DERIVED INVARIANCE OF THE NUMBER OF HOLOMORPHIC 1-FORMS AND VECTOR FIELDS

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ABSTRACT. We prove that smooth projective varieties with equivalent derived categories have isogenous Picard varieties. In particular, their irregularity and number of independent vector fields are the same. This implies that all Hodge numbers are the same for arbitrary derived equivalent threefolds, as well as other consequences of derived equivalence based on numerical criteria.

RÉSUMÉ. Nous montrons que deux variétés projectives lisses dont les catégories dérivées sont équivalentes, ont des variétés de Picard isogènes. En particulier, elles ont la même irrégularité et le même nombre de champs de vecteurs indépendants. On en déduit l’invariance des nombres de Hodge par l’équivalence dérivée pour les variétés de dimension trois, ainsi que quelques autres conséquences numériques.

1. Introduction

Given a smooth projective variety $X$, we denote by $D(X)$ the bounded derived category of coherent sheaves $D^b(Coh(X))$. All varieties we consider below are over the complex numbers. A result of Rouquier, [17] Théorème 4.18, asserts that if $X$ and $Y$ are smooth projective varieties with $D(X) \cong D(Y)$ (as linear triangulated categories), then there is an isomorphism of algebraic groups

$$\text{Aut}^0(X) \times \text{Pic}^0(X) \cong \text{Aut}^0(Y) \times \text{Pic}^0(Y).$$

We refine this by showing that each of the two factors is almost invariant under derived equivalence. According to Chevalley’s theorem $\text{Aut}^0(X)$, the connected component of the identity in $\text{Aut}(X)$, has a unique maximal connected affine subgroup $\text{Aff}(\text{Aut}^0(X))$, and the quotient $\text{Alb}(\text{Aut}^0(X))$ by this subgroup is an abelian variety, the Albanese variety of $\text{Aut}^0(X)$. The affine parts $\text{Aff}(\text{Aut}^0(X))$ and $\text{Aff}(\text{Aut}^0(Y))$, being also the affine parts of the two sides in the isomorphism above, are isomorphic. The main result of the paper is

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THEOREM A. — Let $X$ and $Y$ be smooth projective varieties such that $D(X) \cong D(Y)$. Then

(1) $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ are isogenous; equivalently, $\text{Alb}(\text{Aut}^0(X))$ and $\text{Alb}(\text{Aut}^0(Y))$ are isogenous.

(2) $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$ unless $X$ and $Y$ are étale locally trivial fibrations over isogenous positive dimensional abelian varieties (hence $\chi(D_X) = \chi(D_Y) = 0$).

The key content is part (1), while (2) simply says that $\text{Aut}^0(X)$ and $\text{Aut}^0(Y)$ are affine unless the geometric condition stated there holds (hence the presence of abelian varieties is essentially the only reason for the failure of the derived invariance of the Picard variety).

COROLLARY B. — If $D(X) \cong D(Y)$, then $h^1(X, \Omega^1_X) = h^1(Y, \Omega^1_Y)$ and $h^2(X, \O_X) = h^2(Y, \O_Y)$.

The Hodge number $h^{1,1}(X) = h^0(X, \Omega^1_X)$ is also called the irregularity $q(X)$, the dimension of the Picard and Albanese varieties of $X$. The invariance of the sum $h^0(X, \Omega^1_X) + h^2(X, \O_X)$ was already known, and is a special case of the derived invariance of the Hochschild cohomology of $X$ ([15], [7]; cf. also [9] §6.1). Alternatively, it follows from Rouquier’s result above. Corollary B, together with the derived invariance of Hochschild homology (cf. loc. cit.), implies the invariance of all Hodge numbers for all derived equivalent threefolds. This was expected to hold as suggested by work of Kontsevich [12] (cf. also [1]).

COROLLARY C. — Let $X$ and $Y$ be smooth projective threefolds with $D(X) \cong D(Y)$. Then $h^{p,q}(X) = h^{p,q}(Y)$ for all $p$ and $q$.

