CANONICAL EXTENSIONS OF LOCAL SYSTEMS

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Abstract. A local system, defined on the complement of a divisor $Z \subseteq X$ in a complex manifold, can in general not be extended to all of X because of local monodromy. This paper describes the construction of a normal analytic space that naturally extends the étale space T of the local system, in the case when Z is a divisor with normal crossings and the local system has unipotent local monodromy.

Introduction

The extension problem. In many geometric situations, one has a local system that is defined on the complement of an analytic subset Z in a complex manifold X. A typical example is given by a proper holomorphic map $f\colon Y \to X$ between complex manifolds; outside the analytic subset Z of points where f is not submersive, each sheaf $R^i f_* \mathbb{Z}$ is a local system. The behavior of the local system near Z, which in the example is related to singularities on the fibers of f, is then an important question.

When Z is a divisor (all other cases being trivial), the local system can usually not be extended to all of X, due to the presence of local monodromy. The formalism of perverse sheaves [1] gives a way to extend the local system in an algebraic way; but our interest in this paper is in finding a geometric extension. To achieve this, we focus not on the sheaf $\mathcal{H}$, but on its étale space T, which is a covering space (infinite-sheeted, with countably many connected components) of $U = X - Z$, whose sheaf of sections is the given local system $\mathcal{H}$. What we are looking for is an analytic space $\operatorname{Can}(T)$ over X, with certain properties, that extends the covering space T in a natural way.

A hint as to what can be expected is given by the following example.

Example. Consider the case of a local system on the punctured disk $\Delta^* = \Delta - \{0\}$. Here each connected component of the étale space T is either isomorphic to the upper half plane $\mathbb{H}$ (and of infinite degree over Δ^*), or to a copy of the punctured disk (and of finite degree over Δ^*). There is only one natural way of extending T, namely by filling in the holes, and making the resulting disks into finite branched coverings of Δ. Each point that is added corresponds, in this way, to an element that has finite order under the monodromy transformation.

Since the correct choice of extension is clear in the case of a punctured disk, we shall model the general construction on this particular case, by requiring the following: Given any holomorphic map $f\colon \Delta \to X$ such that $f(\Delta^*) \subseteq U$, we can pull back the local system to Δ^*, and the product $\operatorname{Can}(T) \times_X \Delta$ should give the natural

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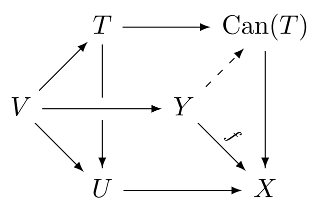
extension of the pullback to Δ. To make this into a good notion, $\mathrm{Can}(T)$ should also be a normal analytic space, and the map $\mathrm{Can}(T) \to X$ should be relatively Stein. These conditions are enough to make $\mathrm{Can}(T)$ functorial for arbitrary maps from normal analytic spaces to X (see Lemma 2 below).

In this paper, we prove the following result concerning the existence of such an extension space.

Theorem 1. *Let X be a complex manifold, $Z \subseteq X$ an analytic divisor with normal crossings, and $U = X - Z$. Let $\mathcal{H}$ be a local system of finitely generated, free $\mathbb{Z}$-modules on U, with associated étale space $T \to U$. If the local monodromy of $\mathcal{H}$ near points of Z is unipotent, then there is a canonically defined extension space* $\mathrm{Can}(T)$ *with the following three properties:*

(A) $\mathrm{Can}(T)$ *is a normal Stein space, containing T as a dense open subset.*
(B) There is a holomorphic map $\mathrm{Can}(T) \to X$ *extending the map $T \to U$.*
(C) Given any map $f\colon Y \to X$ from a normal analytic space such that $V = f^{-1}(U)$ is dense in Y, every section of T over V that is compatible with f extends uniquely to a section of $\mathrm{Can}(T)$ *over Y.*



The condition in (C) is illustrated in the diagram above; by definition, a *section* of $\mathrm{Can}(T)$ over a space Y is simply a holomorphic map $Y \to \mathrm{Can}(T)$ that is compatible with the given map from Y to X.

Discussion. The condition on the local monodromy means that, for any holomorphic map $f\colon \Delta \to X$ with $f(\Delta^*) \subseteq U$, the generator of the fundamental group of Δ^* should act on the fiber of $\mathcal{H}$ by a matrix whose eigenvalues are all equal to 1. A result, due to A. Borel (see [12, Lemma 4.5 on p. 230] for details), shows that the local monodromy of any geometrically defined local system, or in fact of any local system underlying a polarized variation of mixed Hodge structure, is at least quasi-unipotent. Our theorem therefore applies at least after passing to a finite cover and resolving singularities. Most likely, the result could be generalized to allow Z to be an arbitrary divisor, and the local system to have quasi-unipotent monodromy, but some work remains to be done.

In analogy with the case of a punctured disk, points of $\mathrm{Can}(T)$ over Z should be thought of as elements in the fiber of $\mathcal{H}$ that are invariant under single local monodromy transformation. Because of this, and due to the requirement that $\mathrm{Can}(T)$ give the correct extension on any disk contained in X, the fibers of the holomorphic map $\mathrm{Can}(T) \to X$ can be very large; in fact, of dimension up to $\dim X - 1$. This can be seen very clearly in Proposition 5. For the same reason, they are typically no longer abelian groups. However, as a consequence of the condition in (C), the sheaf of sections of $\mathrm{Can}(T) \to X$ is exactly $j_* \mathcal{H}$, where $j\colon U \to X$ is the inclusion map.

