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Two observations about normal functions

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Abstract. Two simple observations are made: (1) If the normal function associated to a Hodge class has a zero locus of positive dimension, then it has a singularity. (2) The intersection cohomology of the dual variety contains the cohomology of the original variety, if the degree of the embedding is large.

This brief note contains two elementary observations about normal functions and their singularities that arose from a conversation with G. Pearlstein. Throughout, X will be a smooth projective variety of dimension $2n$, and ζ a primitive Hodge class of weight $2n$ on X, say with integer coefficients. We shall assume that X is embedded into projective space by a very ample divisor H, and let $\pi\colon \mathfrak{X} \to P$ be the family of hyperplane sections for the embedding. The discriminant locus, which parametrizes the singular hyperplane sections, will be denoted by $X^\vee \subseteq P$; on its complement, the map π is smooth.

1. The zero locus of a normal function

Here we show that if the zero locus of the normal function associated to a Hodge class ζ contains an *algebraic* curve, then the normal function must be singular at one of the points of intersection between $X^\vee$ and the closure of the curve.

Proposition 1. *Let ν_ζ be the normal function on $P \setminus X^\vee$, associated to a non-torsion primitive Hodge class $\zeta \in H^{2n}(X,\mathbb{Z}) \cap H^{n,n}(X)$. Assume that the zero locus of ν_ζ contains an algebraic curve, and that $H = dA$ for A very ample and $d \geq 3$. Then ν_ζ is singular at one of the points where the closure of the curve meets $X^\vee$.*

Before giving the proof, we briefly recall some definitions. In general, a normal function for a variation of Hodge structure of odd weight on a complex manifold Y_0 has an associated *cohomology class.* If $H_\mathbb{Z}$ is the local system underlying the variation, then a normal function ν determines an extension of local systems

$$0 \longrightarrow H_\mathbb{Z} \longrightarrow H'_\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{1}$$

The cohomology class $[\nu] \in H^1\bigl(Y_0, H_\mathbb{Z}\bigr)$ of the normal function is the image of $1 \in H^0\bigl(Y_0, \mathbb{Z}\bigr)$ under the connecting homomorphism for the extension.

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In particular, the normal function ν_ζ associated to a Hodge class ζ determines a cohomology class $[\nu_\zeta] \in H^1\bigl(P \setminus X^\vee, R^{2n-1}\pi_*\mathbb{Z}\bigr)$. With rational coefficients, that class can also be obtained directly from ζ through the Leray spectral sequence

$$E_2^{p,q} = H^p\bigl(P \setminus X^\vee, R^q\pi_*\mathbb{Q}\bigr) \Longrightarrow H^{p+q}\bigl(P \times X \setminus \pi^{-1}(X^\vee), \mathbb{Q}\bigr).$$

That is to say, the pullback of ζ to $P \times X \setminus \pi^{-1}(X^\vee)$ goes to zero in $E_2^{0,2n}$ because ζ is primitive, and thus gives an element of $E_2^{1,2n-1}$; this element is precisely $[\nu_\zeta]$. (Details can be found, for instance, in [6, Section 4].)

LEMMA 2. *Let $C_0 \to P \setminus X^\vee$ be a smooth affine curve mapping into the zero locus of ν_ζ, and let $\psi\colon W_0 \to C_0$ be the pullback of the family $\pi\colon \mathfrak{X} \to P$. Then the image of the Hodge class ζ in $H^{2n}(W_0, \mathbb{Q})$ is zero.*

PROOF. By topological base change, the pullback of $R^{2n-1}\pi_*\mathbb{Q}$ to C_0 is naturally isomorphic to $R^{2n-1}\psi_*\mathbb{Q}$; moreover, when ν_ζ is restricted to C_0, its class is simply the image of $[\nu_\zeta]$ in the group $H^1\bigl(C_0, R^{2n-1}\psi_*\mathbb{Q}\bigr)$. That image has to be zero, because C_0 maps into the zero locus of ν_ζ.

Now let $\zeta_0 \in H^{2n}\bigl(W_0, \mathbb{Q}\bigr)$ be the image of the Hodge class ζ. The Leray spectral sequence for the map ψ gives a short exact sequence

$$0 \longrightarrow H^1\bigl(C_0, R^{2n-1}\psi_*\mathbb{Q}\bigr) \longrightarrow H^{2n}\bigl(W_0, \mathbb{Q}\bigr) \longrightarrow H^0\bigl(C_0, R^{2n}\psi_*\mathbb{Q}\bigr) \longrightarrow 0,$$

and as before, ζ_0 actually lies in $H^1\bigl(C_0, R^{2n-1}\psi_*\mathbb{Q}\bigr)$. Because the spectral sequences for ψ and π are compatible, ζ_0 is equal to the image of $[\nu_\zeta]$; but we have already seen that this is zero. $\square$

