The fundamental group is not a derived invariant

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Abstract. We show that the fundamental group is not invariant under derived equivalence of smooth projective varieties.

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1. Acknowledgements

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Some time after giving the talk, I discovered that Anthony Bak [2] already has a preprint on `arXiv` in which he obtains the same result (that is to say, Theorem 4.1 below). In fact, his proof is more concrete, and therefore much more useful for doing calculations with the derived equivalence. My apology for nevertheless writing this note is that the proof given here is different and, by relying on the theorem of Bridgeland and Maciocia [4], a little bit shorter than Bak's.

2. Introduction

For a smooth complex projective variety X, we denote by $D^b(X) = D^b \operatorname{Coh}(X)$ the bounded derived category of coherent sheaves on X. Recall that two smooth projective varieties X are Y are said to be *derived equivalent* if $D^b(X) \simeq D^b(Y)$ as $\mathbb{C}$-linear triangulated categories. We sometimes write $X \sim Y$ to indicate that X and Y are derived equivalent.

From the work of Bondal, Orlov, Căldăraru, Kawamata, and others, it is known that many of the basic invariants of algebraic varieties are preserved under derived equivalence. These include the dimension, the Kodaira dimension, the canonical ring, and the order of the canonical class. It has also been conjectured that the Hodge structure on the cohomology with rational coefficients is a derived invariant, in the sense that if $X \sim Y$, then one should have $H^k(X, \mathbb{Q}) \simeq H^k(Y, \mathbb{Q})$ as rational

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Hodge structures, for every $k \in \mathbb{Z}$. In particular, it is expected that the Hodge numbers
$$h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X) = \dim_{\mathbb{C}} H^q\big(X, \Omega_X^p\big)$$
are invariant under derived equivalences.

In joint work with Mihnea Popa [10], we showed that if X and Y are derived equivalent, then $H^1(X,\mathbb{Q}) \simeq H^1(Y,\mathbb{Q})$ as rational Hodge structures; in geometric terms, this means that the two Picard varieties $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ are isogenous abelian varieties. Ignoring the choice of base point,
$$H^1(X,\mathbb{Q}) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(\pi_1(X), \mathbb{Q}\big),$$
and so our result naturally leads to the question whether the fundamental group $\pi_1(X)$ is itself a derived invariant. The point of this paper is to show that this is not the case.

More precisely, I will describe an example of a simply connected Calabi-Yau threefold X, with a nontrivial free action by a finite group G, such that the quotient X/G is derived equivalent to X. Since $\pi_1(X/G) = G$, while $\pi_1(X) = \{1\}$, this means that neither the fundamental group, nor the property of being simply connected, are preserved under derived equivalence.

3. A related problem

Before continuing, I should point out that this result is connected to a larger question raised by Daniel Huybrechts and Marc Nieper-Wisskirchen, about derived equivalences of varieties with trivial first Chern class. To set up some notation, suppose that X is a smooth projective variety whose first Chern class $c_1(X)$ is zero as an element of $H^2(X,\mathbb{R})$. By Yau's theorem, X admits a Ricci-flat Kähler metric; by studying the holonomy of this metric, Bogomolov and Beauville [3] have shown that a finite étale cover $X' \to X$ decomposes into a finite product
$$X' \simeq A \times \prod_i Y_i \times \prod_j Z_j$$
with A an abelian variety, each Y_i a simply connected Calabi-Yau manifold of dimension at least three, and each Z_j a holomorphic symplectic manifold. Huybrechts and Nieper-Wisskirchen ask whether the structure of this decomposition is invariant under derived equivalences (see [7], Question 0.2).

In the special case of Calabi-Yau threefolds, the question becomes the following: Suppose that X is a simply connected Calabi-Yau threefold, and that $Y \sim X$. Because of the example in this paper, we cannot expect Y to be simply connected. On the other hand, the first Chern class of Y is also trivial, and so a finite étale cover of Y must be of one of the following three types:

(1) A simply connected Calabi-Yau threefold.

(2) An abelian threefold.

(3) The product of an elliptic curve and a K3-surface.

Note that there are examples of finite quotients of abelian threefolds (or products of an elliptic curve and a K3-surface) with trivial canonical bundle and zero first Betti number; a partial classification may be found in [9]. Nevertheless, it seems likely that such varieties cannot be derived equivalent to a simply connected Calabi-Yau threefold. A proof of this would be a useful step towards answering the general question of Huybrechts and Nieper-Wisskirchen.

