HODGE MODULES AND SINGULAR HERMITIAN METRICS

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Abstract. The purpose of this paper is to study certain notions of metric positivity for the lowest nonzero piece in the Hodge filtration of a Hodge module. We show that the Hodge metric satisfies the minimal extension property. In particular, this singular Hermitian metric has semi-positive curvature.

1. Introduction

Motivated by the Iitaka conjecture, there has been a lot of interest in studying the positivity of direct images of relative pluri-canonical bundles using Hodge theory. Among results about algebraic positivity, it is known that they are weakly positive [V$^+$83]. Recently, the authors in [CP17, HPS18] emphasize a different aspect of positivity from the metric point of view. Let $f : Y \to X$ be a projective and surjective holomorphic map between two complex manifolds. Given a holomorphic line bundle (L, h) on Y where h is a singular hermitian metric with semi-positive curvature, the authors in [HPS18] construct a singular hermitian metric on $f_*(\omega_{Y/X} \otimes L \otimes \mathcal{I}(h))$. Furthermore, they show that this new metric satisfies the "minimal extension property" (see Section 2.6). In particular, it has semi-positive curvature, which generalizes the work of [PT18]. This metric positivity then implies Viehweg's results on weak positivity by [PT18, Theorem 2.5.2].

Let k be the dimension of a general fiber of f and let (L, h) be $\mathcal{O}_Y$ equipped with the trivial metric. The construction in [HPS18] gives a singular Hermitian metric on $f_*(\omega_{Y/X})$, which is the lowest piece in the Hodge filtration of $R^k f_*(\underline{\mathbb{C}})$. This singular Hermitian metric on $f_*(\omega_{Y/X})$ actually comes from the Hodge metric on the smooth part of the fibration. It is natural to ask whether or not the Hodge metric on the lowest pieces in the Hodge filtration of *any* variation of Hodge structures satisfies the "minimal extension property". In this paper, we would like to give an affirmative answer using the language of Hodge modules.

Let X be a complex manifold and let D be an arbitrary divisor on X. Let $\mathcal{V}$ be a polarized variation of rational Hodge structures (VHS) on $X \setminus D$. Saito shows that $\mathcal{V}$ uniquely corresponds to a polarizable Hodge module $\mathcal{M}$ on X with strict support. Let E be the lowest nonzero piece in the Hodge filtration of $\mathcal{V}$ and let $(\mathcal{M}, F_\bullet\mathcal{M})$ be the filtered D_X-module underlying $\mathcal{M}$. Saito also shows that E extends to the lowest nonzero piece of $F_\bullet\mathcal{M}$, which is a torsion-free sheaf on X (see Section 2.4). The Hodge metric h on E extends to a singular Hermitian metric. Our main result is

Theorem A. *Let p be the smallest integer such that $F_p\mathcal{M} \neq 0$. Then the Hodge metric h on $F_p\mathcal{M}$ satisfies the "minimal extension property". In particular, it has semi-positive curvature.*

The advantage of using Hodge modules is that we do not need to assume the singular locus of the VHS is simple normal crossing. If one indeed assumes D is simple normal crossing, the curvature property of the Hodge metric for a $\mathbb{C}$-VHS is proven by Brunebarbe [Bru17, Theorem 1.4].

The notion of "minimal extension property" arises from the work of Ohsawa-Takegoshi with sharp estimates by [Bło13, GZ15] and it is a slight strengthening of Griffiths semi-positivity for singular Hermitian metrics (i.e. with semi-positive curvature). The key point of this notion is about the ability to extend sections over the singular locus of torsion-free sheaves while controlling L^2 norms in a precise way. This notion plays an important role in the proof of the Iitaka conjecture for algebraic fiber spaces over abelian varieties [CP17, HPS18]. One reason to investigate the "minimal extension property" of Hodge metrics is that we hope one can apply it to other related situations.

Let us briefly sketch the idea of the proof. Since it is a local statement, it suffices to assume X is a unit ball in $\mathbb{C}^n$. Because E is the lowest piece in the Hodge filtration of a VHS, Schmid's curvature calculation [Sch73] implies that E is Nakano semi-positive. Starting with any vector in a fiber of E, by Ohsawa-Takegoshi with sharp estimates [Bło13, GZ15], we can extend it to a holomorphic section of E with optimal L^2 bounds. The key step is to show that L^2 sections of E can be identified with sections of $F_p\mathcal{M}$. To prove this statement, first we use the direct image theorem of Hodge modules [Sai88] to reduce to the case where the VHS is supported on the complement of a simple normal crossing divisor. After a finite base change, it suffices to treat the case with unipotent monodromy. To conclude the proof, we use results by Cattani-Kaplan-Schmid [CKS86] on asymptotics of Hodge metrics to analyze the coefficient functions of L^2 sections.

As a corollary of Theorem A, we prove an interesting fact about coherence of sheaves of sections which are locally L^2 near a divisor. Recall that $\mathcal{V}$ is a VHS on $X \setminus D$ and E is the lowest nonzero piece in the Hodge filtration of $\mathcal{V}$. Let $j : X \setminus D \hookrightarrow X$ be the open embedding.

Corollary B. *Let $\mathcal{F}$ be the subsheaf of j_*E consisting of sections of E which are locally L^2 near D with respect to the Hodge metric on E and the standard Lebesgue measure. Then $\mathcal{F}$ is coherent.*

The general philosophy is that the theory of Hodge modules is closely related to L^2 methods. For one thing, by Saito's inductive construction, cohomology of Hodge modules on arbitrary manifolds can be reduced to Hodge modules on curves which are studied by Zucker [Zuc79] using L^2 cohomology. On the other hand, there are lots of work around close relations between D-modules and multiplier ideal sheaves (see [BS03, MP16] for example).

In §2 we will review some background. In §3 we will give the proof of Theorem A and Corollary B.

We will use left D-modules throughout the paper, as they are more natural from the metric point of view.

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2. Preliminaries

In this section, we will set up notation and recall some background.

2.1. **Variation of Hodge Structures.** Let X be a complex manifold. A *polarized variation of rational Hodge structures* (VHS) on X consists of the following data:

(1) A local system $V_{\mathbb{Q}}$ of finite dimensional $\mathbb{Q}$-vector spaces;
(2) A holomorphic vector bundle $\mathcal{V}$ with a flat connection $\nabla : \mathcal{V} \to \Omega^1_X \otimes \mathcal{V}$;
(3) A finite decreasing filtration $F^\bullet\mathcal{V}$ by holomorphic subbundles satisfying the Griffiths transversality

$$\nabla F^p\mathcal{V} \subset \Omega^1_X \otimes F^{p-1}\mathcal{V};$$

(4) A flat non-degenerate bilinear form $S : V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \to \mathbb{Q}$.

