ON THE BEHAVIOR OF KODAIRA DIMENSION UNDER SMOOTH MORPHISMS

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Abstract. We prove several results on the additivity of Kodaira dimension under smooth morphisms of smooth projective varieties.

A. Introduction

The purpose of this paper is to prove some of the conjectures in [Pop21] about the additivity of the Kodaira dimension under smooth morphisms between smooth projective varieties. Our results are unconditional when varieties of general type are involved in some way, or when the base space is either of dimension ≤ 3 or a good minimal model of Kodaira dimension zero; for other varieties, we need a version of the non-vanishing conjecture due to Campana and Peternell [CP11]. We use the convention that an *algebraic fiber space* is a surjective morphism $f \colon X \to Y$ with connected fibers between two smooth projective varieties X and Y (over the field of complex numbers). We usually denote the general fiber by the letter F. We say that an algebraic fiber space is *smooth* if f is a smooth morphism.

A particular case of conjectures 2.1 or 3.4 in [Pop21] predicts that if $f \colon X \to Y$ is a smooth algebraic fiber space, and X is of general type, then Y is of general type as well. Our first result is the following more general statement:

Theorem A. *Let $f \colon X \to Y$ be a smooth algebraic fiber space whose general fiber F satisfies $\kappa(F) \geq 0$. Then Y is of general type if and only if $\kappa(X) = \kappa(F) + \dim Y$.*

By the Easy Addition formula [Mor87, Cor. 1.7], we always have the elementary inequality $\kappa(X) \leq \kappa(F) + \dim Y$. Also recall that Iitaka's conjecture, to the effect that $\kappa(X) \geq \kappa(F) + \kappa(Y)$, is known when Y is of general type [Vie83, Cor. IV]. One half of the theorem is therefore already known. Our contribution is to prove the other half: the identity $\kappa(X) = \kappa(F) + \dim Y$ implies that Y is of general type.

Note. Using the same techniques, one can show that if $f \colon X \to Y$ is an algebraic fiber space with Y not uniruled, $\kappa(F) \geq 0$, and $\kappa(X) = \kappa(F) + \dim Y$, and if $V \subseteq Y$ denotes the smooth locus of f, then V is of log general type. This answers positively [Pop21, Conj. 3.7] when Y is not uniruled, but is weaker than [Pop21, Conj. 4.1]. See also Remark 1.

Our next result deals with smooth algebraic fiber spaces with fibers of general type or of dimension one; compare also to our earlier work on Viehweg hyperbolicity in [PS17].

Theorem B. *Let $f \colon X \to Y$ be a smooth algebraic fiber space whose general fiber F is either of general type, or a curve. If $\kappa(Y) \geq 0$, or if Y is uniruled, then we have $\kappa(X) = \kappa(F) + \kappa(Y)$.*

For technical reasons, the case where $\kappa(Y) = -\infty$ and Y is not uniruled is missing; note however that a well-known conjecture in birational geometry, called the *non-vanishing conjecture*, predicts that $\kappa(Y) = -\infty$ is in fact equivalent to Y being uniruled. Thus up to

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this prediction, our theorem implies the additivity of the Kodaira dimension for any smooth algebraic fiber space with fibers of this type. Unlike the other results in this paper, besides hyperbolicity-type techniques, Theorem B also relies on the existence and properties of moduli spaces of varieties of general type.

We also prove a more general result that depends on the base of the fibration satisfying the following conjecture of Campana-Peternell [CP11, Conj. 2.4], which can be seen as a generalization of the non-vanishing conjecture:

Conjecture (Campana-Peternell)**.** *Let Y be a smooth projective variety, and assume that $K_Y \sim_{\mathbb{Q}} A + B$, where A and B are $\mathbb{Q}$-divisors such that A is effective and B is pseudo-effective. Then $\kappa(Y) \geq \kappa(A)$.*

The non-vanishing conjecture is the case $A = 0$. Concretely, it says that that if K_Y is pseudo-effective – which is equivalent to Y not being uniruled [BDPP13] – then $\kappa(Y) \geq 0$. It can be easily checked [CP11, p. 12–13] that the Campana-Peternell conjecture is implied by the existence of good minimal models; it therefore holds if $\dim Y \leq 3$, since this conjecture in the Minimal Model Program (MMP) is known to hold in this range. On the other hand, the Campana-Peternell conjecture is definitely weaker than the full abundance conjecture; in fact, we believe that it is basically equivalent to the non-vanishing conjecture [Sch22].

