

# SINGULAR HERMITIAN METRICS AND THE DECOMPOSITION THEOREM OF CATANESE, FUJITA, AND KAWAMATA

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ABSTRACT. We prove that a torsion free sheaf  $\mathcal{F}$  endowed with a singular hermitian metric with semi-positive curvature and satisfying the minimal extension property admits a direct-sum decomposition  $\mathcal{F} \simeq \mathcal{U} \oplus \mathcal{A}$  where  $\mathcal{U}$  is a hermitian flat bundle and  $\mathcal{A}$  is a generically ample sheaf. The result applies to the case of direct images of relative pluricanonical bundles  $f_*\omega_{X/Y}^{\otimes m}$  under a surjective morphism  $f: X \rightarrow Y$  of smooth projective varieties with  $m \geq 2$ . This extends previous results of Fujita [Fuj78] and Catanese–Kawamata [CK19].

## 1. INTRODUCTION

Let  $f: X \rightarrow Y$  be a fibration of smooth projective varieties (over the complex numbers) and let  $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$  be the relative canonical bundle. Motivated by earlier work of Fujita, Catanese and Kawamata [CK19, Theorem 1.2] proved a direct-sum decomposition

$$(1) \quad f_*\omega_{X/Y} \simeq \mathcal{U} \oplus \mathcal{A}$$

where  $\mathcal{U}$  is a hermitian flat bundle (possibly trivial), and  $\mathcal{A}$  is a generically ample sheaf (possibly trivial). We recall that a coherent torsion free sheaf  $\mathcal{A}$  on a smooth projective variety is *generically ample* if the restriction to a general complete intersection smooth curve is an ample vector bundle. When  $Y$  is a smooth projective curve, the decomposition (1) is Fujita’s well-known second decomposition from [Fuj78] (see also [CD17, Theorem 3.3]). Motivated by (1), we introduce the following definition.

**Definition 1.** A coherent torsion free sheaf  $\mathcal{F}$  admits a *Catanese–Fujita–Kawamata decomposition* if there exists an isomorphism  $\mathcal{F} \simeq \mathcal{U} \oplus \mathcal{A}$  where  $\mathcal{U}$  is a hermitian flat bundle (possibly trivial), and  $\mathcal{A}$  is a generically ample sheaf (possibly trivial).

In this paper we extend the decomposition (1) to direct images of relative *pluricanonical* bundles.

**Theorem 2.** *If  $f: X \rightarrow Y$  is a surjective morphism of smooth projective complex varieties, then  $f_*\omega_{X/Y}^{\otimes m}$  admits a Catanese–Fujita–Kawamata decomposition for every  $m \geq 2$ .*

While the proof of (1) is Hodge-theoretic, the proof of Theorem 2 is based on the fact that for any  $m \geq 2$  the sheaves  $f_*\omega_{X/Y}^{\otimes m}$  carry canonical singular hermitian metrics with semi-positive curvature. Furthermore, these metrics satisfy the so-called *minimal*

*extension property* (cf. §2, [PT18, Theorem 1.1] and [HPS18, Theorem 27.1]). This is a property that stems from Ohsawa–Takegoshi’s extension theorem with optimal bounds and allows one to extend local sections across subsets of measure zero with a control on the  $L^2$ -norm. Instances of bundles satisfying the minimal extension property are pseudo-effective line bundles and Nakano semi-positive vector bundles. Theorem 2 is an application of the following theorem.

**Theorem 3.** *Let  $\mathcal{F}$  be a coherent torsion free sheaf on a smooth projective variety  $Y$  endowed with a singular hermitian metric with semi-positive curvature and satisfying the minimal extension property. Then  $\mathcal{F}$  admits a Catanese–Fujita–Kawamata decomposition.*

*Proof of Theorem 2.* By [HPS18, Theorem 27.1], for any  $m \geq 2$  the pushforward  $f_*\omega_{X/Y}^{\otimes m}$  admits a singular hermitian metric with semi-positive curvature and the minimal extension property. The result is an application of Theorem 3.  $\square$

An application of Theorem 2 regards the structure of the sheaves  $f_*\omega_{X/Y}^{\otimes m}$  for  $m \geq 2$  in certain circumstances.

