong \mathcal{R}\text{Hom}_{\mathcal{O}_Y}\left(\text{gr}^F_{-p+n} \text{DR}(N_0), \omega_Y^*\right).
\end{equation}

Let us now verify that all the conditions in Theorem 6.11 are satisfied in our setting.

Claim 9.11. One has \(\dim \text{Supp} \mathcal{O}_Y^j \text{DR}(N_Y) \leq -(j + 1)\) for every \(j \geq n + 1\).

Proof of Claim 9.11. Recall that the module $N_0$ has weight $\geq n$, in the sense that $W_{n-1}N_0 = 0$, and that its support is $\text{Supp} N_0 = X$. The dual module $N_Y$ will then have weight $\leq n$, in the sense that $W_{n}N_Y = N_Y$, and $\text{Supp} N_Y = X$. By Lemma 9.3, the perverse sheaf $\text{DR}(N_0)$ has no nontrivial subobjects whose support is contained in $X_{\text{sing}}$. Consequently, the perverse sheaf $\text{DR}(N_Y)$, isomorphic to the Verdier dual of $\text{DR}(N_0)$, has no nontrivial quotients whose support is contained in $X_{\text{sing}}$. Now apply Proposition 5.2. \(\square\) (Claim 9.11)

Claim 9.12. The complex of $\mathcal{O}_Y$-modules $\text{gr}_p^F \text{DR}(N_Y)$ is acyclic for every $p \geq 1$.

Proof of Claim 9.12. Recall that $F_{c-1}N_0 = 0$, where $c = \dim Y - \dim X$. For dimension reasons, the complex $\text{gr}_p^F \text{DR}(N_0)$ is trivial for $p \geq 1$. Now (9.10.1) implies that the complex $\text{gr}_p^F \text{DR}(N_Y)$ is acyclic for every $p \geq 1$. \(\square\) (Claim 9.12)

Claim 9.13. One has $\dim \text{Supp} \mathcal{O}_Y^j \text{gr}_p^F \text{DR}(N_Y) \leq -(j + 2)$ for every $j \geq -n + 1$.

Proof of Claim 9.13. Since $F_{c-1}N_0 = 0$, the formula in (4.3.2) implies that the complex

\begin{equation}
\text{gr}_p^F \text{DR}(N_0) \cong \mathcal{O}_Y^j \text{gr}_p^F \text{DR}(N_0)
\end{equation}

is actually a sheaf in degree 0. Using the assumption that $r, \Omega_X^p(\log E) \cong j_* \Omega_Y^p_{X_{\text{sing}}}$, the following inequalities will therefore hold for all $j \geq -n + 1$:

\(- (j + 2) \geq \text{dim Supp} \mathcal{O}_Y^j \text{gr}_p^F \text{DR}(N_0, \omega_Y^*) \quad \text{by Corollary 6.2}
\)

\begin{equation}
= \text{dim Supp} \mathcal{O}_Y^j \text{gr}_p^F \text{DR}(N_0, \omega_Y^*) \quad \text{by Proposition 9.5}
\end{equation}

\begin{equation}
= \text{dim Supp} \mathcal{O}_Y^j \text{gr}_p^F \text{DR}(N_0, \omega_Y^*) \quad \text{by (9.13.1)}
\end{equation}

\begin{equation}
= \text{dim Supp} \mathcal{O}_Y^j \text{gr}_p^F \text{DR}(N_Y) \quad \text{by (9.10.1)}
\end{equation}

This gives us the desired result. \(\square\) (Claim 9.13)

Having checked all the conditions, we can now apply Theorem 6.11 and conclude the proof of Proposition 9.10. \(\square\)

10. INTRINSIC DESCRIPTION, PROOF OF THEOREMS 1.3 AND 1.7

10.1. Proof of Theorem 1.3. In this section, we prove the criterion for extension of holomorphic forms in Theorem 1.3. In fact, the result is really just a reformulation of Proposition 8.1, although it takes some work to see that this is the case.

Setup. Let $X$ be a reduced complex space of constant dimension $n$. Since the statement to be proved is local on $X$, we may assume that we are in the setting described in Section 7. In particular, $X$ is a complex subspace of an open ball $Y \subseteq \mathbb{C}^{n+1}$, and $f: \tilde{X} \to Y$ denotes the composition of a projective resolution of singularities $r: \tilde{X} \to X$ with the closed embedding $i_X: X \hookrightarrow Y$. Because $Y$ is a Stein manifold, all Kähler differentials on $X$ are restrictions of holomorphic differential forms from $Y$; in particular, if $z_1, \ldots, z_{n+1}$ are holomorphic coordinates on $Y$, then the sheaf $\Omega_X^p$ is generated by the global sections

\[i_X^*(dz_{i_1} \wedge \cdots \wedge dz_{i_p}),\]

where $1 \leq i_1 < i_2 < \cdots < i_p \leq n + c$. 

The intersection complex. As in Section 8, we use the notation $M_X \in \text{HM}(Y, n)$ for the polarisable Hodge module on $Y$ whose underlying perverse sheaf is the intersection complex of $X$, and we let $(M_X, F_c M_X)$ be its underlying filtered $\mathcal{D}_Y$-module. According to Proposition 8.1, we have

$$f_* \Omega^p_X \cong \mathcal{H}^{-(n-p)} \text{gr}^F_p \text{DR}(M_X).$$

Recall from Section 4.1.5 that the de Rham complex

$$\text{DR}(M_X) = \left[ \mathcal{M}_X \xrightarrow{\nabla} \Omega^1_X \otimes M_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{n+c}_X \otimes M_X \right],$$

is concentrated in degrees $-(n+c), \ldots, 0$. Since $\dim Y - \dim X = c$, one has $F_{c-1} M_X = 0$, which means that the complex of coherent $\mathcal{O}_Y$-modules

$$\text{gr}^F_p \text{DR}(M_X) = \left[ \Omega^{p+c}_Y \otimes F_c M_X \xrightarrow{\nabla} \Omega^{p+c+1}_Y \otimes \text{gr}^F_{c+1} M_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{n+c}_Y \otimes \text{gr}_{n-p+c}^F M_X \right]$$

is concentrated in degrees $-(n-p), \ldots, 0$. The result in Proposition 8.1 therefore becomes

$$(10.0.1) \quad f_* \Omega^p_X \cong \ker (\nabla : \Omega^{p+c}_Y \otimes F_c M_X \to \Omega^{p+c+1}_Y \otimes \text{gr}^F_{c+1} M_X).$$

This yields an isomorphism between the space of holomorphic $p$-forms on the resolution $\tilde{X}$, and the space of holomorphic $(p+c)$-forms on $Y$ with coefficients in the coherent $\mathcal{O}_Y$-module $F_c M_X$ whose image under the differential in the de Rham complex is again a holomorphic $(p+c+1)$-form on $Y$ with coefficients in $F_c M_X$. The isomorphism

$$(10.0.2) \quad f_* \Omega^p_X \cong \Omega^{p+c}_Y \otimes F_c M_X$$

is an important special case of this.