Proof. — The fact that the Hochschild homology of $X$ and $Y$ is the same gives

(1.1) $\sum_{p+q+i} h^{p,q}(X) = \sum_{p+q+i} h^{p,q}(Y)$

for all $i$. A straightforward calculation shows that this implies the invariance of all Hodge numbers except for $h^{1,0}$ and $h^{0,1}$, about which we only note that $h^{1,0} + h^{0,1}$ is invariant. We then apply Corollary B.

Corollary C is already known (in arbitrary dimension) for varieties of general type: for these derived equivalence implies $K$-equivalence by a result of Kawamata [11], while $K$-equivalent varieties have the same Hodge numbers according to Batyrev [2] and Kontsevich, Denef-Looijenga [8]. It is also well known for Calabi-Yau threefolds; more generally it follows easily for threefolds with numerically trivial canonical bundle (condition which is preserved by derived equivalence, see [11] Theorem 1.4). Indeed, since for threefolds Hirzebruch-Riemann-Roch gives $\chi(\omega_X) = \frac{1}{2} c_1(X) c_2(X)$, in this case $\chi(\omega_X) = 0$, hence $h^{1,0}(X)$ can be expressed in terms of Hodge numbers that are known to be derived invariant as above. Finally, in general the invariance of $h^{1,0}$ would follow automatically if $X$ and $Y$ were birational, but derived equivalence does not necessarily imply birationality.
By the universal property of Alb(X), it induces an automorphism \( \tilde{g} \in \text{Aut}^0(\text{Alb}(X)) \), making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
\text{Alb}(X) & \xrightarrow{\tilde{g}} & \text{Alb}(X)
\end{array}
\]

commute; in other words, \( \tilde{g}(\text{Alb}(x - x_0)) = \text{Alb}(g(x - x_0)) \). Any such automorphism is translation by an element of \( \text{Alb}(X) \), and the formula shows that this element has to be \( \text{Alb}(g(x_0) - x_0) \). It follows that the map \( G_X \to \text{Alb}(X) \) is given by \( g \to \text{Alb}(g(x_0) - x_0) \). By Chevalley's theorem, it factors through \( \text{Alb}(G_X) \).

A crucial fact is the following theorem of Nishi and Matsumura (cf. also [5]).

**Theorem 2.3 ([13], Theorem 2).** The map \( \text{Alb}(G_X) \to \text{Alb}(X) \) has finite kernel. More generally, any connected algebraic group \( G \) of automorphisms of \( X \) acts on \( \text{Alb}(X) \) by translations, and the kernel of the induced homomorphism \( G \to \text{Alb}(X) \) is affine.

Consequently, the image of \( \text{Alb}(G_X) \) is an abelian subvariety of \( \text{Alb}(X) \) of dimension \( a(X) \). This implies the inequality \( a(X) \leq q(X) \). Briot observed that \( X \) can always be fibered over an abelian variety which is a quotient of \( \text{Alb}(G_X) \) of the same dimension \( a(X) \); the following proof is taken from [5], p. 2 and §3, and is included for later use of its ingredients.

**Lemma 2.4.** There is an abelian subvariety \( \text{Aff}(G_X) \subseteq H \subseteq G_X \) with \( H/\text{Aff}(G_X) \) finite, such that \( H \to G_X \) is the universal abelian variety over \( H/\text{Aff}(G_X) \).

Consequently, \( X \) admits a \( G_X \)-equivariant map \( \psi : X \to G_X/H \). Consequently, \( X \) is isomorphic to the equivariant fiber bundle \( G_X \times H \to G_X/H \) with fiber \( Z = \psi^{-1}(0) \).

**Proof.** By the Poincaré complete reducibility theorem, the map \( \text{Alb}(G_X) \to \text{Alb}(X) \) splits up to isogeny. This means that we can find a subgroup \( H \) containing \( \text{Aff}(G_X) \), such that there is a surjective map \( \text{Alb}(X) \to G_X/H \) with \( \text{Alb}(G_X) \to G_X/H \) an isogeny. It follows that \( H/\text{Aff}(G_X) \) is finite, and hence that \( H \) is an abelian subgroup of \( G_X \) whose identity component is \( \text{Aff}(G_X) \). Let \( \psi : X \to G_X/H \) be the resulting map; it is equivariant by construction. Since \( G_X \) acts transitively on \( G_X/H \), we conclude that \( \psi \) is an equivariant fiber bundle over \( G_X/H \) with fiber \( Z = \psi^{-1}(0) \), and therefore isomorphic to \( G_X \times H \), where \( H \) acts on the product by \( (g, z) \cdot h = (g \cdot h, h^{-1} \cdot z) \).