They are related in the following way:

- The local system of ∇-flat holomorphic sections of $\mathcal{V}$ is isomorphic to $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. In particular,
$$\mathcal{V} \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$$
and S extends to $\mathcal{V}$ in a C^∞ way.
- $\mathcal{V}$ admits the Hodge decomposition $\mathcal{V} = F^p \oplus \overline{F}^{k-p+1}$, where $\overline{F}$ is the conjugate of F relative to $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ and k is the weight of the variation.
- $S^h(\cdot,\cdot) := i^{-k} S(\cdot, \bar{\cdot})$ is a Hermitian form such that the Hodge decomposition is S^h-orthogonal and $(-1)^p S^h$ is positive definite on $\mathcal{V}^{p,k-p}$. In particular, such a polarization determines a smooth Hermitian metric h on $\mathcal{V}$:
$$h := \sum_p (-1)^p S^h|_{\mathcal{V}^{p,k-p}}. \tag{2.1}$$
We will call h the *Hodge metric* on $\mathcal{V}$. For $v \in \mathcal{V}_x$, $|v|_{h,x}$ means the length of this vectors.
- For every $x \in X$, $(\mathcal{V}_x, V_{\mathbb{Q},x}, F^\bullet \mathcal{V}_x)$ defines a rational Hodge structure of weight k which is polarized by the bilinear form S_x.

Remark 2.1. In the rest of the paper, when we say $\mathcal{V}$ is a VHS, we actually mean a polarized variation of rational Hodge structures with data $(V_{\mathbb{Q}}, \mathcal{V}, \nabla, F^\bullet \mathcal{V}, S)$.

2.2. **Deligne's canonical lattice.** In this section, we will assume X is Δ^n and U is $(\Delta^*)^n$. Let $s_1, \ldots, s_n$ be the coordinates on X. Then $D := X \setminus U$ is a simple normal crossing divisor so that $D = \cup D_j$, where $D_j = \{s_j = 0\}$ for $1 \le j \le n$.

Let $\mathcal{V}$ be a holomorphic vector bundle over U with a flat connection $\nabla : \mathcal{V} \to \Omega^1_U \otimes \mathcal{V}$. The fundamental group of U is isomorphic to $\mathbb{Z}^n$ and it acts on the fiber of $\mathcal{V}$ by parallel translation. Let T_j be the operator corresponding to the j-th standard generator of $\mathbb{Z}^n$. We say $(\mathcal{V}, \nabla)$ has *quasi-unipotent monodromy* if each T_j is quasi-unipotent. An interval I contained in $\mathbb{R}$ is said to be of *length one* if under the map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, the interval I maps isomorphically to $\mathbb{R}/\mathbb{Z}$. Deligne [Del70] proves the following fact:

Theorem 2.2. *Let $\mathcal{V}$ be a holomorphic vector bundle over U with a flat connection ∇ so that $(\mathcal{V}, \nabla)$ has quasi-unipotent monodromy. For any interval I of length one, there exists a holomorphic bundle $\mathcal{V}_I$ on X which extends $\mathcal{V}$ such that ∇ extends to*
$$\nabla : \mathcal{V}_I \to \Omega^1_X(\log D) \otimes \mathcal{V}_I,$$
with only logarithmic poles and the eigenvalues of the residue operator
$$\mathrm{Res}_{D_j} \nabla := s_j \nabla_{\frac{\partial}{\partial s_j}} : \mathcal{V}_I|_o \to \mathcal{V}_I|_o$$
are contained in I for each j.

Definition 2.3. We define *Deligne's canonical lattice* to be the associated holomorphic vector bundle $\mathcal{V}_I$ for any interval I of length one. For any $\beta \in \mathbb{R}$, we write $\mathcal{V}^\beta := \mathcal{V}_{[\beta,\beta+1)}$ and $\mathcal{V}^{>\beta} := \mathcal{V}_{(\beta,\beta+1]}$.

For our purposes, the following explicit construction of $\mathcal{V}^{>-1}$ will be useful. First, we need to define the logarithm of a quasi-unipotent operator.

Construction 2.4. Let V be a $\mathbb{C}$-vector space and let T be a quasi-unipotent operator on V. Consider the Jordan decomposition
$$T = T_s \cdot T_u$$

where T_s is semisimple and T_u is unipotent. We define the *quasi-unipotency index* m to be the smallest positive integer such that $T_s^m = \mathrm{Id}$.

(1) Since T_s is semisimple over $\mathbb{C}$, there is a decomposition

$$V = \oplus_\alpha V_\alpha \quad \text{with} \quad T_s|_{V_\alpha} = \lambda_\alpha \cdot \mathrm{Id}|_{V_\alpha},$$

where λ_α are the eigenvalues of T_s. Because $\lambda_\alpha^m = 1$, we can choose integers k_α satisfying $0 \leq k_\alpha \leq m-1$ such that

$$\lambda_\alpha = e^{-2\pi\sqrt{-1}k_\alpha/m}.$$

We define $\log T_s$ by its action on eigenspaces:

$$\log T_s|_{V_\alpha} := -(2\pi\sqrt{-1}k_\alpha/m)\cdot \mathrm{Id}|_{V_\alpha}.$$

(2) Since T_u is unipotent, we can define its logarithm as a convergent series

$$\log T_u = -\sum_{k\geq 1}^{\infty} \frac{1}{k}(\mathrm{Id} - T_u)^k.$$

Then we define the logarithm of T to be

$$\log T := \log T_s + \log T_u,$$

where $\log T_s$ is semisimple and $\log T_u$ is nilpotent.

Now we begin the construction of $\mathcal{V}^{>-1}$. Let us consider the universal covering map of $U = (\Delta^*)^n$:

$$p : \mathbb{H}^n \to (\Delta^*)^n, \quad (z_1, \ldots, z_n) \mapsto (e^{2\pi\sqrt{-1}z_1}, \ldots, e^{2\pi\sqrt{-1}z_n}).$$

The fundamental group $\mathbb{Z}^n$ acts on $\mathbb{H}^n$ by the rule

$$(a_1, \ldots, a_n)\cdot(z_1, \ldots, z_n) = (z_1 + a_1, \ldots, z_n + a_n).$$

Let $V := H^0(\mathbb{H}^n, p^*\mathcal{V})^{p^*\nabla}$ to be the space of $p^*\nabla$-flat sections of $p^*\mathcal{V}$ on $\mathbb{H}^n$, which trivializes $p^*\mathcal{V}$:

$$p^*\mathcal{V} \cong V \otimes \mathcal{O}_{\mathbb{H}^n}.$$

Therefore the sections of $\mathcal{V}$ over $(\Delta^*)^n$ correspond to holomorphic maps $\sigma : \mathbb{H}^n \to V$ with the property that $\sigma(z + e_j) = T_j \cdot \sigma(z)$ for all $z \in \mathbb{H}^n$ and all $j = 1, \ldots, n$.