Claim 10.1. With notation as above, the image of the restriction morphism

$$H^0(Y, \Omega^{p+c}_Y \otimes F_c M_X) \to H^0 \left( Y \setminus X_{\text{reg}}, \Omega^{p+c}_Y \otimes F_c M_X \right)$$

consists exactly of those $(p+c)$-forms with values in $F_c M_X$ whose wedge product with any element of $H^0(Y, \Omega^{n-p}_Y)$ belongs to the image of

$$H^0 \left( Y \setminus X_{\text{sing}}, \Omega^{p+c}_Y \otimes F_c M_X \right) \to H^0 \left( Y \setminus X_{\text{reg}}, \Omega^{p+c}_Y \otimes F_c M_X \right).$$

Proof of Claim 10.1. The isomorphism in (10.0.2) shows that $F_c M_X$ is a rank-one coherent sheaf supported on $X$, whose restriction to $X_{\text{reg}}$ is isomorphic to the line bundle $\det N_{X_{\text{reg}}|Y}$. Using the coordinate functions $z_1, \ldots, z_{n+c}$ on the ball $Y$, we may write any given element of $H^0 \left( Y \setminus X_{\text{reg}}, \Omega^{p+c}_Y \otimes F_c M_X \right)$ uniquely in the form

$$\sum (dz_{i_1} \wedge \cdots \wedge dz_{i_p}) \otimes \lambda_{i_1, \ldots, i_p},$$

with coefficients $\lambda_{i_1, \ldots, i_p} \in H^0 \left( Y \setminus X_{\text{sing}}, F_c M_X \right)$. Clearly such an element belongs to the image of the restriction morphism if and only if all the coefficients are in the image of $H^0(Y, F_c M_X)$. The assertion now follows by taking wedge products with all possible $(n-p)$-forms of the type $dz_{i_1} \wedge \cdots \wedge dz_{i_{n-p}}$. □ (Claim 10.1)

End of proof. Now suppose we are given a holomorphic $p$-form $\alpha \in H^0(X_{\text{reg}}, \Omega^p_Y)$ on the set of nonsingular points of $X$. Using the isomorphism in (10.0.1), it determines a unique element $\tilde{\alpha} \in H^0 \left( Y \setminus X_{\text{sing}}, \Omega^{p+c}_Y \otimes F_c M_X \right)$ with the property that

$$\nabla \tilde{\alpha} \in H^0 \left( Y \setminus X_{\text{sing}}, \Omega^{p+c+1}_Y \otimes F_c M_X \right),$$

and one checks easily that $\nabla \tilde{\alpha}$ corresponds to the $(p+1)$-form $d\alpha$ under the isomorphism in (10.0.1). Again using (10.0.1), we conclude that $\alpha$ extends to a holomorphic $p$-form on $\tilde{X}$ if and only $\tilde{\alpha}$ belongs to the image of

$$H^0 \left( Y \setminus X_{\text{sing}}, \Omega^{p+c}_Y \otimes F_c M_X \right) \to H^0 \left( Y \setminus X_{\text{sing}}, \Omega^{p+c}_Y \otimes F_c M_X \right).$$
and $\nabla \alpha$ belongs to the image of 

$$H^0(Y, \Omega^{p+c+1}_Y \otimes F_c \mathcal{M}_X) \rightarrow H^0(Y \setminus X_{\text{sing}}, \Omega^{p+c+1}_Y \otimes F_c \mathcal{M}_X).$$

According to Claim 10.1, we can test for these two conditions after taking wedge products with elements in $H^0(Y, \Omega_Y^{p-r})$ respectively $H^0(Y, \Omega_Y^{p-r+1})$. Because the restriction mapping from the differentials on $Y$ to the Kähler differentials on $X$ is surjective, we get the desired conclusion. This ends the proof of Theorem 1.3.

10.2. Proof of Theorem 1.7. The proof of Theorem 1.7 is nearly identical to that of Theorem 1.3. The only difference is that one has to work with $\Omega^p_X(\log E)$ instead of $\Omega^p_X$; that one has to use the mixed Hodge module $N_0$ instead of the pure Hodge module $M_X$; and that one should apply Proposition 9.5 instead of Proposition 8.1. We leave the details to the care of the reader.

11. Extension, Proof of Theorems 1.2 and 1.6

11.1. Proof of Theorem 1.2. It clearly suffices to prove Theorem 1.2 only in the case $p = k - 1$, with $1 \leq k \leq n$. Again, we relax the assumptions a little bit and allow $X$ to be any reduced complex space of constant dimension $n$. This makes the entire problem local on $X$. After shrinking $X$, if necessary, we may therefore assume that we are given a holomorphic form $\alpha \in H^0(X_{\text{reg}}, \Omega_X^{k-1})$; our task is to show that $\alpha$ extends holomorphically to the complex manifold $\tilde{X}$. We aim to apply Theorem 1.3, and so we consider an arbitrary open subset $U \subseteq X$ and a pair of Kähler differentials $\beta \in H^0(U, \Omega_X^{n-k+1})$ and $\gamma \in H^0(U, \Omega_X^{n-k})$. We need to check that the holomorphic $n$-forms $\alpha \wedge \beta$ and $\alpha \wedge \gamma$ on $U_{\text{reg}}$ extend to holomorphic $n$-forms on $r^{-1}(U)$. This is again a local problem, and after further shrinking $X$, we may therefore assume without loss of generality that $U = X$ and that we have a closed embedding $i_X : X \hookrightarrow Y$, where $Y$ is an open ball in $\mathbb{C}^{n+c}$. Letting $z_1, \ldots, z_{n+c}$ be holomorphic coordinates on $Y$, the sheaf of Kähler differentials $\Omega^p_X$ is then generated by the global sections

$$i_X^*(dz_{i_1} \wedge \cdots \wedge dz_{i_p}),$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq n + c$. Since $n - k + 1 \geq 1$, we can thus write

$$\beta = \sum_{j=1}^{n+c} i_X^*(dz_{i_j}) \wedge \beta_j$$

for certain Kähler differentials $\beta_j \in H^0(X, \Omega_X^{n-k})$. The holomorphic $k$-forms $\alpha \wedge i_X^*(dz_{i_j})$ and $\alpha \wedge i_X^*(dz_{i_j}) \wedge \beta_j \wedge \gamma$ extend to $\tilde{X}$ as well. It follows that $\alpha \wedge \beta$ and $\alpha \wedge \gamma$ extend to $\tilde{X}$, and this implies that $\alpha$ itself extends to $\tilde{X}$, by another application of Theorem 1.3.