Note that the group \( H \) naturally acts on \( Z \); the proof shows that we obtain \( X \) from the principal \( H \)-bundle \( G_X \to G_X/H \) by replacing the fiber \( H \) by \( Z \) (see [18], §3.2). While \( X \to G_X/H \) is not necessarily locally trivial, it is so in the étale topology.

**Lemma 2.5.** Both \( G_X \to G_X/H \) and \( X \to G_X/H \) are étale locally trivial.

3. **Proof of the main result**

Let \( \Phi : D(X) \to D(Y) \) be an exact equivalence between the derived categories of two smooth projective varieties \( X \) and \( Y \). By Orlov's criterion, \( F \) is uniquely up to isomorphism a Fourier-Mukai functor, i.e. \( \Phi \cong \Phi_{G} \) with \( G \in D(X \times Y) \), where \( \Phi_{G} : D(X) \to D(Y) \). (Here and in what follows all functors are derived.) A result of Rouquier ([17], Théorème 4.18) (see also [9], Proposition 9.45), says that \( \Phi \) induces an isomorphism of algebraic group functors \( F : \text{Aut}^{0}(X) \times \text{Pic}^{0}(X) \cong \text{Aut}^{0}(Y) \times \text{Pic}^{0}(Y) \). In the following manner: A pair of \( \varphi \in \text{Aut}(X) \) and \( L \in \text{Pic}(X) \) defines an auto-equivalence of \( D(X) \) by the formula \( \psi_{\varphi,L}(\cdot \otimes \cdot) \); its kernel is \( (\text{id}, \varphi_{L}) \subseteq D(X \times X) \). When \( (\varphi, L) \in \text{Aut}^{0}(X) \times \text{Pic}^{0}(X) \), Rouquier proves that the composition \( \Phi_{\varphi} \circ \Phi_{G,0} \circ \Phi_{\varphi}^{-1} \) is again of the form \( \Phi_{(\varphi, L),M} \) for a unique pair \( (\varphi, M) \in \text{Aut}^{0}(Y) \times \text{Pic}^{0}(Y) \). We then have \( F(\varphi, L) = (\varphi, M) \). The following interpretation in terms of the kernel \( G \) was proved by Orlov ([15], Corollary 5.1.10) for abelian varieties; the general case is similar, and we include it for the reader's convenience.

**Lemma 3.1.** One has \( F(\psi, L) = (\psi, M) \) if and only if

\[
p_{1}(L) \otimes (\varphi \times \text{id})^{*} \delta \cong p_{2} \delta \otimes (\text{id} \times \psi)^{*} \delta.
\]

**Proof.** By construction, \( F(\psi, L) = (\psi, M) \) is equivalent to the relation

\[
\Phi_{\varphi} \circ \Phi_{(\psi, L)} \circ \Phi_{\varphi}^{-1} = \Phi_{(\psi, M),0} \circ \Phi_{\varphi}.
\]

Since both sides are equivalences, their kernels have to be isomorphic. Mukai's formula for the kernel of the composition of two integral functors (see [9], Proposition 5.10) gives

\[
p_{2} \delta \otimes (\varphi \otimes \text{id}) \otimes p_{1} \delta \cong p_{1} \delta \otimes (\text{id} \otimes \psi) \otimes p_{2} \delta.
\]

(1) Note that in the quoted references the result is stated for the semidirect product of \( \text{Pic}^{0}(X) \) and \( \text{Aut}^{0}(X) \). One can however check that the action of \( \text{Aut}^{0}(X) \) on \( \text{Pic}^{0}(X) \) is trivial. Indeed, \( \text{Aut}^{0}(X) \) acts on \( \text{Pic}^{0}(X) \) by elements of \( \text{Aut}^{0}(\text{Pic}^{0}(X)) \), which are translations. Since the origin is fixed, these must be trivial. This shows in particular that \( \text{Aut}^{0}(X) \) and \( \text{Pic}^{0}(X) \) commute as subgroups of \( \text{Aut}(D(X)) \).