In the situation described in the theorem, we shall see in Section 2 that $\mathrm{Can}(T)$ has toric singularities. One can restate this fact by saying that $\mathrm{Can}(T)$, together with the natural log structure given by the embedding $T \subseteq \mathrm{Can}(T)$, is log regular.

This circumstance, which suggests a connection with logarithmic geometry [11], was pointed out by J. Stix, and will be taken up elsewhere.

Outline of the paper. Because of the functoriality of $\mathrm{Can}(T)$ that is required by condition (C), it is clear that the problem of constructing $\mathrm{Can}(T)$ is a purely local one. In Section 1, we therefore consider the case when X is a polydisk, and investigate basic properties of $\mathrm{Can}(T)$ in a more general setting. In Section 2, we prove the existence of an extension space when $Z \subseteq X$ is a divisor with normal crossing singularities, and $\mathcal{H}$ has unipotent monodromy. In that case, $\mathrm{Can}(T)$ is obtained by taking the closure of T inside Deligne's canonical extension of the corresponding flat vector bundle [6]. The construction has the nice consequence that the singularities of $\mathrm{Can}(T)$ are those of a toric variety.

1. The general local setting

A more general version of the local problem is the following. Let $X \subseteq \mathbb{C}^n$ be a polydisk, let $Z \subseteq X$ be an analytic hypersurface, and $U = X - Z$ its complement. Suppose that we have a covering space $T \to U$, possibly infinite-sheeted and not necessarily connected. We would like to construct, in a natural way, an analytic space $\mathrm{Can}(T)$ extending T that has the three properties in Theorem 1. A first observation is that it suffices to require the condition in (C) for $Y = \Delta$ only.

Lemma 2. *Suppose that an extension space* $\mathrm{Can}(T)$ *exists, subject to the conditions (A), (B), and (C) for* $Y = \Delta$*. Then it automatically satisfies (C) for arbitrary normal spaces* Y*.*

Proof. Let $f\colon Y \to X$ be the given holomorphic map, and $s\colon V \to T$ a compatible section. To show that s extends holomorphically to a map from Y into $\mathrm{Can}(T)$, it suffices to show that s is locally bounded near points of $f^{-1}(Z)$, because Y is normal. But boundedness can be verified by composing s with suitable holomorphic maps $\Delta \to Y$, and therefore follows from condition (C) for the space Δ. □

It is also possible to describe the algebra $\mathscr{O}\big(\mathrm{Can}(T)\big)$ of holomorphic functions on $\mathrm{Can}(T)$, as a closed subalgebra of $\mathscr{O}(T)$. By normality, a holomorphic function $f \in \mathscr{O}(T)$ extends to $\mathrm{Can}(T)$ if and only if it is bounded. This, in turn, can be verified by restricting f to arbitrary holomorphic disks $\Delta \to \mathrm{Can}(T)$ such that $\Delta^* \to T$. Because of (C), each such disk corresponds to a map $\Delta \to X$, together with a compatible section $s\colon \Delta^* \to T$, and the condition is that $s^*(f) \in \mathscr{O}(\Delta)$, where $s^*\colon \mathscr{O}(T) \to \mathscr{O}(\Delta^*)$ is the pullback map. In other words, we have

$$(1)\quad \mathscr{O}\big(\mathrm{Can}(T)\big) = \left\{ f \in \mathscr{O}(T) \;\middle|\; \begin{array}{l} \text{for every map } \Delta \to X, \text{ and every compatible} \\ \text{section } s\colon \Delta^* \to T, \text{ one has } s^*(f) \in \mathscr{O}(\Delta) \end{array} \right\}$$

The right-had side defines a closed subalgebra of $\mathscr{O}(T)$ for the compact-open topology, because $\mathscr{O}(\Delta) \subseteq \mathscr{O}(\Delta^*)$ is a closed subalgebra and each s^* is continuous.

By work of O. Forster, a Stein space is completely determined by its algebra of holomorphic functions [7, Satz 1 on p. 378]. If one knew that the algebra on the right-hand side of (1) was a Stein algebra, it would thus be possible to construct the space $\mathrm{Can}(T)$ in this general setting, using Forster's result. Unfortunately, deciding whether a given topological algebra is Stein appears to be a difficult problem, and we do not pursue this direction any further.

One case where the existence of $\mathrm{Can}(T)$ is known, is when the covering space $T \to U$ has finite degree.

Lemma 3. *If $T \to U$ is a finite unbranched covering, then* $\operatorname{Can}(T)$ *exists.*

Proof. This follows immediately from the “Fortsetzungssatz” for finite branched coverings, due to H. Grauert and R. Remmert [9, Satz 8 on p. 261], and K. Stein [13, Satz 1 on p. 67]. Indeed, their theorem is that $T \to U$ can be extended in a unique way to a finite branched covering $T' \to X$; in particular, T' is a normal Stein space [9, p. 260]. Because the map $T' \to X$ is proper, condition (C) is automatic; we thus have $\operatorname{Can}(T) = T'$. □

2. Local systems with unipotent monodromy on the complement of a normal crossing divisor

In this section, we prove that the extension space $\operatorname{Can}(T)$ exists when T is the total space of a local system $\mathcal{H}$ with unipotent monodromy. More precisely, we consider the following situation:

(i) The divisor $Z \subseteq X$ has at worst normal crossing singularities;
(ii) $\mathcal{H}$ is a local system of finitely generated, free $\mathbb{Z}$-modules on U, and $T \to U$ is the associated étale space; and
(iii) $\mathcal{H}$ has unipotent monodromy.