Theorem C. *Let $f \colon X \to Y$ be a smooth algebraic fiber space, and assume that the Campana-Peternell conjecture holds. Then $\kappa(F) + \kappa(Y) \geq \kappa(X)$.*

Note. More precisely, we need the Campana-Peternell conjecture for the fibers of the Iitaka fibration of Y if $\kappa(Y) \geq 0$; or the non-vanishing conjecture for Y itself if $\kappa(Y) = -\infty$.

Iitaka's conjecture predicts of course the opposite inequality $\kappa(X) \geq \kappa(F) + \kappa(Y)$, for an arbitrary morphism; it is known to be implied by the existence of a good minimal model for the geometric generic fiber of f by a result of Kawamata [Kaw85]. Therefore we obtain the following loosely formulated consequence.

Corollary D. *The conjectures of the MMP imply* [Pop21, Conj. 3.4]*, stating that if $f \colon X \to Y$ is a smooth algebraic fiber space, then $\kappa(X) = \kappa(F) + \kappa(Y)$.*

In any event, we obtain unconditional results – and hence the solution to [Pop21, Conj. 2.1] – for algebraic fiber spaces over surfaces and threefolds.

Corollary E. *If $f \colon X \to S$ is a smooth algebraic fiber space, with S a smooth projective surface, then $\kappa(X) = \kappa(F) + \kappa(S)$.*

Corollary F. *If $f \colon X \to Y$ is a smooth algebraic fiber space, with Y a smooth projective threefold, then $\kappa(F) + \kappa(Y) \geq \kappa(X)$.*

In Corollary E we have equality due to the fact that the Iitaka conjecture is known over surfaces; see [Cao18], and the references therein. This is not yet known when Y is an arbitrary threefold.

Theorem C can be easily reduced to the two cases $\kappa(Y) = -\infty$ and $\kappa(Y) = 0$, and is therefore a consequence of the following two results.

Proposition G. *Let $f \colon X \to Y$ be a smooth morphism between smooth projective varieties, with Y uniruled. Then $\kappa(X) = -\infty$.*

Thus the additivity of Kodaira dimension for smooth morphisms over varieties Y with $\kappa(Y) = -\infty$ holds if one assumes the non-vanishing conjecture. In combination with previous results, the proposition also implies that [Pop21, Conj. 3.4] holds when X is a good minimal model with $\kappa(X) = 0$; in fact then so is Y, see Corollary 2.2. The proof of Proposition G is the first step in the proof of Theorem B. It works by reduction to the case $Y = \mathbb{P}^1$, where a stronger version is a deep result of Viehweg-Zuo [VZ01]. Therefore the main new input is the following:

Theorem H. *Let $f\colon X \to Y$ be a smooth algebraic fiber space with $\kappa(Y) = 0$.*

(i) *If the Campana-Peternell conjecture holds on Y, so for instance if Y has a good minimal model, then $\kappa(F) \geq \kappa(X)$.*
(ii) *If Y is actually a good minimal model, in other words if $K_Y \sim_{\mathbb{Q}} 0$, then $\kappa(X) = \kappa(F)$. Moreover, we have $P_m(F) \geq P_m(X)$ for all $m \geq 1$, with equality if Y is simply connected (e.g. Calabi-Yau).*

Thus [Pop21, Conj. 3.4] holds unconditionally when Y is a good minimal model with $\kappa(Y) = 0$. When Y is an abelian variety, this is a special case of [MP21, Thm. A(2)].

The proofs of the results in this paper rely on a Hodge module construction from [PS14, PS17], which was in turn heavily inspired by techniques of Viehweg-Zuo in their work on hyperbolicity. Because of the smoothness assumption, semistable reduction and the technical passage to logarithmic Higgs bundles are however not needed. The proof of Theorem H is somewhat more delicate (than for instance that of Theorem A), needing both an additional technical ingredient and important analytic results from [PT18, CP17] (see also [HPS18]).

Remark. The disadvantage of this method is the lack of symmetry between the assumptions and the conclusion. Say $f\colon X \to Y$ is a smooth algebraic fiber space, and D a divisor on Y. Roughly speaking, we start from the assumption that $mK_X - f^*D$ is *effective* for some $m \geq 1$; and the conclusion is that $m'K_Y - D$ is *pseudo-effective* for $m' \gg m$. The Campana-Peternell conjecture removes this asymmetry, and allows us to put "pseudo-effective" or "effective" in both places; but it would of course be desirable to have unconditional results.

We conclude by observing that the Campana-Peternell conjecture, together with the techniques above, also leads to the following characterization of $\kappa(Y)$ that can be seen as a generalization of Theorem A.

Corollary I. *Let $f\colon X \to Y$ be a smooth algebraic fiber space, with general fiber F satisfying $\kappa(F) \geq 0$. If the Campana-Peternell conjecture holds on X and Y, then*

$$\kappa(Y) = \max\big\{ \kappa(L) \bigm| mK_X - f^*L \text{ is pseudo-effective for some } m \geq 1 \big\},$$

where the maximum is taken over all line bundles on Y.