Returning to our review of general definitions, let ν be a normal function on a complex manifold Y_0. When $Y_0 \subseteq Y$ is an open subset of a bigger complex manifold, one can look at the behavior of ν near points of $Y \setminus Y_0$. The *singularity* of ν at a point $y \in Y \setminus Y_0$ is by definition the image of $[\nu]$ in the group

$$\lim_{U \ni y} H^1\bigl(U \cap Y_0, H_\mathbb{Z}\bigr),$$

the limit being over all analytic open neighborhoods of the point. If the singularity is non-torsion, ν is said to be *singular* at the point y; this definition from [1] is a generalization of the one by M. Green and P. Griffiths [5].

When ν_ζ is the normal function associated to a non-torsion primitive Hodge class $\zeta \in H^{2n}(X, \mathbb{Z})$, P. Brosnan, H. Fang, Z. Nie, and G. Pearlstein [1, Theorem 1.3], and independently M. de Cataldo and L. Migliorini [2, Proposition 3.7], have proved the following result: Provided the vanishing cohomology of the smooth fibers of π is nontrivial, ν_ζ is singular at a point $p \in X^\vee$ if, and only if, the image of ζ in $H^{2n}\bigl(\pi^{-1}(p), \mathbb{Q}\bigr)$ is nonzero. By recent work of A. Dimca and M. Saito [3, Theorem 6], it suffices to take $H = dA$, with A very ample and $d \geq 3$.

PROOF OF PROPOSITION 1. Let C be the normalization of the closure of the curve in the zero locus. Pulling back the universal family $\pi\colon \mathfrak{X} \to P$ to C and resolving singularities, we obtain a smooth projective $2n$-fold W, together with the two maps shown in the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{\lambda} & X \\ {\scriptstyle\psi}\big\downarrow & & \\ C & & \end{array}$$

This may be done in such a way that the general fiber of ψ is a smooth hyperplane section of X; let $C_0 \subseteq C$ be the open subset where this holds, and $W_0 = \psi^{-1}(C_0)$ its preimage. Assume in addition that, for each $t \in C \setminus C_0$, the fiber $E_t = \psi^{-1}(t)$ is a divisor with simple normal crossing support. The map λ is generically finite, and we let d be its degree.

Let $\zeta_W = \lambda^*(\zeta)$ be the pullback of the Hodge class to W. By Lemma 2, the restriction of ζ_W to W_0 is zero. Consider now the exact sequence

$$H^{2n}(W, W_0, \mathbb{Q}) \xrightarrow{i} H^{2n}(W, \mathbb{Q}) \longrightarrow H^{2n}(W_0, \mathbb{Q}).$$

By what we have just observed, ζ_W belongs to the image of the map i, say $\zeta_W = i(\alpha)$. Under the nondegenerate pairing (given by Poincaré duality)

$$H^{2n}(W, W_0, \mathbb{Q}) \otimes \bigoplus_{t \in C \setminus C_0} H^{2n}(E_t, \mathbb{Q}) \to \mathbb{Q},$$

and the intersection pairing on W, the map i is dual to the restriction map

$$H^{2n}(W, \mathbb{Q}) \to \bigoplus_{t \in C \setminus C_0} H^{2n}(E_t \mathbb{Q}),$$

and so we get that

$$d \cdot \int_X \zeta \cup \zeta = \int_W \zeta_W \cup \zeta_W = \big\langle i(\alpha), \zeta_W \big\rangle = \sum_{t \in C \setminus C_0} \big\langle \alpha, i_t^*(\zeta_W) \big\rangle$$

where $i_t \colon E_t \to W$ is the inclusion. But the first integral is nonzero, because the intersection pairing on X is definite on the subspace of primitive (n,n)-classes. We conclude that the pullback of ζ to at least one of the E_t has to be nonzero.

By construction, E_t maps into one of the singular fibers of π, say to $\pi^{-1}(p)$, where p belongs to the intersection of $X^\vee$ with the closure of the curve. Thus ζ has nonzero image in $H^{2n}\big(\pi^{-1}(p), \mathbb{Q}\big)$; by the result of $[1, 2]$ mentioned above, ν_ζ has to be singular at p, concluding the proof. $\square$

I do not know whether a "converse" to Proposition 1 is true; that is to say, whether the normal function associated to an algebraic cycle on X has to have a zero locus of positive dimension for sufficiently ample H. If it was, this would give one more equivalent formulation of the Hodge conjecture.