In this setting, the construction of $\operatorname{Can}(T)$ is straightforward. Namely, let $E \to U$ be the holomorphic vector bundle determined by the local system, with sheaf of sections $\mathscr{E} = \mathscr{O}_U \otimes_{\mathbb{Z}} \mathcal{H}$. Under the assumptions on U and $\mathcal{H}$, it was shown by P. Deligne [6] that E can be canonically extended to a vector bundle $\widetilde{E} \to X$. The space T is naturally embedded in E, and we prove the (nontrivial) fact that its closure in $\widetilde{E}$ is an analytic subset. We can then define the extension space $\operatorname{Can}(T)$ as the normalization of the closure.

After briefly reviewing Deligne's canonical extension, we first determine the closure of T in $\widetilde{E}$ as a set. We then use this description to obtain finitely many holomorphic equations that define the closure inside of $\widetilde{E}$, and argue that the normalization of the closure has all the properties required of $\operatorname{Can}(T)$. Finally, we show that, in this special setting, $\operatorname{Can}(T)$ always has toric singularities.

Deligne's canonical extension of a flat bundle. Let $E \to U$ be a holomorphic vector bundle, with sheaf of sections $\mathscr{E}$. A connection $\nabla \colon \mathscr{E} \to \Omega^1_U \otimes \mathscr{E}$ is called *flat* if its curvature tensor is zero, meaning that $\nabla \circ \nabla = 0$. The connection is said to have *unipotent monodromy* if the local system $\ker \nabla \subseteq \mathscr{E}$ of $\mathbb{C}$-vector spaces has unipotent monodromy. P. Deligne [6, pp. 91–5] proved the following basic result.

Theorem 4 (Deligne)**.** *Let X be a polydisk, $Z \subseteq X$ a divisor with normal crossing singularities, and $U = X - Z$. Let $E \to U$ be a holomorphic vector bundle with a flat connection ∇ that has unipotent monodromy. Then there is a unique extension of E to a holomorphic vector bundle $\widetilde{E} \to X$, such that the connection ∇ has logarithmic poles with nilpotent residues.*

In other words, letting $\widetilde{\mathscr{E}}$ be the sheaf of sections of the vector bundle $\widetilde{E}$, the connection should extend to a map $\nabla \colon \widetilde{\mathscr{E}} \to \Omega^1_X(\log Z) \otimes \widetilde{\mathscr{E}}$, where $\Omega^1_X(\log Z)$ is the sheaf of differentials with logarithmic poles along Z.

For our purposes, the following explicit description of $\widetilde{E}$ will be useful. Let $t_1, \dots, t_n$ be local coordinates on the polydisk X, and suppose that Z is defined by the equation $t_1 \cdots t_r = 0$. Evidently, it suffices to treat the case when $r = n$; we may thus assume that $X \simeq \Delta^n$, and $U \simeq (\Delta^*)^n$. Let d be the rank of the bundle E. The fundamental group of $(\Delta^*)^n$ is isomorphic to $\mathbb{Z}^n$; it acts on the fiber $\mathbb{C}^d$ of the bundle by parallel translation, and we let T_j be the operator corresponding to the j-th standard generator of $\mathbb{Z}^n$. By assumption, each T_j is a unipotent matrix, and we can define nilpotent matrices

$$N_j = -\log T_j = \sum_{n=1}^{\infty} \frac{1}{n} (\operatorname{id} - T_j)^n$$

by taking logarithms (the minus sign is to keep the conventions of [4]).

We describe the vector bundle $\widetilde{E}$ by giving a collection of sections over U that generate it. To obtain the sections in question, note that the pullback of the flat bundle E to the universal covering space

$$p\colon \mathbb{H}^n \to (\Delta^*)^n, \qquad p(z_1, \dots, z_n) = \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}\right),$$

is trivial. The fundamental group $\mathbb{Z}^n$ acts on $\mathbb{H}^n$ by the rule

$$(a_1, \dots, a_n) \cdot (z_1, \dots, z_n) = (z_1 - a_1, \dots, z_n - a_n),$$

and so sections of E over U correspond to holomorphic maps $\tilde{s}\colon \mathbb{H}^n \to \mathbb{C}^d$ with the property that $\tilde{s}(z - e_j) = T_j \tilde{s}(z)$ for all $z \in \mathbb{H}^n$ and all $j = 1, \dots, n$.

For an arbitrary vector $v \in \mathbb{C}^d$, the holomorphic map

$$\tilde{s}\colon \mathbb{H}^n \to \mathbb{C}^d, \qquad \tilde{s}(z) = e^{\sum z_j N_j} v, \tag{2}$$

has the required invariance property, because $\tilde{s}(z - e_j) = e^{-N_j} \tilde{s}(z) = T_j \tilde{s}(z)$; it thus defines a holomorphic section $s \in \Gamma(U, \mathscr{E})$. Then $\widetilde{E}$ is the vector bundle generated by all such sections. If we let s_i be the section corresponding to the i-th standard basis vector of $\mathbb{C}^d$, we have $\widetilde{E} \simeq X \times \mathbb{C}^d$, with the trivialization given by $s_1, \dots, s_d$.

Although it is not relevant for our purposes, it should be pointed out that the connection ∇ indeed has logarithmic poles with nilpotent residues in that special frame. This is easy to see from (2): on $\mathbb{H}^n$, we have

$$\nabla \tilde{s} = \sum_{j=1}^{n} dz_j \otimes e^{\sum z_j N_j} N_j v,$$

and since $2\pi i \cdot dz_j = dt_j / t_j$, we conclude that

$$\nabla s = \frac{1}{2\pi i} \cdot \sum_{j=1}^{n} \frac{dt_j}{t_j} \otimes N_j s.$$

The poles in this expression are logarithmic, and for each $j = 1, \dots, n$, the residue of the connection along the divisor $t_j = 0$ is the nilpotent matrix N_j.