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B. Proofs

1. Proof of Theorem A. We start by considering a general construction. Let A be an ample line bundle on Y. According to [Mor87, Prop. 1.14], the condition that $\kappa(X) = \kappa(F) + \dim Y$ is equivalent to having

$$H^0(X, \omega_X^{\otimes m} \otimes f^*A^{-1}) \neq 0$$

for some $m \geq 1$. Using the projection formula, this is equivalent to the existence of a nontrivial (hence injective) morphism

$$L := A \otimes (\omega_Y^{-1})^{\otimes m} \hookrightarrow f_*\omega_{X/Y}^{\otimes m}. \tag{1.1}$$

Note that by the invariance of plurigenera, the sheaf $f_*\omega_{X/Y}^{\otimes m}$ is locally free of rank $P_m(F)$ on all of Y.

We now use Viehweg's well-known fiber product trick, in the following form: let $X^{(m)}$ denote the m-fold fiber product $X \times_Y \cdots \times_Y X$, with its induced morphism $f^{(m)} : X^{(m)} \to Y$. Note that this is still a smooth algebraic fiber space, with general fiber isomorphic to the m-fold product $F \times \cdots \times F$. Moreover,

$$f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m} \simeq (f_*\omega_{X/Y}^{\otimes m})^{\otimes m} \tag{1.2}$$

for example by [Mor87, Cor. 4.11]. This gives us an inclusion

$$L^{\otimes m} \hookrightarrow f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m}.$$

By replacing X by $X^{(m)}$, and the fiber space f by $f^{(m)}$, we are therefore allowed to assume from the beginning that we are working with a fiber space f such that

$$H^0(X, B^{\otimes m}) \neq 0 \quad \text{where} \quad B = \omega_{X/Y} \otimes f^*L^{-1}. \tag{1.3}$$

Under this assumption, one can bring into play the machinery used in [PS14, PS17], strongly inspired in turn by the celebrated method developed by Viehweg-Zuo [VZ01, VZ02], based on the existence of so-called Viehweg-Zuo sheaves. Note however that, unlike in the works above, the line bundle L we consider here is usually not positive any more.

First, using [PS17, Thm. 2.2] and denoting $\mathcal{A}_Y = \operatorname{Sym} T_Y$, (1.3) implies that there exists a graded $\mathcal{A}_Y$-module $\mathscr{G}_\bullet$ that is coherent over $\mathcal{A}_Y$, and has the following among its special properties:

- (i) One has $\mathscr{G}_0 \simeq L$.
- (ii) Each $\mathscr{G}_k$ is torsion-free.
- (iii) There exists a regular holonomic $\mathscr{D}_Y$-module $\mathcal{M}$ with good filtration $F_\bullet\mathcal{M}$, and an inclusion of graded $\mathcal{A}_Y$-modules $\mathscr{G}_\bullet \subseteq \operatorname{gr}_\bullet^F \mathcal{M}$.
- (iv) The filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a polarizable Hodge module M on Y, with strict support Y, and $F_k\mathcal{M} = 0$ for $k < 0$.

We next follow closely [PS17, §3.2]; the exact same argument as in the proof of [PS17, Thm. 3.5] shows that at least one of the following two cases must happen:[1]

- (i) The line bundle $L^{-1} = A^{-1} \otimes \omega_Y^{\otimes m}$ is pseudo-effective.
- (ii) There exists a weakly positive sheaf $\mathcal{W}$ on Y and an inclusion

$$\mathcal{W} \otimes L \hookrightarrow (\Omega_Y^1)^{\otimes N}$$

for some positive integer $N > 0$.

In the first case, we immediately get that ω_Y is big, and hence that Y is of general type. In the second case, we argue as follows. We rewrite the above inclusion as

$$\mathcal{W} \otimes A \hookrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m}, \tag{1.4}$$

and observe that $\mathcal{W} \otimes A$ is big in the sense of Viehweg [PS17, Lem. 3.2]. Now $\det \Omega_Y^1 = \omega_Y$, hence there exists also a (split) inclusion

$$\omega_Y^{\otimes m} \hookrightarrow (\Omega_Y^1)^{\otimes m \cdot \dim Y}.$$

Putting everything together we deduce the existence of an inclusion

$$\mathcal{H} \hookrightarrow (\Omega_Y^1)^{\otimes M}$$

[1]The argument is based on iterating generalized Kodaira-Spencer maps, and originated in work of Kovács and Viehweg-Zuo; in this case it requires the negativity of the kernels of generalized Kodaira-Spencer maps for pure Hodge modules with strict support Y, proved in [PW16] using results by Zuo and Brunebarbe.

for some $M > 0$, where $\mathcal{H}$ is a big torsion-free sheaf. The result of Campana-Păun [CP19, Thm. 7.11] implies then that Y is of general type. (Campana and Păun only state the result with $\mathcal{H}$ being a line bundle, but their proof still works when $\mathcal{H}$ has higher rank; see also [Sch17] for a streamlined proof of this important result.)