11.2. Proof of Theorem 1.6. The proof of Theorem 1.6 is nearly identical to the proof of Theorem 1.2. The only difference is that one uses Theorem 1.7 instead of Theorem 1.3.

12. Extension for $(n-1)$-forms, Proof of Theorem 1.8

We maintain the notation and assumptions of Theorem 1.8, but we allow $X$ to be any reduced complex space of constant dimension $n$. Recall that $r : \tilde{X} \rightarrow X$ is a log resolution such that the natural morphism $r_* \Omega^n_{\tilde{X}} \hookrightarrow j_* \Omega^n_{X_{\text{reg}}}$ is an isomorphism. Our task is to show that the natural morphism

$$r_* \Omega^{n-1}_{\tilde{X}}(\log E) \hookrightarrow j_* \Omega^{n-1}_{X_{\text{reg}}}$$
is an isomorphism, or equivalently, that sections of \( f_* \Omega^q_X(\log E)(-E) \) extend uniquely across \( X_{\text{sing}} \). It is easy to see by duality that all the sheaves \( r_* \Omega_j^p(\log E)(-E) \) are independent of the choice of log resolution. Shrinking \( X \) and replacing \( r \) with the canonical strong resolution of singularities, we may assume that we are in the setting described in Section 7 and Section 9. We use the notation introduced there.

The weight filtration on \( N_0 \). The proof relies the results of Section 9.4, where we analysed the weight filtration on the mixed Hodge module \( N_0 = H^0 f_*(j_* Q^H_X[n]) \in \text{MHM}(Y) \). To begin, recall from Proposition 9.6 that we have an isomorphism

\[
R f_* \Omega^1_X(\log E)[n-1] \cong \text{gr}^F_{-1} \text{DR}(N_0).
\]

Using Grothendieck duality for the proper holomorphic mapping \( f : \bar{X} \to Y \), we obtain

\[
R \text{Hom}_{O_Y} \left(R f_* \Omega^{-1}_X(\log E), O_Y^* \right) \cong R f_* \Omega^1_X(\log E)[n] \cong \text{gr}^F_{-1} \text{DR}(N_0)[1].
\]

According to the extension criterion for complexes in Proposition 6.4, it is therefore sufficient to prove the collection of inequalities

\[
\dim \text{Supp} \mathcal{H}^j \text{gr}^F_{-1} \text{DR}(N_0) \leq -(j+1) \quad \text{for every } j \geq -n+2.
\]

On the other hand, recall from Corollary 9.9 that, for all \( j \in \mathbb{Z} \), one has an exact sequence

\[
\mathcal{H}^j \text{gr}^F_{-1} \text{DR}(M_X) \to \mathcal{H}^j \text{gr}^F_{-1} \text{DR}(N_0) \to \bigoplus_{i \in I} R^{n-1+j} f_* \mathcal{O}_{E_i},
\]

The inequalities in (12.0.1) will follow from the analogous inequalities for the dimension of the support of the first and third term in (12.0.2).

The first term in (12.0.2). The first term is easily dealt with. Since we are in the setting of Theorem 1.2, an application of Proposition 8.4 gives the additional inequalities

\[
\dim \text{Supp} \mathcal{H}^j \text{gr}^F_{-1} \text{DR}(M_X) \leq -(j+1) \quad \text{for every } j \geq -n+2.
\]

This is half of what we need to prove (12.0.1).

The third term in (12.0.2). Now we turn to the third term. Fix an index \( i \in I \). Pushing forward the standard short exact sequence

\[
0 \to \mathcal{O}_{\bar{X}}(-E_i) \to \mathcal{O}_{\bar{X}} \to \mathcal{O}_{E_i} \to 0
\]

along \( f : \bar{X} \to Y \) gives us an exact sequence

\[
R^{n-1+j} f_* \mathcal{O}_{\bar{X}} \to R^{n-1+j} f_* \mathcal{O}_{E_i} \to R^{n+1} f_* \mathcal{O}_{\bar{X}}(-E_i).
\]

But then, the following inequalities will hold for every \( j \geq -n+2,

\[
\dim \text{Supp} B \leq -(j+1) \quad \text{for dimension reasons}
\]

\[
\dim \text{Supp} A = \dim \text{Supp} \mathcal{H}^{j-1} \text{gr}^F_{-1} \text{DR}(M_X) \leq -(j-1+2) \quad \text{by Proposition 8.2}
\]

\[
\text{by Proposition 8.4}
\]

In summary, we have \( \dim \text{Supp} R^{n-1+j} f_* \mathcal{O}_{E_i} \leq -(j+1) \) for every \( i \in I \) and every \( j \geq -n+2 \). As discussed above, together with (12.0.3) this suffices to the inequalities in (12.0.1). The proof of Theorem 1.8 is therefore complete.

We again record the following corollary of the proof.

**Corollary 12.1.** In the setting of Theorem 1.8, one has

\[
\dim \text{Supp} R^j f_* \Omega^1_X(\log E) \leq n-2-j, \quad \text{for every } j \geq 1.
\]
13. Local vanishing, proof of Theorem 1.9

We maintain the notation and assumptions of Theorem 1.9, but we allow $X$ to be any reduced complex space of constant dimension $n$. Recall that $r : \tilde{X} \to X$ is a log resolution of singularities such that $R^{n-1}r_*\Omega^1_{\tilde{X}} = 0$. Our goal is to prove that $R^{n-1}r_*\Omega^1_X(\log E) = 0$. Both the assumptions and the conclusion of Theorem 1.9 are independent of the choice of the resolution: the former because complex manifolds have rational singularities, the latter by [MOP18, Lem. 1.1]. We may therefore assume that we are in the setting described in Section 7, and use the notation introduced there.