Set-theoretic description of the closure. We now assume that E is the flat vector bundle associated to the local system $\mathcal{H}$. The space T is then naturally embedded into the canonical extension $\widetilde{E}$, in the following manner. Letting $t_1, \dots, t_n$ be holomorphic coordinates on X, we may again assume that Z is defined by the equation $t_1 \dots t_n = 0$, so that $U \simeq (\Delta^*)^n$. The pullback of the local system to $\mathbb{H}^n$ is trivial, with fiber $\mathbb{Z}^d$, and the description of the canonical extension shows that $T \subseteq \widetilde{E}$ is the image of the holomorphic map

$$f\colon \mathbb{H}^n \times \mathbb{Z}^d \to X \times \mathbb{C}^d,$$

given by the rule

$$(z_1, \dots, z_n, h) \mapsto \bigl(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \dots + z_n N_n)} h\bigr). \tag{3}$$

As it stands, the map f is not one-to-one; when the real parts $x_j = \operatorname{Re} z_j$ are restricted to $0 \le x_1, \dots, x_n < 1$, however, every point in the image is parametrized exactly once.

The first step in the construction of $\operatorname{Can}(T)$ is to determine the closure of the image of f; here is the result.

Proposition 5. *A point in $\Delta^n \times \mathbb{C}^d$ over $(0, \dots, 0) \in \Delta^n$ is in the closure of the image of f if, and only if, it is of the form*

$$\bigl(0, \dots, 0, e^{-(w_1 N_1 + \dots + w_n N_n)} h\bigr);$$

here $h \in \mathbb{Z}^d$ is such that $a_1 N_1 h + \dots + a_n N_n h = 0$ for some choice of positive integers $a_1, \dots, a_n$, while $w_1, \dots, w_n \in \mathbb{C}$ can be arbitrary complex numbers.

As written, the proposition only describes those points in the closure that lie over the origin in Δ^n; this suffices, because we can always move the origin of the coordinate system.

Proof. One half of the proposition is easy to prove: if $h \in \mathbb{Z}^d$ satisfies $a_1 N_1 h + \dots + a_n N_n h = 0$ for positive integers $a_1, \dots, a_n$, then every point of the form

$$\bigl(0, \dots, 0, e^{-(w_1 N_1 + \dots + w_n N_n)} h\bigr)$$

is in the closure of the image of f. Indeed, taking the imaginary part of $z \in \mathbb{H}$ sufficiently large to have $\operatorname{Im}(a_j z + w_j) > 0$ for all j, we get

$$\begin{aligned} f(a_1 z + w_1, \dots, a_n z + w_n, h) &= \bigl(e^{2\pi i a_1 z} e^{2\pi i w_1}, \dots, e^{2\pi i a_n z} e^{2\pi i w_n}, e^{-\sum (a_j z + w_j) N_j} h\bigr) \\ &= \bigl(t^{a_1} e^{2\pi i w_1}, \dots, t^{a_n} e^{2\pi i w_n}, e^{-\sum w_j N_j} e^{-z \sum a_j N_j} h\bigr) \\ &= \bigl(t^{a_1} e^{2\pi i w_1}, \dots, t^{a_n} e^{2\pi i w_n}, e^{-\sum w_j N_j} h\bigr), \end{aligned}$$

having set $t = \exp(2\pi i z)$. As $t \to 0$, these points in the image of f approach the point $\bigl(0, \dots, 0, e^{-(w_1 N_1 + \dots + w_n N_n)} h\bigr)$, which is consequently in the closure.

To prove the remaining half, we take a sequence of points

$$\bigl(z(m), h(m)\bigr) = \bigl(z_1(m), \dots, z_n(m), h(m)\bigr) \in \mathbb{H}^n \times \mathbb{Z}^d$$

such that $f(z(m), h(m))$ converges to a point over $(0, \dots, 0)$. This means that each $y_j(m) = \operatorname{Im} z_j(m)$ is tending to infinity, and that the sequence of vectors

$$e^{-\sum z_j(m) N_j} h(m) \in \mathbb{C}^d$$

is convergent, as $m \to \infty$. We shall prove, in several steps, that the limit is of the required form. In the course of the argument, we shall frequently have to pass to a subsequence of the original sequence; to avoid clutter, this will not be indicated in the notation.

Step 1. To begin with, we may adjust the values of $h(m)$, if necessary, and assume that the real parts $x_j(m) = \operatorname{Re} z_j(m)$ satisfy $0 \le x_j(m) \le 1$. We can then pass to a subsequence along which all $x_j(m)$ converge. The vectors

$$e^{\sum x_j(m) N_j} e^{-\sum z_j(m) N_j} h(m) = e^{-i \sum y_j(m) N_j} h(m)$$

still form a convergent sequence in this case, and so the $x_j(m)$ really play no role for the remainder of the argument.