**Theorem 4.** *Let  $f: X \rightarrow Y$  be a fibration of smooth projective complex varieties and denote  $J = \{m \in \mathbb{N}_{\geq 2} \mid f_*\omega_{X/Y}^{\otimes m} \neq 0\}$ . Suppose there exists an open subset  $U \subset Y$  such that:  $\text{codim}(Y \setminus U) \geq 2$ , the morphism  $f$  is smooth over  $U$ , and  $\omega_{V/U}$  is  $f|_V$ -semi-ample where  $V = f^{-1}(U)$ . Then  $f_*\omega_{X/Y}^{\otimes m}$  is either generically ample for all  $m \in J$ , or hermitian flat for all  $m \in J$ .*

In §4 we collect other instances of Catanese–Fujita–Kawamata decompositions.

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## 2. THE MINIMAL EXTENSION PROPERTY

Let  $Y$  be a complex manifold of dimension  $n$  and let  $(\mathcal{F}, h)$  be a torsion free sheaf endowed with a singular hermitian metric. We refer to [HPS18] for the definition and first properties of singular hermitian metrics and the notion of semi-positive curvature (cf. *loc. cit* Definition 18.1). Denote also by  $B \subset \mathbb{C}^n$  the open unit ball centered at the origin, with volume  $\mu(B) = \pi^n/n!$ . Moreover, let  $F_y$  be the fiber of  $\mathcal{F}$  at a point

$y \in Y$  over which  $\mathcal{F}$  is locally free. We recall the definition of the *minimal extension property* introduced in [HPS18, Definition 20.1].

**Definition 5.** The pair  $(\mathcal{F}, h)$  satisfies the *minimal extension property* if there exists an analytic nowhere dense closed subset  $Z \subset Y$  such that  $\mathcal{F}$  is locally free on  $Y \setminus Z$ , and for every embedding  $\iota: B \hookrightarrow Y$  of the unit ball centered at  $\iota(0) = y \in Y \setminus Z$ , and vector  $v \in F_y$  of length  $|v|_{h,y} = 1$ , there exists a holomorphic section  $s \in H^0(B, \iota^* \mathcal{F})$  such that

$$s(0) = v \quad \text{and} \quad \frac{1}{\mu(B)} \int_B |s|_h^2 d\mu \leq 1.$$

We recall a few properties of the minimal extension property from [HPS18].

**Proposition 6.** *Let  $(\mathcal{F}, h)$  be as in Definition 5 and let  $b: \mathcal{F} \hookrightarrow \mathcal{G}$  be an inclusion of coherent sheaves. If  $b$  is generically an isomorphism, then  $h$  extends to a singular hermitian metric  $h_{\mathcal{G}}$  on  $\mathcal{G}$  satisfying the minimal extension property. Moreover, if  $h$  has semi-positive curvature, then  $h_{\mathcal{G}}$  has semi-positive curvature as well.*

*Proof.* The proposition is essentially proved in [HPS18, Proposition 19.3]. It only remains to note that the minimal extension property holds for  $(\mathcal{G}, h_{\mathcal{G}})$  because every section of  $\mathcal{F}$  is a section of  $\mathcal{G}$ .  $\square$

**Proposition 7.** *Let  $(\mathcal{F}, h)$  be as in Definition 5. If  $\varphi: \mathcal{F} \rightarrow \mathcal{E}$  is a quotient onto a torsion free sheaf and  $h'$  is the induced metric, then  $(\mathcal{E}, h')$  satisfies the minimal extension property.*