**Reduction to a statement about $M_X$.** We have already done pretty much all the necessary work during the proof of Theorem 1.8, and so we shall be very brief. As in the proof of Theorem 1.8, we have an isomorphism

$$R^{n-1}f_*\Omega^1_X(\log E) \cong \mathcal{H}^0 gr^{-1}_F DR(N_0).$$

Corollary 9.9 provides us with an exact sequence

$$\mathcal{H}^0 gr^{-1}_F DR(M_X) \to \mathcal{H}^0 gr^{-1}_F DR(N_0) \to \bigoplus_{i \in I} R^{n-1}f_*\mathcal{O}_{E_i}.$$

The assumption that $R^{n-1}f_*\mathcal{O}_E = 0$ yields $R^{n-1}f_*\mathcal{O}_{E_i} = 0$ for every $i \in I$, because $\mathcal{O}_{E_i}$ is a quotient of $\mathcal{O}_{\tilde{X}}$. To prove Theorem 1.9, it will therefore suffice to prove the vanishing of $\mathcal{H}^0 gr^{-1}_F DR(M_X)$, and this is what we will do next.

**End of proof.** Recall from (8.0.3) that $\mathcal{H}^{-1} gr^F_0 DR(M_X) \cong R^{n-1}f_*\mathcal{O}_X$, which vanishes by assumption. As in the proof of Lemma 6.8, consider the short exact sequence of complexes

$$0 \to F^{-1} DR(M_X) \to F_0 DR(M_X) \to gr^F_0 DR(M_X) \to 0,$$

and the associated sequence of cohomology sheaves

$$\cdots \to \mathcal{H}^{-1} gr^F_0 DR(M_X) \to \mathcal{H}^0 F^{-1} DR(M_X) \to \mathcal{H}^0 F_0 DR(M_X) \to \cdots,$$

$=$ 0 by ass.


$=$ 0 by Cor. 5.3

to see that $\mathcal{H}^0 F^{-1} DR(M_X) = 0$. Next, we look at the sequence

$$0 \to F^{-2} DR(M_X) \to F^{-1} DR(M_X) \to gr^F_{-1} DR(M_X) \to 0$$

and its cohomology,

$$\cdots \to \mathcal{H}^0 F^{-1} DR(M_X) \to \mathcal{H}^0 gr^F_{-1} DR(M_X) \to \mathcal{H}^1 F^{-2} DR(M_X) \to \cdots,$$

$=$ 0 = 0, since concentr. in non-pos. degrees

to conclude the proof.

\[\square\]

14. Pull-back, proof of Theorem 1.10

As promised in Section 1.4, the following result specifies the “natural universal properties” mentioned in Theorem 1.10. With Theorem 1.2 at hand, the proof is almost identical to the proof given in [Keb13b] for spaces with klt singularities.

**Theorem 14.1** (Functorial pull-back for reflexive forms). Let RSing be the category of complex spaces with rational singularities, where morphisms are simply the holomorphic mappings. Then, there exists a unique contravariant functor,

$$d_{\text{rat}} : \text{RSing} \to \text{(C-vector spaces)},$$

$$X \mapsto H^0(X, \Omega^1_X)$$

(14.1.1)
that satisfies the following "compatibility with Kähler differentials". If \( f : Z \to X \) is any morphism in RSing such that the open set \( Z^\circ := Z_{\text{reg}} \cap f^{-1}(X_{\text{reg}}) \) is not empty, then there exists a commutative diagram

\[
\begin{array}{ccc}
H^0(X, \Omega^p_X) & \xrightarrow{\text{res} \cdot f} & H^0(Z, \Omega^p_Z) \\
\text{restriction}_X & & \text{restriction}_Z \\
H^0(X_{\text{reg}}, \Omega^p_{X_{\text{reg}}}) & \xrightarrow{d_{\text{Kähler}}(f|_{Z^\circ})} & H^0(Z^\circ, \Omega^p_{Z^\circ}),
\end{array}
\]

where \( d_{\text{Kähler}}(f|_{Z^\circ}) \) denotes the usual pull-back of Kähler differentials, and where \( d_{\text{Kähler}}(f|_{Z^\circ}) \) denotes the usual pull-back of Kähler differentials, and \( d_{\text{rel}} f \) denotes the linear map of complex vector spaces induced by the contravariant functor (14.1.1).

The universal properties spelled out in Theorem 14.1 above have a number of useful consequences that we briefly mention. Again, statements and proof are similar to the algebraic, klt case. To avoid repetition, we merely mention those consequences and point to the paper [Keb13b] for precise formulations and proofs.

**Fact 14.2** (Additional properties of pull-back, [Keb13b, §5]). The pull-back functor of Theorem 14.1 has the following additional properties.

(14.2.1) Compatibility with open immersions, [Keb13b, Prop. 5.6].
(14.2.2) Compatibility with Kähler differentials for morphisms to smooth targets varieties, [Keb13b, Prop. 5.7].
(14.2.3) Induced pull-back morphisms at the level of sheaves, [Keb13b, Cor. 5.10].
(14.2.4) Compatibility with wedge products and exterior derivatives, [Keb13b, Prop. 5.13].

\( \square \)

14.1. **Sketch of proof for Theorem 14.1.** For quasi-projective varieties with klt singularities, the result has already been shown in [Keb13b, Thm. 5.2]. If \( X \) is a complex space with arbitrary rational singularities, the proof given in [Keb13b] applies with minor modifications once the following obvious adjustments are made.

- Replace all references to the extension theorem [GKKP11, Thm. 1.4], which works for klt spaces only, by references to Theorem 1.2, which also covers the case of rational singularities.
- Equation [Keb13b, (6.10.5)] is shown for klt spaces using Hacon-McKernan’s solution of Shokurov’s rational connectivity conjecture. However, it has been shown by Namikawa, [Nam01, Lem. 1.2], that the equation holds more generally, for arbitrary complex spaces with rational singularities.
- If \( X \) in RSing is a complex space that does not necessarily carry an algebraic structure, then one also needs to modify the proof of [Keb13b, Lem. 6.15], replacing the reference to [GKK10, Cor. 2.12(ii)] by its obvious generalisation to complex spaces.

For the convenience of the reader, we include a sketch of proof that summarises the main ideas and simplifies [Keb13b] a little. Let \( f : Z \to X \) be any holomorphic map between normal complex spaces with rational singularities. Given any \( \sigma \in H^0(X, \Omega^p_X) \), we explain the construction of an appropriate pull-back form \( \tau \in H^0(Z, \Omega^p_Z) \) and leave it to the reader to check that this \( \tau \) is independent of the choices made, and satisfies all required properties.