We set $R_j := \log T_j$ by Construction 2.4. For each $v \in V$, the holomorphic map

$$\sigma : \mathbb{H}^n \to V, \sigma(z) = e^{\sum z_j R_j} v,$$

has the required property because $\sigma(z+e_j) = e^{R_j}\cdot\sigma(z) = T_j \cdot \sigma(z)$. Then $\mathcal{V}^{>-1}$ is defined to be the vector bundle over $X = \Delta^n$ generated by all sections of this form, i.e.

$$\mathcal{V}^{>-1} \cong (e^{\sum z_j R_j} V) \otimes_{\mathbb{C}} \mathcal{O}_X.$$

Note that by the construction,

$$\mathrm{Res}_{D_j} \nabla = \frac{R_j}{2\pi\sqrt{-1}}.$$

Since the eigenvalues of R_j are of the form $-(2\pi\sqrt{-1}k/m)$, where $0 \leq k \leq m-1$, the eigenvalues of $\mathrm{Res}_{D_j}\nabla$ all lies in $(-1, 0]$. Therefore $\mathcal{V}^{>-1}$ satisfies the conditions in Deligne's theorem.

Remark 2.5. If all T_j's are unipotent, we say that $(\mathcal{V}, \nabla)$ has *unipotent monodromy*. In this case, the eigenvalues of $\mathrm{Res}_{D_j}\nabla$ are all zero. Therefore

$$\mathcal{V}^{>-1} = \mathcal{V}^0.$$

2.3. **Asymptotics of Hodge metrics.** In this section, we will review the results on asymptotics of Hodge metrics [CK89] (a variation of rational Hodge structure is in particular a variation of *real* Hodge structures.)

Let $\mathcal{V}$ be a VHS over $(\Delta^*)^n$ with unipotent monodromy operators T_j. To keep the conventions of [CK89], $\log T_j$ is denoted by N_j. Consider the universal covering map

$$p : \mathbb{H}^n \to (\Delta^*)^n, \quad (z_1, \ldots, z_n) \mapsto (e^{2\pi\sqrt{-1}z_1}, \ldots, e^{2\pi\sqrt{-1}z_n}).$$

Let $V := H^0(\mathbb{H}^n, p^*\mathcal{V})^{p^*\nabla}$ be the space of flat sections which trivialize $p^*\mathcal{V}$. Let $\Phi : \mathbb{H}^n \to \mathbf{D}$ be the corresponding period mapping, where $\mathbf{D}$ is the classifying space of polarized Hodge structures of $\mathcal{V}$. Each point $F \in \mathbf{D}$ determines a hermitian inner product H_F on V as in (2.1). Therefore one can represent the pullback of the Hodge metric p^*h on $\mathcal{V}$ by $H_{\Phi(z)}$.

Let $(s_1, \ldots, s_n)$ be the coordinates of $(\Delta^*)^n$. Let $W^{(1)} = W(N_1), \ldots, W^{(n)} = W(N_1 + \ldots + N_n)$ be the sequence of monodromy weight filtrations on V associated to this ordering of variables. Note that each $W^{(j)}$ only depends on the positive cone generated by $N_1, \ldots, N_j$. Cattani, Kaplan and Schmid proved the following theorem [CK89, Theorem 5.1]:

Theorem 2.6. *There is an hermitian inner product Q on V such that over any region of the form*

$$\{z = (z_1, \ldots, z_n) \in \mathbb{H}^n \colon |\operatorname{Re} z_i| < a < 1; \frac{\operatorname{Im} z_j}{\operatorname{Im} z_{j+1}} \geq \epsilon, 1 \leq j \leq n-1, \operatorname{Im} z_n \geq \epsilon\},$$

$|v|^2_{H_{\Phi(z)}}$ is mutually bounded up to a constant with

$$\bigoplus_{l \in \mathbb{Z}^n} \left(\frac{\log|s_1|}{\log|s_2|}\right)^{l_1} \cdots (-\log|s_n|)^{l_n} |v|^2_{Q_l},$$

where $v \in V$ and $Q = \oplus Q_l$ corresponds to the multigrading $V \cong \oplus_{l \in \mathbb{Z}^n} \mathrm{Gr}^{W^{(n)}}_{l_n} \cdots \mathrm{Gr}^{W^{(1)}}_{l_1} V$.

Remark 2.7. By [CK89, Theorem 4.8], up to some explicit factors, the period map converges to $F_{\sqrt{-1}}$ in the period domain. Then Q is the inner product induced by $H_{F_{\sqrt{-1}}}$. Also the multigrading V of $W^{(1)}, \ldots, W^{(n)}$ is Q-orthogonal.

For the proof of the main theorem, we will use the following version: Let $\mathcal{V}$ be a VHS over $\Delta^* \times \Delta^{n-1}$ with unipotent monodromy operator T_1. Let $N := \log T_1$. Let

$$p : \mathbb{H} \times \Delta^{n-1} \to \Delta^* \times \Delta^{n-1}$$

be the universal covering map and let $\Phi : \mathbb{H} \times \Delta^{n-1} \to \mathbf{D}$ be the corresponding period map. Let $V := H^0(\mathbb{H} \times \Delta^{n-1}, p^*\mathcal{V})^{p^*\nabla}$ and let W be the weight filtration on V.

Corollary 2.8. *There is an hermitian inner product Q on V such that over any region of the form*

$$\mathbb{H}_\epsilon \times \Delta^{n-1}$$

where $\mathbb{H}_\epsilon := \{z_1 \in \mathbb{H} : \operatorname{Im} z_1 > \epsilon\}$, $|v|^2_{H_{\Phi(z)}}$ is mutually bounded with

$$\bigoplus_{l \in \mathbb{Z}} (-\log|s_1|)^l |v|^2_{Q_l}.$$

where $v \in V$ and $Q = \oplus Q_l$ corresponds to the grading $V \cong \oplus_{l \in \mathbb{Z}} \mathrm{Gr}^W_l V$.

2.4. **Hodge modules.** Let X be a complex manifold of dimension n. Let $\mathrm{HM}^p(X, w)$ be the category of polarizable Hodge modules of weight w on X with strict support. One of the main results in Saito's theory of Hodge modules [Sai90a, Theorem 3.21] is

Theorem 2.9 (Structure Theorem)**.** *Let X be a complex manifold of dimension n.*

(1) *If $\mathcal{V}$ is a polarizable variation of rational Hodge structures of weight $(w-n)$ on a Zariski open subset of X, then $\mathcal{V}$ extends uniquely to an object of* $\mathrm{HM}^p(X, w)$*.*
(2) *Conversely, every object $\mathcal{M}$ of* $\mathrm{HM}^p(X, w)$ *is obtained this way.*

Assume $X = \Delta^n$ and U is a Zariski open subset of X such that $D := X \setminus U$ is a divisor (If $X \setminus U$ is of higher codimension, the VHS on U extends to a VHS on X). Let $j : U \hookrightarrow X$ be the open embedding. Let $\mathcal{V}$ be a VHS over U and let $\mathcal{M}$ be the polarized Hodge module on X with strict support corresponding to $\mathcal{V}$. We would like to recall the construction of the Hodge filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$. It consists of two steps.