Step 2. While all imaginary parts $y_j(m)$ are going to infinity, this may happen at greatly different rates. To make their behavior more tractable, we use the following technique, borrowed from the paper by E. Cattani, P. Deligne, and A. Kaplan [4, p. 494]. Let $y(m) = \big(y_1(m), \dots, y_n(m)\big)$. After taking a further subsequence, we can find constant vectors $\theta^1, \dots, \theta^r \in \mathbb{R}^n$ with nonnegative components, such that

$$y(m) = \tau_1(m)\theta^1 + \cdots + \tau_r(m)\theta^r + \eta(m),$$

where the remainder term $\eta(m)$ is convergent, and the ratios

$$\frac{\tau_1(m)}{\tau_2(m)}, \frac{\tau_2(m)}{\tau_3(m)}, \dots, \frac{\tau_{r-1}(m)}{\tau_r(m)}, \frac{\tau_r(m)}{1} \tag{4}$$

are all tending to infinity. Moreover, we may assume that

$$0 \leq \theta_j^1 \leq \theta_j^2 \leq \cdots \leq \theta_j^r$$

for all j; because $y_j(m) \to \infty$, all components of the last vector θ^r then have to be positive real numbers. Now define

$$N(m) = \sum_{j=1}^n \big(y_j(m) - \eta_j(m)\big) N_j.$$

As in Step 1, the convergence of $e^{-i\sum \eta_j(m)N_j}$ makes the vectors $\eta_j(m)$ irrelevant to the rest of the argument—the sequence with terms $e^{-iN(m)}h(m)$ is still convergent.

Step 3. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set

$$N^\alpha = \prod_{j=1}^n N_j^{\alpha_j}.$$

Since the N_j are commuting nilpotent operators, $N^\alpha = 0$ whenever $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is sufficiently large. We can thus let $p \geq 0$ be the smallest integer for which there is a subsequence of $\big(z(m), h(m)\big)$ with

$$N^\alpha h(m) = 0 \text{ for all multi-indices } \alpha \text{ with } |\alpha| \geq p+1.$$

Passing to this subsequence, we find that when $|\alpha| = p$, the sequence

$$N^\alpha e^{-i\sum y_j(m)N_j} h(m) = N^\alpha h(m)$$

is convergent. However, it takes its values in a discrete set (in fact, there is an integer $M > 0$ such that each coordinate of $N^\alpha h(m)$ is in $\mathbb{Z}[1/M]$, and M depends only on α and the N_j), and so it has to be eventually constant. If we remove finitely many terms from the sequence, we can therefore achieve that

$$h^\alpha = N^\alpha h(m)$$

is constant whenever $|\alpha| = p$. Moreover, we have $N(m)h^\alpha = 0$ by the choice of p.

Step 4. At this point, we can use an inductive argument to get the same conclusion for all multi-indices α with $|\alpha| \leq p$. Thus let us assume that we already have a subsequence $\big(z(m), h(m)\big)$ for which $h^\alpha = N^\alpha h(m)$ is constant and $N(m) h^\alpha = 0$, whenever α is a multi-index with $p' \leq |\alpha| \leq p$. If $p' > 0$, we now show how to get the same statement with p' replaced by $p' - 1$.

Consider a multi-index α with $|\alpha| = p' - 1$. Then

$$\begin{aligned} N^\alpha e^{-iN(m)} h(m) = N^\alpha h(m) - iN(m) N^\alpha h(m) \\ + \sum_{s=1}^{p-p'} (-i)^{s+1} N(m)^s \cdot N(m) N^\alpha h(m) \end{aligned} \tag{5}$$

is again convergent. Since $\alpha + e_j$ has length p', we find that

$$N(m) N^\alpha h(m) = \sum_{j=1}^n \big(y_j(m) - \eta_j(m)\big) N^{\alpha + e_j} h(m) = \sum_{j=1}^n \big(y_j(m) - \eta_j(m)\big) h^{\alpha + e_j};$$

by the inductive hypothesis, the last term in (5) is therefore zero.

Thus the sequence $N^\alpha h(m) - iN(m) N^\alpha h(m)$ is itself convergent, implying convergence of its real and imaginary parts separately. As before, the sequence of real parts $N^\alpha h(m)$ has to be eventually constant, and after omitting finitely many terms, we can assume that it is constant. Let $h^\alpha = N^\alpha h(m)$ be that constant value. Then the convergence of the imaginary part

$$N(m) h^\alpha = N(m) N^\alpha h(m) = \sum_{i=1}^r \tau_i(m) \sum_{j=1}^n \theta_j^i N^{\alpha + e_j} h(m) = \sum_{i=1}^r \tau_i(m) \sum_{j=1}^n \theta_j^i h^{\alpha + e_j},$$

together with the behavior of the $\tau_i(m)$ described in (4), shows that $N(m) h^\alpha = 0$. The statement is thus proved for all multi-indices α of length $|\alpha| = p' - 1$ as well.

Step 5. From Step 4, we conclude that, on a suitable subsequence, $h^\alpha = N^\alpha h(m)$ is constant for all α, and satisfies $N(m) h^\alpha = 0$. In particular, $h(m)$ is itself constant, equal to a certain element $h = h^{(0,\dots,0)} \in \mathbb{Z}^d$, and we have $N(m) h = 0$ for all m.

On the one hand, we now find that, along the subsequence we have chosen in the previous steps, the original terms simplify to

$$e^{-\sum z_j(m) N_j} h(m) = e^{-\sum (x_j(m) + i\eta_j(m)) N_j} e^{-iN(m)} h = e^{-\sum (x_j(m) + i\eta_j(m)) N_j} h.$$

If we set $w_j = \lim_{m \to \infty} \big(x_j(m) + i\eta_j(m)\big)$, then the limit of the sequence is of the form $e^{-\sum w_j N_j} h$, which was part of the assertion in Proposition 5. On the other hand, we conclude from

$$N(m) h = \sum_{i=1}^r \tau_i(m) \sum_{j=1}^n \theta_j^i N_j h = 0$$

that the vectors $N_j h$ satisfy, for each $i = 1, \dots, r$, the linear relation

$$\sum_{j=1}^n \theta_j^i N_j h = 0. \tag{6}$$

Step 6. By Step 5, we know that the n vectors $N_j h$ are linearly dependent; the coefficients θ_j^r in the relation (6) for $i = r$ are positive real numbers. But as the vectors themselves are in fact in $\mathbb{Q}^d$, we can also find a relation with positive rational coefficients. Taking a suitable multiple, we then obtain positive integers $a_1, \dots, a_n$ satisfying

$$\sum_{j=1}^{n} a_j N_j h = 0.$$

The remaining assertion of the proposition is thereby established. □

For later use, we record one consequence of the proof in the following proposition.