4. The example

Let us now turn to the description of the example, which was suggested to me by Lev Borisov. The Calabi-Yau threefold in question is one of a class of such varieties constructed by Mark Gross and Sorin Popescu [6], and has to do with (1,8)-polarized abelian surfaces.

We shall begin by recalling their construction. Let (A, L) be a (1,8)-polarized abelian surface. In other words, suppose that A is an abelian surface, and that L is an ample line bundle on A such that the isogeny

$$\varphi_L\colon A \to \operatorname{Pic}^0(A), \quad a \mapsto t_a^*(L) \otimes L^{-1},$$

has kernel isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_8$.[1] One can show that L is then automatically very ample with $h^0(L) = 8$; by the Riemann-Roch theorem, it follows that we have $8 = \chi(L) = L^2/2$, which gives $L^2 = 16$. The line bundle therefore embeds A as a surface of degree 16 into $\mathbb{P}^7$, and it is possible to choose the coordinates on the projective space in such a way that the action of $G = \mathbb{Z}_8 \times \mathbb{Z}_8$ on $\mathbb{P}^7$ is given by the formulas

$$\begin{aligned} \sigma(x_0\colon x_1\colon \cdots\colon x_6\colon x_7) &= (x_1\colon x_2\colon \cdots\colon x_7\colon x_0) \\ \tau(x_0\colon x_1\colon \cdots\colon x_6\colon x_7) &= \big(x_0\colon \zeta x_1\colon \cdots\colon \zeta^6 x_6\colon \zeta^7 x_7\big). \end{aligned}$$

Here σ and τ denote the two natural generators of the group G, and ζ is a primitive eighth root of unity.

The idea of Gross and Popescu is to look at quadrics in $\mathbb{P}^7$ that contain the image of A. Provided that the pair (A, L) is general in moduli, they show that the space of such quadrics has dimension four, and that it is generated by the four quadrics f, σf, $\sigma^2 f$, and $\sigma^3 f$, where

$$f = y_1 y_3 (x_0^2 + x_4^2) - y_2^2 (x_1 x_7 + x_3 x_5) + (y_1^2 + y_3^2) x_2 x_6,$$

and $y \in \mathbb{P}^2$ is a general point. The intersection

$$V_{8,y} = Z(f) \cap Z(\sigma f) \cap Z(\sigma^2) f \cap Z(\sigma^3 f)$$

[1] In general, a polarization L is of type (m, n) if the kernel of φ_L is isomorphic to the group $(\mathbb{Z}_m \times \mathbb{Z}_n)^{\oplus 2}$ for positive integers $m|n$.

is then a threefold on which the group G acts freely. Conversely, if we assume that $y \in \mathbb{P}^2$ is chosen sufficiently general, $V_{8,y}$ will be a complete intersection of dimension three which is smooth except for 64 ordinary double points, the G-orbit of the point $(0\colon y_1\colon y_2\colon y_3\colon 0\colon -y_3\colon -y_2\colon -y_1)$. There is always a one-dimensional family of $(1,8)$-polarized abelian surfaces contained in $V_{8,y}$, and every member of the family passes through the 64 distinguished points.

Gross and Popescu discovered that $V_{8,y}$ admits two small resolutions $V_{8,y}^1$ and $V_{8,y}^2$, both Calabi-Yau threefolds. The original abelian surface A is a Weil divisor on $V_{8,y}$ that is not Cartier; blowing up A produces a small resolution $V_{8,y}^2 \to V_{8,y}$. Using the Lefschetz theorem and adjunction, one can easily show that $V_{8,y}^2$ is a simply connected Calabi-Yau threefold. The 64 exceptional curves can be flopped simultaneously to produce another simply connected Calabi-Yau threefold $V_{8,y}^1$, and Gross and Popescu compute that

$$h^{1,1}(V_{8,y}^1) = h^{1,2}(V_{8,y}^1) = 2.$$

In fact, the Picard group of $V_{8,y}^1$ is generated (modulo torsion) by the classes of two divisors: the strict transform A of the original abelian surface, and the preimage H of a hyperplane section. They satisfy

$$H^3 = 16, \quad H^2 \cdot A = 16, \quad H \cdot A^2 = 0, \quad A^3 = 0.$$

The reader can find a concise summary of all the properties of the Calabi-Yau threefolds $V_{8,y}^1$ and $V_{8,y}^2$ in [5]. We shall only list those that are needed below.