Step 1. Assume D is a simple normal crossing divisor, i.e. $D = \cup D_j$ and we can choose coordinates $s_1, \dots, s_n$ on X so that $D_j := \{s_j = 0\}$. Then $U = (\Delta^*)^n$. Denote $\mathcal{V}^{>-1}$ to be the Deligne's canonical lattice associated to $\mathcal{V}$. Schmid's nilpotent orbit theorem [Sch73] guarantees that the Hodge filtration on $\mathcal{V}$ extends to a Hodge filtration on $\mathcal{V}^{>-1}$, i.e.

$$F^k(\mathcal{V}^{>-1}) := j_*(F^k\mathcal{V}) \cap \mathcal{V}^{>-1}$$

are holomorphic subbundles of $\mathcal{V}^{>-1}$. We need to reindex this decreasing filtration to an increasing filtration for D-modules:

$$F_k(\mathcal{V}^{>-1}) := F^{-k}(\mathcal{V}^{>-1}).$$

It is known that ([Sai90a, (3.18.2)]) $\mathcal{M}$ is generated by $\mathcal{V}^{>-1}$ as a D_X-module:

$$\mathcal{M} = D_X \cdot \mathcal{V}^{>-1}.$$

Let $i = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ be a n-tuple of positive integers. We denote $|i| := i_1 + \cdots + i_n$ and $\partial_s^i := \partial_{s_1}^{i_1} \cdots \partial_{s_n}^{i_n}$. The increasing Hodge filtration $F_\bullet\mathcal{M}$ is defined as

$$\begin{aligned} F_k\mathcal{M} &:= \sum_{i \in \mathbb{Z}_+^n} \partial_s^i (F_{k-|i|}\mathcal{V}^{>-1}) \\ &= \sum_{i \in \mathbb{Z}_+^n} \partial_s^i (F^{|i|-k}\mathcal{V}^{>-1}) \\ &= \sum_{i \in \mathbb{Z}_+^n} \partial_s^i (j_*(F^{|i|-k}\mathcal{V}) \cap \mathcal{V}^{>-1}). \end{aligned}$$

Let $q := \max\{k : F^k\mathcal{V} \neq 0\}$ to be the lowest nonzero piece in the Hodge filtration of $\mathcal{V}$ over U. Then the lowest nonzero piece in the Hodge filtration of $\mathcal{M}$ is

$$F_{-q}\mathcal{M} = j_*(F^q\mathcal{V}) \cap \mathcal{V}^{>-1}. \tag{2.2}$$

Remark 2.10. In [Sai90b], Saito has the formula for right Hodge module:

$$F_k\mathcal{M} = \sum_{i \in \mathbb{Z}_+^n} (j_*(F^{|i|-k-n}\mathcal{V}) \cap \mathcal{V}^{>-1})\partial_s^i.$$

Here we convert it to the filtration on the corresponding left D-module, which has a shift by $n = \dim X$.

Step 2. Assume D is an arbitrary divisor. Let $f : (\tilde{X}, \tilde{D}) \to (X, D)$ be a log resolution such that $\tilde{D}$ is a simple normal crossing divisor and $\tilde{X} \setminus \tilde{D} \cong X \setminus D$. Denote $\tilde{U} := \tilde{X} \setminus \tilde{D}$. Consider $\tilde{\mathcal{V}} = (f|_{\tilde{U}})^*\mathcal{V}$ to be the VHS on $\tilde{U}$. Let $\tilde{\mathcal{M}}$ be the Hodge module on $\tilde{X}$ associated to $\tilde{\mathcal{V}}$. Let $F_p\tilde{\mathcal{M}}$ and $F_p\mathcal{M}$ be the lowest nonzero piece in the Hodge filtrations. As a corollary of direct image theorem, Saito [Sai90b, Theorem 1.1] proves that

$$f_*(F_p\tilde{\mathcal{M}} \otimes \omega_{\tilde{X}}) = F_p\mathcal{M} \otimes \omega_X. \tag{2.3}$$

The entire filtration $F^\bullet\mathcal{M}$ is defined in a more complicated way. But for our purpose, it is enough to understand the lowest nonzero piece.

2.5. **Nakano positivity of Hodge bundles.** Let X be a complex manifold. Let E be a holomorphic vector bundle on X with a hermitian metric h, we say that h is *Nakano semi-positive* if the curvature tensor Θ_h is semi-positive definite as a hermitian form on $T_X \otimes E$, i.e. if for every $u \in \Gamma(T_X \otimes E)$, we have $\sqrt{-1}\Theta_h(u,u) \geq 0$ (see [Dem07]).

Example 2.11. Let $\mathcal{V}$ be a VHS on X and let $F^q\mathcal{V}$ be the lowest nonzero piece in the Hodge filtration. Recall that the Hodge metric h on $F^q\mathcal{V}$ is defined as follows: if $v, w \in H^0(X, F^q\mathcal{V})$, then

$$h(v,w) := (-1)^q S^h(v, \bar{w}).$$

By Schmid's curvature calculation [Sch73, Lemma 7.18], h is Nakano semi-positive.

Nakano semi-positive bundles satisfy an optimal version of L^2 extension property proved by Blocki and Guan-Zhou [Bło13, GZ15]. Let $s_1, \ldots, s_n$ be the coordinates on $\mathbb{C}^n$, we write the standard Lebesgue measure to be $d\mu := c_n ds_1 \wedge d\bar{s}_1 \wedge \cdots \wedge ds_n \wedge d\bar{s}_n$ and $c_n = 2^{-n}(-1)^{n^2/2}$.

Theorem 2.12. *Let $B \subset \mathbb{C}^n$ be the open unit ball. Let Z be an analytic subset of $B \setminus \{0\}$. Let (E,h) be a holomorphic vector bundle over $B \setminus Z$ such that h is Nakano semi-positive. Then for every $v \in E_0$ with $|v|_{h,0} = 1$, there is a holomorphic section $\sigma \in H^0(B \setminus Z, E)$ with*

$$\sigma(0) = v \text{ and } \frac{1}{\mu(B)} \int_B |\sigma|_h^2 \, d\mu \leq 1.$$

Remark 2.13. It is easy to see that $(B \setminus Z, \{0\})$ satisfies the condition (ab) in [GZ15, Definition 1.1]. Then we can apply [GZ15, Theorem 2.2] to the pair $(B \setminus Z, \{0\})$, taking $A = 0, c_A(t) \equiv 1$ and $\Psi = n\log(|s_1|^2 + \cdots + |s_n|^2)$. Guan-Zhou's definition of $d\mu$ doesn't involve the constant c_n, but it reduces to the version we write here because the two c_n's in the inequality get cancelled.

2.6. **Singular hermitian metrics and metric positivity.** Let X be a complex manifold. Let $\mathcal{F}$ be a torsion-free coherent sheaf on X. This means that there is an open subset $X(\mathcal{F}) \subset X$ such that $E = \mathcal{F}|_{X(\mathcal{F})}$ is locally free and codimension of $X \setminus X(\mathcal{F})$ is greater or equal than 2.

Definition 2.14. A *singular hermitian metric* on $\mathcal{F}$ is a singular hermitian metric h on the holomorphic vector bundle E.