*Proof.* Let  $Z = Z(\mathcal{F})$  be as in Definition 5 and let  $S(\mathcal{E})$  be the locus where  $\mathcal{E}$  is not locally free. Set  $Z' = Z \cup S(\mathcal{E})$  and let  $y \in Y \setminus Z'$ . For  $w \in E_y$  the induced metric  $h'$  on  $\mathcal{E}$  is defined by

$$(2) \quad |w|_{h',y} = \inf \{ |v|_{h,y} \mid v \in F_y \text{ and } \varphi_y(v) = w \}.$$

(If  $\varphi_y = 0$ , then the metric is  $+\infty$  for all  $w \neq 0$ .) Now let  $w \in E_y$  be such that  $|w|_{h',y} = 1$  and let  $B \subset Y$  be the embedding of the unit ball centered at  $y \in Y \setminus Z'$ . Then there exists  $v \in F_y$  such that  $|v|_{h,y} = 1$  and, by the minimal extension property of  $\mathcal{F}$ , a holomorphic section  $s \in H^0(B, \mathcal{F}|_B)$  such that  $s(0) = v$  and  $\frac{1}{\mu(B)} \int_B |s|_h^2 d\mu \leq 1$ . As  $|\varphi(s)|_{h',y} \leq |s|_{h,y}$  for almost every  $y \in B$ , this yields inequalities

$$\frac{1}{\mu(B)} \int_B |\varphi(s)|_{h'}^2 d\mu \leq \frac{1}{\mu(B)} \int_B |s|_h^2 d\mu \leq 1.$$

$\square$

The following result generalizes [HPS18, Theorem 26.4].

**Proposition 8.** *Suppose  $Y$  is a compact complex manifold and let  $(\mathcal{F}, h)$  be a torsion free sheaf endowed with a singular hermitian metric of semi-positive curvature and satisfying the minimal extension property. If  $f: \mathcal{F} \rightarrow \mathcal{U}$  is a quotient onto a vector bundle  $\mathcal{U}$  endowed with a smooth hermitian flat metric, then there exists a morphism  $s: \mathcal{U} \rightarrow \mathcal{F}$  such that  $f \circ s = \text{id}_{\mathcal{U}}$ .*

*Proof.* Set  $r = \text{rk}(\mathcal{U})$ . The bundle  $\mathcal{U}$  is associated to a representation  $\pi_1(Y) \rightarrow U(r)$  of the fundamental group of  $Y$  to the unitary group  $U(r)$ . As such  $\mathcal{U}$  decomposes as a direct-sum of vector bundles arising from irreducible unitary representations. Without loss of generality we can suppose that  $\mathcal{U}$  is irreducible.

Consider the quotients

$$\mathcal{F} \otimes \mathcal{U}^\vee \rightarrow \mathcal{U} \otimes \mathcal{U}^\vee \xrightarrow{\text{tr}} \mathcal{O}_Y$$

where the first is induced by  $f$  and the second is the trace map of  $\mathcal{U}$ . Since the metric on  $\mathcal{U}$ , and hence on  $\mathcal{U}^\vee$ , is flat, the induced singular hermitian metric on the sheaf  $\mathcal{F} \otimes \mathcal{U}^\vee$  has semi-positive curvature and the minimal extension property. By [HPS18, Theorem 26.4] there exists a splitting  $s': \mathcal{O}_Y \rightarrow \mathcal{F} \otimes \mathcal{U}^\vee$ , and hence a non-trivial morphism  $s'': \mathcal{U} \rightarrow \mathcal{F}$  such that  $f \circ s'' \neq 0$ . By Schur's Lemma the composition  $f \circ s''$  is an isomorphism and  $s := s'' \circ (f \circ s'')^{-1}$  splits  $f$ .  $\square$

Finally we recall the following theorem which is proved in [HPS18, Theorem 26.1].

