EXTENDING HOLOMORPHIC FORMS FROM THE REGULAR LOCUS OF A COMPLEX SPACE TO A RESOLUTION OF SINGULARITIES

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Abstract. We investigate under what conditions holomorphic forms defined on the regular locus of a reduced complex space extend to holomorphic (or logarithmic) forms on a resolution of singularities. We give a simple necessary and sufficient condition for this, whose proof relies on the Decomposition Theorem and Saito's theory of mixed Hodge modules. We use it to generalize the theorem of Greb-Kebekus-Kovács-Peternell to complex spaces with rational singularities, and to prove the existence of a functorial pull-back for reflexive differentials on such spaces. We also use our methods to settle the "local vanishing conjecture" proposed by Mustaţă, Olano, and Popa.

Contents

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1. Overview of the paper

1.1. **Extension of holomorphic forms.** This paper is about the following "extension problem" for holomorphic differential forms on complex spaces. Let X be a reduced complex space, and let $r\colon \widetilde{X} \to X$ be a resolution of singularities. Which holomorphic p-forms on the regular locus X_{reg} extend to holomorphic p-forms on the complex manifold $\widetilde{X}$? Standard facts about resolution of singularities imply that the answer is independent of the choice of resolution. (If the exceptional locus of r is a normal crossing divisor E, one can also ask for an extension with at worst logarithmic poles along E.)

The best existing result concerning this problem is due to Greb, Kebekus, Kovács, and Peternell [GKKP11, Thm. 1.4]. They show that if X underlies a normal algebraic variety with Kawamata log terminal (=klt) singularities, then all p-forms on X_{reg} extend to $\widetilde{X}$, for every $0 \le p \le \dim X$. Their theorem has many applications, including hyperbolicity of moduli, the structure of minimal varieties with trivial canonical class, the nonabelian Hodge correspondence for singular spaces, and quasi-étale uniformisation. Section 1.7 recalls some of these in more detail and gives references.

In this paper, we use the Decomposition Theorem and Saito's theory of mixed Hodge modules to solve the extension problem in general. Our main result is a simple necessary and sufficient condition for a holomorphic p-form on X_{reg} to extend to a holomorphic (or logarithmic) p-form on $\widetilde{X}$. One surprising consequence is that the extension problem for forms of a given degree also controls what happens for forms of smaller degrees. Another consequence is that if X is a complex space with rational singularities, then all p-forms on X_{reg} extend to $\widetilde{X}$, for every $0 \le p \le \dim X$. This result is a crucial step in the recent work of Bakker and Lehn [BL18] on the global moduli theory of symplectic varieties.

1.2. **Main result.** Let X be a reduced complex space of pure dimension n. It is well-known that a holomorphic n-form $\alpha \in H^0(X_{\mathrm{reg}}, \Omega^n_X)$ extends to a holomorphic n-form on any resolution of singularities of X if and only if $\alpha \wedge \overline{\alpha}$ is locally integrable on X. Griffiths [Gri76, §IIa] gave a similar criterion for extension of p-forms in terms of integrals over p-dimensional analytic cycles in X, but his condition is not easy to verify in practice. Our first main result is the following intrinsic description of those holomorphic forms on X_{reg} that extend holomorphically to one (and hence any) resolution of singularities.

Theorem 1.1 (Holomorphic forms)**.** *Let X be a reduced complex space of pure dimension n, and $r\colon \widetilde{X} \to X$ a resolution of singularities. A holomorphic p-form $\alpha \in H^0(X_{\mathrm{reg}}, \Omega^p_X)$ extends to a holomorphic p-form on $\widetilde{X}$ if, and only if, for every open subset $U \subseteq X$, and for every pair of Kähler differentials $\beta \in H^0(U, \Omega^{n-p}_X)$ and $\gamma \in H^0(U, \Omega^{n-p-1}_X)$, the holomorphic n-forms $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ on U_{reg} extend to holomorphic n-forms on $r^{-1}(U)$.*

Our proof of this result is based on the Decomposition Theorem for Hodge modules. We think that it would also be very interesting to have an analytic proof, in terms of L^2-Hodge theory for the $\bar{\partial}$-operator. The following analogue of Theorem 1.1 for forms with logarithmic poles needs some additional results about mixed Hodge modules. Recall that a resolution of singularities $r\colon \widetilde{X} \to X$ of a complex space is called a *(strong) log resolution* if the r-exceptional set is a divisor with (simple) normal crossings on $\widetilde{X}$.

Theorem 1.2 (Logarithmic forms)**.** *Let X be a reduced complex space of pure dimension n, and $r\colon \widetilde{X} \to X$ a log resolution of singularities with exceptional divisor $E \subseteq X$. A holomorphic p-form $\alpha \in H^0(X_{\mathrm{reg}}, \Omega^p_X)$ extends to a holomorphic section of the bundle $\Omega^p_{\widetilde{X}}(\log E)$ on $\widetilde{X}$ if, and only if, for every open subset $U \subseteq X$, and for every pair of Kähler differentials $\beta \in H^0(U, \Omega^{n-p}_X)$ and $\gamma \in H^0(U, \Omega^{n-p-1}_X)$, the holomorphic n-forms $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ on U_{reg} extend to holomorphic sections of the bundle $\Omega^n_{\widetilde{X}}(\log E)$ on $r^{-1}(U)$.*

1.3. **Consequences.** The extension problem for holomorphic (or logarithmic) forms on a complex space X is of course closely related to the singularities of X. Since there might not be any global p-forms on X_{reg}, the effect of the singularities is better captured by the following local version of the problem. Given a resolution of singularities $r\colon \widetilde{X} \to X$ of a reduced complex space X, and an arbitrary open subset $U \subseteq X$, which holomorphic p-forms on U_{reg} extend to holomorphic p-forms on $r^{-1}(U)$? If $j\colon X_{\mathrm{reg}} \hookrightarrow X$ denotes the embedding of the regular locus, this amounts to asking for a description of the subsheaf $r_*\Omega^p_{\widetilde{X}} \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$. This subsheaf is $\mathscr{O}_X$-coherent (by Grauert's theorem) and independent of the choice of resolution (because any two resolutions are dominated by a common third). If the exceptional locus of r is a normal crossing divisor E, one can also ask for a description of the subsheaf $r_*\Omega^p_{\widetilde{X}}(\log E) \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$, which has similar properties.

Example 1.3. When X is reduced and irreducible, it is easy to see that $r_*\mathscr{O}_{\widetilde{X}} \hookrightarrow j_*\mathscr{O}_{X_{\mathrm{reg}}}$ is an isomorphism if and only if $\dim X_{\mathrm{sing}} \leq \dim X - 2$. (Use the normalisation of X.)

One consequence of Theorem 1.1 is that the extension problem for holomorphic forms of a given degree also controls what happens for all forms of smaller degree.

Theorem 1.4 (Extension for p-forms). *Let X be a reduced and irreducible complex space. Let $r\colon \widetilde{X} \to X$ be any resolution of singularities, and $j\colon X_{\mathrm{reg}} \hookrightarrow X$ the inclusion of the regular locus. If the morphism $r_*\Omega^k_{\widetilde{X}} \hookrightarrow j_*\Omega^k_{X_{\mathrm{reg}}}$ is an isomorphism for some $0 \leq k \leq \dim X$, then $\dim X_{\mathrm{sing}} \leq \dim X - 2$, and $r_*\Omega^p_{\widetilde{X}} \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$ is an isomorphism for every $0 \leq p \leq k$.*

An outline of the proof can be found in Section 2 below. The key idea is to use the Decomposition Theorem [BBD82, Sai88], in order to relate the coherent $\mathscr{O}_X$-module $r_*\Omega^p_{\widetilde{X}}$ to the intersection complex of X, viewed as a polarisable Hodge module. In Appendix B, we look at the example of cones over smooth projective varieties; it gives a hint that the extension problem for all p-forms should be governed by what happens for n-forms. When X is normal, an equivalent formulation of Theorem 1.4 is that, if the coherent $\mathscr{O}_X$-module $r_*\Omega^k_{\widetilde{X}}$ is reflexive for some $k \leq \dim X$, then $r_*\Omega^p_{\widetilde{X}}$ is reflexive for every $p \leq k$.

Note. One can easily generalise Theorem 1.4 to arbitrary reduced complex spaces. The precise (but somewhat cumbersome) statement is that if the morphism $r_*\Omega^k_{\widetilde{X}} \hookrightarrow j_*\Omega^k_{X_{\mathrm{reg}}}$ is an isomorphism for some $k \geq 0$, and if $Z \subseteq X$ denotes the union of all the irreducible components of X of dimension $\geq k$, then $\dim Z_{\mathrm{sing}} \leq k - 2$, and the restriction to Z of the morphism $r_*\Omega^p_{\widetilde{X}} \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$ is an isomorphism for every $0 \leq p \leq k$. The reason is that the irreducible components of X are separated in any resolution of singularities, and so one can simply apply Theorem 1.4 one component at a time.

We also establish a version of Theorem 1.4 with log poles, by adapting the techniques in the proof to a certain class of mixed Hodge modules.

Theorem 1.5 (Extension for log p-forms). *Let X be a reduced and irreducible complex space. Let $r\colon \widetilde{X} \to X$ be a log resolution with exceptional divisor $E \subseteq \widetilde{X}$, and $j\colon X_{\mathrm{reg}} \hookrightarrow X$ the inclusion of the regular locus. If the morphism $r_*\Omega^k_{\widetilde{X}}(\log E) \hookrightarrow j_*\Omega^k_{X_{\mathrm{reg}}}$ is an isomorphism for some $0 \leq k \leq \dim X$, then $\dim X_{\mathrm{sing}} \leq \dim X - 2$, and $r_*\Omega^p_{\widetilde{X}}(\log E) \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$ is an isomorphism for every $0 \leq p \leq k$.*

By a result of Kovács, Schwede, and Smith [KSS10, Thm. 1], a complex algebraic variety X that is normal and Cohen-Macaulay has Du Bois singularities if and only if $r_*\omega_{\widetilde{X}}(E)$ is a reflexive $\mathscr{O}_X$-module for some log resolution $r\colon \widetilde{X} \to X$. We think that it would be interesting to know the precise relationship between Du Bois singularities and the extension problem for logarithmic n-forms. The tools we develop for the proof of Theorem 1.5 also lead to a slightly better answer in the case of holomorphic forms of degree $\dim X - 1$.

Theorem 1.6 (Extension for $(n-1)$-forms)**.** *Let X be a reduced and irreducible complex space. Let $r\colon \widetilde{X} \to X$ be a log resolution with exceptional divisor $E \subseteq \widetilde{X}$, and $j\colon X_{\mathrm{reg}} \hookrightarrow X$ the inclusion of the regular locus. If the natural morphism $r_*\Omega^n_{\widetilde{X}} \hookrightarrow j_*\Omega^n_{X_{\mathrm{reg}}}$ is an isomorphism, where $n = \dim X$, then the two morphisms*

$$r_*\big(\Omega^{n-1}_{\widetilde{X}}(\log E)(-E)\big) \hookrightarrow r_*\Omega^{n-1}_{\widetilde{X}} \hookrightarrow j_*\Omega^{n-1}_{X_{\mathrm{reg}}}$$

are also isomorphisms.

1.4. **Rational and weakly rational singularities.** An important class of singular spaces where Theorem 1.4 applies is normal complex spaces with rational singularities. Recall that X has *rational singularities* if the following equivalent conditions hold. We refer to [KM98, §5.1] for details.

(1.7.1) X is normal, and if $r\colon \widetilde{X} \to X$ is any resolution of singularities, then $R^i r_*\mathscr{O}_{\widetilde{X}} = 0$ for every $i \geq 1$.
(1.7.2) X is Cohen-Macaulay and $\omega_X^{\mathrm{GR}} = \omega_X$.
(1.7.3) X is Cohen-Macaulay and ω_X^{GR} is reflexive.

Here $\omega_X^{\mathrm{GR}} = r_*\omega_{\widetilde{X}}$ is sometimes called the *Grauert-Riemenschneider sheaf*, because it appears in the Grauert-Riemenschneider vanishing theorem. In view of Condition (1.7.3), we say that a normal space X has *weakly rational singularities* if the Grauert-Riemenschneider sheaf ω_X^{GR} is reflexive. With this notation, Theorem 1.4 has the following immediate corollary.

Corollary 1.8 (Extension in the case of rational singularities)**.** *Let X be a normal complex space with weakly rational singularities, and let $r\colon \widetilde{X} \to X$ be a resolution of singularities. Then every holomorphic form defined on X_{reg} extends uniquely to a holomorphic form on $\widetilde{X}$.* □

Remark 1.9. Rational singularities are weakly rational by definition. In particular, recall from [KM98, Thm. 5.22 and references there] that klt spaces have rational (and hence weakly rational) singularities. For algebraic klt varieties, the extension result was shown previously in [GKKP11, Thm. 1.4].

As we will see in Section 6.1, having weakly rational singularities turns out to be equivalent to the collection of inequalities

$$\dim \operatorname{Supp} R^i r_*\mathscr{O}_{\widetilde{X}} \leq \dim X - 2 - i \quad \text{for every } i \geq 1.$$

One can also describe the class of weakly rational singularities in more analytic terms: a normal complex space X of dimension n has weakly rational singularities if and only if, for every open subset $U \subseteq X$ and every holomorphic n-form $\omega \in H^0(U_{\mathrm{reg}}, \Omega^n_{U_{\mathrm{reg}}})$, the (n,n)-form $\omega \wedge \overline{\omega}$ on U_{reg} is locally integrable on all of U. Appendix A discusses examples and establishes elementary properties of this class of singularities.

1.5. **Local vanishing conjecture.** The methods developed in this paper also settle the "local vanishing conjecture" proposed by Mustaţă, Olano, and Popa [MOP20, Conj. A]. The original conjecture contained the assumption that X is a normal algebraic variety with rational singularities. In fact, the weaker assumption $R^{\dim X - 1} r_*\mathscr{O}_{\widetilde{X}} = 0$ is sufficient.

Theorem 1.10 (Local vanishing)**.** *Let X be a reduced and irreducible complex space of dimension n. Let $r\colon \widetilde{X} \to X$ be a log resolution, with exceptional divisor $E \subseteq \widetilde{X}$. If $R^{n-1} r_*\mathscr{O}_{\widetilde{X}} = 0$, then $R^{n-1} r_*\Omega^1_{\widetilde{X}}(\log E) = 0$.*

As shown in [MOP20], this result has interesting consequences for the Hodge filtration on the complement of a hypersurface with at worst rational singularities.

1.6. **Functorial pull-back.** One can interpret Theorem 1.4 as saying that any differential form $\sigma \in H^0(X_{\text{reg}}, \Omega^1_{X_{\text{reg}}}) = H^0(X, \Omega^{[1]}_X)$ induces a *pull-back form* $\widetilde{\sigma} \in H^0(\widetilde{X}, \Omega^1_{\widetilde{X}})$. More generally, we show that pull-back exists for reflexive differentials and arbitrary morphisms between varieties with rational singularities. The paper [Keb13b] discusses these matters in detail.

Theorem 1.11 (Functorial pull-back for reflexive differentials)**.** *Let $f: X \to Y$ be any morphism between normal complex spaces with rational singularities. Write $\Omega^{[p]}_X := (\Omega^p_X)^{**}$, ditto for $\Omega^{[p]}_Y$. Then there exists a pull-back morphism*

$$d_{\text{refl}} f : f^* \Omega^{[p]}_Y \to \Omega^{[p]}_X,$$

uniquely determined by natural universal properties.

We refer to Theorem 14.1 and Section 14 for a precise formulation of the "natural universal properties" mentioned in Theorem 1.11. In essence, it is required that the pull-back morphisms agree with the pull-back of Kähler differentials wherever this makes sense, and that they satisfy the composition law.

Note. Theorem 1.11 applies to morphisms $X \to Y$ whose image is entirely contained in the singular locus of Y. Taking the inclusion of the singular set for a morphism, Theorem 1.11 implies that every differential form on Y_{reg} induces a differential form on every stratum on the singularity stratification.

1.6.1. h-*differentials.* One can also reformulate Theorem 1.11 in terms of h-differentials; these are obtained as the sheafification of Kähler differentials with respect to the h-topology on the category of complex spaces, as introduced by Voevodsky. We refer the reader to [HJ14] and to the survey [Hub16] for a gentle introduction to these matters. Using the description of h-differentials found in [HJ14, Thm. 1], the following is an immediate consequence of Theorem 1.11.

Corollary 1.12 (h-differentials on spaces with rational singularities)**.** *Let X be a normal complex space with rational singularities. Write $\Omega^{[p]}_X := (\Omega^p_X)^{**}$. Then,* h*-differentials and reflexive differentials agree: $\Omega^p_{\text{h}}(X) = \Omega^{[p]}_X(X)$.* □

The sheaf Ω^p_{h} of h-differentials appears under a different name in the work of Barlet, [Bar18], who describes it in analytic terms ("integral dependence equations for differential forms") as a subsheaf of $\Omega^{[p]}_X$ and relates it to the normalised Nash transform.

1.7. **Applications.** The main results of this paper allow to study (sheaves of) reflexive differential forms on singular spaces, by pulling them back to a resolution of singularities. At times, this allows to prove results of Hodge-theoretic flavour in settings where classic Hodge-theory is not readily available. We give two immediate application of Theorem 1.4, which can be proven in just a few lines, following [GKKP11, Sect. 6 and 7] verbatim.

Theorem 1.13 (Closedness of forms and Bogomolov-Sommese vanishing)**.** *Let X be a normal complex projective variety. If ω^{GR}_X is reflexive, then any holomorphic differential form on X_{reg} is closed. If $\mathscr{A} \subseteq \Omega^{[p]}_X$ is an invertible subsheaf, then $\kappa(\mathscr{A}) \le p$.* □

Theorem 1.14 (Lipman-Zariski conjecture for weakly rational complex spaces)**.** *Let X be a normal complex space where ω^{GR}_X is reflexive. If the tangent sheaf $\mathscr{T}_X$ is locally free, then X is smooth.* □

To illustrate the range of applicability, we mention some recent (and much more substantial) results that rely on the previously known extension theorem for klt spaces by Greb, Kebekus, Kovács, and Peternell.

- The standard conjectures of minimal model theory predict that the minimal model of any projective manifold X of Kodaira dimension $\kappa(X) = 0$ is a singular space with vanishing first Chern class. A series of papers [GKP16a, Dru18, GGK19, HP19] extended the classic Beauville-Bogomolov Decomposition Theorem to the singular setting. A recent paper [KLSV18] studies degenerations of hyper-Kähler manifolds, using the minimal model program to reduce any degeneration to (singular) "Kulikov type form".
- A series of papers [GKPT19b, GKPT19c, GKPT19a] extends the classic non-abelian Hodge correspondence from Kähler manifolds to singular spaces. For klt spaces, this relates representations of the fundamental group with (singular) Higgs sheaves and yields new results on quasi-étale uniformisation, [GKP16b, LT18, GKT18].
- The extension results are used analysis in the study of (singular) Kähler-Einstein metrics, [BG14, LT19]
- The extension result is used in holomorphic dynamics and foliations, for the classification of foliations [AD14, AD13], but also in the study of compactifications of Drinfeld half-spaces over a finite field, [Lan19].

Very recent work. Our generalisation of the extension theorem to (possibly non-algebraic) complex spaces with rational singularities (in Corollary 1.8 above) is used in a crucial way in the recent work of Bakker and Lehn [BL18] on global moduli for symplectic varieties, where a "symplectic variety" is a normal Kähler space X with a nondegenerate holomorphic 2-form on X_{reg} that extends holomorphically to any resolution of singularities.

1.8. **Earlier results.** As mentioned above, Theorem 1.4 was already known for spaces with Kawamata log terminal (=klt) singularities, where $r_*\omega_{\widetilde{X}}$ is reflexive by definition [GKK10, GKKP11]. If one is only interested in p-forms of small degree (compared to $\dim X$), there are earlier results of Steenbrink-van Straten [vSS85] and Flenner [Fle88]. In the special case where $p = 1$, Graf-Kovács relate the extension problem to the notion of Du Bois singularities [GK14]. For morphisms between varieties with klt singularities, the existence of a pull-back functor was shown in [Keb13b].

We refer the reader to the paper [GKKP11] or to the survey [Keb13a, §4] for a more detailed introduction, and for remarks on the history of the problem. The book [Kol13, §8.5] puts the results into perspective.

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2. Techniques and main ideas

In this section, we sketch some of the ideas that go into the proof of Theorem 1.4. The one-line summary is that it is a consequence of the Decomposition Theorem [BBD82, Sai90]. Appendix B contains a short section on cones over projective manifolds that illustrates the extension problem in a particularly transparent case and explains why one might even expect a result such as Theorem 1.4 to hold true.

2.1. **First proof of Theorem 1.4.** We actually give two proofs for Theorem 1.4. The first proof (in Section 11) relies on Theorem 1.1, which characterises those holomorphic forms on the regular locus of a complex space that extend holomorphically to any resolution of singularities. This proof is very short and, shows clearly why the extension problem for k-forms also controls the extension problem for $(k-1)$-forms (and hence for all forms of smaller degrees).

2.2. **Second proof of Theorem 1.4.** To illustrate the main ideas and techniques used in this paper, we are now going to describe a second, more systematic proof for Theorem 1.4. It is longer, and covers only the case where $k = n$, but it has the advantage of producing a stronger result that has other applications (such as the proof of the local vanishing conjecture). We hope that the description below will make it clear why the Decomposition Theorem is useful in studying the extension problem for holomorphic forms.

Setup. We fix a reduced and irreducible complex space X of dimension n, and a resolution of singularities $r\colon \widetilde{X} \to X$. We denote by $j\colon X_{\mathrm{reg}} \hookrightarrow X$ the embedding of the set of regular points, and assume that the natural morphism $r_*\Omega^n_{\widetilde{X}} \hookrightarrow j_*\Omega^n_{X_{\mathrm{reg}}}$ is an isomorphism. This means concretely that, locally on X, holomorphic n-forms extend from the regular locus to the resolution. Rather than using the given resolution $\widetilde{X}$ to show that p-forms extend, we are going to prove directly that the natural morphism $r_*\Omega^p_{\widetilde{X}} \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$ is an isomorphism for every $p \in \{0, 1, \dots, n\}$. This is a statement about X itself, because the subsheaf $r_*\Omega^p_{\widetilde{X}}$ does not depend on the choice of resolution, as we have seen in the introduction.

Note. Using independence of the resolution, we may assume without loss of generality that the resolution $r\colon \widetilde{X} \to X$ is projective, and an isomorphism over X_{reg}. Such resolutions exist for every reduced complex space by [BM97, Thm. 10.7].

Criteria for extension. The first idea in the proof of Theorem 1.4 is to use duality.[1] Let $\omega^\bullet_X \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ denote the dualizing complex of X; on the n-dimensional complex manifold $\widetilde{X}$, one has $\omega^\bullet_{\widetilde{X}} \cong \omega_{\widetilde{X}}[n]$. The dualizing complex gives rise to a simple numerical criterion for whether sections of a coherent $\mathscr{O}_X$-module extend uniquely over X_{sing}. Indeed, Proposition 6.1 – or rather its generalisation to singular spaces – says that sections of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ extend uniquely over X_{sing} if and only if

$$\dim\bigl(X_{\mathrm{sing}} \cap \operatorname{Supp} R^k\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega^\bullet_X)\bigr) \le -(k+2) \quad \text{for every } k \in \mathbb{Z}.$$

When the support of $\mathscr{F}$ has pure dimension n, as is the case for the $\mathscr{O}_X$-module $r_*\Omega^p_{\widetilde{X}}$ that we are interested in, this amounts to the following two conditions:

(2.0.1) $\dim X_{\mathrm{sing}} \le n-2$

(2.0.2) $\dim \operatorname{Supp} R^k\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega^\bullet_X) \le -(k+2)$ for every $k \ge -n+1$

Unfortunately, there is no good way to compute the dual complex of $r_*\Omega^p_{\widetilde{X}}$. But if we work instead with the entire complex $\mathbf{R}r_*\Omega^p_{\widetilde{X}}$, things get better: Grothendieck duality [RRV71], applied to the proper holomorphic mapping $r\colon \widetilde{X} \to X$, yields

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}\Bigl(\mathbf{R}r_*\Omega^p_{\widetilde{X}}, \omega^\bullet_X\Bigr) \overset{\text{duality}}{\cong} \mathbf{R}r_*\mathbf{R}\mathscr{H}om\Bigl(\Omega^p_{\widetilde{X}}, \omega_{\widetilde{X}}[n]\Bigr) \cong \mathbf{R}r_*\Omega^{n-p}_{\widetilde{X}}[n]. \tag{2.0.3}$$

In Proposition 6.4, we prove the following variant of the criterion for section extension: if $K \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ is a complex with $\mathscr{H}^j K = 0$ for $j < 0$, and if

$$\dim\bigl(X_{\mathrm{sing}} \cap \operatorname{Supp} R^k\mathscr{H}om_{\mathscr{O}_X}(K, \omega^\bullet_X)\bigr) \le -(k+2) \quad \text{for every } k \in \mathbb{Z}, \tag{2.0.4}$$

[1] For the sake of exposition, we work directly on X in this section. In the actual proof, we only use duality for coherent sheaves on complex manifolds, after locally embedding X into a complex manifold.

then sections of the coherent $\mathscr{O}_X$-module $\mathscr{H}^0 K$ extend uniquely over X_{sing}. This observation transforms the problem of showing that sections of $r_*\Omega^p_{\widetilde{X}}$ extend uniquely over X_{sing} into the problem of showing that

$$\dim \operatorname{Supp} R^k r_* \Omega^{n-p}_{\widetilde{X}} \leq n-2-k \quad \text{for every } k \geq 1.$$

In summary, we see that a good upper bound for the dimension of the support of $R^k r_*\Omega^{n-p}_{\widetilde{X}}$ would be enough to conclude that p-forms extend. Or, to put it more simply, "vanishing implies extension".