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To compute the left-hand side of (3.2), let $\lambda: X \times Y \to X \times X \times Y$ be given by $\lambda(x, y) = (x, \psi(x), y)$, making the following diagram commutative:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\lambda} & X \times X \times Y \\
\downarrow{p_{12}} & & \downarrow{p_{12}} \\
X & & X \times X.
\end{array}
$$

By the base-change formula, $p_{12}^*(\text{id} \times \psi) \circ \lambda = \lambda_2 \circ p_{22}$ using the projection formula and the identities $p_{12} \circ \lambda = \text{id}$ and $p_{22} \circ \lambda = \varphi \circ \text{id}$, we then have

$$p_{12}^*(\text{id} \times \psi) \circ \lambda_2 \circ p_{22} = p_{12}^* \circ \lambda_2 \circ p_{22} = p_{12}^* \circ \lambda \circ p_{22} \circ \lambda_2 = (\varphi \times \text{id})^* \delta.
$$

To compute the right-hand side of (3.2), we similarly define $\mu: X \times Y \to X \times X \times Y$ by the formula $\mu(x, y) = (x, y, \psi(y))$, to fit into the diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\mu} & X \times X \times Y \\
\downarrow{p_{12}} & & \downarrow{p_{12}} \\
X & & X \times X.
\end{array}
$$

Since $p_{12} \circ \mu = (\text{id} \times \psi)$ and $p_{12} \circ \lambda = \text{id}$, the same calculation as above shows that

$$p_{12}^*(\text{id} \times \psi) \circ \mu = (\text{id} \times \psi) \circ (\delta \circ p_{22}^* M) = (\text{id} \times \psi) \circ (\delta \circ p_{22}^* M) \circ (\text{id} \times \psi) \circ (\delta \circ p_{22}^* M),$$

where the last step uses that the action of $\text{Aut}_c(Y)$ on $\text{Pic}_c^0(Y)$ is trivial, so $(\text{id} \times \psi)^* \delta \circ p_{22}^* M = (\text{id} \times \psi)^* \delta \circ p_{22}^* M.$

We now give the proof of Theorem A. It is in fact more convenient to start directly with the numerical Corollary B. Note that Rouquier's result (or the invariance of the first Hochschild cohomology) implies the derived invariance of the quantity $h^0(X, \Omega^1_X) + h^0(X, TX)$. Hence it suffices to show that $q(X) = q(Y)$, where we set $q(X) = h^0(X, \Omega^1_X)$, and similarly for $Y$.

We continue to write $G_X = \text{Aut}_c^0(X)$ and $G_Y = \text{Aut}_c^0(Y)$. Let $\delta$ be the kernel defining the equivalence, and let $F: G_X \times \text{Pic}_c^0(X) \to G_Y \times \text{Pic}_c^0(Y)$ be the isomorphism of algebraic groups from Rouquier's theorem, as above. To prove the assertion, we consider the map

$$\beta: \text{Pic}_c^0(X) \to G_Y, \quad \beta(L) = p_1(F(L, L)),
$$

and let $B = \text{Im} \beta$. Similarly, we define

$$\alpha: \text{Pic}_c^0(Y) \to G_X, \quad \alpha(M) = p_1(F^{-1}(\text{id}, M)),
$$

and let $A = \text{Im} \alpha$. One easily verifies that $F$ induces an isomorphism

$$F: A \times \text{Pic}_c^0(X) \to B \times \text{Pic}_c^0(Y).$$

If both $A$ and $B$ are trivial, we immediately obtain $\text{Pic}_c^0(X) \simeq \text{Pic}_c^0(Y)$. Excluding this case from now on, we let the abelian variety $A \times B$ act on $X \times Y$ by automorphisms. Take a point $(x, y)$ in the support of the kernel $\delta$, and consider the orbit map

$$f: A \times B \to X \times Y, \quad (\varphi, \psi) \to (\varphi(x), \psi(y)).$$