**Theorem 9.** *Let  $Y$  be a compact complex manifold and let  $(\mathcal{F}, h)$  be a torsion free sheaf endowed with a singular hermitian metric. Suppose  $h$  has semi-positive curvature and it satisfies the minimal extension property. If  $c_1(\det(\mathcal{F})) = 0$  in  $H^2(Y, \mathbb{R})$ , then  $\mathcal{F}$  is locally free,  $h$  is smooth and  $(\mathcal{F}, h)$  is hermitian flat.*

### 3. CATANESE–FUJITA–KAWAMATA DECOMPOSITIONS

**3.1. Proof of Theorem 3.** Set  $n = \dim X$ . We may suppose that  $\mathcal{F} \neq 0$ . By [HPS18, Proposition 25.1, §26 and Definition 19.1] the line bundle  $\det(\mathcal{F})$  admits a singular hermitian metric with semi-positive curvature. Hence  $\det(\mathcal{F})$  is pseudo-effective and for any very ample line bundle  $A$  on  $X$  the *degree* of  $\mathcal{F}$  satisfies

$$\text{deg}_A(\mathcal{F}) := (A^{n-1} \cdot \mathcal{F}) = (A^{n-1} \cdot \det(\mathcal{F})) \geq 0$$

by Nakai–Moishezon's Theorem. We define the *A-slope* of  $\mathcal{F}$  as

$$\mu^A(\mathcal{F}) := \frac{\text{deg}_A(\mathcal{F})}{\text{rk}(\mathcal{F})}$$

and say that it is *semistable* (with respect to  $A$ ) if for every nonzero coherent submodule  $\mathcal{E} \subset \mathcal{F}$  the inequality  $\mu^A(\mathcal{E}) \leq \mu^A(\mathcal{F})$  holds. Let

$$(3) \quad 0 = \mathcal{N}_0 \subsetneq \mathcal{N}_1 \subsetneq \dots \subsetneq \mathcal{N}_d = \mathcal{F}$$

be the Harder–Narasimhan filtration of  $\mathcal{F}$ . Hence for any  $i = 1, \dots, d$  the quotients  $\mathcal{N}_i/\mathcal{N}_{i-1}$  are torsion free and semistable. Moreover the slopes

$$\mu_i^A := \mu^A(\mathcal{N}_i/\mathcal{N}_{i-1})$$

satisfy

$$\mu_1^A > \mu_2^A > \dots > \mu_d^A$$

[Mar81, Proposition-Definition 1.13]. We denote by

$$\mathcal{Q} := \mathcal{F}/\mathcal{N}_{d-1} = \mathcal{N}_d/\mathcal{N}_{d-1}$$

the minimal destabilizing quotient of  $\mathcal{F}$  and set

$$\mathcal{L} := \det(\mathcal{Q})$$

for the determinant of  $\mathcal{Q}$ . By [PT18, Lemma 2.4.3] and [HPS18, Proposition 25.1] both  $\mathcal{Q}$  and  $\mathcal{L}$  admit a singular hermitian metric with semi-positive curvature. Hence  $\mathcal{L}$  is pseudo-effective and  $\deg_A(\mathcal{L}) \geq 0$ . We distinguish two cases:  $\deg_A(\mathcal{L}) > 0$  and  $\deg_A(\mathcal{L}) = 0$ .

Let's begin with the case  $\deg_A(\mathcal{L}) > 0$ . We are going to show that  $\mathcal{F}$  is already generically ample. Let  $H = A^{\otimes a}$  be a very ample line bundle with  $a \gg 0$  so that Flenner's Theorem [HL10, Theorem 7.1.1] applies to a general complete intersection smooth curve  $C$  cut out by divisors in  $|H|$  and contained in the locus where  $\mathcal{F}$  is locally free. It follows that the Harder–Narasimhan filtration (3) of  $\mathcal{F}$  restricts to the Harder–Narasimhan filtration