Hodge modules and the Decomposition Theorem. The problem with the approach outlined above is that the complex $\mathbf{R}r_*\Omega^{n-p}_{\widetilde{X}}$ has too many potentially nonzero cohomology sheaves, which makes it hard to prove the required vanishing. For example, if the preimage of a singular point $x \in X_{\mathrm{sing}}$ is a divisor in the resolution $\widetilde{X}$, then $R^{n-1}r_*\Omega^{n-p}_{\widetilde{X}}$ might be supported at x, violating the inequality in (2.0.4). Since we are not assuming that the singularities of X are klt, we also do not have enough information about the fibres of $r\colon \widetilde{X} \to X$ to prove vanishing by restricting to fibres as in [GKKP11, §18].

The second idea in the proof, which completely circumvents this problem, is to relate the $\mathscr{O}_X$-module $r_*\Omega^p_{\widetilde{X}}$ to the intersection complex of X, viewed as a polarisable Hodge module[2]. In the process, we make use of the Decomposition Theorem. Roughly speaking, the Decomposition Theorem decomposes the push-forward of the constant sheaf into a "generic part" (that only depends on X) and a "special part" (that is affected by the positive-dimensional fibres of r). The upshot is that the generic part carries all the relevant information, and that the positive-dimensional fibres of r are completely irrelevant for the extension problem. To be more precise, the Decomposition Theorem for the projective morphism r, together with Saito's formalism of Hodge modules, leads to a (non-canonical) decomposition

$$\mathbf{R}r_*\Omega^p_{\widetilde{X}} \cong K_p \oplus R_p \tag{2.0.5}$$

into two complexes $K_p, R_p \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ with the following properties:

(2.0.6) The support of R_p is contained in the singular locus X_{sing}.

(2.0.7) One has $\mathscr{H}^k K_p = 0$ for $k \geq n-p+1$.

(2.0.8) The complexes K_p and K_{n-p} are related by Grothendieck duality in the same way that the complexes $\mathbf{R}r_*\Omega^p_{\widetilde{X}}$ and $\mathbf{R}r_*\Omega^{n-p}_{\widetilde{X}}$ are related in (2.0.3). More precisely, one has $\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(K_p, \omega^\bullet_X) \cong K_{n-p}[n]$.

An improved criterion. As an immediate consequence of the decomposition in (2.0.5), we obtain a decomposition of the 0-th cohomology sheaves

$$r_*\Omega^p_{\widetilde{X}} \cong \mathscr{H}^0 K_p \oplus \mathscr{H}^0 R_p.$$

Because $\mathscr{H}^0 R_p$ is supported inside X_{sing}, whereas $\Omega^p_{\widetilde{X}}$ is torsion free, we deduce that $\mathscr{H}^0 R_p = 0$, and hence that $r_*\Omega^p_{\widetilde{X}} \cong \mathscr{H}^0 K_p$. According to the criterion for section extension in Proposition 6.4, now applied to the complex K_p, all we therefore need for sections of $r_*\Omega^p_{\widetilde{X}}$ to extend uniquely over X_{sing} is to establish the collection of inequalities

$$\dim\big(X_{\mathrm{sing}} \cap \operatorname{Supp} \mathscr{H}^k K_{n-p}\big) \leq n-2-k \quad \text{for all } k \in \mathbb{Z}. \tag{2.0.9}$$

Property (2.0.7) makes this a much more manageable task, compared to the analogous problem for the original complex $\mathbf{R}r_*\Omega^{n-p}_{\widetilde{X}}$. We stress that, except in the case $p = n$, these inequalities are stronger than asking that sections of $r_*\Omega^p_{\widetilde{X}}$ extend uniquely over X_{sing}.

[2]Since the intersection complex is intrinsic to X, this also serves to explain once again why the $\mathscr{O}_X$-module $r_*\Omega^p_{\widetilde{X}}$ does not depend on the choice of resolution.

The case of isolated singularities. We conclude this outline with a brief sketch how (2.0.9) is proved in the case of isolated singularities. In Section 6.2, we more or less reduce the general case to this special case by locally cutting with hypersurfaces; note that this works because we are proving a stronger statement than just extension of p-forms.

Because of Property (2.0.7), we have $\mathscr{H}^k K_{n-p} = 0$ for $k \geq p+1$. Since $\dim X_{\text{sing}} = 0$, the inequality in (2.0.9) is therefore true by default as long as $p \leq n-2$. In this way, we recover the result of Steenbrink and van Straten [vSS85, Thm. 1.3] mentioned in the introduction: on an n-dimensional complex space with isolated singularities, p-forms extend for every $p \leq n-2$. This only leaves two cases, namely $p = n-1$ and $p = n$.

The case $p = n$ is covered by the assumption that n-forms extend. We have $K_n \cong \mathscr{H}^0 K_n \cong r_* \Omega^n_{\widetilde{X}}$, and sections of $r_* \Omega^n_{\widetilde{X}}$ extend uniquely over X_{sing}. Because of the isomorphism $\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(K_n, \omega_X^\bullet) \cong K_0[n]$, Proposition 6.1 gives us the desired inequalities

$$\dim\big(X_{\text{sing}} \cap \operatorname{Supp} \mathscr{H}^k K_0\big) = \dim\Big(X_{\text{sing}} \cap \operatorname{Supp} R^{k-n}\mathscr{H}om_{\mathscr{O}_X}(K_n, \omega_X^\bullet)\Big) \leq n-k-2$$

for every $k \in \mathbb{Z}$.

In the other case $p = n-1$, the inequalities in (2.0.9) are easily seen to be equivalent to the single vanishing $\mathscr{H}^{n-1} K_1 = 0$. Using the fact that $\mathscr{H}^k K_0 = 0$ for $k \geq n-1$, one shows that the $\mathscr{O}_X$-module $\mathscr{H}^{n-1} K_1$ is a quotient of the (constructible) 0-th cohomology sheaf of the intersection complex of X. But the intersection complex is known to be concentrated in strictly negative degrees, and therefore $\mathscr{H}^{n-1} K_1 = 0$.

3. Conventions

3.1. **Global conventions.** Throughout this paper, all complex spaces are assumed to be countable at infinity. All schemes and algebraic varieties are assumed to be defined over the field of complex numbers. We follow the notation used in the standard reference books [Har77, GR84]. In particular, varieties are assumed to be irreducible, and the support of a coherent sheaf $\mathscr{F}$ on X is a closed subset of X, with the induced reduced structure. For clarity, we will always say explicitly when a complex space needs to be reduced, irreducible, or of pure dimension.

3.2. **$\mathscr{D}$-modules.** Unless otherwise noted, we use left $\mathscr{D}$-modules throughout this paper. This choice agrees with the notation of the paper [Sch16], which we will frequently cite. It is, however, incompatible with the conventions of the reference papers [Sai88, Sai90] and of the survey [Sch14] that use right $\mathscr{D}$-modules throughout. We refer the reader to [Sch16, §A.5], where the conversion rules for left and right $\mathscr{D}$-modules are recalled.

3.3. **Complexes.** Let K be a complex of sheaves of Abelian groups on a topological space, for example a complex of sheaves of $\mathscr{O}_X$-modules (or $\mathscr{D}_X$-modules) on a complex manifold X. We use the notation $\mathscr{H}^j K$ for the j-th cohomology sheaf of the complex. We use the notation $K[n]$ for the shift of K. We have $\mathscr{H}^j K[n] = \mathscr{H}^{j+n} K$, and all differentials in the shifted complex are multiplied by $(-1)^n$.

3.4. **The dualizing complex.** If X is any complex space, we write $\omega_X^\bullet \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ for the dualizing complex as introduced by Ramis and Ruget, see [BS76, VII Thm. 2.6]. Given a complex of $\mathscr{O}_X$-modules $K \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ with bounded coherent cohomology, we call the complex $\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(K, \omega_X^\bullet) \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ the *dual complex* of K.

Note. When X is a complex manifold of pure dimension, one has $\omega_X^\bullet \cong \omega_X[\dim X]$.

3.5. **Reflexive sheaves on normal spaces.** Let X be a normal complex space, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Recall that $\mathscr{F}$ is called *reflexive* if the natural morphism from $\mathscr{F}$ to its double dual $\mathscr{F}^{**} := \mathscr{H}om_{\mathscr{O}_X}\big(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X), \mathscr{O}_X\big)$ is an isomorphism. The following notation will be used.

Notation 3.1 (Reflexive hull). Given a normal complex space X and a coherent sheaf $\mathscr{F}$ on X, write $\Omega_X^{[p]} := \left(\Omega_X^p\right)^{**}$, $\mathscr{F}^{[m]} := \left(\mathscr{E}^{\otimes m}\right)^{**}$ and $\det \mathscr{F} := \left(\wedge^{\operatorname{rank} \mathscr{E}} \mathscr{F}\right)^{**}$. Given any morphism $f\colon Y \to X$ of normal complex spaces, write $f^{[*]}\mathscr{F} := (f^*\mathscr{F})^{**}$, etc. Ditto for quasi-projective varieties.

4. Mixed Hodge modules

4.1. **Mixed Hodge modules.** For the convenience of the reader, we briefly recall a number of facts concerning mixed Hodge modules, and lay down the notation that will be used throughout. In a nutshell, a (mixed) Hodge module is something like a variation of (mixed) Hodge structure with singularities, in the sense that the vector bundles with connection (respectively locally constant sheaves) in the definition of a variation of Hodge structure are replaced by regular holonomic $\mathscr{D}$-modules (respectively perverse sheaves). The standard references for mixed Hodge modules are the original papers by Saito [Sai88, Sai90]. The survey articles [Sai89, Sai94, Sch14] review some aspects of the theory in a smaller number of pages. A good reference for $\mathscr{D}$-modules is the book [HTT08]. We consider the following setting throughout the present section.

Setting 4.1. Assume that a complex manifold Y of pure dimension d and a graded-polarisable mixed Hodge module M on Y are given.

More precisely, a *polarisable Hodge module* M of weight w on Y has three components: a regular holonomic left $\mathscr{D}_Y$-module $\mathcal{M}$, called the *underlying $\mathscr{D}$-module*; an increasing good filtration $F_\bullet\mathcal{M}$ by coherent $\mathscr{O}_Y$-modules, called the *Hodge filtration*; and a perverse sheaf of $\mathbb{Q}$-vector spaces $\operatorname{rat} M$, called the *underlying perverse sheaf*. These are subject to a number of conditions, including the existence of a polarisation, which together ensure that M is determined by finitely many polarisable variations of Hodge structure of weight $w - \dim Z_j$ on locally closed submanifolds $Z_j \subseteq Y$. A *graded-polarisable mixed Hodge module* M on Y is an object of the same kind, but with an additional increasing filtration $W_\bullet M$, called the *weight filtration*, such that each subquotient

$$\operatorname{gr}_\ell^W M := W_\ell M \big/ W_{\ell-1} M$$

is a polarisable Hodge module of weight ℓ. The *support* of M, denoted by $\operatorname{Supp} M$, is by definition the support of the $\mathscr{D}_Y$-module $\mathcal{M}$ (or, equivalently, of the perverse sheaf $\operatorname{rat} M$).

Notation 4.2 (Category of mixed Hodge modules). In Setting 4.1, we denote by $\mathrm{MHM}(Y)$ the Abelian category of *graded-polarisable mixed Hodge modules* on Y, and by $\mathrm{HM}(Y, w)$ the Abelian category of *polarisable Hodge modules* of weight w.

Note. In Setting 4.1, we have

$$\operatorname{gr}_\ell^W M := W_\ell M \big/ W_{\ell-1} M \in \mathrm{HM}(Y, \ell), \quad \text{for every } \ell \in \mathbb{Z}.$$

Conversely, every polarisable Hodge module $N \in \mathrm{HM}(Y, w)$ may be viewed as a graded-polarisable mixed Hodge module N with $W_{w-1}N = 0$ and $W_w N = N$.

4.1.1. *Tate twist.* Maintain Setting 4.1. Given any integer $k \in \mathbb{Z}$, define $\mathbb{Q}(k) = (2\pi i)^k \mathbb{Q} \subseteq \mathbb{C}$. The *Tate twist* $M(k)$ is the mixed Hodge module whose underlying perverse sheaf is $\mathbb{Q}(k) \otimes \operatorname{rat} M$, whose underlying filtered $\mathscr{D}_Y$-module is $(\mathcal{M}, F_{\bullet-k}\mathcal{M})$, and whose weight filtration is given by $W_\ell\, M(k) = W_{\ell+2k} M$. When M is pure of weight w, it follows that $M(k)$ is again pure of weight $w - 2k$.

4.1.2. *Decomposition by strict support.* In Setting 4.1, one says that the mixed Hodge module M has *strict support* if the support of every nontrivial subquotient of M is equal to $\operatorname{Supp} M$. Ditto for perverse sheaves and regular holonomic $\mathscr{D}_Y$-modules. Note that the strict support property is generally *not* preserved by restriction to open subsets; for example, $\operatorname{Supp} M$ may be globally irreducible, but locally reducible. We use the symbol

$\mathrm{HM}_X(Y, w)$ to denote the Abelian category of polarisable Hodge modules on Y of weight w with strict support X; this is a full subcategory of $\mathrm{HM}(Y, w)$.

If M is a polarisable Hodge module, then M has strict support if and only if the support of every nontrivial subobject (or quotient object) is equal to $\operatorname{Supp} M$; the reason is that polarisable Hodge modules are semisimple [Sai88, Cor. 5.2.13]. By definition, every polarisable Hodge module admits, on every open subset of X, a *decomposition by strict support* as a (locally finite) direct sum of polarisable Hodge modules with strict support [Sai88, §5.1.6].

4.1.3. *Weight filtration and dual module.* In Setting 4.1, we write $M' = \mathbb{D}M \in \mathrm{MHM}(Y)$ for the *dual mixed Hodge module*. This is again a graded-polarisable mixed Hodge module [Sai90, Prop. 2.6], with the property that

$$\mathbb{D}(\mathrm{gr}^W_\ell M) \cong \mathrm{gr}^W_{-\ell} \mathbb{D}M.$$

In particular, if M is pure of weight ℓ, then $\mathbb{D}M$ is again pure of weight $-\ell$. The underlying perverse sheaf $\operatorname{rat} M'$ is isomorphic to the Verdier dual [HTT08, Def. 4.5.2] of $\operatorname{rat} M$. The regular holonomic left $\mathscr{D}_Y$-module $(\mathcal{M}', F_\bullet \mathcal{M}')$ underlying $M' = \mathbb{D}M$ is isomorphic to the holonomic dual [HTT08, Def. 2.6.1]

$$R^d \mathscr{H}om_{\mathscr{D}_Y}\big(\omega_Y \otimes_{\mathscr{O}_Y} \mathcal{M}, \mathscr{D}_Y\big)$$

of the regular holonomic left $\mathscr{D}_Y$-module $\mathcal{M}$.

4.1.4. *The de Rham complex.* In Setting 4.1, the complex of sheaves of $\mathbb{C}$-vector spaces

$$\mathrm{DR}(\mathcal{M}) = \Big[\mathcal{M} \to \Omega^1_Y \otimes_{\mathscr{O}_Y} \mathcal{M} \to \cdots \to \Omega^d_Y \otimes_{\mathscr{O}_Y} \mathcal{M}\Big][d],$$

concentrated in degrees $-d, \ldots, 0$ is called the *de Rham complex* of $\mathcal{M}$. Since $\mathcal{M}$ is a regular holonomic $\mathscr{D}_Y$-module, the de Rham complex $\mathrm{DR}(\mathcal{M})$ has constructible cohomology sheaves, and is in fact a perverse sheaf on Y by a theorem of Kashiwara [HTT08, Thm. 4.6.6]. In particular, it is always *semiperverse*, which means concretely that

$$\dim \operatorname{Supp} \mathscr{H}^j \mathrm{DR}(\mathcal{M}) \le -j \quad \text{for every } j \in \mathbb{Z}.$$

The perverse sheaf $\operatorname{rat} M$ and the de Rham complex of $\mathcal{M}$ are related through an isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} \operatorname{rat} M \cong \mathrm{DR}(\mathcal{M})$ that is part of the data of a mixed Hodge module.

4.1.5. *Subquotients of the de Rham complex.* Assume Setting 4.1. The filtration $F_\bullet \mathcal{M}$ induces an increasing filtration on the de Rham complex by

$$F_p \,\mathrm{DR}(\mathcal{M}) = \Big[F_p\mathcal{M} \to \Omega^1_Y \otimes_{\mathscr{O}_Y} F_{p+1}\mathcal{M} \to \cdots \to \Omega^d_Y \otimes_{\mathscr{O}_Y} F_{p+d}\mathcal{M}\Big][d]. \tag{4.3.1}$$

The p-th subquotient of this filtration is the complex of $\mathscr{O}_Y$-modules

$$\mathrm{gr}^F_p \,\mathrm{DR}(\mathcal{M}) = \Big[\mathrm{gr}^F_p\mathcal{M} \to \Omega^1_Y \otimes_{\mathscr{O}_Y} \mathrm{gr}^F_{p+1}\mathcal{M} \to \cdots \to \Omega^d_Y \otimes_{\mathscr{O}_Y} \mathrm{gr}^F_{p+d}\mathcal{M}\Big][d]. \tag{4.3.2}$$

For a more detailed discussion of these complexes, see for example [Sch16, §7]. The following simple lemma will be useful later.

Lemma 4.4. *In Setting 4.1, if $\mathrm{gr}^F_p \mathrm{DR}(\mathcal{M})$ is acyclic for every $p \le m$, then $F_{m+\dim Y}\mathcal{M} = 0$.*

Proof. Since $F_\bullet \mathcal{M}$ is a good filtration, there is, at least locally on Y, an integer p_0 such that $F_{p_0}\mathcal{M} = 0$ and, hence, $\mathrm{gr}^F_p \mathcal{M} = 0$ for every $p \le p_0$. To show that $F_{m+d}\mathcal{M} = 0$, it is therefore enough to prove that $\mathrm{gr}^F_p \mathcal{M} = 0$ for every $p \le m + d$. Because $\mathrm{gr}^F_p \mathrm{DR}(\mathcal{M})$ is acyclic for $p \le m$, this follows from (4.3.2) by induction on $p \ge p_0$. □

4.2. **Duality.** Next, we review how the duality functor for mixed Hodge modules affects the subquotients of the de Rham complex. The following nontrivial result by Saito shows that the dual complex of $\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M})$ is nothing but $\operatorname{gr}_{-p}^F \operatorname{DR}(\mathcal{M}')$.

Proposition 4.5 (Duality, mixed case)**.** *Assume Setting 4.1. Then,*

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}\Bigl(\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}), \omega_Y^\bullet\Bigr) \cong \operatorname{gr}_{-p}^F \operatorname{DR}(\mathcal{M}') \quad \textit{for every } p \in \mathbb{Z},$$

where $(\mathcal{M}', F_\bullet \mathcal{M}')$ is the filtered $\mathcal{D}_Y$-module underlying $M' = \mathbb{D}M$.

Proof. This is proved in [Sai88, §2.4.3]; see also [Sch16, Lem. 7.4]. The crucial point in the proof is that $\operatorname{gr}_\bullet^F \mathcal{M}$ is a Cohen-Macaulay module over $\operatorname{gr}_\bullet^F \mathcal{D}_Y$, due to the fact that $(\mathcal{M}, F_\bullet \mathcal{M})$ underlies a mixed Hodge module. □

In the special case where M is a polarisable Hodge module, the de Rham complex is self-dual, up to a shift in the filtration. Duality therefore relates different subquotients of $\operatorname{DR}(\mathcal{M})$, in a way that will be very useful for the proof of Theorem 1.4.

Corollary 4.6 (Duality, pure case)**.** *Let $M \in \operatorname{HM}(Y, w)$ be a polarisable Hodge module of weight w on a complex manifold Y. Any polarisation on M induces an isomorphism*

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}\Bigl(\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}), \omega_Y^\bullet\Bigr) \cong \operatorname{gr}_{-p-w}^F \operatorname{DR}(\mathcal{M}) \quad \textit{for every } p \in \mathbb{Z}.$$

Proof. A polarisation on M induces an isomorphism $\mathbb{D}M \cong M(w)$ [Sai88, §5.2.10], and therefore an isomorphism $(\mathcal{M}', F_\bullet \mathcal{M}') \cong (\mathcal{M}, F_{\bullet - w}\mathcal{M})$. Now apply Proposition 4.5. □

The following proposition contains an acyclicity criterion for subquotients of the de Rham complex, involving both the weight filtration $W_\bullet M$ and the Hodge filtration $F_\bullet \mathcal{M}$.

Proposition 4.7 (Acyclic subquotients)**.** *Assume Setting 4.1. If w, $c \in \mathbb{Z}$ are such that $W_{w-1}M = 0$ and $F_{c-1}\mathcal{M} = 0$, then $\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M})$ is acyclic unless $c - d \leq p \leq d - w - c$.*

Proof. Since $F_{c-1}\mathcal{M} = 0$ and $d = \dim Y$, a look at the formula (4.3.2) for the p-th subquotient of $\operatorname{DR}(\mathcal{M})$ reveals that $\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}) = 0$ for $p \leq c-1-d$. The other inequality is going to follow by duality. Let us first consider the pure case, meaning that $M \in \operatorname{HM}(Y, w')$ is a polarisable Hodge module of weight w'. By Corollary 4.6, we have

$$\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}\Bigl(\operatorname{gr}_{-p-w'}^F \operatorname{DR}(\mathcal{M}), \omega_Y^\bullet\Bigr)$$

and since the complex on the right-hand side is acyclic for $-p - w' \leq c - 1 - d$, we get the result when M is pure. The general case follows from this by considering the subquotients of the weight filtration $W_\bullet M$. □

Proposition 4.7 is especially useful when combined with the following general fact, which an easy consequence of the filtration $F_\bullet \mathcal{M}$ being exhaustive.

Proposition 4.8. *Let Y be a complex manifold. Let $(\mathcal{M}, F_\bullet \mathcal{M})$ be a coherent $\mathcal{D}_Y$-module with a good filtration. If $\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M})$ is acyclic for all $p \geq p_0 + 1$, then the inclusion $F_{p_0} \operatorname{DR}(\mathcal{M}) \hookrightarrow \operatorname{DR}(\mathcal{M})$ is a quasi-isomorphism.* □

4.3. **Direct images and the Decomposition Theorem.** Let $f\colon X \to Y$ be a projective holomorphic mapping between two complex manifolds, and let $M \in \operatorname{MHM}(X)$ be a graded-polarisable mixed Hodge module on X. One of the most important results in Saito's theory is that, in this setting, one can define a direct image functor, compatible with the direct image functor for perverse sheaves and filtered $\mathcal{D}$-modules, and that the i-th higher direct image $H^i f_* M$ is again a graded-polarisable mixed Hodge module on Y. In this section, we briefly review this result and its implications for the underlying filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet \mathcal{M})$ and the de Rham complex $\operatorname{DR}(\mathcal{M})$.