Proposition 6. *Let $\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times \mathbb{Z}^d$ be a sequence of points with $x_j(m) = \operatorname{Re} z_j(m) \in [0,1]$, and assume that $f\bigl(z(m), h(m)\bigr)$ converges to a point in $\Delta^n \times \mathbb{C}^d$ over $(0, \dots, 0) \in \Delta^n$. Then there is a subsequence, still denoted $\bigl(z(m), h(m)\bigr)$, for which $h(m)$ is constant.*

Analytic equations for the closure. We shall now give explicit holomorphic equations for the closure of T inside of $\widetilde{E}$, thus proving that it is an analytic subset.

In general, T will have countably many connected components. Each component is the image of a map $f(\_\,, h) \colon \mathbb{H}^n \to \Delta^n \times \mathbb{C}^d$, for some $h \in \mathbb{Z}^d$. Letting $C(h)$ denote that image, we have

$$T = \bigcup_{h \in H} C(h).$$

We observe that, as a matter of fact, the closure of T satisfies

$$\overline{T} = \overline{\bigcup_{h \in H} C(h)} = \bigcup_{h \in H} \overline{C(h)}.$$

This is because of the description of the closure given in Proposition 5: any point in the closure is already in the closure of one of the components $C(h)$. As would be expected if the closure is an analytic space, only finitely many of the $\overline{C(h)}$ can come together at any given point. This is the content of the following lemma.

Lemma 7. *At most finitely many distinct $\overline{C(h)}$ can meet at any given point in $\Delta^n \times \mathbb{C}^d$.*

Proof. Suppose, to the contrary, that infinitely many distinct $\overline{C(h)}$ met at a certain point P of the closure; then P is in the closure of infinitely many distinct sheets $C(h)$. Moving the center of the coordinate system, if necessary, we may assume that P lies over $(0, \dots, 0) \in \Delta^n$. We can then find a sequence of points $\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times \mathbb{Z}^d$, with $0 \leq \operatorname{Re} z_j(m) \leq 1$ for all $j = 1, \dots, n$, such that $f\bigl(z(m), h(m)\bigr)$ converges to P, but all $h(m)$ are distinct. But such a sequence cannot exist by Proposition 6. This contradiction proves that the number of components meeting at P is indeed finite. □

We are now ready to give finitely many holomorphic equations in $\Delta^n \times \mathbb{C}^d$ that define the closed subset $\overline{C(h)}$. As before, we break the argument down into several steps.

Step 1. Let $S(h) \subseteq \mathbb{Z}^n$ be the subgroup of elements that leave h invariant. As a subgroup of a free group, $S(h)$ is itself free, say of rank $n-k$. If $k=n$, then h is not invariant under any element in $\mathbb{Z}^n$, and so $C(h)$ has to be already closed by Proposition 5. Since no points are added in the closure, there is nothing to prove in this case. We shall assume from now on that $k<n$; then the closure $\overline{C(h)}$ is potentially bigger than the original sheet $C(h)$.

Step 2. The quotient $\mathbb{Z}^n/S(h)$ is a free abelian group. Indeed, $\mathbb{Z}^n$ acts on the fiber $\mathbb{Z}^d$ of the local system by unipotent matrices, and so we have $T_1^{a_1}\cdots T_n^{a_n}h = h$ if and only if $a_1N_1h + \cdots + a_nN_nh = 0$. This means that $S(h)$ is the kernel of the homomorphism

$$\mathbb{Z}^n \to \mathbb{Q}^d, \qquad (a_1,\dots,a_n) \mapsto a_1N_1h + \cdots + a_nN_nh,$$

and so $\mathbb{Z}^n/S(h)$ is free because it embeds into $\mathbb{Q}^d$.

Step 3. Because of Step 2, we can find an $n\times n$ matrix A, with integer entries and $\det A = 1$, whose last $n-k$ columns give a basis for the subgroup $S(h)$. We then introduce new coordinates $(w_1,\dots,w_n) \in \mathbb{C}^n$ by the rule

$$z_i = \sum_{j=1}^n a_{i,j}w_j. \tag{7}$$

Rewriting $z_1N_1 + \cdots z_nN_n$ in the form $w_1M_1 + \cdots + w_nM_n$, where each

$$M_j = \sum_{i=1}^n a_{i,j}N_i$$

is still nilpotent, we now have $M_{k+1}h = \cdots = M_nh = 0$, while the remaining k vectors $M_1h,\dots,M_kh$ are linearly independent. Instead of f, we can then use the parametrization

$$g\colon B \to \Delta^n \times \mathbb{C}^d, \qquad (w_1,\dots,w_n) \mapsto \big(t_1,\dots,t_n, e^{-(w_1M_1+\cdots+w_kM_k)}h\big), \tag{8}$$

of the sheet $C(h)$ under consideration; here

$$t_j = \prod_{s=1}^n e^{2\pi i a_{j,s}w_s}, \tag{9}$$

and the map g is defined on the open subset $B \subseteq \mathbb{C}^n$ where all $|t_j|<1$.