(1) The linear system $|A|$ is one-dimensional, and the resulting morphism

$$p\colon V_{8,y}^1 \to \mathbb{P}^1$$

is an abelian surface fibration with exactly 64 sections. The images of these sections are the 64 exceptional curves of the flop.

(2) Every smooth fiber of p is a (1,8)-polarized abelian surface, with polarization induced by the restriction of the line bundle $\mathscr{O}_X(H)$. The intersection with the images of the 64 sections is precisely the kernel of the polarization.

(3) There are exactly eight singular fibers, each of them an elliptic translation scroll. Such a scroll is obtained from an elliptic normal curve E in $\mathbb{P}^7$ by fixing a point $e \in E$, and letting $T_e(E)$ be the union of all lines through x and $x+e$, for $x \in E$. It is not hard to see that $T_e(E)$ is singular precisely along the elliptic curve E, and that $E \times \mathbb{P}^1$ is a resolution of singularities.

(4) The group G acts freely on $V_{8,y}^1$, and the 64 sections form a single G-orbit. On each smooth fiber, the action of G is the natural action by the kernel of the polarization; on each singular fiber, the action of G is the natural action by the group of 8-torsion points of the elliptic curve.

Now let $X = V_{8,y}^1$. Since $G = \mathbb{Z}_8 \times \mathbb{Z}_8$ acts freely on X, the quotient X/G is again a smooth projective variety with trivial canonical bundle. The following theorem is the main result of this paper.

Theorem 4.1. *The two varieties X and X/G are derived equivalent.*

Since X is simply connected, while the quotient X/G has fundamental group isomorphic to G, it follows that the fundamental group is not invariant under derived equivalences.

Note 4.2. For reasons coming from physics, Mark Gross and Simone Pavanelli [5] conjecture that the quotient of X by one of the two $\mathbb{Z}_8$-factors of G should be the mirror manifold of X, and that X/G should be the mirror of the mirror. Homological mirror symmetry would therefore predict that $D^b(X) \simeq D^b(X/G)$.

5. Proof of the theorem

We now describe one possible proof of Theorem 4.1. An earlier proof, more concrete but also slightly longer, may be found in the preprint by Anthony Bak [2].

To explain the basic idea, let us consider one of the smooth fibers A of the morphism $p\colon X \to \mathbb{P}^1$; it is a $(1,8)$-polarized abelian surface, with polarization L induced by the restriction of H. The group G acts on A, and the 64 points in the kernel of the isogeny

$$\varphi_L\colon A \to \operatorname{Pic}^0(A)$$

form a single G-orbit. Consequently, φ_L gives rise to an isomorphism $A/G \simeq \operatorname{Pic}^0(A)$. Moreover, once we choose one of the 64 points in the kernel as the unit element on A, there is a well-defined Poincaré line bundle on $A \times \operatorname{Pic}^0(A)$, and the associated Fourier-Mukai transform induces an equivalence $D^b(A) \simeq D^b(\operatorname{Pic}^0(A))$. In our proof, we shall generalize these observations by (1) using the theorem of Bridgeland and Maciocia to prove that X is derived equivalent to the compactified relative Picard scheme M, and (2) showing that M is actually isomorphic to X/G.

We begin by introducing the space M. Let s_0 be one of the 64 sections of $p\colon X \to \mathbb{P}^1$. The general fiber of p is an abelian surface, and the eight singular fibers are elliptic translation scrolls, and therefore reduced and irreducible. Thus it makes sense to consider the compactified relative Picard scheme

$$M = \overline{\operatorname{Pic}}^0(X/\mathbb{P}^1)$$

defined by Altman and Kleiman [1]. For any smooth fiber A of $X \to \mathbb{P}^1$, the corresponding fiber of $M \to \mathbb{P}^1$ is also smooth and isomorphic to $\operatorname{Pic}^0(A)$. In general, the compactified relative Picard scheme may be singular, and may fail to be a fine moduli space because the ambiguity in normalizing the Poincaré bundle can prevent the existence of a universal sheaf. But in our case, everything works out nicely.

Lemma 5.1. *M is a nonsingular projective Calabi-Yau threefold. Moreover, a universal sheaf exists on $M \times X$, and induces an equivalence $D^b(M) \simeq D^b(X)$.*

Proof. To begin with, Sawon [11], Lemma 8, has shown that M is projective, because it is an irreducible component of Simpson's moduli space of stable rank-one torsion-free sheaves on X. Next, the existence of a section s_0 implies that there is a universal sheaf on $M \times_{\mathbb{P}^1} X \subseteq M \times X$. Indeed, because X is nonsingular, the image of s_0 has to lie inside the smooth locus of p, and so we can apply [1], Theorem 3.4, to obtain the existence of a universal sheaf. In particular, M is a fine moduli space.