**Step 1.** To find a reflexive form \( \tau \in H^0(Z, \Omega^p_Z) \), it is equivalent to find a big, open subset \( Z^\circ \subseteq Z_{\text{reg}} \) and an honest form \( \tau^\circ \in H^0(Z^\circ, \Omega^p_{Z^\circ}) \). We can therefore assume from the outset that \( Z \) is smooth. Next, let \( T := f(Z) \) denote the Zariski closure of the image, and
let $\tilde{T}$ be a desingularisation. The morphism $f$ factors as

$$
\begin{array}{ccc}
Z & \xrightarrow{\text{meromorphic}} & \tilde{T} \\
\downarrow & & \downarrow \text{desingularisation} \\
&& T \\
& \xrightarrow{\text{inclusion}} & X
\end{array}
$$

Now, if we can find an appropriate pull-back form $\tau_{\tilde{T}} \in H^0(\tilde{T}, \Omega^p_{\tilde{T}})$, we could use the standard fact [Pet94, Rem. 1.8(1)] that the meromorphic map $Z \to \tilde{T}$ is well-defined on a big, Zariski-open subset of $Z$ to find the desired form $\tau$ by pulling back. Replacing $Z$ by $\tilde{T}$, if need be, we may therefore assume without loss of generality that $Z$ is smooth and that the image $T := f(Z)$ is closed in Zariski topology.

**Step 2.** Next, choose a desingularisation $\pi: \tilde{X} \to X$ such that $E := \text{supp} \pi^{-1}(T)$ is an snc divisor. We will then find a Zariski open subset $T^o \subseteq T_{\text{reg}}$ with preimage $E^o := \text{supp} \pi^{-1}(T^o)$ such that $T^o \to T^o$ is relatively snc. The assumption that $X$ has rational singularities is used in the following claim$^3$.

**Claim 14.3.** If $t \in T^o$ is any point with fibre $E_t := \text{supp} \pi^{-1}(t)$, then

$$H^0 \left( E_t, \frac{\Omega^p_{E_t}}{\text{tor}} \right) = 0.$$  

**Proof of Claim 14.3.** In case where $E_t \subset \tilde{X}$ is a divisor, this is a result of Namikawa, [Nam01, Lem. 1.2]. If $E_t$ is not a divisor, we can blow up and apply Namikawa’s result upstairs. The claim then follows from the elementary fact that sheaves of “Kähler differentials modulo torsion” have good pull-back properties, [Keb13b, §2.2]. □ (Claim 14.3)

**Step 3.** Again using that $X$ has rational singularities, Theorem 1.2 yields a form $\tau_{\tilde{X}} \in H^0(\tilde{X}, \Omega^p_{\tilde{X}})$. The following claim asserts that its restriction to $E^o$ comes from a form $\tau_{T^o}$ on $T^o$.

**Claim 14.4.** There exists a unique differential form $\tau_{T^o} \in H^0(T^o, \Omega^p_{T^o})$ such that $\tau_{\tilde{X}}|_{E^o}$ and $d_{\text{Kähler}}(\pi |_{E^o})(\tau_{T^o})$ agree up to torsion.

**Proof of Claim 14.4.** Almost immediate from Claim 14.3 and standard relative differential sequences for sheaves of Kähler differentials modulo torsion, [Keb13b, Prop. 3.11]. □ (Claim 14.4)

Pulling the form $\tau_{T^o}$ back to $Z^o := f^{-1}(T^o)$, we find a form $\tau^o$ on the open set $Z^o := f^{-1}(T^o)$, which is a non-empty subset of $Z$ since $T := f(Z)$ is closed in Zariski topology, but need not be big. We leave it to the reader to follow the arguments in [Keb13b, §6 and 7] to see that this $\tau^o$ extends to a form $\tau$ on all of $Z$ that it is independent of the choices made and satisfies all required properties. □

**Appendix A. Weakly rational singularities**

**A.1. Definition and examples.** Let $X$ be a normal complex space. The main result of this paper asserts that if top-forms on $X_{\text{reg}}$ extend to regular top-forms on one desingularisation, then the same will hold for reflexive $p$-forms, for all values of $p$ and all desingularisations. Spaces whose top-forms extend therefore seem to play an important role. We refer to them as spaces with weakly rational singularities and briefly discuss their main properties in this appendix.

---

$^3$The paper [Keb13b] uses Hacon-McKernan’s solution of Shokurov’s rational connectivity conjecture and the more involved technique “projection to general points of $T$” to prove this result.
Definition A.1 (Weakly rational singularities). Let $X$ be a normal complex space. We say that $X$ has weakly rational singularities if the Grauert-Riemenschneider sheaf $\omega_X^{GR}$ is reflexive. In other words, $X$ has weakly rational singularities if for every (equivalently: one) resolution of singularities, $r: \tilde{X} \to X$, the sheaf $r_*\omega_{\tilde{X}}$ is reflexive. We say that a variety has weakly rational singularities if its underlying complex space does.

Example A.2 (Rational singularities). Recall from Section 1.1.4 that rational singularities are weakly rational. For a concrete example, let $X$ be the affine cone over a Fano manifold $Y$ with conormal bundle $L := \omega_Y^{-1}$, as discussed in [Kol13, §8.8]. By [Kol13, Prop. 3.13], this implies that $X$ has rational singularities because $L^m$ is the tensor product of $\omega_Y$ with the ample line bundle $\omega_Y^{-1} \otimes L^m$. A perhaps more surprising example is that any affine cone over an Enriques surface has rational singularities.

Example A.3 (Varieties with small resolutions). If a normal complex space $X$ admits a small resolution, then $X$ has weakly rational singularities. For a concrete example of a non-rational singularity of this form, consider an elliptic curve $E$ and a very ample line bundle $L \in \text{Pic}(E)$. Let $\tilde{X} \to E$ be the total space of the vector bundle $L^{-1} \oplus L^{-1}$ and identify $E$ with the zero-section in $\tilde{X}$. We claim that there exists a normal, affine variety $X$ and a birational morphism $r: \tilde{X} \to X$ that contracts $E \subset \tilde{X}$ to a normal point $x \in X$ and is isomorphic elsewhere. An elementary computation shows that $R^1r_*\mathcal{O}_{\tilde{X}} \neq 0$, so $X$ does not have rational singularities.