4.3.1. *Filtered $\mathscr{D}$-modules and strictness.* Let X be a complex manifold. Following Saito, we denote by $\mathrm{D}^b_{coh} F(\mathscr{D}_X)$ the derived category of (certain cohomologically bounded and coherent complexes of) filtered $\mathscr{D}_X$-modules, as defined in [Sai88, §2.1.15]. The category of filtered $\mathscr{D}_X$-modules is only an exact category, but it embeds into the larger Abelian category of graded $R_F\mathscr{D}_X$-modules, where

$$R_F\mathscr{D}_X = \bigoplus_{p\in\mathbb{N}} F_p\mathscr{D}_X$$

is the Rees algebra of $\mathscr{D}_X$ with respect to the order filtration. The embedding takes a coherent filtered $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ to the associated Rees module

$$R_F\mathcal{M} = \bigoplus_{p\in\mathbb{Z}} F_p\mathcal{M},$$

which is coherent over $R_F\mathscr{D}_X$. Let $\mathrm{D}^b_{coh} G(R_F\mathscr{D}_X)$ be the derived category of (cohomologically bounded and coherent complexes of) graded $R_F\mathscr{D}_X$-modules. Then the Rees module construction gives an equivalence of categories

$$\mathrm{D}^b_{coh} F(\mathscr{D}_X) \cong \mathrm{D}^b_{coh} G(R_F\mathscr{D}_X),$$

according to [Sai88, Prop. 2.1.16]. The cohomology modules of an object in $\mathrm{D}^b_{coh} F(\mathscr{D}_X)$ are therefore in general not filtered $\mathscr{D}_X$-modules, but graded $R_F\mathscr{D}_X$-modules.

Definition 4.9 (Strictness)**.** *A graded $R_F\mathscr{D}_X$-module is called* strict *if it is isomorphic to the Rees module of a coherent filtered $\mathscr{D}_X$-module. A complex $K \in \mathrm{D}^b_{coh} G(R_F\mathscr{D}_X)$ is called* strict *if all of its cohomology modules $\mathscr{H}^j K$ are strict.*

The functor that takes a coherent filtered $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ to the underlying $\mathscr{D}_X$-module $\mathcal{M}$ extends uniquely to an exact functor

$$\mathrm{D}^b_{coh} G(R_F\mathscr{D}_X) \to \mathrm{D}^b_{coh}(\mathscr{D}_X).$$

Indeed, if we denote by $z \in R_F\mathscr{D}_X$ the degree-one element obtained from $1 \in F_1\mathscr{D}_X$, then the functor is simply the derived tensor product with $R_F\mathscr{D}_X/(1-z)R_F\mathscr{D}_X$. Similarly, the functor that takes a coherent filtered $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ to the coherent graded $\operatorname{Sym}\mathscr{T}_X$-module $\operatorname{gr}^F_\bullet\mathcal{M}$ extends uniquely to an exact functor

$$\operatorname{gr}^F_\bullet\colon \mathrm{D}^b_{coh} G(R_F\mathscr{D}_X) \to \mathrm{D}^b_{coh} G(\operatorname{Sym}\mathscr{T}_X).$$

This time, the functor is given by the derived tensor product with $R_F\mathscr{D}_X/zR_F\mathscr{D}_X$. Lastly, for every $p \in \mathbb{Z}$, the functor that takes a coherent filtered $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ to the complex of coherent $\mathscr{O}_X$-modules $\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M})$ extends uniquely to an exact functor

$$\operatorname{gr}^F_p \operatorname{DR}\colon \mathrm{D}^b_{coh} G(R_F\mathscr{D}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

Indeed, by [Sai88, Prop. 2.2.10], the de Rham functor (which Saito denotes by the symbol $\widetilde{DR}$) defines an equivalence of categories between $\mathrm{D}^b_{coh} F(\mathscr{D}_X)$ and the derived category of filtered differential complexes $\mathrm{D}^b_{coh} F^f(\mathscr{O}_X, \mathrm{Diff})$, and gr^F_p of a filtered differential complex is by construction a (cohomologically bounded and coherent) complex of $\mathscr{O}_X$-modules [Sai88, §2.2.4].

4.3.2. *Direct image functor for filtered $\mathscr{D}$-modules.* Now suppose that $f\colon X \to Y$ is a proper holomorphic mapping between complex manifolds. In this setting, one can construct a direct image functor

$$f_+\colon \mathrm{D}^b_{coh} G(R_F\mathscr{D}_X) \to \mathrm{D}^b_{coh} G(R_F\mathscr{D}_Y);$$

see [Sai88, §2.3.5] for the precise definition. This functor is compatible with the functor $\operatorname{gr}^F_p \operatorname{DR}$ in the following manner [Sai88, §2.3.7].

Proposition 4.10. *Let $f: X \to Y$ be a proper holomorphic mapping between complex manifolds. For every $p \in \mathbb{Z}$, one has a natural isomorphism of functors*

$$\mathbf{R}f_* \circ \mathrm{gr}_p^F \mathrm{DR} \cong \mathrm{gr}_p^F \mathrm{DR} \circ f_+$$

as functors from $\mathrm{D}^b_{coh} G(R_F \mathscr{D}_X)$ to $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. By [Sai88, Lem. 2.3.6], the de Rham functor exchanges the direct image functor $f_+: \mathrm{D}^b_{coh} G(R_F\mathscr{D}_X) \to \mathrm{D}^b_{coh} G(R_F\mathscr{D}_Y)$ and the direct image functor

$$f_*: \mathrm{D}^b_{coh} F^f(\mathscr{O}_X, \mathrm{Diff}) \to \mathrm{D}^b_{coh} F^f(\mathscr{O}_Y, \mathrm{Diff})$$

for filtered differential complexes. But the latter commutes with taking gr_p^F, as is clear from the construction in [Sai88, §2.3.7]. □

Note. In the case of a single coherent filtered $\mathscr{D}_X$-module, this says that

$$\mathbf{R}f_* \, \mathrm{gr}_p^F \mathrm{DR}(\mathcal{M}) \cong \mathrm{gr}_p^F \mathrm{DR}\big(f_+(R_F\mathcal{M})\big),$$

as objects of the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

4.3.3. *Direct image theorem, pure case.* We now assume that the proper holomorphic mapping $f: X \to Y$ is actually projective. Then we have the following important "direct image theorem" due to Saito [Sai88, §5.3].

Theorem 4.11 (Direct image theorem, pure case)**.** *Let $f: X \to Y$ be a projective morphism between complex manifolds, and let $\ell \in H^2(X, \mathbb{Z}(1))$ be the first Chern class of a relatively ample line bundle. If $M \in \mathrm{HM}(X, w)$ is a polarisable Hodge module X, then:*

(4.11.1) The complex $f_+(R_F\mathcal{M})$ is strict, and each $\mathscr{H}^i f_+(R_F\mathcal{M})$ is the filtered $\mathscr{D}_Y$-module underlying a polarisable Hodge module $H^i f_* M \in \mathrm{HM}(Y, w+i)$.

(4.11.2) For every $i \geq 0$, the Lefschetz morphism

$$\ell^i: H^{-i} f_* M \to H^i f_* M(i)$$

is an isomorphism between Hodge modules of weight $w - i$.

(4.11.3) Any polarisation on M induces a polarisation on $\bigoplus_i H^i f_* M$ in the Hodge-Lefschetz sense (= on primitive parts with respect to the action of ℓ). □

One consequence of Theorem 4.11 is a version of the Decomposition Theorem for those filtered $\mathscr{D}$-modules that underlie polarisable Hodge modules.

Corollary 4.12 (Decomposition Theorem)**.** *Let $f: X \to Y$ be a projective morphism between complex manifolds. Let $M \in \mathrm{HM}(X, w)$ be a polarisable Hodge module on X, and let $M_i = H^i f_* M \in \mathrm{HM}(Y, w+i)$. Write $(\mathcal{M}, F_\bullet\mathcal{M})$ respectively $(\mathcal{M}_i, F_\bullet\mathcal{M}_i)$ for the underlying filtered $\mathscr{D}$-modules. Then*

$$f_+(R_F\mathcal{M}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathscr{H}^i f_+(R_F\mathcal{M})\,[-i] \cong \bigoplus_{i\in\mathbb{Z}} R_F\mathcal{M}_i\,[-i],$$

in the derived category $\mathrm{D}^b_{coh} G(R_F\mathscr{D}_Y)$.

Proof. The first isomorphism is a formal consequence of (4.11.2). The second isomorphism follows because the complex $f_+(R_F\mathcal{M})$ is strict. □

4.3.4. *Direct image theorem, mixed case.* In the case of mixed Hodge modules, there are some additional results, having to do with the weight filtration. We summarise them in the following theorem [Sai90, Thm. 2.14 and Prop. 2.15].

Theorem 4.13 (Direct image theorem, mixed case)**.** *Let $f: X \to Y$ be a projective morphism between complex manifolds, and let $M \in \mathrm{MHM}(X)$ be a graded-polarisable mixed Hodge module on X.*

(4.13.1) The complex $f_+(R_F\mathcal{M})$ is strict, and each $\mathscr{H}^i f_+(R_F\mathcal{M})$ is the filtered $\mathscr{D}_Y$-module underlying a graded-polarisable mixed Hodge module $H^i f_* M \in \mathrm{MHM}(Y)$.

(4.13.2) One has a convergent weight spectral sequence

$$E_1^{p,q} = H^{p+q} f_* \operatorname{gr}_{-p}^W M \Longrightarrow H^{p+q} f_* M,$$

and each differential $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ *is a morphism in* $\mathrm{HM}(Y,q)$.

(4.13.3) The weight spectral sequence degenerates at E_2, *and one has*

$$\operatorname{gr}_q^W H^{p+q} f_* M \cong E_2^{p,q} \quad \text{for every } p,q \in \mathbb{Z}. \qquad \square$$

One can use this result to control the range in which the Hodge filtration on the direct image of a graded-polarisable mixed Hodge module is nontrivial.

Proposition 4.14. *Let* $f \colon X \to Y$ *be a projective morphism between complex manifolds, and let* $M \in \mathrm{MHM}(X)$ *be a graded-polarisable mixed Hodge module on* X. *Suppose that the underlying filtered* $\mathscr{D}_X$*-module* $(\mathcal{M}, F_\bullet \mathcal{M})$ *satisfies* $F_{m-1}\mathcal{M} = 0$. *Then one has*

$$F_{m+c-1} \mathscr{H}^i f_+(R_F \mathcal{M}) = 0$$

for every $i \in \mathbb{Z}$, *where* $c = \dim Y - \dim X$.

Proof. One can deduce this from the construction of the direct image functor in [Sai88, §2.3]. Here we outline another proof based on Theorem 4.11 and Theorem 4.13.

We first deal with the case where $M \in \mathrm{HM}(X,W)$ is a polarisable pure Hodge module. By Proposition 4.10 and Corollary 4.12, we have for every $p \in \mathbb{Z}$ an isomorphism

$$\mathbf{R} f_* \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) \cong \operatorname{gr}_p^F \mathrm{DR}\big(f_+(R_F \mathcal{M})\big) \cong \bigoplus_{i \in \mathbb{Z}} \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}_i)[-i],$$

where $(\mathcal{M}_i, F_\bullet \mathcal{M}_i)$ is the filtered $\mathscr{D}_Y$-module underlying $H^i f_* M \in \mathrm{HM}(Y, w+i)$. Since $F_{m-1}\mathcal{M} = 0$, we get $\operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) = 0$ for all $p \leq m-1-\dim X$, and $\operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}_i)$ is therefore acyclic as long as $p \leq m-1-\dim X$. According to Lemma 4.4, this is enough to conclude that $F_{m+c-1}\mathcal{M}_i = F_{m-1-\dim X + \dim Y}\mathcal{M}_i = 0$ for every $i \in \mathbb{Z}$.

Now suppose that $M \in \mathrm{MHM}(X)$ is a graded-polarisable mixed Hodge module. The underlying $\mathscr{D}_X$-module of the Hodge module $\operatorname{gr}_w^F M \in \mathrm{HM}(X,w)$ is $\operatorname{gr}_w^W \mathcal{M}$, with the induced Hodge filtration; because $F_{m-1}\mathcal{M} = 0$, we have $F_{m-1} \operatorname{gr}_w^F \mathcal{M} = 0$. Since we already have the result in the pure case, the assertion now follows by looking at the spectral sequence in (4.13.2). $\square$

4.4. **Non-characteristic restriction to hypersurfaces.** We briefly review the non-characteristic restriction of a mixed Hodge module to a hypersurface. For a more general discussion of non-characteristic restriction, see [Sai88, §3.5] or [Sch16, §8].

Definition 4.15 (Non-characteristic hypersurfaces)**.** *Let* X *be a complex manifold, and let* $D \subseteq X$ *be a smooth hypersurface. The inclusion* $i_D \colon D \hookrightarrow X$ *gives rise to the following morphisms between cotangent bundles:*

(4.15.1)
$$\begin{tikzcd} (T^*X) \times_X D \arrow[r] \arrow[d, "p_1"] & T^*D \\ T^*X & \end{tikzcd}$$

Given a regular holonomic left $\mathscr{D}_X$*-module* $\mathcal{M}$ *on* X, *let* $\mathrm{Ch}(\mathcal{M}) \subseteq T^*X$ *denote its characteristic variety. We say that* $D \subseteq X$ *is* non-characteristic *for* $\mathcal{M}$ *if* $p_1^{-1} \mathrm{Ch}(\mathcal{M})$ *is finite over its image in* T^*D.

Note. As explained for example in [Sch16, §8], $D \subseteq X$ is non-characteristic for $\mathcal{M}$ if and only if D is transverse to every stratum in a Whitney stratification of X that is adapted to the perverse sheaf $\mathrm{DR}(\mathcal{M})$. In particular, generic hyperplane sections (in $\mathbb{P}^n$ or $\mathbb{C}^n$) are always non-characteristic.

The following result of Saito [Sai90, Lem. 2.25] describes what happens to mixed Hodge modules under non-characteristic restriction to smooth hypersurfaces.

Theorem 4.16 (Restriction to non-characteristic hypersurfaces)**.** *Let X be a complex manifold, and let $M \in \mathrm{MHM}(X)$ be a graded-polarisable mixed Hodge module on X, with underlying filtered $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$. Suppose that $i_D: D \hookrightarrow X$ is a smooth hypersurface that is non-characteristic for $\mathcal{M}$. Then there is a graded-polarisable mixed Hodge module $H^{-1}i_D^*M \in \mathrm{MHM}(D)$, whose underlying filtered $\mathscr{D}_D$-module is isomorphic to*

$$\left(\mathscr{O}_D \otimes_{i_D^{-1}\mathscr{O}_X} i_D^{-1}\mathcal{M}, \mathscr{O}_D \otimes_{i_D^{-1}\mathscr{O}_X} i_D^{-1}F_\bullet\mathcal{M}\right),$$

and whose de Rham complex is quasi-isomorphic to

$$i_D^{-1}\,\mathrm{DR}(\mathcal{M})[-1].$$

Moreover, if M is pure of weight w, then $H^{-1}i_D^*M$ is again pure of weight $w-1$.

As the discussion in Saito's paper is rather brief, we include a sketch of the proof of Theorem 4.16 for the convenience of the reader. It relies on the following result of Saito [Sai88, Lem. 3.5.6] whose proof we reproduce here.

Lemma 4.17 (Existence of V-filtration)**.** *In the setting of Definition 4.15, suppose that the smooth hypersurface $D \subseteq X$ is non-characteristic for $\mathcal{M}$. Then the rational V-filtration of $\mathcal{M}$ relative to D exists and is given by*

$$V^\alpha\mathcal{M} = \begin{cases} \mathcal{M} & \text{for } \alpha \leq 0, \\ \mathscr{J}_D^{\lceil\alpha\rceil}\mathcal{M} & \text{for } \alpha \geq 0, \end{cases}$$

where $\mathscr{J}_D \subseteq \mathscr{O}_X$ denotes the coherent ideal sheaf of D.

Proof. The problem is local, and after shrinking X, we may assume that $D = t^{-1}(0)$, where $t: X \to \mathbb{C}$ is holomorphic and submersive. We may also assume that we have a global holomorphic vector field ∂_t with the property that $[\partial_t, t] = 1$. In this situation, the rational V-filtration is the unique exhaustive decreasing filtration $V^\bullet\mathcal{M}$, indexed discretely and left-continuously by the set of rational numbers, with the following properties:

(4.17.1) Each $V^\alpha\mathcal{M}$ is coherent over $V^0\mathscr{D}_X$, the $\mathscr{O}_X$-subalgebra of $\mathscr{D}_X$ preserving $\mathscr{J}_D$.
(4.17.2) One has $t \cdot V^\alpha\mathcal{M} \subseteq V^{\alpha+1}\mathcal{M}$ and $\partial_t \cdot V^\alpha\mathcal{M} \subseteq V^{\alpha-1}\mathcal{M}$ for every $\alpha \in \mathbb{Q}$.
(4.17.3) For $\alpha > -1$, multiplication by t induces an isomorphism $V^\alpha\mathcal{M} \cong V^{\alpha+1}\mathcal{M}$.
(4.17.4) The operator $t\partial_t - \alpha$ acts nilpotently on $\mathrm{gr}_V^\alpha\mathcal{M} = V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$.

If we define the filtration $V^\bullet\mathcal{M}$ as in the statement of the lemma, then the last three properties are immediate; the only thing we need to check is that $\mathcal{M}$ itself is coherent over $V^0\mathscr{D}_X$. After choosing a good filtration $F_\bullet\mathcal{M}$, it is enough to show that $\mathrm{gr}_\bullet^F\mathcal{M}$ is coherent over $\mathrm{gr}_\bullet^F V^0\mathscr{D}_X$. Note that $\mathrm{gr}_\bullet^F\mathcal{M}$ is always coherent over $\mathrm{gr}_\bullet^F\mathscr{D}_X \cong \mathrm{Sym}\,\mathscr{T}_X$.

To prove the required coherence, we denote by $\mathscr{T}_{X/\mathbb{C}}$ the relative tangent sheaf, and by $T^*(X/\mathbb{C})$ the relative cotangent bundle. The fact that t is submersive means that we have a surjective bundle morphism $T^*X \to T^*(X/\mathbb{C})$ on X; its restriction to D is the horizontal arrow in (4.15.1). By assumption, $p_1^{-1}\,\mathrm{Ch}(\mathcal{M})$ is finite over its image in T^*D, and because finiteness is an open condition, we can replace X by a suitable open neighbourhood of D and arrange that $\mathrm{Ch}(\mathcal{M})$ is actually finite over its image in $T^*(X/\mathbb{C})$. By definition of the characteristic variety, the support of the coherent sheaf on T^*X corresponding to $\mathrm{gr}_\bullet^F\mathcal{M}$ is precisely $\mathrm{Ch}(\mathcal{M})$. Because push forward by finite holomorphic mappings preserves coherence, it follows that $\mathrm{gr}_\bullet^F\mathcal{M}$ is coherent over the subalgebra $\mathrm{Sym}\,\mathscr{T}_{X/\mathbb{C}} \subseteq \mathrm{Sym}\,\mathscr{T}_X$. Now it is easy to see that

$$\mathrm{gr}_\bullet^F V^0\mathscr{D}_X \cong \mathrm{Sym}\,\mathscr{T}_{X/\mathbb{C}}[t\partial_t],$$

and so $\mathrm{gr}_\bullet^F\mathcal{M}$ is coherent over this larger $\mathscr{O}_X$-algebra as well. □

We use the above description of the rational V-filtration to prove Theorem 4.16.

Proof of Theorem 4.16. Since all the assertions are local on X, we may assume that $D = t^{-1}(0)$, where $t\colon X \to \mathbb{C}$ is submersive. We keep the notation introduced during the proof of Lemma 4.17. Since $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a mixed Hodge module, multiplication by t induces an isomorphism between $F_pV^\alpha\mathcal{M}$ and $F_pV^{\alpha+1}\mathcal{M}$ for every $\alpha > -1$; see [Sai88, §3.2.1], but keep in mind that we are talking about left $\mathscr{D}$-modules. Specialising to $\alpha = 0$, we conclude that

$$F_p\mathcal{M} \cap t\mathcal{M} = tF_p\mathcal{M}.$$

It follows that $t\colon \operatorname{gr}^F_\bullet \mathcal{M} \to \operatorname{gr}^F_\bullet \mathcal{M}$ is injective, and hence that $\mathscr{O}_D \otimes_{i_D^{-1}\mathscr{O}_X} i_D^{-1} F_\bullet\mathcal{M}$ defines a good filtration of $\mathscr{O}_D \otimes_{i_D^{-1}\mathscr{O}_X} i_D^{-1}\mathcal{M}$ by coherent $\mathscr{O}_D$-submodules. In particular, $i_D\colon D \hookrightarrow X$ is strictly non-characteristic for $(\mathcal{M}, F_\bullet\mathcal{M})$, in the terminology of [Sch16, §8].

According to Lemma 4.17, we have

$$\operatorname{gr}^0_V \mathcal{M} \cong \mathcal{M}/t\mathcal{M} \cong \mathscr{O}_D \otimes_{i_D^{-1}\mathscr{O}_X} i_D^{-1}\mathcal{M},$$

and the action of the (nilpotent) operator $N = t\partial_t$ is trivial. Consequently, the relative weight filtration of N is equal to the filtration $W_\bullet\mathcal{M}/tW_\bullet\mathcal{M} \cong \mathscr{O}_D \otimes_{i_D^{-1}\mathscr{O}_X} i_D^{-1} W_\bullet\mathcal{M}$ induced by the weight filtration of M itself [Sai90, §2.3]. Now Saito's inductive definition of the category of (mixed) Hodge modules in [Sai88, §5.1] and [Sai90, (2.d)] implies the first and third assertion. The second assertion is a special case of Kashiwara's version of the Cauchy-Kovalevskaya theorem [HTT08, Cor. 4.3.4], which says that non-characteristic restriction is compatible with passage to the de Rham complex. □

We end this section by describing the relation between the de Rham complexes of the two mixed Hodge modules M and $H^{-1}i_D^*M$; see [Sch16, (13.3)] for the proof.

Proposition 4.18 (Comparison of de Rham complexes). *In the setting of Theorem 4.16, denote by $(\mathcal{M}_D, F_\bullet\mathcal{M}_D)$ the filtered $\mathscr{D}_D$-module underlying the mixed Hodge module $M_D = H^{-1}i_D^*M$. Given any $p \in \mathbb{Z}$, one has a short exact sequence of complexes*

$$0 \to N^*_{D|X} \otimes_{\mathscr{O}_D} \operatorname{gr}^F_{p+1} \mathrm{DR}(\mathcal{M}_D) \to \mathscr{O}_D \otimes_{\mathscr{O}_Y} \operatorname{gr}^F_p \mathrm{DR}(\mathcal{M}) \to \operatorname{gr}^F_p \mathrm{DR}(\mathcal{M}_D)[1] \to 0,$$

where $N^*_{D|X}$ means the conormal bundle for the inclusion $D \subseteq X$. □

5. A vanishing theorem for intersection complexes

We briefly discuss a vanishing theorem for certain perverse sheaves that applies in particular to intersection complexes. Recall that a perverse sheaf K on a complex manifold Y is, by definition, always *semiperverse*, meaning that

$$\dim \operatorname{Supp} \mathscr{H}^j K \le -j, \quad \text{for every } j \in \mathbb{Z}. \tag{5.1.1}$$

These inequalities can be improved, provided that K does not admit any nontrivial morphisms to perverse sheaves whose support is properly contained in $\operatorname{Supp} K$. This applies for example to the intersection complex on any irreducible complex space, and more generally to the de Rham complex of any polarisable Hodge module with strict support.

Proposition 5.2. *Let K be a perverse sheaf on a complex manifold Y, and assume that* $\operatorname{Supp} K$ *has pure dimension n. Then the following two conditions are equivalent:*

(5.2.1) If L is a perverse sheaf on Y with $\dim \operatorname{Supp} L \le n-1$, *then* $\operatorname{Hom}(K, L) = 0$.
(5.2.2) For every $j \ge -n+1$, one has $\dim \operatorname{Supp} \mathscr{H}^j K \le -(j+1)$.

Proof. Let us show that (5.2.1) implies (5.2.2). Since K is a perverse sheaf, one has $\mathscr{H}^j K = 0$ for $j \le -n-1$, and the inequalities in (5.1.1) imply that $\mathscr{H}^{-n} K$ is supported on all of X, whereas $\dim \operatorname{Supp} \mathscr{H}^j K \le -j$ for every $j \ge -n+1$. If we truncate K with respect to the standard t-structure on $\mathrm{D}^b_c(\mathbb{C}_X)$, the resulting constructible complex $K' := \tau_{\ge -n+1} K$ is still semiperverse, and supported in a complex subspace that is properly contained in X. By (5.2.1), the natural composed morphism

$$K \to K' \to {}^p\mathscr{H}^0 K'$$

to the 0-th cohomology sheaf for the perverse t-structure must therefore be trivial, which implies that the morphism $K \to K'$ factors through $K'' := {}^p\tau_{\leq -1}K'$, truncated with respect to the perverse t-structure on $\mathrm{D}^b_c(\mathbb{C}_X)$. For each $j \geq -n+1$, this gives us a factorisation

$$\mathscr{H}^j K \to \mathscr{H}^j K'' \to \mathscr{H}^j K'$$

of the identity morphism. By construction, $\dim \operatorname{Supp} \mathscr{H}^j K'' \leq -(j+1)$, and therefore also $\dim \operatorname{Supp} \mathscr{H}^j K \leq -(j+1)$ for every $j \geq -n+1$, proving (5.2.2).