Inspired by the optimal L^2 extension theorem discussed in the previous section, it is natural to consider the following "minimal extension property" for singular Hermitian metrics on torsion-free sheaves [HPS18]. Let $B \subset \mathbb{C}^n$ be the open unit ball.

Definition 2.15. We say that a singular hermitian metric h on $\mathcal{F}$ has the *minimal extension property* if there exists a nowhere dense closed analytic subset $Z \subset X$ with the following two properties

(1) $\mathcal{F}$ is locally free on $X \setminus Z$.

(2) For every embedding $\iota : B \hookrightarrow X$ with $x = \iota(0) \in X \setminus Z$, and every $v \in E_x$ with $|v|_{h,x} = 1$, there is a holomorphic section $\sigma \in H^0(B, \iota^*\mathcal{F})$ such that

$$\sigma(0) = v \text{ and } \frac{1}{\mu(B)} \int_B |\sigma|_h^2 \, d\mu \leq 1.$$

Example 2.16. By Theorem 2.12, if E is a holomorphic vector bundle with a Nakano semi-positive metric h, then h has the minimal extension property. In particular, the Hodge metric on the lowest nonzero piece in the Hodge filtration of any VHS has the minimal extension property.

For consequences and applications of minimal extension properties, readers may refer to [HPS18].

3. Proofs

Let X be a complex manifold of dimension n and let D be an arbitrary divisor on X. Let $\mathcal{V}$ be a VHS over U and let $\mathcal{M}$ be the polarized Hodge module on X with strict support corresponding to $\mathcal{V}$ via Saito's structure theorem 2.9. Let $F^q\mathcal{V}$ denote the lowest nonzero piece in the Hodge filtration of $\mathcal{V}$. By the construction in section 2.4, $F_{-q}\mathcal{M}$ is the lowest nonzero piece in the Hodge filtration of $\mathcal{M}$ and

$$F_{-q}\mathcal{M}\big|_U = F^q\mathcal{V}.$$

Let h be the Hodge metric on $F^q\mathcal{V}$ as in Example 2.11. Saito proves that $F_{-q}\mathcal{M}$ is a torsion-free coherent sheaf [Sai90b, Proposition 2.6], so h is a singular Hermitian metric on $F_{-q}\mathcal{M}$.

Proof of Theorem A . It suffices to show that h has the minimal extension property. It follows that h has semi-positive curvature by the proof of [HPS18, Theorem 21.1].

Note that the divisor $D \subset X$ satisfies condition (1), as in definition 2.15. For condition (2), let us fix an embedding $\iota : B \hookrightarrow X$ with $x = \iota(0) \in X \setminus D$, so that $F_{-q}\mathcal{M}$ can be identified with $\iota^*(F_{-q}\mathcal{M})$. Choose $v \in (F^q\mathcal{V})_x$ with $|v|_{h,x} = 1$. By Example 2.11, h is Nakano semi-positive over $B \setminus D$ so Theorem 2.12 implies that there exists a holomorphic section $\sigma \in H^0(B \setminus D, F^q\mathcal{V}|_{B\setminus D})$ satisfying

$$\sigma(x) = v \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |\sigma|_h^2 \, d\mu \leq 1. \tag{3.1}$$

It suffices to show that σ extends to a holomorphic section in $H^0(B, F_{-q}\mathcal{M})$.

We divide the proof into two steps.

Step 1. We reduce to the situation where D is a simple normal crossing divisor. Let

$$f : (\tilde{B}, \tilde{D}) \to (B, D)$$

be a log resolution such that $\tilde{B} \setminus \tilde{D} \cong B \setminus D$ and $\tilde{D}$ is a simple normal crossing divisor. $\mathcal{V}$ pulls back isomorphically to a VHS on $\tilde{B} \setminus \tilde{D}$ which is denoted by $\tilde{\mathcal{V}}$. By abuse of notation, we use h to denote the Hodge metric on $\tilde{\mathcal{V}}$. Let $\tilde{\mathcal{M}}$ be the Hodge module on $\tilde{B}$ corresponding to $\tilde{\mathcal{V}}$.

Fixing coordinates $s_1, \ldots, s_n$ on B, then $\beta := \sigma \otimes ds_1 \wedge \cdots \wedge ds_n$ is a section of $H^0(B, F^q\mathcal{V} \otimes \omega_B)$. Define $\beta \wedge \overline{\beta} := |\sigma|_h^2 d\mu$, (3.1) implies that

$$\int_B \beta \wedge \overline{\beta} < \infty.$$

Since we have chosen a log resolution that does not change the open subset, we have

$$\int_{\tilde{B}} f^*\beta \wedge \overline{f^*\beta} < \infty. \tag{3.2}$$

In Step 2, we will show that this integrability condition implies that

$$f^*\beta \in H^0(\tilde{B}, F_{-q}\tilde{\mathcal{M}} \otimes \omega_{\tilde{B}}).$$

Granting this for now, since (2.3) says

$$H^0(\tilde{B}, F_{-q}\tilde{\mathcal{M}} \otimes \omega_{\tilde{B}}) = H^0(B, F_{-q}\mathcal{M} \otimes \omega_B),$$

we may then conclude that

$$\beta \in H^0(B, F_{-q}\mathcal{M} \otimes \omega_B).$$

Therefore

$$\sigma \in H^0(B, F_{-q}\mathcal{M}).$$

Step 2. Working locally, we may replace $\tilde{B}$ by Δ^n and $\tilde{D}$ by $\Delta^n \setminus (\Delta^*)^n$. Let $s_1, \ldots, s_n$ be coordinates on Δ^n. Write $f^*\beta = \tilde{\sigma} \otimes ds_1 \wedge \ldots \wedge ds_n$. (3.2) implies that

$$\int_{(\Delta^*)^n} |\tilde{\sigma}|_h^2 \, d\mu < \infty,$$

Thus $\tilde{\sigma} \in H^0(\Delta^n, \tilde{V}^{>-1})$ by Proposition 3.1. Over Δ^n, by (2.2) we have

$$F_{-q}\tilde{\mathcal{M}} = j_*(F^{-q}\tilde{\mathcal{V}}) \cap \tilde{\mathcal{V}}^{>-1},$$

then $\sigma \in H^0(\Delta^n, F_{-q}\tilde{M})$. Therefore,

$$f^*\beta \in H^0(\Delta^n, F_{-q}\tilde{M} \otimes \omega_{\Delta^n}).$$

□

3.1. L^2 **sections are holomorphic.** In this section, we give the main technical result:

Proposition 3.1. *Let $\mathcal{V}$ be a VHS over $(\Delta^*)^n$, let $F^q\mathcal{V}$ be the lowest nonzero piece in the Hodge filtration, and let h be the Hodge metric on $F^q\mathcal{V}$. If a section $\sigma \in H^0((\Delta^*)^n, F^q\mathcal{V})$ satisfies*

$$\int_{(\Delta^*)^n} |\sigma|_h^2 \, d\mu < \infty,$$

then $\sigma \in H^0(\Delta^n, \mathcal{V}^{>-1})$.