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \dots \subsetneq \mathcal{M}_d = \mathcal{F}|_C$$

of  $\mathcal{F}|_C$ . Here the sheaves  $\mathcal{M}_i := \mathcal{N}_i|_C$  are locally free and semistable, and

$$\frac{\deg(\mathcal{M}_i/\mathcal{M}_{i-1})}{\text{rk}(\mathcal{M}_i/\mathcal{M}_{i-1})} > \frac{\deg(\mathcal{M}_{i+1}/\mathcal{M}_i)}{\text{rk}(\mathcal{M}_{i+1}/\mathcal{M}_i)}$$

for all  $i = 1, \dots, d-1$ . Since

$$\deg(\mathcal{M}_i/\mathcal{M}_{i-1}) = (H^{n-1} \cdot \mathcal{N}_i/\mathcal{N}_{i-1}) = a^{n-1} (A^{n-1} \cdot \mathcal{N}_i/\mathcal{N}_{i-1}),$$

the *minimal slope* of  $\mathcal{F}|_C$  satisfies

$$\mu_{\min}(\mathcal{F}|_C) := \frac{\deg(\mathcal{M}_d/\mathcal{M}_{d-1})}{\text{rk}(\mathcal{M}_d/\mathcal{M}_{d-1})} = a^{n-1} \mu_d^A = a^{n-1} \frac{\deg_A(\mathcal{L})}{\text{rk}(\mathcal{Q})} > 0.$$

By [Bre04, Theorem 2.1]  $\mathcal{F}|_C$  is an ample bundle and in this case we set  $\mathcal{U} = 0$  and  $\mathcal{A} = \mathcal{F}$ .

Now let us suppose that  $\deg_A(\mathcal{L}) = 0$ . We will first show that  $c_1(\mathcal{L}) = 0$  in  $H^2(X, \mathbb{R})$ . Let  $H = A^{\otimes a}$  be a very ample line bundle as before with  $a$  sufficiently large. Let  $D_1, \dots, D_{n-2} \in |H|$  be general members such that for all  $i = 2, \dots, n-2$  the partial intersections

$$V_i := D_1 \cap D_2 \cap \dots \cap D_i$$

are smooth and irreducible ample divisors in  $V_{i-1}$ . Hence  $S := V_{n-2}$  is a smooth surface with  $[S] = H^{n-2}$  in  $H_4(X; \mathbb{Z})$  and such that the restriction map

$$(4) \quad H^2(X; \mathbb{Z}) \rightarrow H^2(S; \mathbb{Z}) \quad \text{is injective}$$

(see Lefschetz's Hyperplane Theorem [Laz04, Theorem 3.1.15]). Moreover, we can choose the general divisors  $D_i \in |H|$  such that the restrictions  $\mathcal{L}|_{V_i}$  are pseudo-effective. Hence

$$0 = (H^{n-1} \cdot \mathcal{L}) = (H|_S \cdot \mathcal{L}|_S)$$

from which we deduce that  $\mathcal{L}|_S$  is numerically trivial as  $H|_S$  is ample and  $\mathcal{L}|_S$  is a limit of effective classes (*cf.* [Laz04, Theorem 1.4.26]). By (4) the claim follows.

Since  $c_1(\mathcal{L}) = 0$ , by Proposition 7 and Theorem 9 the bundle  $\mathcal{Q}$  is hermitian flat. Moreover, by Proposition 8 there exists a decomposition  $\mathcal{F} \simeq \mathcal{Q} \oplus \mathcal{N}_{d-1}$  so that

we only need to prove that  $\mathcal{N}_{d-1}$  is generically ample (or trivial). Let  $C$  be a general complete intersection smooth curve cut out by divisors in  $|H|$  and consider the Harder–Narasimhan filtration