It remains to show that, conversely, (5.2.2) implies (5.2.1). Suppose we are given a morphism of perverse sheaves $\varphi \colon K \to L$ with $\dim \operatorname{Supp} L \leq n-1$. After replacing L by $\operatorname{img} \varphi$, we can assume that φ is surjective. As before, we have $\mathscr{H}^j L = 0$ for $j \leq -n$. Now fix some $j \geq -n+1$, and consider the short exact sequence

$$\mathscr{H}^j K \to \mathscr{H}^j L \to \mathscr{H}^{j+1}(\ker \varphi).$$

We have $\dim \operatorname{Supp} \mathscr{H}^j K \leq -(j+1)$ by (5.2.2), and $\dim \operatorname{Supp} \mathscr{H}^{j+1}(\ker \varphi) \leq -(j+1)$ by (5.1.1). Consequently, $\dim \operatorname{Supp} \mathscr{H}^j L \leq -(j+1)$ for every $j \in \mathbb{Z}$, and since L is a perverse sheaf, the properties of the perverse t-structure imply that $L = 0$. □

The following vanishing theorem for the de Rham complex plays a crucial role in the proof of our main theorem, and so we state it as a corollary.

Corollary 5.3. *Let Y be a complex manifold, and let $M \in \mathrm{HM}_X(Y, w)$ be a polarisable Hodge module of weight w with strict support an irreducible complex subspace $X \subseteq Y$. If $F_{c-1}\mathcal{M} = 0$ for some $c \in \mathbb{Z}$, one has $\mathscr{H}^0 F_{\dim Y-(w+c)} \operatorname{DR}(\mathcal{M}) = 0$.*

Proof. According to Proposition 4.7, the complex $\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M})$ is acyclic for $p \geq \dim Y - (w+c)+1$. By Proposition 4.8, this implies that the inclusion of the subcomplex $F_{p_0} \operatorname{DR}(\mathcal{M})$ into $\operatorname{DR}(\mathcal{M})$ is a quasi-isomorphism for $p_0 = \dim Y - (w+c)$. In particular, the inclusion induces an isomorphism $\mathscr{H}^0 F_{p_0} \operatorname{DR}(\mathcal{M}) \cong \mathscr{H}^0 \operatorname{DR}(\mathcal{M})$. But now M has strict support X, and so the perverse sheaf $\operatorname{DR}(\mathcal{M})$ does not have nontrivial quotient objects whose support is properly contained in X. We conclude that $\mathscr{H}^0 \operatorname{DR}(\mathcal{M}) = 0$, by Proposition 5.2. □

6. Coherent sheaves and Mixed Hodge modules

The present section forms the technical core of the present paper. Its main results, Theorem 6.6 and Theorem 6.11, as well as Corollary 6.7 and Corollary 6.12 are criteria to guarantee that sections of certain coherent sheaves derived from the de Rham complex of certain (mixed) Hodge modules on X extend across the singular locus X_{sing}.

6.1. **Extending sections of coherent sheaves.** In this paragraph, we give a homological formulation of the property that sections of a coherent sheaf extend uniquely over a given complex subspace. The material covered here will be known to experts.

Proposition and Definition 6.1 (Extension across subsets)**.** *Let Y be a complex manifold. Let $A \subseteq Y$ be a complex subspace, and let $j \colon Y \setminus A \hookrightarrow Y$ be the open embedding. If $\mathscr{F}$ is a coherent sheaf of $\mathscr{O}_Y$-modules, then the following conditions are equivalent:*

- *(6.1.1) The natural morphism $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism.*
- *(6.1.2) For every $k \in \mathbb{Z}$, one has $\dim\big(A \cap \operatorname{Supp} R^k \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \omega_Y^\bullet)\big) \leq -(k+2)$.*

If these conditions are satisfied, we say that sections of $\mathscr{F}$ extend uniquely across A.

We will often apply Proposition 6.1 in the following form.

Corollary 6.2. *Let Y be a complex manifold, and let $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_Y$-modules. If $\operatorname{Supp} \mathscr{F}$ has pure dimension n, then the following conditions are equivalent:*

- *(6.2.1) Sections of $\mathscr{F}$ extend uniquely across any $A \subseteq Y$ with $\dim A \leq n-2$.*
- *(6.2.2) For every $k \geq -n+1$, one has $\dim \operatorname{Supp} R^k \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \omega_Y^\bullet) \leq -(k+2)$.*

Proof. According to [Sta18, Tag 0A7U], one has $R^k \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \omega_Y^\bullet) = 0$ for every $k \le -n$. If $A \subseteq Y$ is a complex subspace with $\dim A \le n-2$, then of course

$$\dim\bigl(A \cap \operatorname{Supp} R^{-n} \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \omega_Y^\bullet)\bigr) \le n-2,$$

and so the condition in (6.2.2) is equivalent to the condition in (6.1.2). The assertion now follows from Proposition 6.1. □

Before giving the proof of Proposition 6.1, we briefly review some facts about singular sets of coherent sheaves. Let Y be a complex manifold, and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_Y$-modules. Recall that the *singular sets* of $\mathscr{F}$ are defined as

$$S_m(\mathscr{F}) := \bigl\{ y \in Y \bigm| \operatorname{depth}_y \mathscr{F} \le m \bigr\}.$$

The singular sets $S_m(\mathscr{F})$ are closed complex subspaces of Y; we refer the reader to [BS76, Chapt. II.2] for a detailed discussion. The following homological fact about regular local rings [Sta18, Tag 0A7U] relates the singular sets to the dualizing complex. In the smooth case at hand, observe that the dualizing complex of [Sta18, Tag 0A7U] agrees with the analytic dualizing complex, as both equal the canonical bundle shifted by the dimension.

Proposition 6.3 (Singular sets and duality). *If $\mathscr{F}$ is a coherent sheaf of $\mathscr{O}_Y$-modules on a complex manifold Y, then the singular sets of $\mathscr{F}$ are described as*

$$S_m(\mathscr{F}) = \bigcup_{k \ge 0} \operatorname{Supp} R^{k-m} \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \omega_Y^\bullet),$$

where $\omega_Y^\bullet$ is the dualizing complex. □

Proof of Proposition 6.1. We consider the standard exact sequence for sheaves of local cohomology with supports, see for example [BS76, II Cor. 1.10].

$$0 \to \mathscr{H}^0_A \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} \to \mathscr{H}^1_A \mathscr{F} \to 0$$

Because of this sequence, (6.1.1) is equivalent to the condition that $\mathscr{H}^0_A \mathscr{F} = \mathscr{H}^1_A \mathscr{F} = 0$. The vanishing theorem for local cohomology of Scheja-Trautmann [BS76, II Thm. 3.6] relates this to the singular sets of $\mathscr{F}$: it asserts that $\mathscr{H}^0_A \mathscr{F} = \mathscr{H}^1_A \mathscr{F} = 0$ is equivalent to the collection of inequalities

$$\dim\bigl(A \cap S_m(\mathscr{F})\bigr) \le m-2 \quad \text{for all } m \in \mathbb{Z}.$$

But Proposition 6.3 shows that this last line is in turn equivalent to (6.1.2). □

We will later need the following variant of Proposition 6.1 that works for complexes of $\mathscr{O}_Y$-modules rather than single sheaves. We stress that, in the case of a complex with two or more nonzero cohomology sheaves, the condition below is *stronger* than asking that sections of $\mathscr{H}^0 K$ extend uniquely across A.

Proposition 6.4. *Let Y be a complex manifold, let $A \subseteq Y$ be a complex subspace, and let $K \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$ be a complex with $\mathscr{H}^j K = 0$ for $j < 0$. If*

$$\dim\bigl(A \cap \operatorname{Supp} R^k \mathscr{H}om_{\mathscr{O}_Y}(K, \omega_Y^\bullet)\bigr) \le -(k+2) \quad \textit{for every } k \in \mathbb{Z},$$

then sections in $\mathscr{H}^0 K$ extend uniquely across A.

Proof. Let $\tau_{\ge 1} K$ denote the truncation of the complex K in cohomological degree ≥ 1. In the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$, one has a distinguished triangle

$$\mathscr{H}^0 K \to K \to \tau_{\ge 1} K \to \bigl(\mathscr{H}^0 K\bigr)[1].$$

After applying the functor $\mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}(-, \omega_Y^\bullet)$ and taking cohomology, we obtain the following exact sequence:

$$R^k \mathscr{H}om_{\mathscr{O}_Y}\bigl(K, \omega_Y^\bullet\bigr) \to R^k \mathscr{H}om_{\mathscr{O}_Y}\bigl(\mathscr{H}^0 K, \omega_Y^\bullet\bigr) \to R^{k+1} \mathscr{H}om_{\mathscr{O}_Y}\bigl(\tau_{\ge 1} K, \omega_Y^\bullet\bigr)$$

Thus $A \cap \operatorname{Supp} R^k \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{H}^0 K, \omega_Y^\bullet)$ is contained in the union of the two sets

$$A \cap \operatorname{Supp} R^k \mathscr{H}om_{\mathscr{O}_Y}(K, \omega_Y^\bullet) \quad \text{and} \quad \operatorname{Supp} R^{k+1} \mathscr{H}om_{\mathscr{O}_Y}(\tau_{\geq 1} K, \omega_Y^\bullet)$$

By assumption, the dimension of the first set is at most $-(k+2)$ for every $k \in \mathbb{Z}$. As $\tau_{\geq 1} K \in \mathrm{D}^{\geq 1}_{coh}(\mathscr{O}_Y)$, the same is true for the second set; this follows from [Sta18, Tag 0A7U] by considering the spectral sequence

$$E_2^{p,q} = R^p \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{H}^{-q} \tau_{\geq 1} K, \omega_Y^\bullet) \Longrightarrow R^{p+q} \mathscr{H}om_{\mathscr{O}_Y}(\tau_{\geq 1} K, \omega_Y^\bullet).$$

We conclude the proof by applying Proposition 6.1 to the coherent $\mathscr{O}_Y$-module $\mathscr{H}^0 K$. □

6.2. **The case of Hodge modules.** In this section, we apply the criteria from Section 6.1 to certain coherent sheaves derived from the de Rham complex of certain Hodge modules. We specify the precise setting first.

Setting 6.5. Let Y be a complex manifold, and let $X \subseteq Y$ be a reduced and irreducible complex subspace of dimension n. Let c be the codimension of the closed embedding $i_X \colon X \hookrightarrow Y$, so that $\dim Y = n + c$. Suppose that $M \in \mathrm{HM}_X(Y, n)$ is a polarisable Hodge module of weight n with strict support equal to X. We denote the underlying filtered left $\mathscr{D}_Y$-module by $(\mathcal{M}, F_\bullet \mathcal{M})$, and make the following assumptions about M.

(6.5.1) One has $F_{c-1} \mathcal{M} = 0$.

(6.5.2) One has $\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}_0^F \mathrm{DR}(\mathcal{M}) \leq -(j+2)$ for every $j \geq -n+1$.

Note. By [Sai90, Thm. 3.21], there is a dense Zariski-open subset of X on which M is a polarisable variation of Hodge structure of weight 0. The condition $F_{c-1}\mathcal{M} = 0$ is equivalent to asking that the variation of Hodge structure is entirely of type $(0,0)$; being polarisable, it must therefore be a unitary flat bundle. Now $F_c \mathcal{M}$ is a certain extension of this unitary flat bundle to a coherent $\mathscr{O}_Y$-module, and (6.5.2) is equivalent to asking that sections of $F_c \mathcal{M}$ extend uniquely over any complex subspace of X of dimension at most $n-2$.

Theorem 6.6 (Inequalities for Hodge modules)**.** *Assume Setting 6.5 and let $p \in \mathbb{Z}$ be any integer. Then one has*

(6.6.1) $$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) \leq -(p+j+2) \quad \textit{for every } j \textit{ with } p+j \geq -n+1.$$

A proof of Theorem 6.6 is given in Section 6.2.1 and Section 6.2.2 below. First, however, we note that the dimension estimates in Theorem 6.6 imply the promised extension property for certain coherent sheaves derived from the de Rham complex.

Corollary 6.7 (Extending sections)**.** *Assume Setting 6.5. Then for any $p \in \mathbb{Z}$, sections of $\mathscr{H}^{-(n-p)} \operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M})$ extend uniquely across any complex subspace of dimension $\leq n-2$.*

Proof. Recall from Proposition 4.7 that $\operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M})$ is acyclic, unless $0 \leq p \leq n$. Assuming that p is in this range, we aim to apply Proposition 6.4 to the complex

$$K_p := \operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M})[p-n],$$

which requires first of all that K_p is contained in $\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$. To this end recall from Assumption (6.5.1) that $F_{c-1}\mathcal{M} = 0$. An application of Formula (4.3.2) for the subquotients of the de Rham complex then shows that

$$\mathscr{H}^j K_p = \mathscr{H}^{j+p-n} \operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M}) \overset{(4.3.2)}{=} 0, \quad \text{for every } j \leq -1.$$

So $K \in \mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$, as desired. Next, choose a polarisation on the Hodge module M, in order to obtain an isomorphism as follows,

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\left(K_p, \omega_Y^\bullet\right) \overset{\text{Corollary 4.6}}{\cong} \operatorname{gr}_{-(n-p)}^F \mathrm{DR}(\mathcal{M})[n-p].$$

The Inequalities (6.6.1) of Theorem 6.6 therefore take the form

$$\dim \operatorname{Supp} R^j \mathscr{H}om_{\mathscr{O}_X}(K_p, \omega_Y^\bullet) = \dim \operatorname{Supp} \mathscr{H}^{j+n-p} \operatorname{gr}_{-(n-p)}^F \mathrm{DR}(\mathcal{M}) \leq -(j+2)$$

for every $j \geq -n+1$. We conclude from Proposition 6.4 that sections of the coherent $\mathscr{O}_Y$-module $\mathscr{H}^0 K_p = \mathscr{H}^{-(n-p)} \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M})$ extend uniquely across any complex subspace $A \subseteq Y$ with $\dim A \leq n-2$. □

6.2.1. *Preparation for proof of Theorem 6.6.* In cases where $p + j \geq \max(-n+1, -1)$, the inequality (6.6.1) in Theorem 6.6 is claiming that $\mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) = 0$. As it turns out, the proof of this special case is the core of the argument; the other cases follow quickly from the following lemma by induction, taking repeated hyperplane sections.

Lemma 6.8. *Assume Setting 6.5. If $p + j \geq \max(-n+1, -1)$, then $\mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) = 0$.*

Proof. The complex $\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M})$ is concentrated in non-positive degrees, and acyclic for $p \geq 1$ by Proposition 4.7 and by Assumption (6.5.1). This means that $\mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) = 0$ whenever $j \geq 1$ or $p \geq 1$. Assumption (6.5.2) implies the claim when $p = 0$. This leaves only one case to consider, namely $p = -1$ and $j = 0$. We shall argue that $\mathscr{H}^0 \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{M}) = 0$, too.

Recall that M has strict support X. Assumption (6.5.1) therefore allows us to apply Corollary 5.3. We obtain $\mathscr{H}^0 F_0 \operatorname{DR}(\mathcal{M}) = 0$. Now consider the short exact sequence of complexes (of sheaves of $\mathbb{C}$-vector spaces)

$$0 \to F_{-1} \operatorname{DR}(\mathcal{M}) \to F_0 \operatorname{DR}(\mathcal{M}) \to \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}) \to 0.$$

Since $\mathscr{H}^j \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}) = 0$ for $j \geq -1$, we get

$$\mathscr{H}^0 F_{-1} \operatorname{DR}(\mathcal{M}) \cong \mathscr{H}^0 F_0 \operatorname{DR}(\mathcal{M}) \overset{\text{Cor. 5.3}}{=} 0 \tag{6.8.1}$$

from the long exact sequence in cohomology. By the same logic, the short exact sequence of complexes (of sheaves of $\mathbb{C}$-vector spaces)

$$0 \to F_{-2} \operatorname{DR}(\mathcal{M}) \to F_{-1} \operatorname{DR}(\mathcal{M}) \to \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{M}) \to 0$$

gives us an exact sequence

$$\cdots \to \underbrace{\mathscr{H}^0 F_{-1} \operatorname{DR}(\mathcal{M})}_{=0 \text{ by (6.8.1)}} \to \mathscr{H}^0 \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{M}) \to \underbrace{\mathscr{H}^1 F_{-2} \operatorname{DR}(\mathcal{M})}_{=0, \text{ since concentr. in non-pos. degrees}} \to \cdots.$$

As a consequence, we obtain the desired vanishing $\mathscr{H}^0 \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{M}) = 0$. □

6.2.2. *Proof of Theorem 6.6.* We prove Theorem 6.6 by induction on $n = \dim X$. If $n = 1$ or $n = 2$, then the desired statement follows from Lemma 6.8 above, and we are done. We will therefore assume for the remainder of the proof that $n \geq 3$, and that Theorem 6.6 is already known for all strictly smaller values of n.

Cutting down. The statement we are trying to prove is local on Y, and so we can assume for the remainder of this proof that Y is an open ball in $\mathbb{C}^{n+c}$. (If the restriction of M no longer has strict support, for example because X was locally reducible, then we simply replace M by any of the summands in the decomposition by strict support, and X by the support of that summand.) Let $H \subseteq Y$ be the intersection of Y with a generic hyperplane in $\mathbb{C}^{n+c}$. The intersection $H \cap X$ is then reduced and irreducible of dimension $n-1 \geq 2$. The inclusion mapping $i_H \colon H \hookrightarrow Y$ is non-characteristic for M, and the inverse image $M_H = H^{-1} i_H^* M$ is a polarisable Hodge module of weight $(n-1)$ with strict support $H \cap X$; see Section 4.4 for a discussion of non-characteristic restriction to smooth hypersurfaces. Denoting the underlying filtered $\mathscr{D}_H$-module by $(\mathcal{M}_H, F_\bullet \mathcal{M}_H)$, we have moreover

$$\mathcal{M}_H \cong \mathscr{O}_H \otimes_{i_H^{-1}\mathscr{O}_Y} i_H^{-1} \mathcal{M} \quad \text{and} \quad F_\bullet \mathcal{M}_H \cong \mathscr{O}_H \otimes_{i_H^{-1}\mathscr{O}_Y} i_H^{-1} F_\bullet \mathcal{M}. \tag{6.9.1}$$

This is explained in Theorem 4.16.

Properties of M_H. The isomorphisms in (6.9.1) imply that $F_{c-1}\mathcal{M}_H = 0$, and so M_H also satisfies Assumption (6.5.1). We claim that M_H also satisfies Assumption (6.5.2). To this end, recall from Proposition 4.18 that there exists a short exact sequence of complexes,

$$0 \to N^*_{H|Y} \otimes_{\mathscr{O}_H} \operatorname{gr}^F_{p+1} \operatorname{DR}(\mathcal{M}_H) \to \mathscr{O}_H \otimes_{\mathscr{O}_Y} \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) \to \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}_H)[1] \to 0, \tag{6.9.2}$$

where $N^*_{H|Y}$ is the conormal bundle for the inclusion $H \subseteq Y$. As $F_{c-1}\mathcal{M}_H = 0$, one shows as before that the complex $\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}_H)$ is acyclic for every $p \geq 1$. This gives us

$$\mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathscr{H}^{j-1} \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}) \cong \mathscr{H}^j \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}_H),$$

and because Assumption (6.5.2) holds for M, we obtain that

$$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}_H) = -1 + \dim \operatorname{Supp} \mathscr{H}^{j-1} \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}) \leq -(j+2)$$

for every $j \geq -\dim(H \cap X) + 1$. But this is exactly (6.5.2) for M_H.

Conclusion. We have established that $M_H \in \operatorname{HM}_{H\cap X}(H, n-1)$ again satisfies the two assumptions in (6.5.1) and (6.5.2). Since $\dim(H \cap X) = n - 1$, we can therefore conclude by induction that

$$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}_H) \leq -(p+j+2), \quad \text{whenever } p + j \geq -(n-1) + 1.$$

Taking cohomology, (6.9.2) gives us an exact sequence of $\mathscr{O}_H$-modules,

$$N^*_{H|Y} \otimes \mathscr{H}^j \operatorname{gr}^F_{p+1} \operatorname{DR}(\mathcal{M}_H) \to \mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) \to \mathscr{H}^{j+1} \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}_H),$$

and therefore the inequality

$$\dim \operatorname{Supp}\big(\mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M})\big) \leq -(p+j+3), \quad \text{whenever } p + j \geq -n + 1.$$

Since $H \subseteq Y$ was a generic hyperplane section of Y, this inequality clearly implies that

$$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) \leq -\min(p+j+2, 0), \quad \text{whenever } p + j \geq -n + 1.$$

This is enough for our purposes, because we have already shown in Lemma 6.8 that $\mathscr{H}^j \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) = 0$ whenever $p + j \geq -1$. The proof of Theorem 6.6 is thus complete. □

6.3. **The case of mixed Hodge modules.** In this section, we generalise Theorem 6.6 and Corollary 6.7 to a certain class of mixed Hodge modules. The results presented here will later be relevant to establish the extension results for logarithmic forms, Theorem 1.2, Theorem 1.5 and Theorem 1.6, as well as the proof of local vanishing, Theorem 1.10. The reader who is primarily interested in the extension for p-forms, Theorem 1.4, might wish to avoid the additional complications arising from the use of mixed Hodge modules and skip this section on first reading.

The main line of argument follows Section 6.2, though there are some noteworthy differences. To keep the text readable, we chose to include full arguments, at the cost of introducing some repetition.

Setting 6.10. Let Y be a complex manifold of pure dimension $n + c$, and let $X \subseteq Y$ be a complex subspace of pure dimension n. As before, c is equal to the codimension of the closed embedding $i_X : X \hookrightarrow Y$. Suppose that $M \in \operatorname{MHM}(Y)$ is a graded-polarisable mixed Hodge module with support equal to X. We denote the underlying filtered left $\mathscr{D}_Y$-module by $(\mathcal{M}, F_\bullet\mathcal{M})$, and make the following assumptions about M:

(6.10.1) One has $\dim \operatorname{Supp} \mathscr{H}^j \operatorname{DR}(\mathcal{M}) \leq -(j+1)$ for every $j \geq -n+1$.
(6.10.2) The complex of $\mathscr{O}_Y$-modules $\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M})$ is acyclic for every $p \geq 1$.
(6.10.3) One has $\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}) \leq -(j+2)$ for every $j \geq -n+1$.

These are the natural generalisations of (6.5.1) and (6.5.2) to the mixed case, formulated in a way that is convenient for a proof by induction on the dimension. As before, write $M' := \mathbb{D}M \in \operatorname{MHM}(Y)$ to denote the dual mixed Hodge module, which is again graded-polarisable, and write $(\mathcal{M}', F_\bullet\mathcal{M}')$ for its underlying filtered left $\mathscr{D}_Y$-module. Recall that the support does not change when taking duals, so $\operatorname{Supp} M' = \operatorname{Supp} M = X$.

Note. The cohomology sheaves of the de Rham complex $\mathrm{DR}(\mathcal{M})$ are constructible sheaves on Y. Since $\mathrm{DR}(\mathcal{M})$ is a perverse sheaf, the dimension of the support of $\mathscr{H}^j \mathrm{DR}(\mathcal{M})$ is always at most $-j$ for every $j \in \mathbb{Z}$. In light of Proposition 5.2, the condition in (6.10.1) is saying that $\mathrm{DR}(\mathcal{M})$ does not admit nontrivial quotients whose support has dimension $\leq n-1$.

Theorem 6.11 (Inequalities for mixed Hodge modules)**.** *Assume Setting 6.10 and let $p \in \mathbb{Z}$ be any integer. Then one has*

(6.11.1) $$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) \leq -(p+j+2) \quad \text{for every } j \text{ with } p+j \geq -n+1.$$

The proof of Theorem 6.11 is given in Section 6.3.1 and Section 6.3.2 below. As before, Theorem 6.11 leads to extension theorems for certain coherent sheaves derived from the de Rham complex.