We will first prove a reduction lemma.

Lemma 3.2. *If the Proposition 3.1 holds for VHS with unipotent monodromy, then it holds for VHS with quasi-unipotent monodromy.*

Proof. Let $\mathcal{V}$ be a VHS with quasi-unipotent monodromy over $(\Delta^*)^n$ with $\operatorname{rk}\mathcal{V} = r$. Let T_j be the j-th monodromy operator with Jordan decomposition $T_j = (T_j)_s(T_j)_u$. Let m_j be the quasi-unipotency index of T_j such that $(T_j)_s^{m_j} = \mathrm{Id}$.

Let p be the universal covering map of $(\Delta^*)^n$:

$$p : \mathbb{H}^n \to (\Delta^*)^n, \quad (z_1, \ldots, z_n) \mapsto (e^{2\pi\sqrt{-1}z_1}, \ldots, e^{2\pi\sqrt{-1}z_n}).$$

Set $V := H^0(\mathbb{H}^n, p^*\mathcal{V})^{p^*\nabla}$, which is isomorphic to the fiber of $\mathcal{V}$. Since T_j commute with each other, there is a basis $(v_\alpha)_{1\le\alpha\le r}$ of V so that it simultaneously diagonalizes all $(T_j)_s$. We choose integer $0 \le k_{\alpha j} \le m_j - 1$ such that

$$(T_j)_s \cdot v_\alpha = e^{-2\pi\sqrt{-1}k_{\alpha j}/m_j} \cdot v_\alpha.$$

By Construction 2.4, let $R_j := \log T_j$ with the decomposition

$$R_j = H_j + N_j,$$

where $H_j = \log(T_j)_s$ and $N_j = \log(T_j)_u$. By the construction we have

$$H_j \cdot v_\alpha = -(2\pi\sqrt{-1}k_{\alpha j}/m_j) \cdot v_\alpha.$$

and N_j is nilpotent for each j.

To reduce to the unipotent situation, consider the following unramified covering map:

$$f : (\Delta^*)^n \to (\Delta^*)^n, (\tilde{s}_1, \ldots, \tilde{s}_n) \mapsto (\tilde{s}_1^{m_1}, \ldots, \tilde{s}_n^{m_n}).$$

Let $(\tilde{\mathcal{V}}, \tilde{\nabla}) := f^*(\mathcal{V}, \nabla)$ to be the VHS polarized by f^*h. Note that $(\tilde{\mathcal{V}}, \tilde{\nabla})$ has unipotent monodromy because the j-th monodromy operators $\tilde{T}_j$ equals to $T_j^{m_j} = (T_j)_u^{m_j}$. Set $\tilde{R}_j := \log \tilde{T}_j$, then $\tilde{R}_j = m_j N_j$. The finite base change comes with the following diagram:

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\tilde{f}} & \mathbb{H}^n \\ {\scriptstyle \tilde{p}}\Big\downarrow & & {\scriptstyle p}\Big\downarrow \\ (\Delta^*)^n & \xrightarrow{f} & (\Delta^*)^n \end{array}$$

where

$$\tilde{p}(\tilde{z}_1, \ldots, \tilde{z}_n) = (e^{2\pi\sqrt{-1}\tilde{z}_1}, \ldots, e^{2\pi\sqrt{-1}\tilde{z}_n}),$$
$$\tilde{f}(\tilde{z}_1, \ldots, \tilde{z}_n) = (m_1\tilde{z}_n, \ldots, m_n\tilde{z}_n).$$

Let $\tilde{V}$ to be $H^0(\mathbb{H}^n, \tilde{p}^*\tilde{\mathcal{V}})^{\tilde{p}^*\tilde{\nabla}}$. Then as in section 2.2,

$$\mathcal{V}^{>-1} \cong (e^{\sum z_j R_j} V) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta^n}, \quad \tilde{\mathcal{V}}^{>-1} \cong (e^{\sum z_j \tilde{R}_j} \tilde{V}) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta^n}.$$

Now we want to calculate the pull back of generating sections of $\mathcal{V}^{>-1}$ via f. Since they are identified with holomorphic functions on $\mathbb{H}^n$, it suffices to pull back via $\tilde{f}$. Let $v \in V$ and $w = e^{\sum z_j R_j} v$, then

$$\begin{aligned} \tilde{f}^*(w) &= \exp(\sum \tilde{z}_j m_j R_j)\tilde{f}^*v \\ &= \exp(\sum \tilde{z}_j m_j H_j) \cdot \exp(\sum \tilde{z}_j m_j N_j)\tilde{f}^*v \\ &= \exp(\sum \tilde{z}_j m_j H_j) \cdot \exp(\sum \tilde{z}_j \tilde{R}_j)\tilde{f}^*v \\ &= \exp(\sum \tilde{z}_j m_j H_j)\tilde{w}. \end{aligned}$$

where $\tilde{w} = e^{\sum \tilde{z}_j \tilde{R}_j} \tilde{f}^* v \in H^0(\Delta^n, \tilde{\mathcal{V}}^{>-1})$.

Since $(v_\alpha)_{1 \le \alpha \le r}$ is the basis of V such that

$$H_j \cdot v_\alpha = -(2\pi\sqrt{-1}k_{\alpha j}/m_j) \cdot v_\alpha.$$

Set $w_\alpha = e^{\sum z_j R_j} v_\alpha$, then

$$\tilde{f}^*(w_\alpha) = e^{\sum \tilde{z}_j m_j H_j} \tilde{w} = \prod e^{-2\pi\sqrt{-1}\tilde{z}_j k_{\alpha j}} \tilde{w} = \prod \tilde{s}_j^{-k_{\alpha j}} \tilde{w}_\alpha. \tag{3.3}$$

where $\tilde{w}_\alpha = e^{\sum \tilde{z}_j \tilde{R}_j} \tilde{f}^* v_\alpha$.

Remark 3.3. The generating sections of $\mathcal{V}^{>-1}$ pull back to generating sections of $\tilde{\mathcal{V}}^{>-1}$ with an extra monomial factor.

Now we start the proof of this lemma. Let σ be a section of $F^q\mathcal{V}$ over $(\Delta^*)^n$ such that $\int_{(\Delta^*)^n} |\sigma|_h^2 \, d\mu < +\infty$. Let $j : (\Delta^*)^n \hookrightarrow \Delta^n$ be the open embedding. By the construction of Deligne's canonical lattice,

$$H^0\left((\Delta^*)^n, F^q\mathcal{V}\right) \subset H^0\left((\Delta^*)^n, \mathcal{V}^{>-1}|_{(\Delta^*)^n}\right).$$

Therefore we can write

$$\sigma = \sum_{\alpha=1}^{r} h_\alpha w_\alpha,$$

where $h_\alpha \in \mathbb{C}(s_1, \ldots, s_n)$ is a holomorphic function over $(\Delta^*)^n$. To show $\sigma \in H^0(\Delta^n, \mathcal{V}^{>-1})$, it suffices to show that each h_α is actually a holomorphic function over Δ^n.