2. Cohomology of the discriminant locus

G. Pearlstein pointed out that the singularities of the discriminant locus should be complicated enough to capture all the primitive cohomology of the original variety, once $H = dA$ is a sufficiently big multiple of a very ample class. In this section, we give an elementary proof of this fact for $d \geq 3$.

To do this, we need a simple lemma, used to estimate the codimension of loci in $X^\vee$ where the fibers of π have a singular set of positive dimension. Let

$$V_d = H^0\big(X, \mathscr{O}_X(dA)\big)$$

be the space of sections of dA, for A very ample.

Lemma 3. *Let $Z \subseteq X$ be a closed subvariety of positive dimension $k > 0$. Write $V_d(Z)$ for the subspace of sections that vanish along Z. Then*

$$\operatorname{codim}\big(V_d(Z), V_d\big) \geq \binom{d+k}{k}.$$

Proof. Since A is very ample, we may find $(k+1)$ points $P_0, P_1, \dots, P_k$ on Z, together with $(k+1)$ sections $s_0, s_1, \dots, s_k \in V_1$, such that each s_i vanishes at all points P_j with $j \neq i$, but does not vanish at P_i. Then all the sections

$$s_0^{\otimes i_0} \otimes s_1^{\otimes i_1} \otimes \dots \otimes s_k^{\otimes i_k} \in V_d,$$

for $i_0 + i_1 + \dots + i_k = d$, are easily seen to be linearly independent on Z. The lower bound on the codimension follows immediately. □

We now use this estimate to make the above idea about the cohomology of $X^\vee$ precise. As one further bit of notation, let $\mathfrak{X}_{sing} \subseteq \mathfrak{X}$ stand for the union of all the singular points in the fibers of π. It is well-known that $\mathfrak{X}_{sing}$ is a projective space bundle over X, and in particular smooth.

Proposition 4. *Let $H = dA$ for a very ample class A. If $d \geq 3$, then the map $\phi\colon \mathfrak{X}_{sing} \to X^\vee$ is a small resolution of singularities, and therefore*

$$IH^*(X^\vee, \mathbb{Q}) \simeq H^*(\mathfrak{X}_{sing}, \mathbb{Q}).$$

In particular, $H^*(X, \mathbb{Q})$ is a direct summand of $IH^*(X^\vee, \mathbb{Q})$ once $d \geq 3$.

Proof. By [3, p. Theorem 6], the discriminant locus is a divisor in P once $d \geq 3$. This means that there are hyperplane sections of X with exactly one ordinary double point [8, p. 317]. The map ϕ is then birational, and therefore a resolution of singularities of $X^\vee$. To prove that it is a small resolution, take a stratification of $X^\vee$ with smooth strata, and such the fibers of the map ϕ have constant dimension over each stratum; this is easily done, using the constructibility of the higher direct image sheaves $R^k\phi_*\mathbb{Q}$.

Let $S \subseteq X^\vee$ be an arbitrary stratum along which the singular set of the fiber has dimension $k > 0$. At a general point $t \in S$, there then has to be an irreducible component Z in the singular locus of $pi^{-1}(t)$ that remains singular to first order along S. Now a tangent vector to S may be represented by a section s of $\mathscr{O}_X(dA)$, and a simple calculation in local coordinates shows that, in order for Z to remain singular to first order, the section s has to vanish along Z. By Lemma 3, the space of such sections has codimension at least $\binom{d+k}{k}$, and a moment's thought shows that, therefore,

$$\operatorname{codim}(S, X^\vee) \geq \binom{d+k}{k} - 1.$$

This quantity is evidently a lower bound for the codimension of the locus where the fibers of ϕ have dimension k. In order for ϕ to be a small resolution, it is thus sufficient that

$$\binom{d+k}{k} - 1 > 2k$$

for all $k > 0$. Now one easily sees that this condition is satisfied provided that $d \geq 3$. This proves the first assertion; the second one is a general fact about intersection cohomology [4, pp. 120–1]. Finally, $H^*(X, \mathbb{Q})$ is a direct summand of $H^*(\mathfrak{X}_{sing}, \mathbb{Q})$ because $\mathfrak{X}_{sing}$ is a projective space bundle over X, and the third assertion follows. □

The proof shows that, as in the theorem by A. Dimca and M. Saito, $d \geq 2$ is sufficient in most cases. A result related to Proposition 4, and also showing the effect of taking H sufficiently ample, was pointed out to me by H. Clemens;

he noticed that, as a consequence of M. Nori's connectivity theorem, one has an isomorphism

$$H^{2n}(X, \mathbb{Q})_{\mathrm{prim}} \simeq H^1\big(P \setminus X^\vee, (R^{2n-1}\pi_*\mathbb{Q})_{\mathrm{van}}\big),$$

once H is sufficiently ample [7, Corollary 4.4 on p. 364].

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