Remark 1 (**The non-smooth case.**). As mentioned in the introduction, when Y is not uniruled the proof can be extended without much effort to the general case, when f is only smooth over an open set $V \subseteq Y$, to deduce that V is of log general type. By blowing up it is immediate to reduce to the case where $Y \smallsetminus V = D$, a simple normal crossing divisor. The only difference in the argument is that (1.2) is now an isomorphism only away from D, which in practice means that in the proof above we have to replace L by $L(-rD)$ for some integer $r > 0$. The same steps now lead to the bigness of a line bundle of the form

$$\omega_Y^{\otimes a} \otimes \mathscr{O}_Y(bD)$$

for some $a, b > 0$, and the pseudo-effectivity of ω_Y is then needed to deduce the bigness of $\omega_Y(D)$ by multiplying by a suitable multiple of either ω_Y or $\mathscr{O}_Y(D)$. (Compare also with the proof of [PS17, Thm. 4.1].)

Note also that if $\kappa(Y) \geq 0$ and the complement of V in Y has codimension at least 2, it is not hard to check that $\kappa(V) = \kappa(Y)$; see e.g. [MP21, Lem. 2.6]. Thus under this assumption the conclusion is still that Y is of general type.

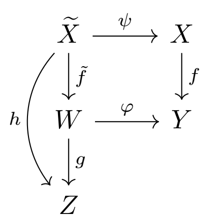
2. Proof of Theorem B. We present the proof in the case when the fibers are of general type, and specify at the very end the minor modifications needed when the fibers are curves. We divide the proof into a few steps.

Step 1. In this step we deal with the case when Y is uniruled,[2] so in particular $\kappa(Y) = -\infty$, by easy reduction to the case $Y = \mathbb{P}^1$, in which case a theorem of Viehweg-Zuo applies. We are in fact proving Proposition G; no assumption on the fibers of f is necessary. See also Remark 2 below for an alternative proof based on the techniques used in Theorem A.

The hypothesis means that there exists a variety Z, which can be assumed to be smooth and projective, and a dominant rational map

$$Z \times \mathbb{P}^1 \dashrightarrow Y.$$

By resolving the indeterminacies of this map, we obtain a generically finite surjective morphism $\varphi\colon W \to Y$, such that W admits a morphism $g\colon W \to Z$ with general fiber $\mathbb{P}^1$. We consider the diagram



where $h = g \circ \tilde{f}$. Since ψ is generically finite we have $\kappa(\widetilde{X}) \geq \kappa(X)$ (see [Mor87, Cor. 2.3(i)]), and therefore it suffices to show that $\kappa(\widetilde{X}) = -\infty$. Note now that the general fiber H of h admits a smooth morphism (with fiber F) to the general fiber of g, i.e. to $\mathbb{P}^1$. By [VZ01, Thm. 2] it follows that $\kappa(H) = -\infty$, and therefore by Easy Addition (see [Mor87, Cor. 1.7]) we also have $\kappa(\widetilde{X}) = -\infty$. As promised, this proves Proposition G.

Step 2. In this step we assume that $\kappa(Y) \geq 0$, and show that we can reduce to the case $\kappa(Y) = 0$. First, since the fibers of f are of general type, the subadditivity

$$\kappa(X) \geq \kappa(F) + \kappa(Y)$$

[2] As mentioned in the introduction, conjecturally this is equivalent to $\kappa(Y) = -\infty$.

conjectured by Iitaka holds by [Kol87]. To prove Theorem B, it therefore suffices to show

$$\kappa(F) + \kappa(Y) \geq \kappa(X). \tag{2.1}$$

We show the reduction for this inequality.

To this end, we consider the Iitaka fibration $g\colon Y \to Z$, which after base change can be assumed to be a morphism, with Z smooth and projective. We again denote $h = g \circ f$, with general fiber H, and we also denote the general fiber of g by G. Thus we have a smooth morphism $H \to G$, with fiber F, and since $\kappa(G) = 0$, we may assume that $\kappa(F) \geq \kappa(H)$. On the other hand, by definition we have $\dim Z = \kappa(Y)$, and Easy Addition applied to h gives

$$\kappa(H) + \dim Z \geq \kappa(X).$$

Putting the two inequalities together, we obtain (2.1).