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To conclude the proof of part (1), note that we have extensions
\[ 0 \to \ker \beta \to \text{Pic}^0(X) \to B \to 0 \quad \text{and} \quad 0 \to \ker \alpha \to \text{Pic}^0(Y) \to A \to 0. \]
By definition, $\ker \beta$ consists of those $L \in \text{Pic}^0(X)$ for which $P(L, id) = (id, M)$; obviously, this implies an isomorphism $\ker \beta \cong \ker \alpha$, and therefore $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ are isogenous. Now by Rouquier’s isomorphism (3.1) and the uniqueness of $\text{Aff}(G)$ in Chevalley’s theorem we have $\text{Aff}(G_X) \cong \text{Aff}(G_Y)$ and
\[ \text{Alb}(G_X) \times \text{Pic}^0(X) \cong \text{Alb}(G_Y) \times \text{Pic}^0(Y). \]
Therefore we also have equivalently that $\text{Alb}(G_X)$ and $\text{Alb}(G_Y)$ are isogenous.

It remains to check part (2). Clearly $a(X) = a(Y)$. If $a(X) = 0$, we obviously have $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$. On the other hand, if $a(X) > 0$, Lemmas 2.4 and 2.5 show that $X$ can be written as an étale locally trivial fiber bundle over a quotient of $\text{Alb}(G_X)$ by a finite subgroup, so an abelian variety isogenous to $\text{Alb}(G_X)$ and $\text{Alb}(G_Y)$ is the same holds for $Y$ by symmetry. Note that in this case we have $\chi(U_X) = \chi(U_Y) = 0$ by Corollary 2.6.

Remark 3.2. Results of Mukai [14], §5 and §6, imply that each $H^4(T)$ on $X \times B$ in the proof above has a filtration with simple semi-homogeneous quotients, all of the same slope, associated to a subvariety $\text{Im} \gamma$. In line with Orlov’s work on derived equivalences of abelian varieties [15][8], one may guess that these simple bundles induce derived equivalences between $A$ and $B$, and that $\text{Im} \gamma$ induces an isomorphism between $A \times B \times \mathbb{B}$. But we have not been able to prove this.

Remark 3.3 (Further numerical applications). In the case of fourfolds, in addition to the Hodge numbers that are equal due to the general invariance of Hochschild homology (namely $h^{0,0}$ and $h^{1,1}$), Corollary B implies:

**Corollary 3.4.** Let $X$ and $Y$ be smooth projective fourfolds with $D(X) \cong D(Y)$. Then $h^{0,0}(X) = h^{1,1}(Y)$. If in addition $\text{Aut}^0(X)$ is not affine, then $h^{0,0}(X) = h^{0,0}(Y)$ and $h^{1,1}(X) = h^{1,1}(Y)$.

**Proof.** The analogue of (1.1) for fourfolds implies that $h^{3,1}$ is invariant if and only if $h^{0,3}$ is invariant and $h^{2,0}$ is invariant if and only if $h^{0,3}$ is invariant. On the other hand, if $\text{Aut}^0(X)$ is not affine, then $\chi(U_X) = 0$ (cf. Lemma 2.6), which implies that $h^{0,0}$ is invariant if and only if $h^{0,3}$ is invariant. We apply Corollary B.

It is also worth noting that Corollary B can help in verifying the invariance of classification properties characterized numerically. We exemplify with a quick proof of the following statement ([10] Proposition 3.1): If $D(X) \cong D(Y)$, and $X$ is an abelian variety, then so is $Y$. Indeed, the derived invariance of the pluricanonical series [15] Corollary 2.1.9 and Theorem A imply that $P(Y) = P(Y) = 1$ and $q(Y) = q(Y) = \dim Y$. The main result of [6] implies that $Y$ is birational, so it actually has a birational morphism, to an abelian variety $B$. But $\omega_X = O_X$, so $\omega_Y = O_Y$ as well (see e.g. [9] Proposition 4.1), and therefore $Y \cong B$.

**REFERENCES**


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