Since $\int_{(\Delta^*)^n} |\sigma|_h^2 \, d\mu < +\infty$, local calculation shows that

$$\int_{(\Delta^*)^n} f^*(|\sigma|_h^2 \, d\mu) = \int_{(\Delta^*)^n} \big| \prod_{j=1}^{n} m_j \tilde{s}_j^{m_j-1} \cdot f^*(\sigma) \big|_{f^*h}^2 d\tilde{\mu} < +\infty.$$

Here

$$d\mu = c_n ds_1 \wedge d\bar{s}_1 \wedge \cdots \wedge ds_n \wedge d\bar{s}_n, d\tilde{\mu} = c_n d\tilde{s}_1 \wedge d\bar{\tilde{s}}_1 \wedge \cdots \wedge d\tilde{s}_n \wedge d\bar{\tilde{s}}_n$$

are the standard Lebesgue measure on Δ^n and $c_n = 2^{-n}(-1)^{n^2/2}$. Since we assume that Proposition 3.1 is true for VHS with unipotent monodromy, it follows that

$$\prod_{j=1}^{n} \tilde{s}_j^{m_j-1} \cdot f^*(\sigma)$$

must belongs to $H^0(\Delta^n, \tilde{\mathcal{V}}^{>-1})$, which equals to $H^0(\Delta^n, \tilde{\mathcal{V}}^0)$ by remark 2.5. By (3.3),

$$\begin{aligned} f^*(\sigma) &= \sum_{\alpha=1}^{r} f^*(h_\alpha w_\alpha) \\ &= \sum_{\alpha=1}^{r} f^*(h_\alpha) \cdot \prod_{j=1}^{n} \tilde{s}_j^{-k_{\alpha j}} \tilde{w}_\alpha \end{aligned}$$

Hence

$$\prod_{j=1}^{n} \tilde{s}_j^{m_j-1} \cdot f^*(\sigma) = \sum_{\alpha=1}^{r} \left(f^*(h_\alpha) \cdot \prod_{j=1}^{n} \tilde{s}_j^{m_j-1-k_{\alpha j}} \right) \tilde{w}_\alpha.$$

Therefore we conclude that for each $1 \leq \alpha \leq r$, the corresponding coefficient function

$$f^*(h_\alpha) \cdot \prod_{j=1}^{n} \tilde{s}_j^{m_j-1-k_{\alpha j}}$$

is holomorphic in $\tilde{s}_1, \ldots, \tilde{s}_n$. In particular, this implies that h_α must be a meromorphic function. For each α, we let $n_{\alpha j}$ be the lowest power of s_j in the Laurent expansion of h_α, then the lowest power of $\tilde{s}_j$ in $f^*(h_\alpha)$ is $m_j n_{\alpha j}$. Holomorphicity imply that

$$m_j n_{\alpha j} + m_j - 1 - k_{\alpha j} \geq 0.$$

Because $k_{\alpha j} \geq 0$, we have

$$n_{\alpha j} \geq -1 + \frac{1 + k_{\alpha j}}{m_j} \geq -1 + \frac{1}{m_j},$$

Since $n_{\alpha j}$ is an integer, it follows that $n_{\alpha j} \geq 0$, i.e. h_α is actually a holomorphic function in $s_1, \ldots, s_n$. Therefore we conclude that $\sigma \in H^0(\Delta^n, \mathcal{V}^{>-1})$.

□

Proof of Proposition 3.1 . Let $\mathcal{V}$ be a VHS over $(\Delta^*)^n$ and $D := \Delta^n \setminus (\Delta^*)^n = \cup D_j$ is the simple normal crossing divisor. By Schmid's monodromy theorem [Sch73, Theorem 6.1], $(\mathcal{V}, \nabla)$ has quasi-unipotent monodromy. Then we can assume that $(\mathcal{V}, \nabla)$ has unipotent monodromy by Lemma 3.2, . Let σ be a section of $H^0((\Delta^*)^n, F^q\mathcal{V})$ such that

$$\int_{(\Delta^*)^n} |\sigma|_h^2 d\mu < +\infty.$$

By Hartog's theorem, to show $\sigma \in H^0(\Delta^n, \mathcal{V}^{>-1})$, it suffices to show that σ extends over any generic point of each divisor D_j.

Without loss of generality, let us assume $j = 1$. Around a generic point of D_1, we can choose a coordinate neighborhood U_1 such that it is isomorphic to $\Delta^* \times \Delta^{n-1}$. Let s_1 be the coordinate of Δ^*. Let T be the monodromy operator of $\mathcal{V}|_{U_1}$ and $N := \log T$. Let $p : \mathbb{H} \times \Delta^{n-1} \to \Delta^* \times \Delta^{n-1}$ be the universal covering map and let $\Phi : \mathbb{H} \times \Delta^{n-1} \to \mathbf{D}$ be the corresponding period map. Let $V = H^0(\mathbb{H} \times \Delta^{n-1}, p^*\mathcal{V})^{p^*\nabla}$ and let $W_\bullet V$ be the weight filtration associated with N.

Let $\Delta_r^* := \{s_1 \in \Delta^* : |s_1| < r\}$ be the punctured disk of radius $r < 1$. By Corollary 2.8, there exists a Hermitian inner product Q on V and over $\Delta_r^* \times \Delta^{n-1}$, there exists $C_1 > 0$ such that

$$|\sigma|^2_{h,s} = |p^*\sigma|^2_{\Phi(z)} \geq C_1 \sum_l (-\log|s_1|)^l \cdot |p^*\sigma|^2_{Q_l}.$$

Here $Q = \oplus Q_l$ corresponds to the grading $V = \oplus_l \mathrm{Gr}_l^W V$ and we write $p^*\sigma$ in terms of basis of V using the trivialization of $p^*\mathcal{V}$ by V.

Let (v_α) be an Q-orthogonal basis of V with respect to $V = \oplus_l \mathrm{Gr}_l^W V$. Then $(w_\alpha = e^{z_1 N} v_\alpha)$ is a basis of generating sections of $\mathcal{V}^{>-1}|_{U_1}$. Set

$$\sigma\big|_{U_1} = \sum_\alpha h_\alpha w_\alpha,$$

where h_α is a holomorphic function over $\Delta_r^* \times \Delta^{n-1}$. It suffices to prove that h_α extends to a holomorphic function over $\Delta_r \times \Delta^{n-1}$

Since N is nilpotent and $|s_1|$ is bounded, there exists a constant $C_2 > 0$ such that for all l,

$$|p^*\sigma|^2_{Q_l} \geq C_2 |e^{-z_1 N} p^*\sigma|^2_{Q_l} = C_2 \big|\sum_\alpha h_\alpha v_\alpha\big|^2_{Q_l}.$$

Therefore

$$\begin{aligned}
\infty &> \int_{\Delta_r^* \times \Delta^{n-1}} |\sigma|^2_h d\mu \\
&> C_1 \int_{\Delta_r^* \times \Delta^{n-1}} \sum_l (-\log|s_1|)^l |p^*\sigma|^2_{Q_l} d\mu \\
&> C_1 C_2 \int_{\Delta_r^* \times \Delta^{n-1}} \sum_l (-\log|s_1|)^l \big|\sum_\alpha h_\alpha v_\alpha\big|^2_{Q_l} d\mu.
\end{aligned}$$

Since (v_α) is Q-orthogonal, for each α there exists l such that

$$\int_{\Delta_r^* \times \Delta^{n-1}} (-\log|s_1|)^l |h_\alpha|^2 d\mu < \infty.$$

We can rescale r to 1, then by lemma 3.4, h_α is holomorphic over $\Delta_r \times \Delta^{n-1}$. Therefore σ extends to a holomorphic section of $\mathcal{V}^{>-1}$ over Δ^n.