Step 3. Under the assumption that the fibers of f are of general type, by Taji [Taj20, Theorem 1.2] we also know the validity of the Kebekus-Kovács conjecture, stating that if $\kappa(Y) \geq 0$, then $\kappa(Y) \geq \mathrm{Var}(f)$. Using this in combination with Step 2, we may therefore assume that $\mathrm{Var}(f) = 0$, i.e. that f is birationally isotrivial.[3]

Step 4. In this final step, we show the assertion in the theorem under the assumption that $\mathrm{Var}(f) = 0$ (with no assumptions on Y). This is a consequence of the existence and properties of the moduli space of canonically polarized varieties with canonical singularities. We thank Ziquan Zhuang for suggesting this proof.

Concretely, we consider a relative canonical model $f'\colon X' \to Y$ of f, which exists by [BCHM10]. By Siu's deformation invariance of plurigenera [Siu98], f' is a flat morphism whose fibers are canonical models of the fibers of f, with constant Hilbert polynomial. Therefore there exists a morphism

$$\varphi\colon Y \to \mathbf{M}$$

to the appropriate coarse moduli space of canonically polarized varieties with canonical singularities [Vie95, Kol21]. Since f is birationally isotrivial, it follows that φ is constant on an open set, and therefore constant everywhere; in other words the family f' is isotrivial. Since canonical models of varieties of general type have finite automorphism group, and indeed $\mathbf{M}$ is the coarse moduli space of a Deligne-Mumford stack, it is therefore well-known that there is a finite étale base change $\widetilde{Y} \to Y$ such that

$$\widetilde{X} := X' \times_Y \widetilde{Y} \simeq F' \times \widetilde{Y},$$

with F' the fiber of f'. We thus have

$$\kappa(X) = \kappa(X') = \kappa(\widetilde{X}) = \kappa(F') + \kappa(\widetilde{Y}) = \kappa(F) + \kappa(Y),$$

where the first and last inequality are due to the fact that X' and F' have canonical singularities (in which case the Kodaira dimension can be defined equivalently either using pluricanonical forms for sufficiently divisible multiplies, or as that of any resolution of singularities). This concludes the proof in the case of fibers of general type.

When the fibers are curves, the only potentially different case is that of fibers of genus 1; however the argument is essentially the same. Iitaka's conjecture in Step 2 is known for curve fibrations by [Vie77], and Taji's result used in Step 3 is proved more generally in [Taj20] when the fiber F has a good minimal model. In Step 4, passing to the dual elliptic fibration $\mathrm{Pic}^0(X/Y) \to Y$, which has a section, allows us to consider the associated morphism $\varphi\colon Y \to M_{1,1}$ to the coarse moduli space of elliptic curves, which then has to be constant if

[3] It is worth noting that in this proof, the hyperbolicity-type techniques present in the other results in this paper are hidden in the results from [VZ01] and [Taj20].

$\mathrm{Var}(f) = 0$. This implies that all the fibers of the original f are isomorphic as well. Finally, under this assumption it is also known that there exists a finite étale cover $\widetilde{Y} \to Y$ such that

$$\widetilde{X} := X \times_Y \widetilde{Y} \simeq F \times \widetilde{Y}.$$

See for instance [KL09, Lem. 17] for a more general statement. This finishes the proof.

Remark 2. Proposition G can be rephrased as saying that if $f\colon X \to Y$ is smooth and $\kappa(X) \geq 0$, then K_Y is pseudo-effective. Here is an alternative proof of this fact, resembling that of Theorem A. We consider the construction in the proof of that theorem, only now we take $A = \mathscr{O}_Y$, which is possible because $\kappa(X) \geq 0$. The exact same proof shows then that either we have directly that K_Y is pseudo-effective (case (i)), or that there is an inclusion

$$\mathcal{W} \hookrightarrow (\Omega^1_Y)^{\otimes N} \otimes \omega_Y^{\otimes m},$$

where $\mathcal{W}$ has pseudo-effective determinant (case (ii)). But [CP19, Thm. 7.6] says precisely that in the latter case K_Y is again pseudo-effective.

Remark 3. In fact, recent work in symplectic geometry can be used to prove the following strengthening of Proposition G: *Let $f\colon X \to Y$ be a smooth algebraic fiber space. If K_X is pseudo-effective, then K_Y is also pseudo-effective.* Here is a sketch of the proof. If K_Y is not pseudo-effective, then Y is covered by rational curves. After restricting to a smooth rational curve whose normal bundle has nonnegative degree, we obtain a smooth algebraic fiber space $g\colon Z \to \mathbb{P}^1$ such that K_Z is pseudo-effective. But a very recent theorem by Pieloch [Pie21, Thm. 1.1] – answering a question by Starr [Sta15] – says that Z is covered by sections of g, and therefore uniruled. This is a contradiction.