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \dots \subsetneq \mathcal{M}_{d-2} \subsetneq \mathcal{M}_{d-1},$$

of  $\mathcal{M}_{d-1} := (\mathcal{N}_{d-1})|_C$  where, as before,  $\mathcal{M}_i := \mathcal{N}_i|_C$ . As the minimal slope of  $\mathcal{M}_{d-1}$  satisfies

$$\mu_{\min}(\mathcal{M}_{d-1}) := \frac{\deg(\mathcal{M}_{d-1}/\mathcal{M}_{d-2})}{\mathrm{rk}(\mathcal{M}_{d-1}/\mathcal{M}_{d-2})} = a^{n-1}\mu_{d-1}^A > a^{n-1}\mu_d^A = a^{n-1}\frac{\deg_{\mathcal{A}}(\mathcal{L})}{\mathrm{rk}(\mathcal{Q})} = 0,$$

the bundle  $\mathcal{M}_{d-1}$  is ample. In this case we set  $\mathcal{U} = \mathcal{Q}$  and  $\mathcal{A} = \mathcal{N}_{d-1}$ .

#### 4. OTHER APPLICATIONS

**Corollary 10.** *Let  $f: X \rightarrow Y$  be a surjective morphism of smooth projective varieties and  $L$  be a semiample line bundle on  $X$ . Then  $f_*(\omega_{X/Y} \otimes L)$  admits a Catanese–Fujita–Kawamata decomposition.*

*Proof.* Since  $L$  is semiample, it admits a smooth hermitian metric  $h$  with semi-positive curvature; in particular, the multiplier ideal  $\mathcal{I}(h) = \mathcal{O}_X$  is trivial. By Păun–Takayama’s theorem [PT18, Theorem 1.1] and [HPS18, Theorem 21.1],  $f_*(\omega_{X/Y} \otimes L)$  admits a singular hermitian metric with semi-positive curvature and the minimal extension property. The corollary follows from Theorem 3.  $\square$

**Corollary 11.** *If  $f: X \rightarrow Y$  is a smooth fibration of smooth projective varieties and  $E$  is a Nakano semi-positive vector bundle on  $X$ , then  $R^j f_*(\omega_{X/Y} \otimes E)$  admits a Catanese–Fujita–Kawamata decomposition for every  $j \geq 0$ .*

*Proof.* The main result of [MT08, Theorem 1.1] proves that  $R^j f_*(\omega_{X/Y} \otimes E)$  is a Nakano semi-positive vector bundle for all  $j \geq 0$ . On the other hand a Nakano semi-positive vector bundle satisfies the minimal extension property (cf. [SY20, Example 2.16]). The result follows by Theorem 3.  $\square$

#### 5. PROOF OF THEOREM 4

In this section we prove Theorem 4. In order to do so, we begin by recalling the following result of Esnault and Viehweg, which in fact holds under more general assumptions.

Given a fibration  $f: X \rightarrow Y$  of varieties, a line bundle  $L$  on  $X$  is *f-semi-ample* if for some positive integer  $N > 0$  the natural morphism  $f^* f_* L^{\otimes N} \rightarrow L^{\otimes N}$  is surjective.

**Theorem 12.** *Let  $g: X \rightarrow C$  be a smooth fibration from a smooth projective variety to a smooth projective curve such that  $\omega_{X/C}$  is  $g$ -semi-ample. If  $\det(g_* \omega_{X/C}^{\otimes m})$  is an ample line bundle for some  $m \geq 1$ , then for all  $m \geq 2$  the bundle  $g_* \omega_{X/C}^{\otimes m}$  is ample, if not trivial.*

*Proof.* The proof is a special case of [EV91, Theorem 0.1].  $\square$

**Corollary 13.** *Let  $g: X \rightarrow C$  be a smooth fibration from a smooth projective variety to a smooth projective curve such that  $\omega_{X/C}$  is  $g$ -semi-ample. Denote  $J = \{m \in \mathbb{N}_{\geq 2} \mid g_*\omega_{X/C}^{\otimes m} \neq 0\}$ . Then either  $g_*\omega_{X/C}^{\otimes m}$  is ample for every  $m \in J$ , or hermitian flat for every  $m \in J$ .*