To construct the contraction in detail, one might either invoke [AT82, Thm. 3 on p. 59], or argue directly as follows. Write $\mathcal{L}$ for the sheaf of holomorphic sections in $L$ and consider the nef, locally free sheaf $\mathcal{E} := \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{O}_E$. The space $\mathbb{P}(\mathcal{E})$ is a natural compactification of $\tilde{X}$, the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef and big on $\mathbb{P}(\mathcal{E})$, and its restriction to $\tilde{X}$ is trivial. We can therefore identify sections in $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ with functions on $\tilde{X}$, set

$$X := \text{Spec} \bigoplus_{m \in \mathbb{N}} H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m))$$

and obtain the desired map $r: \tilde{X} \to X$. Denoting the ideal sheaf of $E \subset V$ by $\mathcal{J}_E$ and the $m$th infinitesimal neighbourhood of $E$ in $\tilde{X}$ by $E_m$, the cohomology of the standard sequence

$$0 \to \mathcal{J}^m_E \to \mathcal{J}^{m+1}_E \to \mathcal{O}_{E_m} \to \mathcal{O}_{E_{m-1}} \to 0$$

then shows that the restrictions $H^1(E, \mathcal{O}_{E_m}) \to H^1(E, \mathcal{O}_{E_{m-1}})$ are isomorphic for all $m$, so that

$$(R^1r_*\mathcal{O}_{\tilde{X}})_x \cong \lim_{\to} H^1(E, \mathcal{O}_{E_m}) \neq 0,$$

as required.

Perhaps somewhat counter-intuitively, there are example of log-canonical varieties $X$ whose singularities are weakly rational but not rational. If $K_X$ is Cartier and $\omega_X$ is locally generated by one element, this can of course not happen, so that the canonical divisors of the examples will never be Cartier.

Example A.4 (Some log canonical singularities are weakly rational, not rational). To start, let $E$ be a smooth projective variety of positive irregularity whose canonical divisor is torsion, but not linearly trivial. Let $L \in \text{Pic}(E)$ be very ample, and let $X$ be the affine cone over $E$ with conormal bundle $L$. By [Kol13, §3.8], $X$ is log canonical and does not have rational singularities. Yet, Proposition B.2 asserts that the singularities of $X$ are weakly rational.

For a concrete example, let $S$ be a K3 surface obtained as a double cover of the projective plane branched along a non-singular degree six. Observe that the Galois involution $\sigma \in$
Aut(S) acts non-trivially on $H^0(S, \omega_S) \cong \mathbb{C}$. Let $C$ be an elliptic curve, and let $r \in \text{Aut}(C)$ be a translation by a torsion element of degree two, so that $r$ is again an involution. Consider the involution $(\sigma, r) \in \text{Aut}(S \times C)$, which is fixed point free, and choose $E$ to be the quotient, $E := (S \times C)/\mathbb{Z}_2$. The threefold $E$ admits no global top-form by choice of $\sigma$, and has positive irregularity since it admits a morphism to the elliptic curve $C/\mathbb{Z}_2$.

**Remark A.5** (Incompatible definitions in the literature). There already exists a notion of "weakly rational" in the literature. Andreotti-Silva [AS84] call a variety $X$ "weakly rational" in the literature. Andrea/ta-Silva call a variety $X$ to be a translation by a torsion element of degree two, so that $\tau$ is again an involution. Consider the involution $X$ be a quasi-projective variety with weakly rational singularities, let $L \in \text{Pic}(X)$ be a line bundle and $L \subseteq |L|$ be a finite-dimensional, basepoint free linear system whose general member is connected. Then, there exists a dense, Zariski-open subset $L^o \subseteq L$ such that any hyperplane $H \in L^o$ has weakly rational singularities, and satisfies the adjunction formula

\begin{equation}
\omega^\text{GR}_H \cong \omega^\text{GR}_X \otimes \mathcal{O}_X(H) \otimes \mathcal{O}_H.
\end{equation}

**Proposition A.6** (Stability under general hyperplane sections). Let $X$ be a quasi-projective variety with weakly rational singularities, let $L \in \text{Pic}(X)$ be a line bundle and $L \subseteq |L|$ be a finite-dimensional, basepoint free linear system whose general member is connected. Then, there exists a dense, Zariski-open subset $L^o \subseteq L$ such that any hyperplane $H \in L^o$ has weakly rational singularities, and satisfies the adjunction formula

\begin{equation}
\text{(A.6.1)}
\omega^\text{GR}_H \cong \omega^\text{GR}_X \otimes \mathcal{O}_X(H) \otimes \mathcal{O}_H.
\end{equation}

**Proof.** Choose a resolution of singularities, $r: \tilde{X} \to X$. There exists a dense, Zariski-open $L^o \subseteq L$ such that any hyperplane $H \in L^o$ satisfies the following properties.

\begin{enumerate}
\item[(A.6.2)] The hypersurface $H$ is normal, connected and $H_{\text{sing}} = X_{\text{sing}} \cap H$: Seidenberg’s theorem, [Sei50], and the fact that a variety is smooth along a Cartier divisor if the divisor itself is smooth.
\item[(A.6.3)] The preimage $\tilde{H} := r^{-1}H$ is smooth: Bertini’s theorem.
\item[(A.6.4)] The restriction $\omega^\text{GR}_{\tilde{H}}$ is reflexive: [Gro66, Thm. 12.2.1].
\end{enumerate}

We claim that the adjunction formula (A.6.1) holds for $H$, which together with (A.6.4) implies that $H \in L^o$ has weakly rational singularities. The setup is summarised in the following diagram

\[
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\text{r}} & \tilde{X} \\
\downarrow & & \downarrow \\
H & \xrightarrow{\text{r}} & X
\end{array}
\]

We obtain an adjunction morphism,

\begin{equation}
\text{(A.6.5)}
\iota^*(\omega^\text{GR}_X(H)) \cong \iota^* r_!(\omega^\text{GR}_{\tilde{H}}(\tilde{H}))
\end{equation}

Projection formula

\begin{equation}
\rightarrow (r_H)_! \iota^* (\omega^\text{GR}_{\tilde{H}}(\tilde{H}))
\end{equation}

Cohomology and base change

\begin{equation}
\cong (r_H)_! \omega^\text{GR}_{\tilde{H}} \cong \omega^\text{GR}_H
\end{equation}

Adjunction and smoothness of $\tilde{H}$

which is clearly an isomorphism over the big open subset of $H$ where $H$ and $X$ are both smooth. More can be said. Item (A.6.4) implies that the left hand side of (A.6.5) is reflexive, while the right hand side of (A.6.5) is a push forward of a torsion free sheaf, hence torsion free. As a morphism from a reflexive to a torsion free sheaf that is isomorphic in codimension one, the adjunction morphism must then in fact be isomorphic. \qed
As a second positive result, we show that images of weakly rational singularities under arbitrary finite morphisms are again weakly rational. This can be seen as an analogue of the fact that quotients of rational singularities under the actions of finite groups are again rational.