Remark 4. As pointed out to us by Fanjun Meng, Proposition G (or Step 1 in the proof above), combined with a result from [Zha96], has the following interesting consequence:

Corollary 2.2. *Let $f\colon X \to Y$ be a smooth morphism between smooth projective varieties. If X is a good minimal model with $\kappa(X) = 0$ (i.e. $K_X \sim_{\mathbb{Q}} 0$), then so is Y.*

Proof. By [Zha96, Thm. 2] (which only needs f to be surjective), since $-K_X$ is nef, it follows that either Y is uniruled, or $K_Y \sim_{\mathbb{Q}} 0$. On the other hand, Proposition G implies that we cannot have $\kappa(Y) = -\infty$. □

Note in particular that under the hypothesis of the Corollary we again have the additivity $\kappa(X) = \kappa(F) + \kappa(Y)$, with both sides being of course equal to 0.

3. Proof of Theorem C. If $\kappa(Y) = -\infty$ the result follows from the non-vanishing conjecture together with Proposition G, while if $\kappa(Y) = 0$, it follows from Theorem H(i). If $\kappa(Y) > 0$, we can reduce to the case $\kappa = 0$ using the standard argument involving the Iitaka fibration: we may assume that this is a morphism $g\colon Y \to Z$, with general fiber G satisfying $\kappa(G) = 0$. Therefore the general fiber H of $h = g \circ f$ has a smooth morphism $H \to G$ with fiber F, and consequently $\kappa(F) \geq \kappa(H)$. On the other hand, Easy Addition for h gives

$$\kappa(H) + \kappa(Y) = \kappa(H) + \dim Z \geq \kappa(X).$$

Hence the remaining point is to prove Theorem H.

4. Proof of Theorem H. If $\kappa(F) = -\infty$, then we also have $\kappa(X) = -\infty$ by Easy Addition. We may therefore assume throughout that $\kappa(F) \geq 0$. The key players are again the vector bundles

$$F_m := f_*\omega_{X/Y}^{\otimes m}, \quad \text{for} \ \ m \geq 1,$$

We also consider their twists

$$E_m := F_m \otimes \omega_Y^{\otimes m} \simeq f_*\omega_X^{\otimes m} \quad \text{for} \ \ m \geq 1,$$

for which we have

$$r_m := \mathrm{rk}(E_m) = \mathrm{rk}(F_m) = P_m(F) \quad \text{and} \quad h^0(Y, E_m) = P_m(X).$$

We have for each m a (split) inclusion

$$\det F_m \hookrightarrow F_m^{\otimes r_m},$$

and therefore using the Viehweg fiber product trick just as in the proof of Theorem A, we obtain an inclusion

$$(\det F_m)^{\otimes m} \hookrightarrow f_*^{(mr_m)} \omega_{X^{(mr_m)}/Y}^{\otimes m}.$$

Proceeding precisely as in the proof of that theorem (where the role of L there is now played by $\det F_m$), the arguments from [PS17] lead then to one of the following two possibilities:

(1) $(\det F_m)^{-1}$ is pseudo-effective.

(2) There exists a weakly positive sheaf $\mathcal{W}$ on Y, and an inclusion

$$\mathcal{W} \otimes \det F_m \hookrightarrow (\Omega_Y^1)^{\otimes N}$$

for some positive integer $N > 0$.

Recall on the other hand that F_m is a weakly positive sheaf by [Vie83, Thm. III], and therefore $\det F_m$ is a pseudo-effective line bundle, e.g. by [Vie83, Lem. 1.6 (6)]. If (1) holds, then both $\det F_m$ and $(\det F_m)^{-1}$ are pseudo-effective, and so $\det F_m \equiv 0$ (i.e. it is numerically trivial).

On the other hand, if (2) holds, by taking the saturation of the left hand side inside $(\Omega_Y^1)^{\otimes N}$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{W}' \otimes \det F_m \longrightarrow (\Omega_Y^1)^{\otimes N} \longrightarrow \mathscr{Q} \longrightarrow 0$$

where $\mathcal{W}'$ is a weakly positive sheaf, and $\mathscr{Q}$ is torsion-free. We also know that $\det \mathscr{Q}$ is pseudo-effective; this follows from [CP19, Thm. 1.3], which says that if K_Y is pseudo-effective, then any quotient of any tensor power of Ω_Y^1 has pseudo-effective determinant. Passing to determinants, we then obtain an isomorphism of the form

$$\omega_Y^{\otimes a} \simeq (\det F_m)^{\otimes b} \otimes P, \tag{4.1}$$

where $a, b > 0$ and P is a pseudo-effective line bundle.