Step 4. We now analyze the term $e^{-(w_1M_1+\cdots+w_kM_k)}h$ in the parametrization g. As a matter of fact, the map

$$\mathbb{C}^k \to \mathbb{C}^d, \qquad (w_1,\dots,w_k) \mapsto v = e^{-(w_1M_1+\cdots+w_kM_k)}h,$$

is a closed embedding, because the vectors $M_1h,\dots,M_kh$ are linearly independent. We will prove this by constructing an inverse: we show that there are polynomials $p_1(v),\dots,p_k(v)$ in $v=(v_1,\dots,v_d)$, such that whenever v is in the image, one has

$$(w_1,\dots,w_k) = \big(p_1(v),\dots,p_k(v)\big).$$

Proof. We construct suitable polynomials by induction on the number k of variables. If $k=0$, there is nothing to do. So let us assume that the existence of such polynomials is known for $k-1 \geq 0$ variables, and let us establish it for k.

For any multi-index $\alpha = (\alpha_1,\dots,\alpha_k) \in \mathbb{N}^k$, we write

$$M^\alpha = M_1^{\alpha_1}\cdots M_k^{\alpha_k};$$

these matrices are zero whenever $|\alpha|$ is sufficiently large. Among all multi-indices α with $M^\alpha h \neq 0$, select one of maximal length $|\alpha|$. Then $|\alpha| \geq 1$, because the vectors $M_j h$ are in particular nonzero, and without loss of generality we may assume that $\alpha_k \geq 1$. We have

$$M^{\alpha - e_k} v = \big(\mathrm{id} - w_1 M_1 - \cdots - w_{k-1} M_{k-1}\big) M^{\alpha - e_k} h - w_k M^\alpha h.$$

Because at least one of the components of $M^\alpha h$ is non-zero, we can now solve for w_k in the form

$$w_k = c_1 w_1 + \cdots + c_{k-1} w_{k-1} + l(v),$$

with $c_1, \dots, c_{k-1} \in \mathbb{Q}$, and $l(v)$ a degree-one polynomial in v. Substituting back, we obtain

$$e^{l(v) M_k} v = e^{-w_1 (M_1 + c_1 M_k) - \cdots - w_{k-1}(M_{k-1} + c_{k-1} M_k)} h,$$

and, by the inductive hypothesis, $w_1, \dots, w_{k-1}$ can be expressed as polynomials in the coordinates of the vector $e^{l(v) M_k} v$, since the vectors $(M_i + c_i M_k) h$ are of course still linearly independent. It is thus possible to find polynomials in v such that

$$(w_1, \dots, w_{k-1}) = \big(p_1(v), \dots, p_{k-1}(v)\big).$$

Then $w_k = c_1 p_1(v) + \cdots + c_{k-1} p_{k-1}(v) + l(v)$ is also a polynomial in v, and the assertion is proved. □

Step 5. The result of Step 4 now gives us half of the equations for the closed subset $\overline{C(h)}$. Indeed, we have seen that if $(t, v) \in \Delta^n \times \mathbb{C}^d$ is a point of $C(h)$, then it is in the image of g, and so its v-coordinates satisfy the relation

$$v = e^{-(p_1(v) M_1 + \cdots + p_k(v) M_k)} h. \tag{10}$$

In components, these are d polynomial equations for $v = (v_1, \dots, v_d)$. The same equations obviously have to hold for every point in the closure $\overline{C(h)}$.

Step 6. Next, we turn our attention to the remaining n coordinates $(t_1, \dots, t_n)$ of the parametrization g, as given in (9). Letting $u_j = \exp(2\pi i w_j)$, for $j = k+1, \dots, n$, we have

$$t_j = u_{k+1}^{a_{j,k+1}} \cdots u_n^{a_{j,n}} \cdot e^{2\pi i (a_{j,1} w_1 + \cdots + a_{j,k} w_k)}.$$

The shape of these formulas leads us to consider the algebraic map

$$(\mathbb{C}^*)^{n-k} \to \mathbb{C}^n, \qquad (u_{k+1}, \dots, u_n) \mapsto (x_1, \dots, x_n), \tag{11}$$

whose coordinates are the products

$$x_j = \prod_{i=k+1}^{n} u_i^{a_{j,i}}. \tag{12}$$

Because the map is given by polynomials in the variables u_i and u_i^{-1}, the topological closure of its image is actually a closed algebraic subvariety of $\mathbb{C}^n$, and as such defined by finitely many polynomial equations

$$f_1(x_1, \dots, x_n) = \cdots = f_e(x_1, \dots, x_n) = 0.$$

In fact, because the original map is monomial, each $f_b(x)$ can be taken as a binomial in the variables $x_1, \dots, x_n$; the closure of the image of (11) is therefore a (possibly non-normal) toric variety.

Step 7. From Step 6, we can now read off the remaining equations for $\overline{C(h)}$. Indeed, a point (t, v) in the image of g has to satisfy the equations

$$f_b\Big(t_1 e^{-2\pi i \sum_{s\leq k} a_{1,s}w_s}, \dots, t_n e^{-2\pi i \sum_{s\leq k} a_{n,s}w_s}\Big) = 0$$

for $b = 1, \dots, e$. From Step 4 we know, moreover, that $w_s = p_s(v)$; therefore

$$f_b\Big(t_1 e^{-2\pi i \sum_{s\leq k} a_{1,s}p_s(v)}, \dots, t_n e^{-2\pi i \sum_{s\leq k} a_{n,s}p_s(v)}\Big) = 0 \tag{13}$$

is another set of e holomorphic equations satisfied by the closure $\overline{C(h)}$.