**Proposition A.7** (Stability under finite quotients). Let \( \gamma : X \to Y \) be a proper, surjective morphism between normal complex spaces. Assume that \( \gamma \) is finite, or that it bimeromorphic and small. If \( X \) has weakly rational singularities, then so does \( Y \).

**Proof.** The case of a small morphism is rather trivial, so we consider finite morphisms only. We assume without loss of generality \( Y \) is Stein. Let \( r_Y : \bar{Y} \to Y \) be a log-resolution, with exceptional set \( E \subset \bar{Y} \).

Since \( Y \) is Stein, to prove that \( Y \) has weakly rational singularities, it suffices to show that for any given section \( \sigma \in H^0(\bar{Y}, \omega_{\bar{Y}}) \), the associated rational form \( \tilde{\sigma} \) on \( \bar{X} \), might a priori have poles along \( E \), does in fact not have any poles. To this end, let \( \bar{X} \) be a strong resolution of the normalised fibre product \( X \times_Y \bar{Y} \). The following diagram summarises the situation:

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\Gamma, \text{generically finite}} & \bar{Y} \\
\downarrow r_X, \text{desing.} & & \downarrow r_Y, \text{desing.} \\
X & \xrightarrow{\gamma, \text{finite}} & Y.
\end{array}
\]

Set \( F := \text{supp} \Gamma^{-1}E \) and consider the rational differential form \( \tau \) on \( \bar{X} \), which might a priori have poles along \( F \). Since \( \Gamma \) is generically finite, [GKK10, Cor. 2.12(ii)] applies\(^4\) to show that \( \tilde{\sigma} \) is without poles along \( E \) if and only if \( \tau \) is without poles along \( F \), or more precisely: without poles along those components of \( F \) that dominate components of \( E \).

To show that \( \tilde{\tau} \) has no pole indeed, observe that finiteness of \( \gamma \) and reflexivity of \( \omega_X \) imply that there exists a section \( \tau \in H^0(X, \omega_X) \) that agrees with \( d\gamma(\sigma) \) wherever \( X \) and \( Y \) are smooth. The assumption that \( X \) has weakly rational singularities will then give a regular differential form on \( \bar{X} \), without poles, that agrees with \( dr_X(\tau) \) wherever \( X \) is smooth. This form clearly equals \( \tilde{\tau} \). \( \square \)

A.2.2. **Negative results.** In spite of the positive results above, the following examples show that the class of varieties with weakly rational singularities does not remain invariant when taking quasi-étales covers or special hyperplane sections, even in the simplest cases.

**Example A.8** (Instability under special hyperplane sections). Grauert-Riemenschneider construct a normal, two-dimensional, isolated hypersurface singularity where \( \omega_X \) is not reflexive, [GR70, p. 280f]. In particular, \( X \) does not have weakly rational singularities and a naive adjunction formula for the Grauert-Riemenschneider sheaf as in (A.6.1) does not hold in this case.

**Example A.9** (Instability under quasi-étales covers). Any cone \( Y \) over an Enriques surface has rational singularities and admits a quasi-étales cover by a cone \( X \) over a K3 surface, which is Cohen-Macaulay, but does not have rational singularities, [Kol13, Ex. 3.6]. As we saw in Section 1.1.4, this implies that \( X \) does not have weakly rational singularities. We obtain examples of quasi-étales maps \( X \to Y \) between isolated, log-canonical singularities where \( Y \) is weakly rational while \( X \) is not.

---

\(^4\)The reference [GKK10] works in the algebraic setting. However, the result quoted here (and its proof) will also be true for complex spaces.
Cones over projective manifolds are a useful class of examples to illustrate how the extension problem for $p$-forms is related to the behaviour of the canonical sheaf. We follow the notation introduced in Kollár’s book [Kol13] and work in the following setting.

**Setting B.1 (Cones over projective manifolds, compare [Kol13, §3.1])**. Fix a number $n \geq 2$ and a smooth projective variety $Y$ of dimension $\dim Y = n - 1$, together with an ample line bundle $L \in \text{Pic}(Y)$. Following [Kol13, §3.8], we define the affine cone over $Y$ with conormal bundle $L$ as the affine algebraic variety

$$X := \text{Spec} \bigoplus_{m \geq 0} H^0(Y, L^m)$$

The ring is finitely generated since $L$ is ample. The variety $X$ is normal of dimension $n$ and smooth outside of the vertex $\tilde{v}$, which is the point corresponding to the zero ideal. Unless $Y = \mathbb{P}^{n-1}$ and $L = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, the vertex will always be an isolated singular point.

Since $Y$ is smooth, the partial resolution of singularities constructed in [Kol13, §3.8], say $r: \tilde{X} \to X$, is in fact a log resolution of singularities. The variety $\tilde{X}$ is isomorphic to the total space of the line bundle $L^{-1}$ and the $r$-exceptional set $E \subseteq \tilde{X}$ is identified with the zero-section of that bundle.

**Figure B.1.** Cone over a smooth variety

Note. The definition is motivated by the geometric construction of cones, as illustrated in Figure B.1. Suppose that $Y$ is a submanifold of $\mathbb{P}^d$. The affine cone over $Y$, with vertex the origin in $\mathbb{C}^{d+1}$, is the union of all the lines in $\mathbb{C}^{d+1}$ corresponding to the points of $Y$. Its coordinate ring is the graded $\mathbb{C}$-algebra

$$\mathbb{C}[x_0, x_1, \ldots, x_d]/I_Y,$$

where $I_Y$ is the homogeneous ideal of $Y$. The affine cone is not always normal, but it is easy to see that the coordinate ring of its normalisation is $\text{Spec} R$, where $R$ is the section ring of the very ample line bundle $\mathcal{O}_Y(1)$. Our definition is slightly more general, because $L$ is only assumed to be ample.

**B.1. Extension of differential forms.** Now we turn out attention to the extension problem for differential forms. The following result can be summarised very neatly by saying that if $n$-forms extend, then $p$-forms extend for every $0 \leq p \leq n$.