Remark 5. Up to here we only used that $\kappa(F) \geq 0$ and K_Y is pseudo-effective.

If we assume that $K_Y \sim_{\mathbb{Q}} 0$, from (4.1) we again deduce that $(\det F_m)^{-1}$ is pseudo-effective, and therefore as above $\det F_m \equiv 0$. It follows from [CP17, Thm. 5.2] (see also [HPS18, Cor. 27.2]) that F_m is a hermitian flat vector bundle, and therefore comes from a unitary representation of the fundamental group. If we assume in addition that Y is simply connected, we then obtain $F_m \simeq \mathscr{O}_Y^{\oplus r_m}$, or equivalently

$$E_m \simeq (\omega_Y^{\otimes m})^{\oplus r_m}.$$

Note that by hypothesis $\omega_Y^{\otimes m} \simeq \mathscr{O}_X$ when m is sufficiently divisible, and otherwise vanishes. Consequently $P_m(X) = P_m(F)$ when $P_m(Y) = 1$, and $P_m(X) = 0$ otherwise. In general we only have $r_m \geq h^0(Y, E_m)$ (because unitary representations are semisimple), and therefore by the same argument we obtain

$$P_m(F) \geq P_m(X) \quad \text{for all } m \geq 1,$$

hence also $\kappa(F) \geq \kappa(X)$. On the other hand, a result of Cao-Păun [CP17, Thm. 5.6] states, for any algebraic fiber space f, that if $c_1(\det F_m) = 0 \in H^2(X, \mathbb{R})$ for some $m \geq 2$, then Iitaka's conjecture holds for f. Cao and Păun prove this result under the assumption that $c_1(\det F_m) = 0 \in H^2(X, \mathbb{Z})$, but because of [HPS18, Thm. 27.2], the weaker assumption is sufficient to conclude that F_m is a hermitian flat bundle; the rest of the argument then proceeds as in [CP17]. In our case, this gives the opposite inequality $\kappa(X) \geq \kappa(F)$, and concludes the proof of (ii).

We are thus left with proving (i). We need the following lemma.

Lemma 4.2. *Let E be a nonzero vector bundle on a projective variety X. If $h^0(X,E) \geq \operatorname{rk} E + 1$, then there is a line bundle L with $h^0(X,L) \geq 2$ and an inclusion $L \hookrightarrow E^{\otimes r}$ for some $1 \leq r \leq \operatorname{rk} E$.*

Proof. Let $\mathscr{F} \subseteq E$ be the torsion-free subsheaf generated by the global sections of E, and let $1 \leq r \leq \operatorname{rk} E$ be the generic rank of $\mathscr{F}$. Choose r global sections $s_1, \ldots, s_r \in H^0(X,E)$ that generate the stalk of $\mathscr{F}$ at a general point of X. The resulting morphism $\mathscr{O}_X^{\oplus r} \to \mathscr{F}$ is injective; it cannot be surjective, for otherwise it would be an isomorphism, and then $h^0(X,E) = h^0(X,\mathscr{F}) = r \leq \operatorname{rk} E$, contradicting the assumption that $h^0(X,E) \geq \operatorname{rk} E + 1$. Let $x \in X$ be one of the points where the morphism fails to be surjective on stalks. Since $\mathscr{F}$ is globally generated, depending on the situation we can do one of the following two things:

• If $s_1, \ldots, s_r$ remain linearly independent at x, we can find an additional global section $s \in H^0(X,E)$ such that such that $s(x)$ does not lie in the linear span of $s_1(x), \ldots, s_r(x)$ inside the stalk E_x. It follows that, among the $r+1$ sections

$$s_1 \wedge \cdots \wedge s_r, \quad s_1 \wedge \cdots \wedge s_{i-1} \wedge s \wedge s_{i+1} \wedge \cdots \wedge s_r \quad \text{for } i = 1, \ldots, r$$

of the bundle $\bigwedge^r E$, at least 2 are linearly indepedent.

• If $s_1, \ldots, s_r$ do not remain linearly independent at x, we can in any case find other global sections $t_1, \ldots, t_r \in H^0(X,E)$ such that $t_1(x), \ldots, t_r(x) \in E_x$ form part of a minimal system of generators of $\mathscr{F}_x$. Then

$$s_1 \wedge \cdots \wedge s_r \quad \text{and} \quad t_1 \wedge \cdots \wedge t_r$$

are linearly independent global sections of the bundle $\bigwedge^r E$.