The theorem of Bridgeland and Maciocia [4], Theorem 1.2, now allows us to conclude that M is also a nonsingular Calabi-Yau threefold, and that the universal sheaf induces an equivalence between the derived categories of M and X. $\square$

The remainder of the proof consists in showing that M is, in fact, isomorphic to the quotient X/G. Our argument is based on the observation, explained above, that the smooth fibers of X/G and M are canonically isomorphic. The main issue is to extend this isomorphism to the singular fibers.

We begin by constructing a rational map from X/G to M, using the universal property of the fine moduli space M. The idea is the following: Let (A, L) be a polarized abelian variety, and let $\mathcal{P}_A$ denote the normalized Poincaré bundle on $A \times \operatorname{Pic}^0(A)$. The pullback of $\mathcal{P}_A$ under the morphism

$$\mathrm{id} \times \varphi_L \colon A \times A \to A \times \operatorname{Pic}^0(A)$$

satisfies

$$(\mathrm{id} \times \varphi_L)^* \mathcal{P}_A \simeq m^* L \otimes pr_1^* L^{-1} \otimes pr_2^* L^{-1}, \tag{1}$$

where $m \colon A \times A \to A$ is the multiplication on A. This allows us to describe the morphism φ_L to the moduli space $\operatorname{Pic}^0(A)$ in terms of a line bundle on $A \times A$.

To extend this construction to X, let $B \subseteq \mathbb{P}^1$ be the complement of the eight singular values of p, and set $U = p^{-1}(B)$. Then $p \colon U \to B$ is smooth, and our choice of section s_0 makes it into a group scheme over B. We also denote the multiplication morphism by $m \colon U \times_B U \to U$.

Lemma 5.2. *There is a morphism $f \colon U/G \to M$, commuting with the projections to $\mathbb{P}^1$, whose restriction to any smooth fiber agrees with the natural isomorphism $A/G \to \operatorname{Pic}^0(A)$ induced by the $(1,8)$-polarization $\mathscr{O}_X(H)|_A$.*

Proof. We begin by constructing a morphism $U \to M$ whose restriction to every smooth fiber A agrees with the natural morphism $\varphi_L \colon A \to \operatorname{Pic}^0(A)$ induced by the polarization $L = \mathscr{O}_X(H)|_A$. By the universal property of M, it suffices to construct a line bundle on the product $U \times_B U$ whose restriction to $A \times A$ is isomorphic to the pullback of the Poincaré bundle $\mathcal{P}_A$. But clearly

$$m^* \mathscr{O}_X(H) \otimes pr_1^* \mathscr{O}_X(-H) \otimes pr_2^* \mathscr{O}_X(-H)$$

is such a line bundle, by virtue of (1). The same identity shows that this line bundle is invariant under the action of G on the second factor of $U \times_B U$, and therefore descends to a line bundle on $U \times_B (U/G)$ whose restriction to $A \times (A/G)$ equals the pullback of $\mathcal{P}_A$. The universal property of M now gives us the desired morphism $f \colon U/G \to M$. $\square$

In particular, f is an isomorphism onto its image, and so X/G and M are birational. This is already sufficient to conclude that $\pi_1(M) \simeq G$; but in fact, we can use the geometry of both varieties to show that they must be isomorphic.

Lemma 5.3. *$f\colon U/G \to M$ extends to an isomorphism $X/G \simeq M$.*

Proof. We consider f as a birational map from X/G to M. Since both are smooth Calabi-Yau threefolds, any birational map between X/G and M is either an isomorphism, or a composition of flops [8], Theorem 5.3. The second possibility is easily ruled out: Indeed, since f is an isomorphism over U, the exceptional locus is contained in the eight singular fibers; moreover, it must be a union of rational curves [8], Lemma 4.3. Now each singular fiber is an elliptic translation scroll, which means that the only rational curves on it are the one-dimensional family of lines on the scroll. Since these lines cover the singular fiber, which is a divisor in X/G, they cannot be flopped. Consequently, the birational map f must extend to an isomorphism $X/G \simeq M$. $\square$

Note 5.4. Following the argument in [5], Lemma 1.2, one can show more generally that X/G does not admit any flops relative to $\mathbb{P}^1$ at all.

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