**Proposition B.2** (Extension of differential forms on cones). Assume Setting B.1. Then, $p$-forms extend for all $p \leq n - 2$. The following equivalences hold in addition.

(B.2.1) $$(n-1)\text{-forms extend } \iff H^0(Y, \omega_Y \otimes L^{-m}) = 0, \forall m \geq 1.$$  

(B.2.2) $$n\text{-forms extend } \iff H^0(Y, \omega_Y \otimes L^{-m}) = 0, \forall m \geq 0.$$
Proof. Since \( \widetilde{X} \setminus E \) is isomorphic to \( X_{\text{reg}} \), the question is simply under what conditions on \( Y \) and \( L \) the restriction mapping

\[
H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) \to H^0(\widetilde{X} \setminus E, \Omega^p_{\widetilde{X}})
\]

is an isomorphism for different values of \( p \in \{0, 1, \ldots, n\} \). We use the identification of \( \widetilde{X} \) with the total space of the line bundle \( L^{-1} \) and denote the projection by \( q: \widetilde{X} \to Y \). The sequence of differentials and the sequence of \( p \)th exterior powers now read as follows,

\[
0 \to q^* \Omega^1_Y \to \Omega^1_{\widetilde{X}} \to q^* L \to 0 \quad \text{and} \quad 0 \to q^* \Omega^p_Y \to \Omega^p_{\widetilde{X}} \to q^*(\Omega^p_{Y} \otimes L) \to 0.
\]

Now both \( q: \widetilde{X} \to Y \) and its restriction \( q|_{\widetilde{X} \setminus E} \) are affine, and

\[
q_* \mathcal{O}_{\widetilde{X}} \cong \bigoplus_{m \geq 0} L^m \quad \text{and} \quad (q|_{\widetilde{X} \setminus E})_* \mathcal{O}_{\widetilde{X} \setminus E} \cong \bigoplus_{m \in \mathbb{Z}} L^m.
\]

We therefore obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & \bigoplus_{m \geq 0} H^0(Y, \Omega^1_Y \otimes L^m) & \to & H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) & \to & H^0(\widetilde{X} \setminus E, \Omega^1_{\widetilde{X}}) & \to & 0 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & \bigoplus_{m \in \mathbb{Z}} H^0(Y, \Omega^p_Y \otimes L^m) & \to & H^0(\widetilde{X} \setminus Y, \Omega^p_{\widetilde{X}}) & \to & H^0(\widetilde{X} \setminus E, \Omega^p_{\widetilde{X}}) & \to & 0
\end{array}
\]

Consider the first vertical arrow, labelled \( \alpha \), in the commutative diagram above. By the Nakano vanishing theorem, we have \( H^0(Y, \Omega^p_Y \otimes L^m) = 0 \) for \( m \leq -1 \) and \( p \leq \dim Y - 1 \), and so \( \alpha \) is an isomorphism if and only if

\[
\text{(B.2.3)} \quad H^0(Y, \omega_Y \otimes L^m) = 0, \quad \forall m \leq -1.
\]

Consider next the third vertical arrow, labelled \( \beta \), in the commutative diagram. For the same reason as before, we have \( H^0(Y, \Omega_Y^{p-1} \otimes L^m) = 0 \) for \( m \leq -1 \) and \( p - 1 \leq \dim Y - 1 \). For \( m = 0 \), the horizontal arrow

\[
H^0(Y, \Omega_Y^{p-1}) \to H^1(Y, \omega_Y^p)
\]

in the second row is cup product with the first Chern class of the ample line bundle \( L \); by the Hard Lefschetz Theorem, it is injective as long as \( p - 1 \leq \dim Y - 1 \). Consequently, \( \beta \) is an isomorphism if and only if

\[
\text{(B.2.4)} \quad H^0(Y, \omega_Y \otimes L^m) = 0, \quad \forall m \leq 0.
\]

The conclusion is that \( p \)-forms extend for \( p \leq n - 2 \) without any extra assumptions on \( (Y, L) \); since the cone over \( (Y, L) \) has an isolated singularity at the vertex, this is consistent with the result by Steenbrink and van Straten [vSS85, Thm. 1.3]. Moreover, \( (n - 1) \)-forms extend iff the condition in (B.2.3) is satisfied, and \( n \)-forms extend iff the condition in (B.2.4) is satisfied.

\[\square\]

B.2. Characterisation of standard singularity types. The following summary of several well-known results relates different classes of singularities to properties of the line bundle \( L \), in particular to the vanishing of higher cohomology for \( L \) and its powers.

**Proposition B.3** (Classes of singularities on cones). Assume Setting B.1. Then, the following equivalences hold.

\[
\begin{align*}
\text{(B.3.1)} & \quad X \text{ has rational singularities} \iff H^i(Y, L^m) = 0, \forall i > 0, \forall m \geq 0. \\
\text{(B.3.2)} & \quad X \text{ has Du Bois singularities} \iff H^i(Y, L^m) = 0, \forall i > 0, \forall m > 0. \\
\text{(B.3.3)} & \quad X \text{ is Cohen-Macaulay} \iff H^i(Y, L^m) = 0, \forall \dim Y > i > 0, \forall m \geq 0.
\end{align*}
\]
The singularity types of the minimal model program are described as follows.

(B.3.4) \( X \) is \( \mathbb{Q} \)-Gorenstein \( \iff \exists m : K_Y \sim_{\mathbb{Q}} L^m. \)
(B.3.5) \( X \) is klt \( \iff \exists m < 0 : K_Y \sim L^m. \)
(B.3.6) \( X \) is log canonical \( \iff \exists m \leq 0 : K_Y \sim L^m. \)

Proof. See [Kol13, Lem. 3.1, Cor. 3.11, Prop. 3.13 and Prop. 3.14] and [GK14, Thm 2.5]. \( \square \)

Comparing Proposition B.2 and B.3, we find that the extension property of \( p \)-forms is a comparatively mild condition on \( Y, L \). It is not as cohomological in nature as "rational", "Du Bois" and "Cohen-Macaulay", and certainly not nearly as restrictive as being klt, which only happens in the special case where \( Y \) is a Fano manifold and \( L \) is \( \mathbb{Q} \)-linearly equivalent to a positive multiple of \( -K_Y \). This suggests looking for an extension theorem that goes beyond the class of singularities used in the Minimal Model Program.

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Appendix

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