*Proof.* By Theorem 3  $g_*\omega_{X/C}^{\otimes m}$  decomposes as  $g_*\omega_{X/C}^{\otimes m} \simeq \mathcal{U}_m \oplus \mathcal{A}_m$  with  $\mathcal{U}_m$  hermitian flat (or trivial), and  $\mathcal{A}_m$  ample (or trivial). If  $\mathcal{A}_m = 0$  for all  $m \in J$ , then  $g_*\omega_{X/C}^{\otimes m}$  is hermitian flat for every  $m \in J$  and the proof is complete. On the other hand, if  $\mathcal{A}_m \neq 0$  for some  $m \in J$ , then  $\det(g_*\omega_{X/C}^{\otimes m})$  is ample, and by Theorem 12 the bundle  $g_*\omega_{X/C}^{\otimes m}$  is itself ample for every  $m \in J$ .  $\square$

**Remark 14.** Corollary 13 admits the following stronger formulation in which a certain kind of singular fibers is allowed. Let  $g: X \rightarrow C$  be a surjective Gorenstein morphism from a smooth projective variety to a smooth projective curve such that the fibers are normal irreducible varieties with at most rational singularities. If  $\omega_{X/C}$  is  $g$ -semi-ample, then  $g_*\omega_{X/C}^{\otimes m}$  is either ample for all  $m \in J$ , or hermitian flat for all  $m \in J$ .

**Lemma 15.** *Let  $f: X \rightarrow Y$  be a fibration of smooth projective varieties. If  $m$  is a positive integer, then for any sufficiently general hyperplane section  $H$  of  $Y$  the variety  $X_H := f^{-1}(H)$  is smooth and irreducible. Moreover, there is an isomorphism  $f_*\omega_{X/Y}^{\otimes m}|_H \simeq g_*\omega_{X_H/H}^{\otimes m}$  where  $g := f|_{X_H}$ .*

*Proof.* The fact that  $X_H$  is smooth and irreducible follows by Bertini theorem, as stated in [Jou83, Theorem 6.3]. The following argument is inspired by [Kol86]. Consider the following cartesian diagram

$$\begin{array}{ccc} X_H & \xrightarrow{j} & X \\ \downarrow g & & \downarrow f \\ H & \xrightarrow{i} & Y \end{array}$$

where  $i$  and  $j$  are the natural inclusions. There is a short exact sequence for any  $m \geq 1$

$$0 \rightarrow \omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_X(-X_H) \rightarrow \omega_{X/Y}^{\otimes m} \rightarrow j_*\omega_{X_H/H}^{\otimes m} \rightarrow 0$$

since  $N_{X_H/X} \simeq g^*N_{H/Y}$  and  $\omega_{X/Y}^{\otimes m}|_{X_H} \simeq \omega_{X_H/H}^{\otimes m}$ . By taking higher direct images, there is a long exact sequence

$$\begin{aligned} 0 &\rightarrow f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \rightarrow f_*\omega_{X/Y}^{\otimes m} \rightarrow i_*g_*\omega_{X_H/H}^{\otimes m} \rightarrow \\ &\rightarrow R^1f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \rightarrow R^1f_*\omega_{X/Y}^{\otimes m} \rightarrow i_*R^1g_*\omega_{X_H/H}^{\otimes m} \rightarrow \dots \\ &\dots \rightarrow R^j f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \rightarrow R^j f_*\omega_{X/Y}^{\otimes m} \rightarrow i_*R^j g_*\omega_{X_H/H}^{\otimes m} \rightarrow \dots \end{aligned}$$

For every index  $j \geq 0$  the morphism  $\psi_j: R^j f_*\omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \rightarrow R^j f_*\omega_{X/Y}^{\otimes m}$  is obtained by tensoring the natural inclusion  $\mathcal{O}_Y(-H) \rightarrow \mathcal{O}_Y$  with  $R^j f_*\omega_{X/Y}^{\otimes m}$ . Hence, for a general

hyperplane section  $H$ , which does not contain any associated subvariety of  $R^j f_* \omega_{X/Y}^{\otimes m}$ , the morphism  $\psi_j$  is injective. For  $j = 0$  it follows that the following sequence

$$(5) \quad 0 \rightarrow f_* \omega_{X/Y}^{\otimes m} \otimes \mathcal{O}_Y(-H) \rightarrow f_* \omega_{X/Y}^{\otimes m} \rightarrow i_* g_* \omega_{X_H/H}^{\otimes m} \rightarrow 0$$

is short exact. The desired isomorphism is obtained by restricting (5) to  $H$ .  $\square$

Repeated applications of the previous lemma yield the following corollary.