Corollary 6.12 (Extending sections)**.** *Assume Setting 6.10. Then for any $p \in \mathbb{Z}$, sections of $\mathscr{H}^p \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}')$ extend uniquely across any complex subspace of dimension $\leq n-2$.*

Proof. Write $K_p := \operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M}')[-p]$. As in the proof of Corollary 6.7, we begin by showing that $K_p \in \mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$. To this end, Proposition 4.5, implies that

$$\operatorname{gr}_\ell^F \mathrm{DR}(\mathcal{M}') \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\left(\operatorname{gr}_{-\ell}^F \mathrm{DR}(\mathcal{M}), \omega_Y^\bullet\right) \quad \text{for every } \ell \in \mathbb{Z}.$$

By (6.10.2), this complex is acyclic for all $\ell \leq -1$. In particular, it follows from Lemma 4.4 that $F_{d-1}\mathcal{M}' = 0$. The description (4.3.2) of the graded pieces in the de Rham complex then implies that $\mathscr{H}^j \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}') = 0$ for $j < -p$. In other words, we obtain that $K_p \in \mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$ as desired.

As before, Proposition 4.5 gives isomorphisms

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\left(K_p, \omega_Y^\bullet\right) = \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\left(\operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M}')[-p], \omega_Y^\bullet\right) \cong \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M})[p]$$

With these identifications, the inequalities (6.11.1) in Theorem 6.11 take the form

$$\dim \operatorname{Supp} R^j \mathscr{H}om_{\mathscr{O}_X}\left(K_p, \omega_Y^\bullet\right) = \dim \operatorname{Supp} \mathscr{H}^{j+p} \operatorname{gr}_{-p}^F \mathrm{DR}(\mathcal{M}) \leq -(j+2)$$

for every $j \geq -n+1$. As before, we conclude from Proposition 6.4 that sections of the coherent $\mathscr{O}_Y$-module $\mathscr{H}^0 K_p = \mathscr{H}^p \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}')$ extend uniquely across any complex subspace $A \subseteq Y$ with $\dim A \leq n-2$. □

6.3.1. *Preparation for proof of Theorem 6.11.* In cases where $p+j \geq \max(-n+1,-1)$, the inequality (6.11.1) in Theorem 6.11 is claiming that $\mathscr{H}^j \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) = 0$. We begin by proving that this is indeed the case.

Lemma 6.13. *Assume Setting 6.10. If $p+j \geq \max(-n+1,-1)$, then $\mathscr{H}^j \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) = 0$.*

Proof. The complex $\operatorname{gr}_p^F \mathrm{DR}(\mathcal{M})$ is concentrated in non-positive degrees, and is acyclic for $p \geq 1$ by Assumption (6.10.2). This means that $\mathscr{H}^j \operatorname{gr}_p^F \mathrm{DR}(\mathcal{M}) = 0$ whenever $j \geq 1$ or $p \geq 1$. Assumption (6.10.3) implies the claim when $p = 0$. This leaves only one case to consider, namely $p = -1$ and $j = 0$. We show that $\mathscr{H}^0 \operatorname{gr}_{-1}^F \mathrm{DR}(\mathcal{M}) = 0$, too.

The inclusion $F_0 \mathrm{DR}(\mathcal{M}) \subseteq \mathrm{DR}(\mathcal{M})$ is a quasi-isomorphism; this follows from Assumption (6.10.2) and Proposition 4.8. In particular, the inclusion induces an isomorphism $\mathscr{H}^0 F_0 \mathrm{DR}(\mathcal{M}) \cong \mathscr{H}^0 \mathrm{DR}(\mathcal{M})$. The inequality in (6.10.1) shows that $\mathscr{H}^0 \mathrm{DR}(\mathcal{M}) = 0$, and therefore $\mathscr{H}^0 F_0 \mathrm{DR}(\mathcal{M}) = 0$. Now consider the short exact sequence of complexes (of sheaves of $\mathbb{C}$-vector spaces)

$$0 \to F_{-1} \mathrm{DR}(\mathcal{M}) \to F_0 \mathrm{DR}(\mathcal{M}) \to \operatorname{gr}_0^F \mathrm{DR}(\mathcal{M}) \to 0.$$

Since $\mathscr{H}^j \operatorname{gr}_0^F \mathrm{DR}(\mathcal{M}) = 0$ for $j \geq -1$, we obtain

$$\mathscr{H}^0 F_{-1} \mathrm{DR}(\mathcal{M}) \cong \mathscr{H}^0 F_0 \mathrm{DR}(\mathcal{M}) = 0$$

from the long exact sequence in cohomology. The rest of the proof now proceeds exactly as in Lemma 6.8. □

6.3.2. *Proof of Theorem 6.11.* We prove Theorem 6.11 by induction on $n = \dim X$. If $n = 1$ or $n = 2$, then the desired statement follows from Lemma 6.13 above, and we are done. We will therefore assume for the remainder of the proof that $n \geq 3$, and that Theorem 6.11 is already known for smaller values of n.

Cutting down. The statement we are trying to prove is local on Y, and so we can assume for the remainder of the argument that Y is an open ball in $\mathbb{C}^{n+c}$, and that $X \subseteq Y$ is connected. Let $H \subseteq Y$ be the intersection of Y with a generic hyperplane in $\mathbb{C}^{n+c}$. The intersection $H \cap X$ is then a connected complex subspace of pure dimension $n-1 \geq 2$. The inclusion mapping $i_H \colon H \hookrightarrow Y$ is non-characteristic for M, and the inverse image $M_H = H^{-1} i_H^* M$ is again a graded-polarisable mixed Hodge module with support $H \cap X$; see Theorem 4.16 for the details. Note that the support of $M_H \in \mathrm{MHM}(H)$ still has codimension c in the ambient complex manifold H. Denoting the underlying filtered $\mathscr{D}_H$-module by $(\mathcal{M}_H, F_\bullet \mathcal{M}_H)$, Theorem 4.16 give

$$\mathcal{M}_H \cong \mathscr{O}_H \otimes_{i_H^{-1}\mathscr{O}_Y} i_H^{-1}\mathcal{M} \quad \text{and} \quad F_\bullet \mathcal{M}_H \cong \mathscr{O}_H \otimes_{i_H^{-1}\mathscr{O}_Y} i_H^{-1} F_\bullet \mathcal{M},$$

as well as an isomorphism of perverse sheaves

$$\mathrm{DR}(\mathcal{M}_H) \cong i_H^{-1}\,\mathrm{DR}(\mathcal{M})[-1]. \tag{6.13.1}$$

Properties of M_H. As before, we claim that $M_H \in \mathrm{MHM}(H)$ satisfies all assumptions made in Setting 6.10. We consider the assumptions one by one. Because M satisfies Assumption (6.10.1) and because of the choice of H as a *generic* hyperplane section, (6.13.1) yields

$$\dim \operatorname{Supp} \mathscr{H}^j\,\mathrm{DR}(\mathcal{M}_H) = \dim\Bigl(H \cap \operatorname{Supp} \mathscr{H}^{j-1}\,\mathrm{DR}(\mathcal{M})\Bigr) \leq -(j+1),$$

for every $j \geq -\dim(H \cap X) + 1$. In other words, M_H satisfies (6.10.1) as well.

According Proposition 4.18, one has a short exact sequence of complexes

$$0 \to N^*_{H|Y} \otimes_{\mathscr{O}_H} \mathrm{gr}^F_{p+1}\,\mathrm{DR}(\mathcal{M}_H) \to \mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}) \to \mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}_H)[1] \to 0, \tag{6.13.2}$$

where $N^*_{H|Y}$ is the conormal bundle for the inclusion $H \subseteq Y$. Since $\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}_H)$ is acyclic for $p \gg 0$, and since Assumption (6.10.2) holds for M, we can use descending induction on p to show that $\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}_H)$ is acyclic for every $p \geq 1$, and hence that M_H satisfies (6.10.2). It also follows that

$$\mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathscr{H}^{j-1}\,\mathrm{gr}^F_0\,\mathrm{DR}(\mathcal{M}) \cong \mathscr{H}^j\,\mathrm{gr}^F_0\,\mathrm{DR}(\mathcal{M}_H),$$

and because of Assumption (6.10.3), we get

$$\dim \operatorname{Supp} \mathscr{H}^j\,\mathrm{gr}^F_0\,\mathrm{DR}(\mathcal{M}_H) = -1 + \dim \operatorname{Supp} \mathscr{H}^{j-1}\,\mathrm{gr}^F_0\,\mathrm{DR}(\mathcal{M}) \leq -(j+2)$$

for every $j \geq -\dim(H \cap X) + 1$. But this is exactly (6.10.3) for M_H.

Conclusion. In summary, we have established that $M_H \in \mathrm{MHM}(H)$ also has the three properties in (6.10.1) to (6.10.3), but with $\dim \operatorname{Supp} M_H = \dim(H \cap X) = n-1$. We can therefore conclude by induction on the dimension of the support that

$$\dim \operatorname{Supp} \mathscr{H}^j\,\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}_H) \leq -(p+j+2) \quad \text{whenever } p+j \geq -\dim(H \cap X) + 1.$$

Taking cohomology in the short exact in (6.13.2), we obtain an exact sequence of coherent $\mathscr{O}_H$-modules

$$N^*_{H|Y} \otimes \mathscr{H}^j\,\mathrm{gr}^F_{p+1}\,\mathrm{DR}(\mathcal{M}_H) \to \mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathscr{H}^j\,\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}) \to \mathscr{H}^{j+1}\,\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}_H),$$

and therefore the inequality

$$\dim \operatorname{Supp}\Bigl(\mathscr{O}_H \otimes_{\mathscr{O}_Y} \mathscr{H}^j\,\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M})\Bigr) \leq -(p+j+3) \quad \text{whenever } p+j \geq -\dim X + 1.$$

Since $H \subseteq Y$ was a generic hyperplane section of Y, this inequality clearly implies that

$$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}) \leq -\min(p+j+2, 0) \quad \text{for } p+j \geq -\dim X + 1.$$

This is enough for our purposes, because we have already shown that $\mathscr{H}^j \operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}) = 0$ whenever $p + q \geq -1$. The proof of Theorem 6.11 is thus complete. □

7. Setup for the proof

We will prove the main results of the present paper in the following sections. Since we want to work locally, and since an irreducible complex space is not necessarily locally irreducible, we relax the assumptions a little bit and allow any reduced complex space of pure dimension. Except for Theorem 1.11, the proofs all work in essentially the same setup. We will therefore fix the setup here and introduce notation that will be consistently be used throughout the following sections.

Setup 7.1. *Consider a reduced complex space X of pure dimension n, together with an embedding $i_X : X \hookrightarrow Y$ into an open ball. Choose a strong log resolution $r : \widetilde{X} \to X$ that is projective as a morphism of complex spaces.*

Notation 7.2. We denote dimensions and codimensions by

$$n := \dim X \quad \text{and} \quad c := \operatorname{codim}_Y X,$$

which means that Y is an open ball in $\mathbb{C}^{n+c}$. The assumption that r is a *strong* log resolution implies that X_{reg} is isomorphic to its preimage $r^{-1}(X_{\text{reg}})$. Finally, let $E := r^{-1}(X_{\text{sing}})$ be the reduced r-exceptional set. The assumption that r is a strong log resolution implies that $E \subsetneq \widetilde{X}$ is a divisor with simple normal crossings; we write its irreducible components as $E = \cup_{i \in I} E_i$. The following diagram summarises the relevant morphisms in our setting.

$$\begin{tikzcd}
\widetilde{X} \setminus E \arrow[r, "{j,\ \text{open embedding}}"] \arrow[d, "{r|_{\widetilde{X}^\circ},\ \text{isomorphism}}"'] & \widetilde{X} \arrow[d, "{r,\ \text{strong log resolution}}"'] \arrow[dr, bend left, "f:=i_X\circ r"] & \\
X_{\text{reg}} \arrow[r, "{j,\ \text{open embedding}}"'] & X \arrow[r, "{i_X,\ \text{closed embedding}}"'] & Y
\end{tikzcd}$$

8. Pure Hodge modules and differentials on the resolution

Maintaining the assumptions and notation of Setting 7.1, we explain in this section how the (higher) direct images of $\Omega^p_{\widetilde{X}}$ are related to the intersection complex on X. We begin with a discussion of the constant Hodge module on the complex manifold $\widetilde{X}$.

8.1. **The constant Hodge module on the resolution.** On the complex manifold $\widetilde{X}$, consider the locally constant sheaf $\mathbb{Q}_{\widetilde{X}}$, viewed as a polarised variation of Hodge structure of type $(0,0)$. Following Saito [Sai88, Thm. 5.4.3], we denote by $\mathbb{Q}^H_{\widetilde{X}}[n] \in \operatorname{HM}(\widetilde{X}, n)$ the corresponding polarised Hodge module of weight n; see also [Pop18, Sect. 2, Ex. 4]. Its underlying regular holonomic left $\mathscr{D}_{\widetilde{X}}$-module is $\mathscr{O}_{\widetilde{X}}$, with the usual action by differential operators, and the Hodge filtration $F_\bullet \mathscr{O}_{\widetilde{X}}$ is given by

$$F_p \mathscr{O}_{\widetilde{X}} = \begin{cases} 0 & \text{if } p \leq -1 \\ \mathscr{O}_{\widetilde{X}} & \text{otherwise.} \end{cases}$$

The de Rham complex $\operatorname{DR}(\mathscr{O}_{\widetilde{X}})$, which is quasi-isomorphic to $\mathbb{C}_{\widetilde{X}}[n]$, is

$$\operatorname{DR}(\mathscr{O}_{\widetilde{X}}) = \Big[\mathscr{O}_{\widetilde{X}} \xrightarrow{d} \Omega^1_{\widetilde{X}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\widetilde{X}} \Big][n].$$

It is filtered in the usual way, by degree, and the $(-p)$-th graded piece is then

$$gr^F_{-p} DR(\mathscr{O}_{\widetilde{X}}) \cong \Omega^p_{\widetilde{X}}[n-p]. \tag{8.0.1}$$

Following the discussion in Section 4.3, we consider the direct image $f_+(R_F \mathscr{O}_{\widetilde{X}})$ of the filtered $\mathscr{D}_{\widetilde{X}}$-module $(\mathscr{O}_{\widetilde{X}}, F_\bullet \mathscr{O}_{\widetilde{X}})$, as an object of the bounded derived category of coherent graded $R_F \mathscr{D}_Y$-modules. The direct image functor commutes with taking the associated graded of the de Rham complex by Proposition 4.10, which allows us to identify the graded pieces of the de Rham complex for $f_+(R_F \mathscr{O}_{\widetilde{X}})$ as

$$\operatorname{gr}^F_{-p} \operatorname{DR}\Big(f_+(R_F \mathscr{O}_{\widetilde{X}})\Big) \cong \mathbf{R} f_* \operatorname{gr}^F_{-p} \operatorname{DR}(\mathscr{O}_{\widetilde{X}}) \cong \mathbf{R} f_* \Omega^p_{\widetilde{X}}[n-p]. \tag{8.0.2}$$

8.2. **The intersection complex of X.** Consider the constant variation of Hodge structure of type $(0,0)$ on X_{reg}. By Saito's fundamental theorem [Sai90, Thm. 3.21], applied to each irreducible component of the complex space X, it determines a polarised Hodge module $M_X \in \operatorname{HM}(Y, n)$ of weight $n = \dim X$ on the complex manifold Y, with support equal to X. Its underlying perverse sheaf is the intersection complex of X. Denoting the filtered regular holonomic $\mathscr{D}_Y$-module underlying M_X by $(\mathcal{M}_X, F_\bullet \mathcal{M}_X)$, we have $F_{c-1}\mathcal{M}_X = 0$ by construction. The de Rham complex $\operatorname{DR}(\mathcal{M}_X)$ is again filtered, and its subquotients are

$$\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}_X) = \Big[\operatorname{gr}^F_p \mathcal{M}_X \to \Omega^1_Y \otimes \operatorname{gr}^F_{p+1} \mathcal{M}_X \to \cdots \to \Omega^{n+c}_Y \otimes \operatorname{gr}^F_{p+n+c} \mathcal{M}_X\Big][n+c].$$

Note that this complex is concentrated in degrees $-(n+c), \ldots, 0$.

8.3. **Decomposition.** As discussed in Section 4.3, the fact that the holomorphic mapping $f\colon \widetilde{X} \to Y$ is projective implies that each $H^\ell f_* \mathbb{Q}^H_{\widetilde{X}}[n]$ is again a polarisable Hodge module of weight $n+\ell$ on Y. Using the decomposition by strict support, we obtain moreover

$$H^\ell f_* M_{\widetilde{X}} \cong \begin{cases} M_X \oplus M_0 & \text{if } \ell = 0, \\ M_\ell & \text{if } \ell \neq 0, \end{cases}$$

where $M_X \in \operatorname{HM}(Y, n)$ is as above, and where the other summands $M_\ell \in \operatorname{HM}(Y, n+\ell)$ are polarisable Hodge modules on Y whose support is contained inside X_{sing}. Denoting the associated $\mathscr{D}_Y$-modules by $\mathcal{M}_\ell$, the properties of the direct image functor imply that $F_c \mathcal{M}_\ell = 0$, as a special case of Proposition 4.14.

Note. For dimension reasons, one has $M_\ell = 0$ once $|\ell|$ is greater than the "defect of semismallness" of $r\colon \widetilde{X} \to X$; in particular, this holds for $|\ell| \geq n-1$.

8.4. **Relation with differential forms.** Saito's version of the Decomposition Theorem, Corollary 4.12, together with the isomorphism in (8.0.2), allows us to identify, for every $p \in \mathbb{Z}$, the derived push forward of the sheaf of p-forms on $\widetilde{X}$ as

$$\mathbf{R} f_* \Omega^p_{\widetilde{X}}[n-p] \cong \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M}_X) \oplus \bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M}_\ell)\,[-\ell]. \tag{8.0.3}$$

In the situation at hand, the relation between $f_* \Omega^p_{\widetilde{X}}$ and the intersection complex of X is an almost direct consequence of the isomorphism in (8.0.3) above.

Proposition 8.1. *Maintaining Setting 7.1 and using the notation introduced above, we have*

$$f_* \Omega^p_{\widetilde{X}} \cong \mathscr{H}^{-(n-p)} \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M}_X) \quad \text{for every } p \in \mathbb{Z}.$$

Proof. Recall from (8.0.3) that we have a decomposition

$$\mathbf{R} f_* \Omega^p_{\widetilde{X}}[n-p] \cong \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M}_X) \oplus \operatorname{Rest}_p,$$

in which the support of the complex $\operatorname{Rest}_p \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$ is contained inside X_{sing}. Taking cohomology in degree $-(n-p)$, we get

$$f_* \Omega^p_{\widetilde{X}} \cong \underbrace{\mathscr{H}^{-(n-p)} \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M}_X)}_{=:A} \oplus \underbrace{\mathscr{H}^{-(n-p)} \operatorname{Rest}_p}_{=:B},$$

and therefore $f_* \Omega^p_{\tilde{X}}$ is the direct sum of A and a coherent $\mathscr{O}_X$-module B supported on X_{sing}. The claim follows because $\Omega^p_{\tilde{X}}$ is torsion free: the functor f^* is a left adjoint for f_*, and the adjoint morphism $f^*\mathrm{B} \to \Omega^p_{\tilde{X}}$ vanishes because $f^*\mathrm{B}$ is supported on $f^{-1}(X_{\mathrm{sing}})$. □

Note. The proof shows once again that $F_c\mathcal{M}_\ell = 0$ for every $\ell \in \mathbb{Z}$. (Use Lemma 4.4.) This fact is also proved in much greater generality in [Sai91, Prop. 2.6].

The two values $p = n$ and $p = 0$ are special, because there is no contribution from the Hodge modules M_ℓ in those cases.

Proposition 8.2. *Maintaining Setting 7.1 and using the notation introduced above, we have*

$$\mathbf{R}f_*\Omega^n_{\tilde{X}} \cong \mathrm{gr}^F_{-n} \mathrm{DR}(\mathcal{M}_X) \quad \text{and} \quad \mathbf{R}f_*\mathscr{O}_{\tilde{X}}[n] \cong \mathrm{gr}^F_0 \mathrm{DR}(\mathcal{M}_X).$$

Proof. By Proposition 4.14, we have $F_c\mathcal{M}_\ell = 0$ for every $\ell \in \mathbb{Z}$, and so $\mathrm{gr}^F_{-n}\mathrm{DR}(\mathcal{M}_\ell) = 0$. Together with (8.0.3), this implies the first isomorphism. The second isomorphism follows by duality, using Corollary 4.6 and the fact that $M_X \in \mathrm{HM}_X(Y, n)$. □

The higher direct images of $\Omega^p_{\tilde{X}}$ can of course also be computed from (8.0.3), but they generally involve some of the other terms M_ℓ. We give one example, in the special case $p = 1$, that will serve to illustrate the general technique.

Proposition 8.3. *Maintaining Setting 7.1 and using the notation introduced above, we have*

$$R^{n-1}f_*\Omega^1_{\tilde{X}} \cong \mathscr{H}^0\, \mathrm{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_X) \oplus \mathscr{H}^0\, \mathrm{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_0).$$

Proof. Formula (8.0.3) identifies the left side of the desired equality as

$$R^{n-1}f_*\Omega^1_{\tilde{X}} \cong \mathscr{H}^0\, \mathrm{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_X) \oplus \bigoplus_{\ell \geq 0} \mathscr{H}^{-\ell}\, \mathrm{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_\ell).$$

To prove Proposition 8.3, it is therefore enough to show that $\mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{M}_\ell)$ is acyclic for every $\ell \geq 1$. But using the fact that the Hodge modules $M_\ell \in \mathrm{HM}(Y, n+\ell)$ are polarisable of weight $n + \ell$, Corollary 4.6 yields

$$\mathrm{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_\ell) \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\Big(\mathrm{gr}^F_{1-(n+\ell)} \mathrm{DR}(\mathcal{M}_\ell), \omega^\bullet_Y\Big).$$

Now a look back at the description of the filtration on the de Rham complex, in (4.3.1), reveals that the complex $\mathrm{gr}^F_{1-(n+\ell)}\mathrm{DR}(\mathcal{M}_\ell)$ only involves the $\mathscr{O}_Y$-modules $\mathrm{gr}^F_k \mathcal{M}_\ell$ with $k \leq c + 1 - \ell$. As $F_c\mathcal{M}_\ell = 0$, it follows that $\mathrm{gr}^F_{1-(n+\ell)}\mathrm{DR}(\mathcal{M}_\ell) = 0$ for every $\ell \geq 1$. □

8.5. **Application to the extension problem.** We conclude this section with a brief discussion of the effect that extendability of n-forms has on $\mathrm{DR}(\mathcal{M}_X)$ and its subquotients. The following result, together with Corollary 6.7, can be used to prove that if n-forms extend, then all forms extend. As explained in Section 2.2, this gives another proof for Theorem 1.4 in the (most important) case $k = n$.

Proposition 8.4 (Extension of n-forms and M_X)**.** *Maintaining Setting 7.1 and using the notation introduced above, assume that $r_*\Omega^n_{\tilde{X}} \hookrightarrow j_*\Omega^n_{X_{\mathrm{reg}}}$ is an isomorphism. Then one has*

$$\dim \mathrm{Supp}\, \mathscr{H}^j\, \mathrm{gr}^F_p \mathrm{DR}(\mathcal{M}_X) \leq -(j + p + 2)$$

for all integers $p, j \in \mathbb{Z}$ with $p + j \geq -n + 1$.

Proof. After replacing the Hodge module $M_X \in \mathrm{HM}(Y, n)$ by any of the summands in its decomposition by strict support, and X by the support of that summand, we may assume without loss of generality that X is reduced, irreducible, and n-dimensional, and that M_X has strict support X; in symbols, $M_X \in \mathrm{HM}_X(Y, w)$. We aim to apply Theorem 6.6. Recalling from Section 8.2 that $F_{c-1}\mathcal{M}_X = 0$, where $c = \dim Y - \dim X$, all the conditions in Theorem 6.6 hold in our context, provided we manage to prove the inequalities

$$\dim \mathrm{Supp}\, \mathscr{H}^\ell\, \mathrm{gr}^F_0 \mathrm{DR}(\mathcal{M}_X) \leq -(\ell + 2)$$

for every number $\ell \geq -n+1$. But we have

$$\begin{aligned} -(\ell+2) &\geq \dim \operatorname{Supp} R^\ell \mathscr{H}om_{\mathscr{O}_Y}\big(f_*\Omega^n_{\widetilde{X}}, \omega^\bullet_Y\big) && \text{by Corollary 6.2} \\ &= \dim \operatorname{Supp} R^\ell \mathscr{H}om_{\mathscr{O}_Y}\big(\operatorname{gr}^F_{-n} \operatorname{DR}(\mathcal{M}_X), \omega^\bullet_Y\big) && \text{by Proposition 8.2} \\ &= \dim \operatorname{Supp} \mathscr{H}^\ell \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}_X) && \text{by Corollary 4.6} \end{aligned}$$

This completes the proof. □

9. Mixed Hodge modules and log differentials on the resolution

We maintain the assumptions and notation of Setting 7.1. While the direct images of $\Omega^p_{\widetilde{X}}$ are described in terms of the pure Hodge modules discussed in the previous Section 8, the study of logarithmic differentials requires us to look at certain mixed Hodge modules. As with Section 6.3, we feel that readers who are primarily interested in extension results for (non-logarithmis) p-forms, Theorem 1.4 and related results, might consider skipping this section on first reading.