Either way, by construction all these sections come from global sections of the sheaf $\bigwedge^r \mathscr{F}$. Let L be the reflexive hull of $\bigwedge^r \mathscr{F}$; then L is a line bundle with $h^0(X,L) \geq 2$. We also have inclusions

$$L \hookrightarrow \bigwedge^r E \hookrightarrow E^{\otimes r}$$

because E is locally free. □

Let us now suppose for the sake of contradiction that $P_m(X) \geq P_m(F) + 1$. The lemma, applied to the vector bundle E_m, then gives us an inclusion

$$L \hookrightarrow E_m^{\otimes r} \simeq F_m^{\otimes r} \otimes \omega_Y^{\otimes mr}$$

for some integer $1 \leq r \leq r_m$, where L is a line bundle with $h^0(X,L) \geq 2$, and so $\kappa(L) \geq 1$. Arguing as above, using the fiber product trick, we obtain one of the following two possibilities:

(1) $L^{-1} \otimes \omega_Y^{\otimes mr}$ is pseudo-effective.

(2) There exists a weakly positive sheaf $\mathcal{W}$ on Y, and a short exact sequence

$$0 \to \mathcal{W} \otimes L \to (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes mr} \to \mathscr{Q} \to 0$$

for some positive integer $N > 0$.

In case (1), since $\kappa(L) \geq 1$, the Campana-Peternell conjecture implies that $\kappa(Y) \geq 1$, contradicting our assumption that $\kappa(Y) = 0$.

In case (2), as above $\det \mathscr{Q}$ must be a pseudo-effective line bundle by [CP19, Thm. 1.3]. Passing to determinants in the short exact sequence in (2), we now obtain an isomorphism

$$\omega_Y^{\otimes a} \simeq L^{\otimes b} \otimes P$$

where $a, b > 0$ and P is a pseudo-effective line bundle. We then conclude just as in case (1).

5. Proof of Corollary I. Let $f\colon X \to Y$ be a smooth algebraic fiber space whose general fiber F satisfies $\kappa(F) \geq 0$. We use the notation

$$\kappa_f(Y) := \max\big\{\, \kappa(Y,L) \;\big|\; mK_X - f^*L \text{ is pseudo-effective for some } m \geq 1 \,\big\},$$

and show that $\kappa(Y) = \kappa_f(Y)$ provided that $\kappa(F) \geq 0$. One inequality is easy. Our assumption that $\kappa(F) \geq 0$ implies that the relative canonical divisor $K_{X/Y}$ is pseudo-effective; the reason is that $\omega_{X/Y}$ has a canonical singular hermitian metric with semi-positive curvature [PT18, Thm. 4.2.2] (see also [HPS18, Thm. 27.1]). Since $K_X - f^*K_Y \equiv K_{X/Y}$, we get $\kappa_f(Y) \geq \kappa(Y)$.

To prove the other inequality, let L be a line bundle on Y such that $mK_X - f^*L$ is pseudo-effective, and such that $\kappa_f(Y) = \kappa(Y,L)$. After replacing Y by a log resolution of the base locus of the linear system $|nL|$ (for some $n \geq 1$), and $f\colon X \to Y$ by the fiber product, we can assume that $nL \sim B + E$ with B semi-ample and E effective; consequently, $nmK_X - f^*B \sim n(mK_X - f^*L) + E$ is still pseudo-effective. After replacing L by B, we may therefore assume from the beginning that L is semi-ample. The Campana-Peternell conjecture (on X) implies that $mK_X - f^*L$ is effective for $m \gg 0$; this is proved in [Sch22]. We can now apply exactly the same argument as in the proof of Theorem A, up to the line containing (1.4), and obtain a short exact sequence

$$0 \to \mathcal{W} \otimes L \to (\Omega^1_Y)^{\otimes N} \otimes \omega_Y^{\otimes m} \to \mathcal{Q} \to 0$$

for some $N \geq 1$, with $\mathcal{W}$ weakly positive. (Concretely, $\mathcal{W}$ is a torsion-free sheaf with a semi-positively curved singular hermitian metric.) Since $m \geq 1$, both K_Y and $\det \mathcal{Q}$ must therefore be pseudo-effective: this follows by combining [CP19, Thm. 7.6] (to conclude that K_Y is pseudo-effective) and [CP19, Thm. 1.3] (to conclude that any quotient of any tensor power of Ω^1_Y has pseudo-effective determinant). Passing to determinants in the short exact sequence above, we see that there are two integers $a, b \geq 1$ such that

$$aK_Y \sim bL + \det \mathcal{Q} + \det \mathcal{W}.$$

As both $\det \mathcal{Q}$ and $\det \mathcal{W}$ are pseudo-effective, the Campana-Peternell conjecture (on Y) now gives us the desired inequality $\kappa(Y) \geq \kappa(Y,L) = \kappa_f(Y)$.

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