Step 8. It remains to see that the $d + e$ equations in (10) and (13) really define $\overline{C(h)}$, and not a bigger set; for this, we use the set-theoretic description of the closure in Proposition 5. The trivial case is when h is not invariant under any part of the monodromy; here $k = n$, and as pointed out in Step 1, $C(h)$ is then already a closed set, and there is nothing to prove. In the remaining case, when $k < n$, it suffices to consider solutions of the equations over $(0, \dots, 0) \in \Delta^n$, since we can always move the center of the coordinate system.

So consider a point $(0, v) \in \Delta^n \times \mathbb{C}^d$ that satisfies the equations. On the one hand, the equations in (10) define the image of a closed embedding, as explained in Step 4; therefore, $v = e^{-(w_1 M_1 + \cdots + w_k M_k)} h$ for a unique point $(w_1, \dots, w_k) \in \mathbb{C}^k$. Letting $w = (w_1, \dots, w_k, 0, \dots, 0)$, and going back to the original coordinates z in (7), we get a point $(z_1, \dots, z_n) \in \mathbb{C}^n$ such that

$$v = e^{-(z_1 N_1 + \cdots + z_n N_n)} h.$$

On the other hand, the equations in (13) arose from the map defined in Step 6. Now (12) shows that the point $(0, \dots, 0)$ can only be in the closure of the image when some linear combination of the exponent vectors $(a_{1,i}, \dots, a_{n,i})$, for $i = k+1, \dots, n$, has positive coordinates. Since these vectors generate the subgroup $S(h)$, we thus get positive integers $a_1, \dots, a_n$ with

$$a_1 N_1 h + \cdots + a_n N_n h = 0$$

But by the description in Proposition 5, this says exactly that the point $(0, v)$ belongs to $\overline{C(h)}$.

In summary, we have established the following result.

Proposition 8. *Let $\overline{T}$ be the closure of T inside the vector bundle $\widetilde{E}$. Then $\overline{T}$ is an analytic subset with countably many irreducible components, each of the form $\overline{C(h)}$ for some $h \in \mathbb{Z}^d$. Moreover, each $\overline{C(h)}$ can be defined by finitely many explicit holomorphic equations in the coordinates $(t_1, \dots, t_n, v_1, \dots, v_d)$ of $\widetilde{E} \simeq \Delta^n \times \mathbb{C}^d$, as in* (10) *and* (13) *above.*

Existence of the extension space. We now let $\mathrm{Can}(T)$ be the normalization of $\overline{T}$; from the discussion above, it should be apparent that $\overline{T}$ itself is generally not normal. We need to verify that $\mathrm{Can}(T)$ satisfies the three defining conditions. For (A) and (B), this is obvious; for (C), it is the content of the following lemma.

Lemma 9. *The normalization of $\overline{T}$ satisfies the condition in (C).*

Proof. This is an immediate consequence of the functoriality of Deligne's canonical extension. By Lemma 2, we only need to verify the condition when $Y = \Delta$. So let $f\colon \Delta \to X$ be the given map, and $s\colon \Delta^* \to T$ a compatible section of T. Since $T \subseteq E$, we can view s as a holomorphic section of the pullback bundle f^*E over

Δ^*. Now the local system $\mathcal{H}$ is unipotent, and so its pullback $f^{-1}\mathcal{H}$ to Δ^* still has unipotent monodromy around $0 \in \Delta$. By [6, Proposition 5.4 on p. 94], the canonical extension of the associated flat vector bundle on Δ^* coincides with $f^*\widetilde{E}$. From the description of the canonical extension, it is then obvious that the section s of $f^{-1}\mathcal{H}$ extends to a holomorphic section of $f^*\widetilde{E}$. Since $\overline{T} \subseteq \widetilde{E}$ is the closure of T, this means that s extends to a holomorphic map $\Delta \to \overline{T}$, and hence to a map from Δ to the normalization of $\overline{T}$. □

Thus the total space T of a local system with unipotent monodromy admits an extension space $\mathrm{Can}(T)$. It follows from the construction that this space is only mildly singular, as we now explain.

Toric singularities. We conclude this section by describing the singularities of the extension space $\mathrm{Can}(T)$, in the case when T is the étale space of a local system with unipotent monodromy, and Z a divisor with normal crossings.

When passing from $\overline{T}$ to its normalization $\mathrm{Can}(T)$, two things are happening. Firstly, the individual components $\overline{C(h)}$ are separated at points where they meet, and become disjoint. Secondly, each component $\overline{C(h)}$ itself is normalized. From Step 6 on p. 11 in Section 2, we see that $\overline{C(h)}$ is locally isomorphic to a (non-normal) toric variety. Indeed, the map g in (8), whose image is the sheet $C(h)$, is locally the product of a closed immersion and a map defined by monomials. As explained in the article by D. Cox [5, p. 402], the closure of the image of a monomial map as in (11) is a non-normal toric variety; after taking the normalization, one gets a toric variety in the usual sense. It follows that the normalization of each $\overline{C(h)}$ is locally, in the analytic topology, isomorphic to a toric variety. It is known [8, Proposition on p. 76] that toric varieties have only mild singularities; in particular, the singularities are always Cohen-Macaulay and rational [10]. The same is therefore true for the extension space $\mathrm{Can}(T)$. Given that the construction involves taking a closure, this is quite remarkable.

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