9.1. **The mixed Hodge module on the complement of the exceptional divisor.** Recall that X is a reduced complex space of pure dimension n, and that $r\colon \widetilde{X} \to X$ is a log resolution with exceptional divisor E. We denote by $j\colon \widetilde{X} \setminus E \hookrightarrow \widetilde{X}$ the open embedding of the complement of the normal crossing divisor E. By analogy with the argument in Section 8.1, we consider the constant Hodge module $\mathbb{Q}^H_{\widetilde{X}\setminus E}[n]$ on the complement of E, and its extension to a mixed Hodge module

$$j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n] \in \operatorname{MHM}(\widetilde{X})$$

on $\widetilde{X}$, as discussed in [Sai90, Thm. 3.27]. For the reader's convenience, we summarise its main properties, properly translated to our convention of using left $\mathscr{D}$-modules.

9.1.1. *Perverse sheaf and filtered $\mathscr{D}$-module.* The underlying perverse sheaf of the mixed Hodge module $j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n]$ is, by construction, $\mathbf{R}j_*\mathbb{Q}_{\widetilde{X}\setminus E}[n]$. The underlying regular holonomic $\mathscr{D}_{\widetilde{X}}$-module is $\mathscr{O}_{\widetilde{X}}(*E)$, the sheaf of meromorphic functions on the complex manifold $\widetilde{X}$ that are holomorphic outside the normal crossing divisor E. The Hodge filtration is given by

$$F_p\mathscr{O}_{\widetilde{X}}(*E) = \begin{cases} 0 & \text{if } p \leq -1, \\ F_p\mathscr{D}_{\widetilde{X}} \cdot \mathscr{O}_{\widetilde{X}}(E) & \text{if } p \geq 0. \end{cases}$$

The de Rham complex of $\mathscr{O}_{\widetilde{X}}(*E)$ is the complex of meromorphic differential forms

$$\operatorname{DR}\big(\mathscr{O}_{\widetilde{X}}(*E)\big) = \Big[\mathscr{O}_{\widetilde{X}}(*E) \xrightarrow{d} \Omega^1_{\widetilde{X}}(*E) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\widetilde{X}}(*E)\Big][n],$$

placed in degrees $-n, \dots, 0$ as always. Saito [Sai90, Prop. 3.11] has shown that this complex, with the filtration induced by $F_p\mathscr{O}_{\widetilde{X}}(*E)$, is filtered quasi-isomorphic to the log de Rham complex $\Omega^\bullet_{\widetilde{X}}(\log E)[n]$, with the usual filtration by degree; in fact, the Hodge filtration on $\mathscr{O}_{\widetilde{X}}(*E)$ is defined so as to make this true.

Proposition 9.1. *Maintaining Setting 7.1 and using the notation introduced above, the natural inclusion $\Omega^\bullet_{\widetilde{X}}(\log E)[n] \hookrightarrow \operatorname{DR}\big(\mathscr{O}_{\widetilde{X}}(*E)\big)$ is a filtered quasi-isomorphism. In particular, we have canonical isomorphisms*

$$\Omega^p_{\widetilde{X}}(\log E)[n-p] \cong \operatorname{gr}^F_{-p}\Omega^\bullet_{\widetilde{X}}(\log E)[n] \cong \operatorname{gr}^F_{-p}\operatorname{DR}\big(\mathscr{O}_{\widetilde{X}}(*E)\big). \qquad \square$$

9.1.2. *Weight filtration.* The weight filtration on the mixed Hodge module $j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n]$ is governed by how the components of the normal crossing divisor E intersect. Since this fact is not explicitly mentioned in [Sai90, Thm. 3.27], we include a precise statement and a proof.

Proposition 9.2 (Description of weight filtration)**.** *Maintaining Setting 7.1 and using the notation introduced above, the first pieces of the weight filtration on the mixed Hodge module $j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n]$ of the filtrations are given by*

$$W_{n-1}\, j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n] = 0 \qquad \textit{and} \qquad W_n\, j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n] \cong \mathbb{Q}^H_{\widetilde{X}}[n].$$

Likewise, for $\ell \geq 1$, the Hodge module $\mathrm{gr}^W_{n+\ell}\, j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n] \in \mathrm{HM}(\widetilde{X}, n+\ell)$ is isomorphic to the direct sum, over all subsets $J \subseteq I$ of size ℓ, of the Hodge modules

$$\mathbb{Q}^H_{E_J}(-\ell)[n-\ell] \in \mathrm{HM}(E_J, n+\ell),$$

pushed forward from the complex submanifold $E_J := \bigcap_{i \in J} E_i$ into $\widetilde{X}$.

Proof. One possibility is to factor j_* as a composition of open embeddings over the irreducible components of the simple normal crossing divisor E, as in [Sai90, Thm. 3.27]. Here, we explain a different argument, based on Saito's computation of the nearby cycles functor in the normal crossing case [Sai90, Thm. 3.3].

To begin with, we observe that the weight filtration on a graded-polarisable mixed Hodge module is, even locally, unique: the reason is that there are no nontrivial morphisms between polarisable Hodge modules of different weights. This reduces the problem to the case where $\widetilde{X}$ is a polydisk, say with coordinates $x_1, \dots, x_n$, and where E is the divisor $g = x_1 \cdots x_r = 0$. Moreover, it is enough to prove the statement for the underlying $\mathscr{D}$-modules. Indeed, by [Sai88, Thm. 3.21], every polarisable Hodge module on $\widetilde{X}$, whose underlying $\mathscr{D}$-module is the direct image of $\mathscr{O}_{E_J}$, comes from a polarisable variation of Hodge structure on E_J, hence must be isomorphic to the push forward of $\mathbb{Q}^H_{E_J}(k)$ for some $k \in \mathbb{Z}$. The Tate twist is then determined by the weight, because $n + \ell = \dim E_J + k$.

After embedding $\widetilde{X}$ into $\widetilde{X} \times \mathbb{C}$, via the graph of $g = x_1 \cdots x_r$, we have, according to [Sai90, (2.11.10)], that

$$\mathrm{gr}^W_{n+\ell}\, j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n] \cong \begin{cases} 0 & \text{if } \ell < 0, \\ \mathbb{Q}^H_{\widetilde{X}}[n] & \text{if } \ell = 0, \\ P_N\, \mathrm{gr}^W_{n+\ell-2}\, \psi_{g,1} \mathbb{Q}^H_{\widetilde{X}}[n](-1) & \text{if } \ell > 0, \end{cases}$$

where $\psi_{g,1}$ denotes the nearby cycles functor (with respect to the coordinate function t on $\widetilde{X} \times \mathbb{C}$). In our normal crossing setting, the nearby cycles functor is computed explicitly in [Sai90, Thm. 3.3]. In the notation introduced in [Sai90, §3.4], the right $\mathscr{D}_{\widetilde{X}}$-module associated to $\mathscr{O}_{\widetilde{X}}$ is isomorphic to $M(\mu, \varnothing)$, where $\mu = (-1, \dots, -1) \in \mathbb{Z}^n$. By [Sai90, (3.5.4)], the right $\mathscr{D}_{\widetilde{X}}$-module underlying $P_N\, \mathrm{gr}^W_{n+\ell-2}\, \psi_{g,1} \mathbb{Q}^H_{\widetilde{X}}[n](-1)$ is therefore isomorphic to the direct sum of $M(\mu, J)$, where $J \subseteq \{1, \dots, r\}$ runs over all subsets of size ℓ. But $M(\mu, J)$ is exactly the right $\mathscr{D}_{\widetilde{X}}$-module associated to the push forward of $\mathscr{O}_{E_J}$, and so we get the desired result. □

9.2. **Push forward to Y.** Recall that $f : \widetilde{X} \to Y$ is the projective holomorphic mapping obtained by composing our resolution of singularities $r : \widetilde{X} \to X$ with the closed embedding $i_X : X \hookrightarrow Y$. We now define a family of mixed Hodge modules $N_\ell \in \mathrm{MHM}(Y)$, indexed by $\ell \in \mathbb{Z}$, by setting

$$N_\ell := H^\ell f_* \Big(j_* \mathbb{Q}^H_{\widetilde{X} \setminus E}[n] \Big).$$

Note that each N_ℓ is again a graded-polarisable mixed Hodge module on Y, due to the fact that f is a projective morphism (see Theorem 4.11). Clearly, $\operatorname{Supp} N_0 = X$, and $\operatorname{Supp} N_\ell \subseteq X_{\mathrm{sing}}$ for $\ell \neq 0$.

Lemma 9.3. *Maintaining Setting 7.1 and using the notation introduced above, we have $N_\ell = 0$ for $\ell \leq -1$. The mixed Hodge module N_0 has no nontrivial subobjects whose support is contained in X_{sing}.*

Proof. It suffices to prove this for the underlying perverse sheaves $\operatorname{rat} N_\ell$. By construction, $\operatorname{rat} N_\ell$ is the ℓ-th perverse cohomology sheaf of the constructible complex

$$\mathbf{R}f_*\big(j_*\mathbb{Q}_{\widetilde{X}\setminus E}[n]\big) \cong \mathbf{R}j_*\mathbb{Q}_{X_{\mathrm{reg}}}[n].$$

Now, if $K \in \mathrm{D}^b_c(\mathbb{Q}_X)$ is any constructible complex, then

$$\operatorname{Hom}_{\mathrm{D}^b_c(\mathbb{Q}_X)}\Big(K,\, \mathbf{R}j_*\mathbb{Q}_{X_{\mathrm{reg}}}[n]\Big) \cong \operatorname{Hom}_{\mathrm{D}^b_c(\mathbb{Q}_{X_{\mathrm{reg}}})}\Big(j^{-1}K,\, \mathbb{Q}_{X_{\mathrm{reg}}}[n]\Big),$$

and the right-hand side vanishes if $\operatorname{Supp} K \subseteq X_{\mathrm{sing}}$. The first assertion of Lemma 9.3 thus follows by taking $K = \operatorname{rat} N_\ell[-\ell]$ for $\ell \leq -1$. Once it is known that $N_\ell = 0$ for $\ell \leq -1$, the second assertion follows by taking K to be any subobject of $\operatorname{rat} N_0$. □

Each mixed Hodge module N_ℓ has weight $\geq n + \ell$, in the following sense.

Lemma 9.4. *Maintaining Setting 7.1 and using the notation introduced above, we have $W_{n+\ell-1}N_\ell = 0$. The module $W_{n+\ell}N_\ell$ is a quotient of $H^\ell f_*\mathbb{Q}^H_{\widetilde{X}}[n]$.*

Proof. This is proved in [Sai90, Prop. 2.26]. For the convenience of the reader, we explain how to deduce it from the degeneration of the weight spectral sequence in Theorem 4.13. Since f is a projective morphism, the weight spectral sequence

$$E_1^{p,q} = H^{p+q}f_*\operatorname{gr}^W_{-p} j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n] \Longrightarrow N_{p+q}$$

degenerates at E_2, and the induced filtration on N_ℓ is the weight filtration $W_\bullet N_\ell$. More precisely, $E_1^{p,q}$ and $E_2^{p,q}$ are Hodge modules of weight q, and

$$\operatorname{gr}^W_q N_{p+q} \cong E_2^{p,q}.$$

As $j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n]$ has weight $\geq n$, we have $E_1^{p,q} = 0$ for $p \geq -n+1$, whence $\operatorname{gr}^W_w N_\ell = 0$ for $w \leq n+\ell-1$. This also shows that $W_{n+\ell}N_\ell$ is a quotient of $E_1^{-n,n+\ell} = H^\ell f_*\mathbb{Q}^H_{\widetilde{X}}[n]$. □

9.3. **Relation with logarithmic differentials on the resolution.** Now we can relate the coherent $\mathscr{O}_Y$-module $f_*\Omega^p_{\widetilde{X}}(\log E)$ to the de Rham complex of the mixed Hodge module N_0. In line with the notation used before, write $(\mathcal{N}_\ell, F_\bullet\mathcal{N}_\ell)$ for the filtered regular holonomic $\mathscr{D}_Y$-module underlying the mixed Hodge module N_ℓ.

Proposition 9.5. *Maintaining Setting 7.1 and using the notation introduced above, we have $f_*\Omega^p_{\widetilde{X}}(\log E) \cong \mathscr{H}^{p-n}\operatorname{gr}^F_{-p}\operatorname{DR}(\mathcal{N}_0)$ for every $p \in \mathbb{Z}$.*

Proof. Fix an integer $p \in \mathbb{Z}$. Proposition 9.1, together with Proposition 4.10 about the compatibility of the de Rham complex with direct images, implies that

$$\mathbf{R}f_*\Omega^p_{\widetilde{X}}(\log E)[n-p] \cong \operatorname{gr}^F_{-p}\operatorname{DR}\Big(f_+\big(R_F\mathscr{O}_{\widetilde{X}}(*E)\big)\Big).$$

Because the complex computing the direct image is strict by Theorem 4.13, we have a convergent spectral sequence

$$E_2^{a,b} = \mathscr{H}^a\operatorname{gr}^F_{-p}\operatorname{DR}(\mathcal{N}_b) \Longrightarrow \mathscr{H}^{a+b}\operatorname{gr}^F_{-p}\operatorname{DR}\Big(f_+\big(R_F\mathscr{O}_{\widetilde{X}}(*E)\big)\Big),$$

and we are interested in the terms with $a + b = p - n$. Proposition 4.14 guarantees that $F_{c-1}\mathcal{N}_\ell = 0$ for every $\ell \in \mathbb{Z}$, whence $E_2^{a,b} = 0$ for $a \leq p - n - 1$. Also, $\mathcal{N}_\ell = 0$ for $\ell \leq -1$ by Lemma 9.3, and so $E_2^{a,b} = 0$ for $b \leq -1$. The spectral sequence therefore gives us the desired isomorphism. □

The analysis of the higher direct images quickly gets complicated. For that reason, we shall only consider what happens in the case of 1-forms with log poles. Here, one has the following simple relation between $\mathbf{R}f_*\Omega^1_{\tilde{X}}(\log E)$ and the complex $\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0)$.

Proposition 9.6. *Maintaining Setting 7.1 and using the notation introduced above, we have a canonical isomorphism*

$$\mathbf{R}f_*\Omega^1_{\tilde{X}}(\log E)[n-1] \cong \mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0).$$

In particular, $R^{n-1}f_*\Omega^1_{\tilde{X}}(\log E) \cong \mathscr{H}^0\,\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0)$.

The proof of Proposition 9.6 relies on the following lemma, which we discuss first.

Lemma 9.7. *Maintaining Setting 7.1 and using the notation introduced above, the complex* $\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_\ell)$ *is acyclic for every* $\ell \neq 0$.

Proof. Recall from Lemma 9.4 that $N_\ell \in \mathrm{MHM}(Y)$ has weight $\geq n+\ell$, which means that $\mathrm{gr}^W_w N_\ell = 0$ for $w \leq n+\ell-1$. Proposition 4.14 guarantees that $F_{c-1}\mathcal{N}_\ell = 0$ for every $\ell \geq 0$. This implies that $F_{c-1}\,\mathrm{gr}^W_w \mathcal{N}_\ell = 0$ for every $w \in \mathbb{Z}$. According to Corollary 4.6, we have

$$\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_\ell) \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\Big(\mathrm{gr}^F_{1-w}\,\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_\ell), \omega^\bullet_Y\Big),$$

and the complex $\mathrm{gr}^F_{1-w}\,\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_\ell)$ only uses the $\mathscr{O}_Y$-modules $\mathrm{gr}^F_p\,\mathrm{gr}^W_w \mathcal{N}_\ell$ in the range

$$1-w \leq p \leq 1-w+\dim Y = c+1-\ell-\big(w-(n+\ell)\big).$$

As $F_{c-1}\,\mathrm{gr}^W_w \mathcal{N}_\ell = 0$ and $\ell \geq 1$, we see that $\mathrm{gr}^F_{1-w}\,\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_\ell) = 0$, except maybe in the special case $w = n+\ell$. But by the E_2-degeneration of the weight spectral sequence, $\mathrm{gr}^W_{n+\ell} N_\ell$ is a quotient of $M_\ell = H^\ell f_*\mathbb{Q}^H_{\tilde{X}}[n]$, and since we already know that $F_c\mathcal{M}_\ell = 0$, we also have $F_c\,\mathrm{gr}^W_{n+\ell}\mathcal{N}_\ell = 0$ for $\ell \geq 1$. This proves that $\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_\ell)$ is acyclic for every $\ell \geq 1$ and every $w \in \mathbb{Z}$. Since the functor $\mathrm{gr}^F_{-1}\,\mathrm{DR}$ is exact on mixed Hodge modules, it follows that the complex $\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_\ell)$ is also acyclic. □

Proof of Proposition 9.6. Because $\mathcal{N}_j = 0$ for $j \leq -1$, and because the complex computing the direct image is strict by Theorem 4.13, we have a canonical morphism

$$(\mathcal{N}_0, F_\bullet\mathcal{N}_0) \to f_+\big(R_F\mathscr{O}_{\tilde{X}}(*E)\big)$$

in the derived category $\mathrm{D}^b_{coh}G(R_F\mathscr{D}_Y)$. As a first step, we are going to show that the induced morphism

$$\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0) \to \mathrm{gr}^F_{-1}\,\mathrm{DR}\Big(f_+\big(R_F\mathscr{O}_{\tilde{X}}(*E)\big)\Big) \tag{9.7.1}$$

between complexes of $\mathscr{O}_Y$-modules is a quasi-isomorphism. Lemma 9.7 implies that the spectral sequence

$$E_2^{a,b} = \mathscr{H}^a\,\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_b) \Longrightarrow \mathscr{H}^{a+b}\,\mathrm{gr}^F_{-1}\,\mathrm{DR}\Big(f_+\big(R_F\mathscr{O}_{\tilde{X}}(*E)\big)\Big),$$

degenerates at E_2, and so we have a collection of isomorphisms

$$\mathscr{H}^a\,\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0) \cong \mathscr{H}^a\,\mathrm{gr}^F_{-1}\,\mathrm{DR}\Big(f_+\big(R_F\mathscr{O}_{\tilde{X}}(*E)\big)\Big).$$

These isomorphisms are induced by the morphism in (9.7.1), which is therefore a quasi-isomorphism. Now the compatibility of the de Rham complex with direct images, together with Proposition 9.1, implies that

$$\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0) \cong \mathbf{R}f_*\,\mathrm{gr}^F_{-1}\,\mathrm{DR}\big(\mathscr{O}_{\tilde{X}}(*E)\big) \cong \mathbf{R}f_*\Omega^1_{\tilde{X}}(\log E)[n-1],$$

as asserted by the proposition. □

9.4. **The weight filtration on N_0.** We describe how the weight filtration interacts with the complex $\mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{N}_0)$.

Proposition 9.8 (The complex $\mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{N}_0)$)**.** *Maintaining Setting 7.1 and using the notation introduced above, the complex $\mathrm{gr}^F_{-1}\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_0)$ is acyclic for $w \notin \{n, n+1\}$ and*

$$\mathrm{gr}^F_{-1}\mathrm{DR}(\mathrm{gr}^W_n \mathcal{N}_0) \cong \mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{M}_X) \tag{9.8.1}$$

$$\mathrm{gr}^F_{-1}\mathrm{DR}(\mathrm{gr}^W_{n+1}\mathcal{N}_0) \cong \bigoplus_{i\in I} \mathbf{R}f_*\mathscr{O}_{E_i}[n-1]. \tag{9.8.2}$$

Proof. Consider again the weight spectral sequence

$$E_1^{p,q} = H^{p+q} f_* \mathrm{gr}^W_{-p} j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n] \Longrightarrow N_{p+q}.$$

Because f is projective, the spectral sequence degenerates at E_2, and the induced filtration on N_ℓ is the weight filtration $W_\bullet N_\ell$, see Theorem 4.13. More precisely, what happens is that $E_1^{p,q}$ and $E_2^{p,q}$ are polarisable Hodge modules of weight q, and

$$\mathrm{gr}^W_q N_{p+q} \cong E_2^{p,q}.$$

Now $j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n]$ has weight $\geq n$, and so $E_1^{p,q} = 0$ for $p \geq -n+1$, and $W_{n-1}N_0 = 0$. Moreover, $W_n N_0$ is the cokernel of the morphism $d_1 \colon E_1^{-n-1,n} \to E_1^{-n,n}$. Using the description of the weight filtration in Proposition 9.2, we compute that

$$E_1^{-n,n} = H^0 f_* \mathrm{gr}^W_n j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n] \cong H^0 f_*\mathbb{Q}^H_{\widetilde{X}}[n] \cong M_X \oplus M_0$$

and that the support of $E_1^{-n+1,n}$ is contained inside X_{sing}. Because N_0 has no subobjects that are supported inside X_{sing} (by Lemma 9.3) , and M_X has neither subobjects nor quotient objects that are supported inside X_{sing} (by construction), we conclude that $W_n N_0 \cong M_X$. This already proves (9.8.1).

Likewise, $\mathrm{gr}^W_{n+1} N_0$ is the cohomology of the complex of Hodge modules of weight $n+1$

$$E_1^{-n-2,n+1} \xrightarrow{d_1} E_1^{-n-1,n+1} \xrightarrow{d_1} E_1^{-n,n+1}. \tag{9.8.3}$$

By a similar computation as above, we have $E_1^{-n,n+1} \cong M_1$ and

$$E_1^{-n-1,n+1} \cong \bigoplus_{i\in I} H^0 f_*\mathbb{Q}^H_{E_i}(-1)\,[n-1]$$
$$E_1^{-n-2,n+1} \cong \bigoplus_{i,j\in I} H^{-1} f_*\mathbb{Q}^H_{E_i\cap E_j}(-2)\,[n-2].$$

We showed during the proof of Proposition 8.3 that $\mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{M}_1)$ is acyclic. At the same time, using the compatibility of the de Rham complex with direct images, we have

$$\mathrm{gr}^F_{-1}\mathrm{DR}(\mathscr{E}_1^{-n-1,n+1}) \cong \bigoplus_{i\in I} \mathbf{R}f_* \mathrm{gr}^F_0 \mathrm{DR}(\mathscr{O}_{E_i}) \cong \bigoplus_{i\in I} \mathbf{R}f_*\mathscr{O}_{E_i}[n-1].$$

By a similar calculation and the Decomposition Theorem, the complex $\mathrm{gr}^F_{-1}\mathrm{DR}(\mathscr{E}_1^{-n-2,n+1})$ is isomorphic to a direct summand in

$$\bigoplus_{i,j\in I} \mathbf{R}f_* \mathrm{gr}^F_1 \mathrm{DR}(\mathscr{O}_{E_i\cap E_j})$$

and therefore acyclic. Since morphisms between mixed Hodge modules strictly preserve the Hodge filtration, it now follows from (9.8.3) that

$$\mathrm{gr}^F_{-1}\mathrm{DR}(\mathrm{gr}^W_{n+1}\mathcal{N}_0) \cong \bigoplus_{i\in I} \mathbf{R}f_*\mathscr{O}_{E_i}[n-1],$$

proving (9.8.2).

Since $W_{n-1}N_0 = 0$, the complex $\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathrm{gr}^W_w \mathcal{N}_0)$ is certainly acyclic for $w \leq n-1$. It remains to show that it is also acyclic for $w \geq n+2$. The proof of this fact is the same as that of Lemma 9.7, and so we omit it. □

Corollary 9.9. *Maintaining Setting 7.1 and using the notation introduced above, we obtain a long exact sequence*

$$\cdots \to \mathscr{H}^j\,\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{M}_X) \to \mathscr{H}^j\,\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0) \to \bigoplus_{i\in I} R^{n-1+j} f_*\mathscr{O}_{E_i} \to \cdots$$

Proof. Proposition 9.8 implies that the complex $\mathrm{gr}^F_{-1}\,\mathrm{DR}(W_{n-1}\mathcal{N}_0)$ is acyclic, and that the natural morphism

$$\mathrm{gr}^F_{-1}\,\mathrm{DR}(W_{n+1}\mathcal{N}_0) \to \mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0)$$

is a quasi-isomorphism. We therefore get a distinguished triangle

$$\mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{M}_X) \to \mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0) \to \bigoplus_{i\in I} \mathbf{R}f_*\mathscr{O}_{E_i}[n-1] \to \mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{M}_X)[1]$$

in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$. The claim follows by passing to cohomology. □

9.5. **Application to the extension problem.** In analogy with Section 8.5, we conclude with a brief discussion of the effect that extendability of log n-forms has on $\mathrm{DR}(\mathcal{N}_0)$. Once again, Corollary 6.12 and the result below can be used to show if n-forms extend with log poles, then all forms extend with log poles. This gives another proof for Theorem 1.5 in the (most important) case $k = n$. Since we are now working with mixed Hodge modules, the reader may find it instructive to compare the proof below with that of the analogous result for pure Hodge modules in Section 8.5

Proposition 9.10 (Extension of log n-forms and N_Y)**.** *Maintaining Setting 7.1 and using the notation introduced above, assume that the morphism $r_*\Omega^n_{\widetilde{X}}(\log E) \hookrightarrow j_*\Omega^n_{X_{\mathrm{reg}}}$ is an isomorphism. Then one has*

$$\dim \mathrm{Supp}\,\mathscr{H}^j\,\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{N}_Y) \leq -(j+p+2)$$

for all integers $j, p \in \mathbb{Z}$ with $p + j \geq -n+1$.