**Corollary 16.** *Let  $f: X \rightarrow Y$  be a fibration of smooth projective varieties. If  $m$  is a positive integer, then for a general complete intersection smooth curve  $C$  in  $Y$  the variety  $X_C = f^{-1}(C)$  is smooth and irreducible. Moreover, there is an isomorphism  $f_* \omega_{X/Y}^{\otimes m}|_C \simeq g_* \omega_{X_C/C}^{\otimes m}$  where  $g := f|_{X_C}$ .*

*Proof of Theorem 4.* Set  $n = \dim Y$ . Thanks to Corollary 13 we can assume that  $n \geq 2$ . Moreover, without loss of generality we can suppose that  $J \neq \emptyset$ . Recall that for every  $m \in J$  there exists a decomposition

$$(6) \quad \mathcal{F}_m := f_* \omega_{X/Y}^{\otimes m} \simeq \mathcal{U}_m \oplus \mathcal{A}_m$$

where  $\mathcal{U}_m$  is hermitian flat (or trivial), and  $\mathcal{A}_m$  is generically ample (or trivial). If  $\mathcal{A}_m = 0$  for all  $m \in J$ , then  $\mathcal{F}_m$  is hermitian flat for every  $m \in J$  and the proof is complete. We may suppose that there exists an index  $m_0 \in J$  such that  $\mathcal{A}_{m_0} \neq 0$ . We aim to prove that  $\mathcal{U}_m = 0$  for all  $m \in J$ .

Denote by  $U \subset Y$  an open subset whose complement is of codimension at least two as in the statement of the theorem and fix a very ample line bundle  $H$  on  $Y$ . Moreover, let  $C \subset U$  be a general complete intersection smooth curve cut out by divisors in  $|H|$  such that  $\deg(\mathcal{A}_{m_0}|_C) > 0$ . If  $g: X_C \rightarrow C$  denotes the restriction of  $f$  to  $X_C := f^{-1}(C)$ , then by Corollary 16 there is an isomorphism

$$(7) \quad 0 \neq \mathcal{F}_{m_0}|_C \simeq g_* \omega_{X_C/C}^{\otimes m_0} \simeq \mathcal{A}_{m_0}|_C \oplus \mathcal{U}_{m_0}|_C.$$

Hence we have

$$\deg(\det(g_* \omega_{X_C/C}^{\otimes m_0})) = (\det(\mathcal{A}_{m_0}) \cdot C) > 0$$

and by Corollary 13 we conclude that

$$(8) \quad g_* \omega_{X_C/C}^{\otimes k} \text{ is ample for all } k \in J.$$

If  $m \in J$  is an arbitrary index, we can select a curve  $C$  (depending on  $m$ ) as above, and general enough so that also the following isomorphism

$$(9) \quad 0 \neq \mathcal{F}_m|_C \simeq g_* \omega_{X_C/C}^{\otimes m}$$

holds. By (6), (8) and (9) the bundle  $g_* \omega_{X_C/C}^{\otimes m} \simeq \mathcal{U}_m|_C \oplus \mathcal{A}_m|_C$  is ample, and therefore  $\mathcal{U}_m|_C$  itself is ample, if not trivial. Since  $(\det(\mathcal{U}_m) \cdot C) = 0$ , this forces  $\det(\mathcal{U}_m)|_C = 0$ , and thus  $\mathcal{U}_m = 0$ .  $\square$



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