□

To conclude the proof, we need the following

Lemma 3.4. *Let f is a holomorphic function on $\Delta^* \times \Delta^{n-1}$. Let s_1 be the coordinate of Δ^*. Suppose there exists an integer k such that*

$$\int_{\Delta^* \times \Delta^{n-1}} (-\log|s_1|)^k \cdot |f|^2 d\mu < \infty,$$

Then f must be a holomorphic function on $\Delta \times \Delta^{n-1}$.

Proof. We denote $d\mu_n = c_n ds_1\wedge d\bar{s}_1\wedge\cdots\wedge ds_n\wedge d\bar{s}_n$ to be the standard Lebesgue measure on Δ^n and $c_n = 2^{-n}(-1)^{n^2/2}$. Then $d\mu_n = d\mu_1\times d\mu_{n-1}$. Expand f as a Laurent series in s_1:

$$f=\sum_{i\in\mathbb{Z}} f_i s_1^i,$$

where f_i is a holomorphic function on Δ^{n-1}.

To simplify the presentation, we denote $-\log|s_1|$ by $L(s_1)$. Let (r,θ) be the polar coordinate such that $s_1 = re^{\sqrt{-1}\theta}$. Observe that $L(s_1)$ is a function only depending on $|s_1| = r$, then for any integers i,j such that $i\neq j$,

$$\int_{\Delta^*} L(s_1)^k s_1^i\bar{s}_1^j d\mu_1 = \int_0^1 L(r)^k r^{i+j+1}dr\int_0^{2\pi} e^{\sqrt{-1}\theta(i-j)}d\theta = 0.$$

Since the Laurent series converges on any anulus $\{\epsilon_1\leq|s_1|\leq\epsilon_2\}\times\Delta^{n-1}$, so

$$\begin{aligned}\int_{\Delta^*\times\Delta^{n-1}} L(s_1)^k\cdot|f|^2 d\mu &= \int_{\Delta^{n-1}}\left(\int_{\Delta^*} L(s_1)^k\cdot\sum_{i,j} s_1^i\bar{s}_1^j d\mu_1\right) f_i\bar{f}_j d\mu_{n-1}\\ &= \sum_{i\in\mathbb{Z}}\left(\int_{\Delta^*} L(s_1)^k|s_1|^{2i}d\mu_1\right)\cdot\left(\int_{\Delta^{n-1}}|f_i|^2 d\mu_{n-1}\right)\end{aligned}$$

Since $\int_{\Delta^*} L(s_1)^k|s_1|^{2i}d\mu<\infty$ if and only if $i\geq 0$ and $\int_{\Delta^{n-1}}|f_i|^2 d\mu_{n-1} = 0$ if and only if $f_i\equiv 0$, we must have $f_i\equiv 0$ for all $i<0$. In particular we conclude that f is holomorphic over $\Delta\times\Delta^{n-1}$.

□

3.2. **Consequences.** In this section, we would like to prove Corollary B. Let X be a complex manifold and let D be an arbitrary divisor. Let (E,h) be a holomorphic vector bundle on $X\setminus D$ with a smooth Hermitian metric. A section σ of E is said to be *locally* L^2 *near* D if for any point in D, there exists a coordinate neighborhood U such that $|\sigma|_h^2$ is integrable with respect to the standard Lebesgue measure on U.

Let $\mathcal{V}$ be a VHS on $X\setminus D$ and $F^q\mathcal{V}$ is the lowest nonzero piece in the Hodge filtration. Let $j: X\setminus D\hookrightarrow X$ be the open embedding. Let $\mathcal{F}$ be the subsheaf of $j_*(F^q\mathcal{V})$ consisting of sections of $F^q\mathcal{V}$ which are locally L^2 near D. We want to show that $\mathcal{F}$ is coherent.

Proof of Corollary B. Let $\mathcal{M}$ be the Hodge module on X with strict support corresponding to $\mathcal{V}$. The proof of Theorem A implies that

$$\mathcal{F}\subset F_{-q}\mathcal{M}.$$

We will show

$$\mathcal{F}\supset F_{-q}\mathcal{M}. \tag{3.4}$$

In particular $\mathcal{F} = F_{-q}\mathcal{M}$. Since the filtered D-module $(\mathcal{M}, F_\bullet\mathcal{M})$ underlying $\mathcal{M}$ is good, $\mathcal{F}$ must be coherent.

Since (3.4) is a local statement, we can assume $X=\Delta^n$. Let σ be a section of $F_{-q}\mathcal{M}$, then $\sigma\in H^0(X\setminus D, F^q\mathcal{V})$. As usual, we divide the proof into two steps.

Step 1. If D is a simple normal crossing divisor, we can choose $s_1,\dots,s_n$ to be the coordinates on X such that $D=\cup\{s_i=0\}$. By Theorem 2.6, over any region of the form

$$\{\underline{s}\in(\Delta^*)^n : |\underline{s}|<a<1; \frac{\log|s_j|}{\log|s_{j+1}|}\geq\epsilon, 1\leq j\leq n-1\},$$

$|\sigma|_h^2$ is bounded above by sums of products of logarithm functions in $|s_i|$. In particular, it is integrable with respect to the standard Lebesgue measure on this region. Since we can cover the neighborhood of any point of D using finite regions of this type, it follows that σ is locally L^2 near D.

Step 2. If D is an arbitrary divisor, we can choose a log resolution

$$f : (\tilde{X}, \tilde{D}) \to (X, D),$$

such that $\tilde{D}$ is simple normal crossing and $\tilde{X} \setminus \tilde{D} \cong X \setminus D$. The VHS $\mathcal{V}$ on $X \setminus D$ pulls back isomorphically to a VHS $\tilde{\mathcal{V}}$ on $\tilde{X} \setminus \tilde{D}$. Let $\tilde{\mathcal{M}}$ be the Hodge module on $\tilde{X}$ corresponding to $\tilde{\mathcal{V}}$.

Since $f^*\sigma \in H^0(\tilde{X}, F_{-q}\tilde{\mathcal{M}})$, by the previous step it follows that $f^*\sigma$ is locally L^2 near $\tilde{D}$. Because $\tilde{X} \setminus \tilde{D} \cong X \setminus D$, we can conclude that σ is locally L^2 near D. $\square$

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