Proof. This time, we aim to apply Theorem 6.11. Recall that X is reduced of pure dimension n; that the mixed Hodge module $N_0 \in \mathrm{MHM}(Y)$ has support equal to X; and that we defined $N_Y := \mathbb{D}(N_0)(-n) \in \mathrm{MHM}(Y)$ by taking the $(-n)$-th Tate twist of the dual mixed Hodge module. Taking into account the Tate twist, the formula for the de Rham complex of the dual mixed Hodge module in Proposition 4.5 becomes

$$\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{N}_Y) \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\Big(\mathrm{gr}^F_{-(p+n)}\,\mathrm{DR}(\mathcal{N}_0), \omega^\bullet_Y\Big). \tag{9.10.1}$$

Let us now verify that all the conditions in Theorem 6.11 are satisfied in our setting.

Claim 9.11. One has $\dim \mathrm{Supp}\,\mathscr{H}^j\,\mathrm{DR}(\mathcal{N}_Y) \leq -(j+1)$ for every $j \geq -n+1$.

Proof of Claim 9.11. Recall that the module N_0 has weight $\geq n$, in the sense that $W_{n-1}N_0 = 0$, and that its support is $\mathrm{Supp}\,N_0 = X$. The dual module N_Y will then have weight $\leq n$, in the sense that $W_n N_Y = N_Y$, and $\mathrm{Supp}\,N_Y = X$. By Lemma 9.3, the perverse sheaf $\mathrm{DR}(\mathcal{N}_0)$ has no nontrivial subobjects whose support is contained in X_{sing}. Consequently, the perverse sheaf $\mathrm{DR}(\mathcal{N}_Y)$, isomorphic to the Verdier dual of $\mathrm{DR}(\mathcal{N}_0)$, has no nontrivial quotient objects whose support is contained in X_{sing}. Now apply Proposition 5.2. □ (Claim 9.11)

Claim 9.12. The complex of $\mathscr{O}_Y$-modules $\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{N}_Y)$ is acyclic for every $p \geq 1$.

Proof of Claim 9.12. Recall that $F_{c-1}\mathcal{N}_0 = 0$, where $c = \dim Y - \dim X$. For dimension reasons, the complex $\mathrm{gr}^F_{-(p+n)}\,\mathrm{DR}(\mathcal{N}_0)$ is trivial for $p \geq 1$. Now (9.10.1) implies that the complex $\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{N}_Y)$ is acyclic for every $p \geq 1$. □ (Claim 9.12)

Claim 9.13. One has $\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}_0^F \operatorname{DR}(\mathcal{N}_Y) \leq -(j+2)$ for every $j \geq -n+1$.

Proof of Claim 9.13. Since $F_{c-1}\mathcal{N}_0 = 0$, the formula in (4.3.2) implies that the complex

$$\operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{N}_0) \cong \mathscr{H}^0 \operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{N}_0) \tag{9.13.1}$$

is actually a sheaf in degree 0. Using the assumption that $r_*\Omega^n_{\widetilde{X}}(\log E) \cong j_*\Omega^n_{X_{\mathrm{reg}}}$, the following inequalities will therefore hold for all $j \geq -n+1$:

$$\begin{aligned}
-(j+2) &\geq \dim \operatorname{Supp} R^j \mathscr{H}om_{\mathscr{O}_Y}\big(f_*\Omega^n_{\widetilde{X}}(\log E), \omega_Y^\bullet\big) && \text{by Corollary 6.2}\\
&= \dim \operatorname{Supp} R^j \mathscr{H}om_{\mathscr{O}_Y}\big(\mathscr{H}^0 \operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{N}_0), \omega_Y^\bullet\big) && \text{by Proposition 9.5}\\
&= \dim \operatorname{Supp} R^j \mathscr{H}om_{\mathscr{O}_Y}\big(\operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{N}_0), \omega_Y^\bullet\big) && \text{by (9.13.1)}\\
&= \dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}_0^F \operatorname{DR}(\mathcal{N}_Y) && \text{by (9.10.1)}
\end{aligned}$$

This gives us the desired result. □ (Claim 9.13)

Having checked all the conditions, we can now apply Theorem 6.11 and conclude the proof of Proposition 9.10. □

10. Intrinsic description, proof of Theorems 1.1 and 1.2

10.1. **Proof of Theorem 1.1.** In this section, we prove the criterion for extension of holomorphic forms in Theorem 1.1. In fact, the result is really just a reformulation of Proposition 8.1, although it takes some work to see that this is the case.

Setup. Let X be a reduced complex space of pure dimension n. Since the statement to be proved is local on X, we may assume that we are in the setting described in Section 7. In particular, X is a complex subspace of an open ball $Y \subseteq \mathbb{C}^{n+c}$, and $f\colon \widetilde{X} \to Y$ denotes the composition of a projective resolution of singularities $r\colon \widetilde{X} \to X$ with the closed embedding $i_X\colon X \hookrightarrow Y$. Because Y is a Stein manifold, all Kähler differentials on X are restrictions of holomorphic differential forms from Y; in particular, if $z_1, \dots, z_{n+c}$ are holomorphic coordinates on Y, then the sheaf Ω^p_X is generated by the global sections

$$i_X^*(dz_{i_1} \wedge \cdots \wedge dz_{i_p}),$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq n+c$. Having set up the notation, we can now prove the following (slightly more precise) local version of Theorem 1.1.

Theorem 10.1 (Local version of Theorem 1.1)**.** *In the setting above, a holomorphic p-form $\alpha \in H^0(X_{\mathrm{reg}}, \Omega^p_X)$ extends to a holomorphic p-form on $\widetilde{X}$ if, and only if, the holomorphic n-forms $\alpha \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_{n-p}}$ and $d\alpha \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_{n-p-1}}$ on X_{reg} extend to holomorphic n-forms on $\widetilde{X}$, for every choice of indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-p} \leq n+c$.*

The intersection complex. As in Section 8, we use the notation $M_X \in \mathrm{HM}(Y, n)$ for the polarisable Hodge module on Y whose underlying perverse sheaf is the intersection complex of X, and we let $(\mathcal{M}_X, F_\bullet\mathcal{M}_X)$ be its underlying filtered $\mathscr{D}_Y$-module. According to Proposition 8.1, we have

$$f_*\Omega^p_{\widetilde{X}} \cong \mathscr{H}^{-(n-p)} \operatorname{gr}_{-p}^F \operatorname{DR}(\mathcal{M}_X).$$

Recall from Section 4.1.5 that the de Rham complex

$$\operatorname{DR}(\mathcal{M}_X) = \Big[\mathcal{M}_X \xrightarrow{\nabla} \Omega^1_Y \otimes \mathcal{M}_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{n+c}_Y \otimes \mathcal{M}_X\Big],$$

is concentrated in degrees $-(n+c), \dots, 0$. Since $\dim Y - \dim X = c$, one has $F_{c-1}\mathcal{M}_X = 0$, which means that the complex of coherent $\mathscr{O}_Y$-modules

$$\operatorname{gr}_{-p}^F \operatorname{DR}(\mathcal{M}_X) = \Big[\Omega^{p+c}_Y \otimes F_c\mathcal{M}_X \xrightarrow{\nabla} \Omega^{p+c+1}_Y \otimes \operatorname{gr}^F_{c+1}\mathcal{M}_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{n+c}_Y \otimes \operatorname{gr}^F_{n-p+c}\mathcal{M}_X\Big]$$

is concentrated in degrees $-(n-p), \dots, 0$. The result in Proposition 8.1 therefore becomes

$$f_* \Omega^p_{\widetilde{X}} \cong \ker\Bigl(\nabla \colon \Omega_Y^{p+c} \otimes F_c \mathcal{M}_X \to \Omega_Y^{p+c+1} \otimes \operatorname{gr}^F_{c+1} \mathcal{M}_X\Bigr). \tag{10.1.1}$$

This yields an isomorphism between the space of holomorphic p-forms on the resolution $\widetilde{X}$, and the space of holomorphic $(p+c)$-forms on Y with coefficients in the coherent $\mathscr{O}_Y$-module $F_c\mathcal{M}_X$ whose image under the differential in the de Rham complex is again a holomorphic $(p+c+1)$-form on Y with coefficients in $F_c\mathcal{M}_X$. The isomorphism

$$f_* \Omega^n_{\widetilde{X}} \cong \Omega_Y^{n+c} \otimes F_c \mathcal{M}_X \tag{10.1.2}$$

is an important special case of this.

Claim 10.2. With notation as above, the image of the restriction morphism

$$H^0\bigl(Y, \Omega_Y^{p+c} \otimes F_c\mathcal{M}_X\bigr) \to H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{p+c} \otimes F_c\mathcal{M}_X\bigr)$$

consists exactly of those $(p+c)$-forms with values in $F_c\mathcal{M}_X$ whose wedge product with any element of $H^0(Y, \Omega_Y^{n-p})$ belongs to the image of

$$H^0\bigl(Y, \Omega_Y^{n+c} \otimes F_c\mathcal{M}_X\bigr) \to H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{n+c} \otimes F_c\mathcal{M}_X\bigr).$$

Proof of Claim 10.2. The isomorphism in (10.1.2) shows that $F_c\mathcal{M}_X$ is a rank-one coherent sheaf supported on X, whose restriction to X_{reg} is isomorphic to the line bundle $\det N_{X_{\mathrm{reg}}|Y}$. Using the coordinate functions $z_1, \dots, z_{n+c}$ on the ball Y, we may write any given element of $H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{p+c} \otimes F_c\mathcal{M}_X\bigr)$ uniquely in the form

$$\sum (dz_{i_1} \wedge \dots \wedge dz_{i_{p+c}}) \otimes \lambda_{i_1,\dots,i_{p+c}},$$

with coefficients $\lambda_{i_1,\dots,i_{p+c}} \in H^0\bigl(Y \setminus X_{\mathrm{sing}}, F_c\mathcal{M}_X\bigr)$. Clearly such an element belongs to the image of the restriction morphism if and only if all the coefficients are in the image of $H^0(Y, F_c\mathcal{M}_X)$. The assertion now follows by taking wedge products with all possible $(n-p)$-forms of the type $dz_{i_1} \wedge \dots \wedge dz_{i_{n-p}}$. □ (Claim 10.2)

End of proof. Now suppose we are given a holomorphic p-form $\alpha \in H^0(X_{\mathrm{reg}}, \Omega_X^p)$ on the set of nonsingular points of X. Using the isomorphism in (10.1.1), it determines a unique element $\widetilde{\alpha} \in H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{p+c} \otimes F_c\mathcal{M}_X\bigr)$ with the property that

$$\nabla\widetilde{\alpha} \in H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{p+c+1} \otimes F_c\mathcal{M}_X\bigr),$$

and one checks easily that $\nabla\widetilde{\alpha}$ corresponds to the $(p+1)$-form $d\alpha$ under the isomorphism in (10.1.1). Again using (10.1.1), we conclude that α extends to a holomorphic p-form on $\widetilde{X}$ if and only $\widetilde{\alpha}$ belongs to the image of

$$H^0\bigl(Y, \Omega_Y^{p+c} \otimes F_c\mathcal{M}_X\bigr) \to H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{p+c} \otimes F_c\mathcal{M}_X\bigr)$$

and $\nabla\widetilde{\alpha}$ belongs to the image of

$$H^0\bigl(Y, \Omega_Y^{p+c+1} \otimes F_c\mathcal{M}_X\bigr) \to H^0\bigl(Y \setminus X_{\mathrm{sing}}, \Omega_Y^{p+c+1} \otimes F_c\mathcal{M}_X\bigr).$$

According to Claim 10.2, we can test for these two conditions after taking wedge products with elements in $H^0(Y, \Omega_Y^{n-p})$ respectively $H^0(Y, \Omega_Y^{n-p-1})$. Because the restriction mapping from the differentials on Y to the Kähler differentials on X is surjective, we get the desired conclusion. This ends the proof of Theorem 1.1. □

10.2. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is nearly identical to that of Theorem 1.1. The only difference is that one has to work with $\Omega^p_{\widetilde{X}}(\log E)$ instead of $\Omega^p_{\widetilde{X}}$; that one has to use the mixed Hodge module N_0 instead of the pure Hodge module M_X; and that one should apply Proposition 9.5 instead of Proposition 8.1. We leave the details to the care of the reader. □

11. Extension, proof of Theorems 1.4 and 1.5

11.1. **Proof of Theorem 1.4.** It clearly suffices to prove Theorem 1.4 only in the case $p = k - 1$, with $1 \le k \le n$. Again, we relax the assumptions a little bit and allow X to be any reduced complex space of pure dimension n. This makes the entire problem local on X. After shrinking X, if necessary, we may therefore assume that we are given a holomorphic form $\alpha \in H^0(X_{\mathrm{reg}}, \Omega_X^{k-1})$; our task is to show that α extends holomorphically to the complex manifold $\widetilde{X}$. We aim to apply Theorem 1.1, and so we consider an arbitrary open subset $U \subseteq X$ and a pair of Kähler differentials $\beta \in H^0(U, \Omega_X^{n-k+1})$ and $\gamma \in H^0(U, \Omega_X^{n-k})$. We need to check that the holomorphic n-forms $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ on U_{reg} extend to holomorphic n-forms on $r^{-1}(U)$. This is again a local problem, and after further shrinking X, we may therefore assume without loss of generality that $U = X$ and that we have a closed embedding $i_X : X \hookrightarrow Y$, where Y is an open ball in $\mathbb{C}^{n+c}$. Letting $z_1, \dots, z_{n+c}$ be holomorphic coordinates on Y, the sheaf of Kähler differentials Ω_X^p is then generated by the global sections

$$i_X^*(dz_{i_1} \wedge \cdots \wedge dz_{i_p}),$$

where $1 \le i_1 < i_2 < \cdots < i_p \le n + c$. Since $n - k + 1 \ge 1$, we can thus write

$$\beta = \sum_{j=1}^{n+c} i_X^*(dz_j) \wedge \beta_j$$

for certain Kähler differentials $\beta_j \in H^0(X, \Omega_X^{n-k})$. The holomorphic k-forms $\alpha \wedge i_X^*(dz_j)$ and $d\alpha$ extend holomorphically to $\widetilde{X}$, by assumption, and so Theorem 1.1 guarantees that the holomorphic n-forms $\alpha \wedge i_X^*(dz_j) \wedge \beta_j$ and $d\alpha \wedge \gamma$ extend to $\widetilde{X}$ as well. It follows that $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ extend to $\widetilde{X}$, and this implies that α itself extends to $\widetilde{X}$, by another application of Theorem 1.1. □

11.2. **Proof of Theorem 1.5.** The proof of Theorem 1.5 is nearly identical to the proof of Theorem 1.4. The only difference is that one uses Theorem 1.2 instead of Theorem 1.1. □

12. Extension for $(n-1)$-forms, proof of Theorem 1.6

We maintain the notation and assumptions of Theorem 1.6, but we allow X to be any reduced complex space of pure dimension n. Recall that $r : \widetilde{X} \to X$ is a log resolution such that the natural morphism $r_*\Omega^n_{\widetilde{X}} \hookrightarrow j_*\Omega^n_{X_{\mathrm{reg}}}$ is an isomorphism. Our task is to show that the natural morphism

$$r_*\Omega^{n-1}_{\widetilde{X}}(\log E)(-E) \hookrightarrow j_*\Omega^{n-1}_{X_{\mathrm{reg}}}$$

is an isomorphism, or equivalently, that sections of $f_*\Omega^{n-1}_{\widetilde{X}}(\log E)(-E)$ extend uniquely across X_{sing}. It is easy to see by duality that all the sheaves $r_*\Omega^p_{\widetilde{X}}(\log E)(-E)$ are independent of the choice of log resolution. Shrinking X and replacing r with the canonical strong resolution of singularities, we may assume that we are in the setting described in Section 7 and Section 9. We use the notation introduced there.

The weight filtration on N_0. The proof relies the results of Section 9.4, where we analysed the weight filtration on the mixed Hodge module $N_0 = H^0 f_*\big(j_*\mathbb{Q}^H_{\widetilde{X}\setminus E}[n]\big) \in \mathrm{MHM}(Y)$. To begin, recall from Proposition 9.6 that we have an isomorphism

$$\mathbf{R}f_*\Omega^1_{\widetilde{X}}(\log E)[n-1] \cong \mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0).$$

Using Grothendieck duality for the proper holomorphic mapping $f : \widetilde{X} \to Y$, we obtain

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}\Big(\mathbf{R}f_*\Omega^{n-1}_{\widetilde{X}}(\log E)(-E), \omega_Y^\bullet\Big) \cong \mathbf{R}f_*\Omega^1_{\widetilde{X}}(\log E)\,[n] \cong \mathrm{gr}^F_{-1}\,\mathrm{DR}(\mathcal{N}_0)\,[1].$$

According to the extension criterion for complexes in Proposition 6.4, it is therefore sufficient to prove the collection of inequalities

$$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{N}_0) \leq -(j+1) \quad \text{for every } j \geq -n+2. \tag{12.0.1}$$

On the other hand, recall from Corollary 9.9 that, for all $j \in \mathbb{Z}$, one has an exact sequence

$$\mathscr{H}^j \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{M}_X) \to \mathscr{H}^j \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{N}_0) \to \bigoplus_{i \in I} R^{n-1+j} f_* \mathscr{O}_{E_i}, \tag{12.0.2}$$

The inequalities in (12.0.1) will follow from the analogous inequalities for the dimension of the support of the first and third term in (12.0.2).

The first term in (12.0.2). The first term is easily dealt with. Since we are in the setting of Theorem 1.4, an application of Proposition 8.4 gives the additional inequalities

$$\dim \operatorname{Supp} \mathscr{H}^j \operatorname{gr}^F_{-1} \operatorname{DR}(\mathcal{M}_X) \leq -(j+1) \quad \text{for every } j \geq -n+2. \tag{12.0.3}$$

This is half of what we need to prove (12.0.1).

The third term in (12.0.2). Now we turn to the third term. Fix an index $i \in I$. Pushing forward the standard short exact sequence

$$0 \to \mathscr{O}_{\widetilde{X}}(-E_i) \to \mathscr{O}_{\widetilde{X}} \to \mathscr{O}_{E_i} \to 0$$

along $f : \widetilde{X} \to Y$ gives us an exact sequence

$$\underbrace{R^{n-1+j} f_* \mathscr{O}_{\widetilde{X}}}_{=:\mathrm{A}} \to R^{n-1+j} f_* \mathscr{O}_{E_i} \to \underbrace{R^{n+j} f_* \mathscr{O}_{\widetilde{X}}(-E_i)}_{=:\mathrm{B}}.$$

But then, the following inequalities will hold for every $j \geq -n+2$,

$$\begin{aligned} \dim \operatorname{Supp} \mathrm{B} &\leq -(j+1) && \text{for dimension reasons} \\ \dim \operatorname{Supp} \mathrm{A} &= \dim \operatorname{Supp} \mathscr{H}^{j-1} \operatorname{gr}^F_0 \operatorname{DR}(\mathcal{M}_X) && \text{by Proposition 8.2} \\ &\leq -(j-1+2) && \text{by Proposition 8.4} \end{aligned}$$

In summary, we have $\dim \operatorname{Supp} R^{n-1+j} f_* \mathscr{O}_{E_i} \leq -(j+1)$ for every $i \in I$ and every $j \geq -n+2$. As discussed above, together with (12.0.3) this suffices to the inequalities in (12.0.1). The proof of Theorem 1.6 is therefore complete. □

We again record the following corollary of the proof.

Corollary 12.1. *In the setting of Theorem 1.6, one has*

$$\dim \operatorname{Supp} R^j f_* \Omega^1_{\widetilde{X}}(\log E) \leq n-2-j, \quad \textit{for every } j \geq 1.$$ □

13. Local vanishing, proof of Theorem 1.10

We maintain the notation and assumptions of Theorem 1.10, but we allow X to be any reduced complex space of pure dimension n. Recall that $r : \widetilde{X} \to X$ is a log resolution of singularities such that $R^{n-1} r_* \mathscr{O}_{\widetilde{X}} = 0$. Our goal is to prove that $R^{n-1} r_* \Omega^1_{\widetilde{X}}(\log E) = 0$. Both the assumptions and the conclusion of Theorem 1.10 are independent of the choice of the resolution: the former because complex manifolds have rational singularities, the latter by [MOP20, Lem. 1.1]. We may therefore assume that we are in the setting described in Section 7, and use the notation introduced there.

Reduction to a statement about M_X. We have already done pretty much all the necessary work during the proof of Theorem 1.6, and so we shall be very brief. As in the proof of Theorem 1.6, we have an isomorphism

$$R^{n-1}f_*\Omega^1_{\widetilde{X}}(\log E) \cong \mathscr{H}^0 \operatorname{gr}^F_{-1} \mathrm{DR}(\mathcal{N}_0).$$

Corollary 9.9 provides us with an exact sequence

$$\mathscr{H}^0 \operatorname{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_X) \to \mathscr{H}^0 \operatorname{gr}^F_{-1} \mathrm{DR}(\mathcal{N}_0) \to \bigoplus_{i \in I} R^{n-1}f_*\mathscr{O}_{E_i}.$$

The assumption that $R^{n-1}f_*\mathscr{O}_{\widetilde{X}} = 0$ yields $R^{n-1}f_*\mathscr{O}_{E_i} = 0$ for every $i \in I$, because $\mathscr{O}_{E_i}$ is a quotient of $\mathscr{O}_{\widetilde{X}}$. To prove Theorem 1.10, it will therefore suffice to prove the vanishing of $\mathscr{H}^0 \operatorname{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_X)$, and this is what we will do next.

End of proof. Recall from (8.0.3) that $\mathscr{H}^{-1} \operatorname{gr}^F_0 \mathrm{DR}(\mathcal{M}_X) \cong R^{n-1}f_*\mathscr{O}_{\widetilde{X}}$, which vanishes by assumption. As in the proof of Lemma 6.8, consider the short exact sequence of complexes

$$0 \to F_{-1}\mathrm{DR}(\mathcal{M}_X) \to F_0\mathrm{DR}(\mathcal{M}_X) \to \operatorname{gr}^F_0 \mathrm{DR}(\mathcal{M}_X) \to 0,$$

and the associated sequence of cohomology sheaves

$$\cdots \to \underbrace{\mathscr{H}^{-1} \operatorname{gr}^F_0 \mathrm{DR}(\mathcal{M}_X)}_{=0 \text{ by ass.}} \to \mathscr{H}^0 F_{-1}\mathrm{DR}(\mathcal{M}_X) \to \underbrace{\mathscr{H}^0 F_0\mathrm{DR}(\mathcal{M}_X)}_{=0 \text{ by Cor. } 5.3} \to \cdots$$

to see that $\mathscr{H}^0 F_{-1}\mathrm{DR}(\mathcal{M}_X) = 0$. Next, we look at the sequence

$$0 \to F_{-2}\mathrm{DR}(\mathcal{M}_X) \to F_{-1}\mathrm{DR}(\mathcal{M}_X) \to \operatorname{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_X) \to 0$$

and its cohomology,

$$\cdots \to \underbrace{\mathscr{H}^0 F_{-1}\mathrm{DR}(\mathcal{M}_X)}_{=0} \to \mathscr{H}^0 \operatorname{gr}^F_{-1} \mathrm{DR}(\mathcal{M}_X) \to \underbrace{\mathscr{H}^1 F_{-2}\mathrm{DR}(\mathcal{M}_X)}_{=0, \text{ since concentr. in non-pos. degrees}} \to \cdots,$$

to conclude the proof. □

14. Pull-back, proof of Theorem 1.11

As promised in Section 1.6, the following result specifies the "natural universal properties" mentioned in Theorem 1.11. With Theorem 1.4 at hand, the proof is almost identical to the proof given in [Keb13b] for spaces with klt singularities.

