Thomas Krämer and Claude Sabbah pointed out to me that the published proof of Lemma 20.2 only works in the regular case. The purpose of this note is to give a correct proof for the general case. Here is the statement again.

**Lemma.** Let $f: A \to B$ be a surjective morphism of abelian varieties, with connected fibers. If $\mathcal{N}$ is a nontrivial simple holonomic $\mathcal{D}_B$-module, then $f^*\mathcal{N}$ is a simple holonomic $\mathcal{D}_A$-module.

**Proof.** Since $f$ is smooth, $f^*\mathcal{N} = \mathcal{O}_A \otimes f^{-1}\mathcal{O}_B f^{-1}\mathcal{N}$ is a holonomic $\mathcal{D}_A$-module, and so there is a surjective morphism $f^*\mathcal{N} \to \mathcal{M}$ to a nontrivial simple holonomic $\mathcal{D}_A$-module $\mathcal{M}$. We will prove the assertion by showing that it is an isomorphism.

The support $X = \text{Supp} \mathcal{N}$ is an irreducible subvariety of $B$. As $\mathcal{N}$ is holonomic, there is a dense Zariski-open subset $U \subseteq B$ such that $X \cap U$ is nonsingular and such that the restriction $\mathcal{N}|_U$ is the direct image of a holomorphic vector bundle with integrable connection $(\mathcal{E}, \nabla)$ on $X \cap U$ [HTT08, Proposition 3.1.6]. This means that $f^*\mathcal{N}$ is supported on $f^{-1}(X)$, and that its restriction to $f^{-1}(U)$ is the direct image of $(f^*\mathcal{E}, f^*\nabla)$. We observe that, on the fibers of $f$ over points of $X \cap U$, the latter is a trivial bundle of rank $n = \text{rk} \mathcal{E}$.

Since $\mathcal{M}$ is a quotient of $f^*\mathcal{N}$, its restriction to $f^{-1}(U)$ is also the direct image of a holomorphic vector bundle with integrable connection on $f^{-1}(X \cap U)$; as a quotient of $(f^*\mathcal{E}, f^*\nabla)$, the restriction of this bundle to the fibers of $f$ must be trivial of some rank $k \leq n$. Now let $r = \dim A - \dim B$ be the relative dimension of $f$. By adjunction [HTT08, Corollary 3.2.15], the surjective morphism

$$f^*\mathcal{N} \to \mathcal{M}$$

gives rise to a nontrivial morphism

$$\mathcal{N} \to \mathcal{H}^{-r}f_+\mathcal{M}$$

which must be injective because $\mathcal{N}$ is simple. Over $U$, the left-hand side is a vector bundle of rank $n$ and the right-hand side a vector bundle of rank $k \leq n$; this is only possible if $k = n$. But then $f^*\mathcal{N} \to \mathcal{M}$ is an isomorphism over $f^{-1}(U)$, and since $\mathcal{M}$ is simple, we obtain a short exact sequence

$$0 \to \mathcal{K} \to f^*\mathcal{N} \to \mathcal{M} \to 0$$

where $\mathcal{K}$ is a holonomic $\mathcal{D}_A$-module whose support is contained in $f^{-1}(X \setminus X \cap U)$.

Now recall that $f^+\mathcal{N} = f^*\mathcal{N}[r]$. By adjunction [HTT08, Corollary 3.2.15], the inclusion $\mathcal{K} \to f^*\mathcal{N}$ determines a morphism $f_+\mathcal{K}[r] \to \mathcal{N}$, which factors as

$$f_+\mathcal{K}[r] \to \mathcal{H}^r f_+\mathcal{K} \to \mathcal{N}.$$ 

Since the support of $\mathcal{H}^r f_+\mathcal{K}$ is contained in $X \setminus X \cap U$, the morphism in question must be zero, and so $\mathcal{K} = 0$. We conclude that $f^*\mathcal{N}$ is isomorphic to $\mathcal{M}$. $\Box$

**References**