Theorem 14.1 (Functorial pull-back for reflexive forms)**.** *Let* RSing *be the category of complex spaces with rational singularities, where morphisms are simply the holomorphic mappings. Then, there exists a unique contravariant functor,*

$$d_{\mathrm{refl}} : \begin{array}{ccc} \mathsf{RSing} & \to & \{\mathbb{C}\text{-vector spaces}\}, \\ X & \mapsto & H^0\big(X,\, \Omega^{[p]}_X\big) \end{array} \tag{14.1.1}$$

that satisfies the following "compatibility with Kähler differentials". If $f : Z \to X$ is any morphism in RSing *such that the open set $Z^\circ := Z_{\mathrm{reg}} \cap f^{-1}(X_{\mathrm{reg}})$ is not empty, then there exists a commutative diagram*

$$\begin{array}{ccc} H^0\big(X,\, \Omega^{[p]}_X\big) & \xrightarrow{\ d_{\mathrm{refl}} f\ } & H^0\big(Z,\, \Omega^{[p]}_Z\big) \\ \big\downarrow{\scriptstyle \mathrm{restriction}_X} & & \big\downarrow{\scriptstyle \mathrm{restriction}_Z} \\ H^0\big(X_{\mathrm{reg}},\, \Omega^p_{X_{\mathrm{reg}}}\big) & \xrightarrow[d_{\text{Kähler}}(f|_{Z^\circ})]{} & H^0\big(Z^\circ,\, \Omega^p_{Z^\circ}\big), \end{array}$$

where $d_{\text{Kähler}}(f|_{Z^\circ})$ denotes the usual pull-back of Kähler differentials, and where $d_{\text{Kähler}}(f|_{Z^\circ})$ denotes the usual pull-back of Kähler differentials, and $d_{\mathrm{refl}} f$ denotes the linear map of complex vector spaces induced by the contravariant functor (14.1.1)*.*

Remark 14.2 (Rational vs. weakly rational singularities in Theorem 14.1). We do not expect Theorem 14.1 to hold true if one replaces "rational" by "weakly rational" singularities. As we will see in Step 2 of the sketched proof, the result relies on a theorem of Namikawa which is specific to rational singularities.

The universal properties spelled out in Theorem 14.1 above have a number of useful consequences that we briefly mention. Again, statements and proof are similar to the algebraic, klt case. To avoid repetition, we merely mention those consequences and point to the paper [Keb13b] for precise formulations and proofs.

Fact 14.3 (Additional properties of pull-back, [Keb13b, §5])**.** *The pull-back functor of Theorem 14.1 has the following additional properties.*

(14.3.1) Compatibility with open immersions, [Keb13b, Prop. 5.6].
(14.3.2) Compatibility with Kähler differentials for morphisms to smooth targets varieties, [Keb13b, Prop. 5.7].
(14.3.3) Induced pull-back morphisms at the level of sheaves, [Keb13b, Cor. 5.10].
(14.3.4) Compatibility with wedge products and exterior derivatives, [Keb13b, Prop. 5.13].

□

14.1. **Sketch of proof for Theorem 14.1.** For quasi-projective varieties with klt singularities, the result has already been shown in [Keb13b, Thm. 5.2]. If X is a complex space with arbitrary rational singularities, the proof given in [Keb13b] applies with minor modifications once the following obvious adjustments are made.

- Replace all references to the extension theorem [GKKP11, Thm. 1.4], which works for klt spaces only, by references to Theorem 1.4, which also covers the case of rational singularities.
- Equation [Keb13b, (6.10.5)] is shown for klt spaces using Hacon-McKernan's solution of Shokurov's rational connectivity conjecture. However, is has been shown by Namikawa, [Nam01, Lem. 1.2], that the equation holds more generally, for arbitrary complex spaces with rational singularities.
- If X in RSing is a complex space that does not necessarily carry an algebraic structure, then one also needs to modify the proof of [Keb13b, Lem. 6.15], replacing the reference to [GKK10, Cor. 2.12(ii)] by its obvious generalisation to complex spaces.

For the convenience of the reader, we include a sketch of proof that summarises the main ideas and simplifies [Keb13b] a little. Let $f\colon Z \to X$ be any holomorphic map between normal complex spaces with rational singularities. Given any $\sigma \in H^0\big(X,\, \Omega_X^{[p]}\big)$, we explain the construction of an appropriate pull-back form $\tau \in H^0\big(Z,\, \Omega_Z^{[p]}\big)$ and leave it to the reader to check that this τ is independent of the choices made, and satisfies all required properties.

Step 1. To find a reflexive form $\tau \in H^0\big(Z,\, \Omega_Z^{[p]}\big)$, it is equivalent to find a big, open subset $Z^\circ \subseteq Z_{\mathrm{reg}}$ and an honest form $\tau^\circ \in H^0\big(Z^\circ,\, \Omega^p_{Z^\circ}\big)$. We can therefore assume from the outset that Z is smooth. Next, let $T := \overline{f(Z)}$ denote the Zariski closure of the image, and let $\widetilde{T}$ be a desingularisation. The morphism f factors as

$$
\begin{tikzcd}
Z \arrow[r, dashed, "\text{meromorphic}"'] \arrow[rrr, bend left, "f"] & \widetilde{T} \arrow[r, "\text{desingularisation}"'] & T \arrow[r, "\text{inclusion}"'] & X
\end{tikzcd}
$$

Now, if we can find an appropriate pull-back form $\tau_{\widetilde{T}} \in H^0\big(\widetilde{T},\, \Omega^p_{\widetilde{T}}\big)$, we could use the standard fact [Pet94, Rem. 1.8(1)] that the meromorphic map $Z \dashrightarrow \widetilde{T}$ is well-defined on a big, Zariski-open subset of Z to find the desired form τ by pulling back. Replacing Z by $\widetilde{T}$, if need be, we may therefore assume without loss of generality that Z is smooth and that the image $T := f(Z)$ is closed in Zariski topology.

Step 2. Next, choose a desingularisation $\pi: \widetilde{X} \to X$ such that $E := \operatorname{supp} \pi^{-1}(T)$ is an snc divisor. We will then find a Zariski open subset $T^\circ \subseteq T_{\mathrm{reg}}$ with preimage $E^\circ := \operatorname{supp} \pi^{-1}(T^\circ)$ such that $E^\circ \to T^\circ$ is relatively snc. The assumption that X has rational singularities is used in the following claim[3].

Claim 14.4. If $t \in T^\circ$ is any point with fibre $E_t := \operatorname{supp} \pi^{-1}(t)$, then

$$H^0\left(E_t,\ \Omega^p_{E_t}\middle/\mathrm{tor}\right) = 0.$$

Proof of Claim 14.4. In case where $E_t \subset \widetilde{X}$ is a divisor, this is a result of Namikawa, [Nam01, Lem. 1.2]. If E_t is not a divisor, we can blow up and apply Namikawa's result upstairs. The claim then follows from the elementary fact that sheaves of "Kähler differentials modulo torsion" have good pull-back properties, [Keb13b, §2.2]. □ (Claim 14.4)

Step 3. Again using that X has rational singularities, Theorem 1.4 yields a form $\tau_{\widetilde{X}} \in H^0\big(\widetilde{X},\ \Omega^p_{\widetilde{X}}\big)$. The following claim asserts that its restriction to E° comes from a form τ_{T° on T°.

Claim 14.5. There exists a unique differential form $\tau_{T^\circ} \in\in H^0\big(T^\circ,\ \Omega^p_{T^\circ}\big)$ such that $\tau_{\widetilde{X}}|_{E^\circ}$ and $d_{\text{Kähler}}(\pi|_{E^\circ})(\tau_{T^\circ})$ agree up to torsion.

Proof of Claim 14.5. Almost immediate from Claim 14.4 and standard relative differential sequences for sheaves of Kähler differentials modulo torsion, [Keb13b, Prop. 3.11]. □ (Claim 14.5)

Pulling the form τ_{T° back to $Z^\circ := f^{-1}(T^\circ)$, we find a form τ° on the open set $Z^\circ := f^{-1}(T^\circ)$, which is a non-empty subset of Z since $T := f(Z)$ is closed in Zariski topology, but need not be big. We leave it to the reader to follow the arguments in [Keb13b, §6 and 7] to see that this τ° extends to a form τ on all of Z that it is independent of the choices made and satisfies all required properties. □

Appendix A. Weakly rational singularities

A.1. **Definition and examples.** Let X be a normal complex space. The main result of this paper asserts that if top-forms on X_{reg} extend to regular top-forms on one desingularisation, then the same will hold for reflexive p-forms, for all values of p and all desingularisations. Spaces whose top-forms extend therefore seem to play an important role. We refer to them as spaces with *weakly rational* singularities and briefly discuss their main properties in this appendix.

Definition A.1 (Weakly rational singularities)**.** *Let X be a normal complex space. We say that X has* weakly rational singularities *if the Grauert-Riemenschneider sheaf ω^{GR}_X is reflexive. In other words, X has weakly rational singularities if for every (equivalently: one) resolution of singularities, $r: \widetilde{X} \to X$, the sheaf $r_*\omega_{\widetilde{X}}$ is reflexive. We say that a variety has weakly rational singularities if its underlying complex space does.*

Example A.2 (Rational singularities). Recall from Section 1.4 that rational singularities are weakly rational. For a concrete example, let X be the affine cone over a Fano manifold Y with conormal bundle $L := \omega_Y^{-1}$, as discussed in [Kol13, §3.8]. By [Kol13, Prop. 3.13], this implies that X has rational singularities because L^m is the tensor product of ω_Y with the ample line bundle $\omega_Y^{-1} \otimes L^m$. A perhaps more surprising example is that any affine cone over an Enriques surface has rational singularities.

[3] The paper [Keb13b] uses Hacon-McKernan's solution of Shokurov's rational connectivity conjecture and the more involved technique "projection to general points of T" to prove this result.

Example A.3 (Varieties with small resolutions). If a normal complex space X admits a small resolution, then X has weakly rational singularities. For a concrete example of a non-rational singularity of this form, consider an elliptic curve E and a very ample line bundle $L \in \operatorname{Pic}(E)$. Let $\widetilde{X} \to E$ be the total space of the vector bundle $L^{-1} \oplus L^{-1}$ and identify E with the zero-section in $\widetilde{X}$. We claim that there exists a normal, affine variety X and a birational morphism $r\colon \widetilde{X} \to X$ that contracts $E \subset \widetilde{X}$ to a normal point $x \in X$ and is isomorphic elsewhere. An elementary computation shows that $R^1 r_* \mathscr{O}_{\widetilde{X}} \neq 0$, so X does not have rational singularities. The preprint version of this paper spells out more details.

Perhaps somewhat counter-intuitively, there are example of log-canonical varieties X whose singularities are weakly rational but not rational. If K_X is Cartier and ω_X is locally generated by one element, this can of course not happen, so that the canonical divisors of the examples will never be Cartier.

Example A.4 (Some log canonical singularities are weakly rational, not rational). To start, let E be a smooth projective variety of positive irregularity whose canonical divisor is torsion, but not linearly trivial. Let $L \in \operatorname{Pic}(E)$ be very ample, and let X be the affine cone over E with conormal bundle L. By [Kol13, §3.8], X is log canonical and does *not* have rational singularities. Yet, Proposition B.2 asserts that the singularities of X are weakly rational. The preprint version of this paper discusses a concrete example.

Remark A.5 (Incompatible definitions in the literature). There already exists a notion of "weakly rational" in the literature. Andreatta-Silva [AS84] call a variety X weakly rational if $R^{\dim X - 1} r_* \mathscr{O}_{\widetilde{X}} = 0$ for one (or equivalently, any) resolution of singularities. They seem to be assuming implicitly that X has isolated singularities, although they do not include this assumption into the definition. (For a complex space with isolated singularities, both definitions are equivalent.)

A.2. **Behaviour with respect to standard constructions.** In view of their importance for our result, we briefly review the main properties of weakly rational singularities, in particular their behaviour under standard operations of birational geometry.

A.2.1. *Positive results.* In the positive direction, we show that weakly rational singularities are stable under general hyperplane sections, and that a space has weakly rational singularities if it is covered by a space with weakly rational singularities.

Proposition A.6 (Stability under general hyperplane sections)**.** *Let X be a quasi-projective variety with weakly rational singularities, let $L \in \operatorname{Pic}(X)$ be a line bundle and $\mathbb{L} \subseteq |L|$ be a finite-dimensional, basepoint free linear system whose general member is connected. Then, there exists a dense, Zariski-open subset $\mathbb{L}^\circ \subseteq \mathbb{L}$ such that any hyperplane $H \in \mathbb{L}^\circ$ has weakly rational singularities, and satisfies the adjunction formula*

$$\omega_H^{\mathrm{GR}} \cong \omega_X^{\mathrm{GR}} \otimes \mathscr{O}_X(H) \otimes \mathscr{O}_H. \tag{A.6.1}$$

Proof. Choose a resolution of singularities, $r\colon \widetilde{X} \to X$. There exists a dense, Zariski-open $\mathbb{L}^\circ \subseteq L$ such that any hyperplane $H \in \mathbb{L}^\circ$ satisfies the following properties.

(A.6.2) The hypersurface H is normal, connected and $H_{\mathrm{sing}} = X_{\mathrm{sing}} \cap H$: Seidenberg's theorem, [Sei50], and the fact that a variety is smooth along a Cartier divisor if the divisor itself is smooth.

(A.6.3) The preimage $\widetilde{H} := r^{-1} H$ is smooth: Bertini's theorem.

(A.6.4) The restriction $\omega_X^{\mathrm{GR}}|_H$ is reflexive: [Gro66, Thm. 12.2.1].

We claim that the adjunction formula (A.6.1) holds for H, which together with (A.6.4) implies that $H \in \mathbb{L}^\circ$ has weakly rational singularities. The setup is summarised in the following diagram

$$\begin{tikzcd} \widetilde{H} \arrow[r, "\widetilde{\iota}\text{, closed embedding}"] \arrow[d, "r_H\text{, resolution}"'] & \widetilde{X} \arrow[d, "r\text{, resolution}"] \\ H \arrow[r, "\iota\text{, closed embedding}"'] & X \end{tikzcd}$$

We obtain an adjunction morphism,

$$\begin{aligned} \iota^*\big(\omega_X^{\mathrm{GR}}(H)\big) &\cong \iota^* r_*\big(\omega_{\widetilde{X}}(\widetilde{H})\big) && \text{Projection formula} \\ &\to (r_H)_* \widetilde{\iota}^*\big(\omega_{\widetilde{X}}(\widetilde{H})\big) && \text{Cohomology and base change} \\ &\cong (r_H)_* \omega_{\widetilde{H}} \cong \omega_H^{\mathrm{GR}} && \text{Adjunction and smoothness of } \widetilde{H} \end{aligned} \tag{A.6.5}$$

which is clearly an isomorphism over the big open subset of H where H and X are both smooth. More can be said. Item (A.6.4) implies that the left hand side of (A.6.5) is reflexive, while the right hand side of (A.6.5) is a push forward of a torsion free sheaf, hence torsion free. As a morphism from a reflexive to a torsion free sheaf that is isomorphic in codimension one, the adjunction morphism must then in fact be isomorphic. □

As a second positive result, we show that images of weakly rational singularities under arbitrary finite morphisms are again weakly rational. This can be seen as an analogue of the fact that quotients of rational singularities under the actions of finite groups are again rational. The proof follows along the lines of [GKK10, proof of Cor. 3.2] and is therefore omitted here. The preprint version of this paper spells out all details.

Proposition A.7 (Stability under finite quotients)**.** *Let $\gamma: X \to Y$ be a proper, surjective morphism between normal complex spaces. Assume that γ is finite, or that it bimeromorphic and small. If X has weakly rational singularities, then so does Y.* □

A.2.2. *Negative results.* In spite of the positive results above, the following examples show that the class of varieties with weakly rational singularities does not remain invariant when taking quasi-étale covers or special hyperplane sections, even in the simplest cases.

Example A.8 (Instability under special hyperplane sections)*.* Grauert-Riemenschneider construct a normal, two-dimensional, isolated hypersurface singularity where ω_X^{GR} is not reflexive, [GR70, p. 280f]. In particular, X does *not* have weakly rational singularities and a naive adjunction formula for the Grauert-Riemenschneider sheaf as in (A.6.1) does not hold in this case.

Example A.9 (Instability under quasi-étale covers)*.* Any cone Y over an Enriques surface has rational singularities and admits a quasi-étale cover by a cone X over a K3 surface, which is Cohen-Macaulay, but does *not* have rational singularities, [Kol13, Ex. 3.6]. As we saw in Section 1.4, this implies that X does *not* have weakly rational singularities. We obtain examples of quasi-étale maps $X \to Y$ between isolated, log-canonical singularities where Y is weakly rational while X is not.

Appendix B. Cones over projective manifolds

Cones over projective manifolds are a useful class of examples to illustrate how the extension problem for p-forms is related to the behaviour of the canonical sheaf. We follow the notation introduced in Kollár's book [Kol13] and work in the following setting.

Setting B.1 (Cones over projective manifolds, compare [Kol13, §3.1])*.* Fix a number $n \geq 2$ and a smooth projective variety Y of dimension $\dim Y = n - 1$, together with an ample line bundle $L \in \operatorname{Pic}(Y)$. Following [Kol13, §3.8], we define the *affine cone over Y with conormal bundle L* as the affine algebraic variety

$$X := \operatorname{Spec} \bigoplus_{m \ge 0} H^0\big(Y,\, L^m\big)$$

The ring is finitely generated since L is ample. The variety X is normal of dimension n and smooth outside of the *vertex* $\vec{v}$, which is the point corresponding to the zero ideal. Unless $Y = \mathbb{P}^{n-1}$ and $L = \mathscr{O}_{\mathbb{P}^{n-1}}(1)$, the vertex will always be an isolated singular point.

Since Y is smooth, the partial resolution of singularities constructed in [Kol13, §3.8], say $r\colon \widetilde{X} \to X$, is in fact a log resolution of singularities. The variety $\widetilde{X}$ is isomorphic to the total space of the line bundle L^{-1} and the r-exceptional set $E \subsetneq \widetilde{X}$ is identified with the zero-section of that bundle.

B.1. **Extension of differential forms.** Now we turn out attention to the extension problem for differential forms. The following result can be summarised very neatly by saying that if n-forms extend, then p-forms extend for every $0 \le p \le n$.

Proposition B.2 (Extension of differential forms on cones)**.** *Assume Setting B.1. Then, p-forms extend for all $p \le n-2$. The following equivalences hold in addition.*

$$(n-1)\text{-forms extend} \Leftrightarrow H^0\big(Y,\, \omega_Y \otimes L^{-m}\big) = 0, \forall m \ge 1. \tag{B.2.1}$$

$$n\text{-forms extend} \Leftrightarrow H^0\big(Y,\, \omega_Y \otimes L^{-m}\big) = 0, \forall m \ge 0. \tag{B.2.2}$$

Proof. Since $\widetilde{X} \setminus E$ is isomorphic to X_{reg}, the question is simply under what conditions on Y and L the restriction mapping

$$H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) \to H^0\big(\widetilde{X} \setminus E, \Omega^p_{\widetilde{X}}\big)$$

is an isomorphism for different values of $p \in \{0, 1, \dots, n\}$. We use the identification of $\widetilde{X}$ with the total space of the line bundle L^{-1} and denote the projection by $q\colon \widetilde{X} \to Y$. The sequence of differentials and the sequence of pth exterior powers now read as follows,

$$0 \to q^*\Omega^1_Y \to \Omega^1_{\widetilde{X}} \to q^*L \to 0 \quad \text{and} \quad 0 \to q^*\Omega^p_Y \to \Omega^p_{\widetilde{X}} \to q^*\big(\Omega^{p-1}_Y \otimes L\big) \to 0.$$

Now both $q\colon \widetilde{X} \to Y$ and its restriction $q|_{\widetilde{X}\setminus E}$ are affine, and

$$q_*\mathscr{O}_{\widetilde{X}} \cong \bigoplus_{m\ge 0} L^m \quad \text{and} \quad (q|_{\widetilde{X}\setminus E})_*\mathscr{O}_{\widetilde{X}\setminus E} \cong \bigoplus_{m \in \mathbb{Z}} L^m.$$

We therefore obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{m\ge 0} H^0(Y, \Omega^p_Y \otimes L^m) \arrow[r] \arrow[d, "\alpha"] & H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) \arrow[r] \arrow[d] & \bigoplus_{m\ge 1} H^0(Y, \Omega^{p-1}_Y \otimes L^m) \arrow[r] \arrow[d, "\beta"] & \bigoplus_{m\ge 0} H^1(Y, \Omega^p_Y \otimes L^m) \arrow[d] \\
0 \arrow[r] & \bigoplus_{m\in\mathbb{Z}} H^0(Y, \Omega^p_Y \otimes L^m) \arrow[r] & H^0(\widetilde{X} \setminus Y, \Omega^p_{\widetilde{X}}) \arrow[r] & \bigoplus_{m\in\mathbb{Z}} H^0(Y, \Omega^{p-1}_Y \otimes L^m) \arrow[r] & \bigoplus_{m\in\mathbb{Z}} H^1(Y, \Omega^p_Y \otimes L^m)
\end{tikzcd}$$

Consider the first vertical arrow, labelled α, in the commutative diagram above. By the Nakano vanishing theorem, we have $H^0\big(Y,\, \Omega^p_Y \otimes L^m\big) = 0$ for $m \le -1$ and $p \le \dim Y - 1$, and so α is an isomorphism if and only if

$$H^0\big(Y,\, \omega_Y \otimes L^m\big) = 0, \quad \forall m \le -1. \tag{B.2.3}$$

Consider next the third vertical arrow, labelled β, in the commutative diagram. For the same reason as before, we have $H^0\big(Y,\, \Omega^{p-1}_Y \otimes L^m\big) = 0$ for $m \le -1$ and $p-1 \le \dim Y - 1$. For $m = 0$, the horizontal arrow

$$H^0\big(Y,\, \Omega^{p-1}_Y\big) \to H^1\big(Y,\, \Omega^p_Y\big)$$

in the second row is cup product with the first Chern class of the ample line bundle L; by the Hard Lefschetz Theorem, it is injective as long as $p - 1 \le \dim Y - 1$. Consequently, β is an isomorphism if and only if

$$H^0\big(Y,\, \omega_Y \otimes L^m\big) = 0, \quad \forall m \le 0. \tag{B.2.4}$$

The conclusion is that p-forms extend for $p \le n - 2$ without any extra assumptions on (Y, L); since the cone over (Y, L) has an isolated singularity at the vertex, this is consistent with the result by Steenbrink and van Straten [vSS85, Thm. 1.3]. Moreover, $(n-1)$-forms extend iff the condition in (B.2.3) is satisfied, and n-forms extend iff the condition in (B.2.4) is satisfied. □

B.2. **Characterisation of standard singularity types.** The following summary of several well-known results relates different classes of singularities to properties of the line bundle L, in particular to the vanishing of higher cohomology for L and its powers.

Proposition B.3 (Classes of singularities on cones)**.** *Assume Setting B.1. Then, the following equivalences hold.*

$$X \text{ has rational singularities} \Leftrightarrow H^i\big(Y,\, L^m\big) = 0, \forall i > 0, \forall m \ge 0. \tag{B.3.1}$$

$$X \text{ has Du Bois singularities} \Leftrightarrow H^i\big(Y,\, L^m\big) = 0, \forall i > 0, \forall m > 0. \tag{B.3.2}$$

$$X \text{ is Cohen-Macaulay} \Leftrightarrow H^i\big(Y,\, L^m\big) = 0, \forall \dim Y > i > 0, \forall m \ge 0. \tag{B.3.3}$$

The singularity types of the minimal model program are described as follows.

$$X \text{ is } \mathbb{Q}\text{-Gorenstein} \Leftrightarrow \exists m : K_Y \sim_{\mathbb{Q}} L^m. \tag{B.3.4}$$

$$X \text{ is klt} \Leftrightarrow \exists m < 0 : K_Y \sim_{\mathbb{Q}} L^m. \tag{B.3.5}$$

$$X \text{ is log canonical} \Leftrightarrow \exists m \le 0 : K_Y \sim_{\mathbb{Q}} L^m. \tag{B.3.6}$$

Proof. See [Kol13, Lem. 3.1, Cor. 3.11, Prop. 3.13 and Prop. 3.14] and [GK14, Thm 2.5]. □

Comparing Proposition B.2 and B.3, we find that the extension property of p-forms is a comparatively mild condition on (Y, L). It is not as cohomological in nature as "rational", "Du Bois" and "Cohen-Macaulay", and certainly not nearly as restrictive as being klt, which only happens in the special case where Y is a Fano manifold and L is $\mathbb{Q}$-linearly equivalent to a positive multiple of $-K_Y$. This suggests looking for an extension theorem that goes beyond the class of singularities used in the Minimal Model Program.

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