

# SINGULAR METRICS AND A CONJECTURE BY CAMPANA AND PETERNELL

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ABSTRACT. A conjecture by Campana and Peternell says that if a positive multiple of  $K_X$  is linearly equivalent to an effective divisor  $D$  plus a pseudo-effective divisor, then the Kodaira dimension of  $X$  should be at least as big as the Iitaka dimension of  $D$ . This is a very useful generalization of the non-vanishing conjecture (which is the case  $D = 0$ ). We use recent work about singular metrics on pluri-adjoint bundles to show that the Campana-Peternell conjecture is almost equivalent to the non-vanishing conjecture.

## A. INTRODUCTION

**1.** Let  $X$  be a smooth projective variety over the complex numbers, and denote by  $K_X$  the canonical divisor class. The non-vanishing conjecture predicts that if  $K_X$  is pseudo-effective, then some positive multiple of  $K_X$  is effective.<sup>1</sup> This is a special case of the famous abundance conjecture from the minimal model program. At some point, Campana and Peternell [CP11, Conj. on p. 43] suggested the following generalization of the non-vanishing conjecture.

**Conjecture 1.1** (Campana-Peternell). *Let  $X$  be a smooth projective variety, and  $D$  an effective divisor on  $X$ . Suppose that, for some  $m \geq 1$ , the divisor class  $mK_X - D$  is pseudo-effective. Then one should have  $\kappa(X) \geq \kappa(X, D)$ .*

Here  $\kappa(X, D)$  is the Iitaka dimension of the divisor  $D$ , and  $\kappa(X) = \kappa(X, K_X)$  the Kodaira dimension of  $X$ . The Campana-Peternell conjecture contains the non-vanishing conjecture as the special case  $D = 0$ , and therefore sits somewhere in between the non-vanishing conjecture and the abundance conjecture. It is obviously weaker than the full abundance conjecture, but still extremely useful in practice, for example for questions related to the behavior of Kodaira dimension in algebraic fiber spaces [PS17].

**2.** The purpose of this note is to show that the Campana-Peternell conjecture is almost equivalent to the non-vanishing conjecture. (I believe that the two conjectures are in fact equivalent; I write “almost” because there is one special case involving rationally connected varieties where the question remains unresolved.) The main tool is the existence of certain singular hermitian metrics on pluri-adjoint bundles, introduced by Păun and Takayama [PT18]. For a non-expert like myself, one mysterious aspect of the non-vanishing conjecture – and of Conjecture 1.1 more generally – is how adding multiples of  $K_X$  can possibly make things better. We will see below that the metric techniques provide at least one mechanism for this.

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<sup>1</sup>Hashizume [Has18] has shown that this version is equivalent to the full non-vanishing conjecture for lc pairs.

**3.** I thank Christopher Hacon for answering some questions about the abundance conjecture. I also had a useful email exchange about the Campana-Peternell conjecture with Behrouz Taji. While writing this paper, I have been partially supported by NSF grant DMS-1551677 and a Simons Fellowship from the Simons Foundation. I thank both of these institutions for their support, and the Max-Planck-Institute for providing me with excellent working conditions during my stay in Bonn.

#### B. EQUIVALENT FORMULATIONS OF THE CONJECTURE

**4.** We begin by formulating Conjecture 1.1 in a way that is closer to the non-vanishing conjecture. Campana and Peternell observed that, up to birational modifications of  $X$ , one can always assume that the divisor  $D$  is base-point free. Here is why. Choose  $n$  large enough, in order for the linear system  $|nD|$  to give the Iitaka fibration of  $D$ , and let  $\mu: X' \rightarrow X$  be an embedded resolution of singularities of the linear system  $|nD|$ . Then one has a decomposition  $\mu^*|nD| = |F| + E$ , where  $F$  is base-point free and  $E$  is effective. But now

$$nmK_{X'} - F \equiv n\mu^*(mK_X - D) + nmK_{X'/X} + E$$

is the sum of a pseudo-effective divisor and an effective divisor, hence pseudo-effective. This reduces Conjecture 1.1 to the case where  $D$  is base-point free.

**5.** Now suppose that  $D$  is base-point free. After replacing  $D$  by a multiple, we may assume that the linear system  $|D|$  defines a surjective morphism  $f: X \rightarrow Y$  to a projective variety  $Y$  with  $\dim Y = \kappa(X, D)$ . We also get a very ample divisor  $H$  such that  $f^*H = D$ ; consequently,  $mK_X - f^*H$  is pseudo-effective. After replacing  $f: X \rightarrow Y$  by its Stein factorization, we may assume that  $Y$  is normal, and that  $f$  has connected fibers. Let  $\nu: Y' \rightarrow Y$  be a resolution of singularities, and choose a compatible resolution of singularities  $\mu: X' \rightarrow X$ , giving us a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{\nu} & Y. \end{array}$$

The divisor  $\nu^*H$  is big and nef; after replacing  $H$  by a big enough multiple, we can therefore find an ample divisor  $H'$  on  $Y'$  such that  $\nu^*H - H'$  is effective. Now

$$mK_{X'} - (f')^*H' \equiv mK_{X'/X} + \mu^*(mK_X - f^*H) + (f')^*(\nu^*H - H')$$

is still pseudo-effective, and since  $\kappa(X') = \kappa(X)$ , Conjecture 1.1 is reduced to the case where  $D$  is the pullback of an ample divisor along an algebraic fiber space.

**6.** This leads to an equivalent formulation of Conjecture 1.1. Let  $f: X \rightarrow Y$  be an algebraic fiber space; by this I mean that  $X$  and  $Y$  are smooth projective varieties, and that  $f$  is surjective with connected fibers. Let  $H$  be an ample divisor on  $Y$ , and suppose that for some  $m \geq 1$ , the divisor class  $mK_X - f^*H$  is pseudo-effective. The Campana-Peternell conjecture is equivalent to the statement that  $\kappa(X) \geq \dim Y$ ; this is clear from the discussion above.

**7.** In fact, one can get an even stronger prediction out of Conjecture 1.1. To state it, let  $F$  denote the general fiber of the algebraic fiber space  $f: X \rightarrow Y$ . If the divisor class  $mK_X - f^*H$  is pseudo-effective, then  $K_F$  is also pseudo-effective; according to the non-vanishing conjecture, this should imply that  $\kappa(F) \geq 0$ .

**Lemma 7.1.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space with general fiber  $F$ . Let  $H$  be an ample divisor on  $Y$ , and suppose that  $mK_X - f^*H$  is pseudo-effective for some  $m \geq 1$ . Then Conjecture 1.1 predicts that*

$$(7.2) \quad \kappa(X) = \kappa(F) + \dim Y.$$

*Proof.* Since  $K_F$  is pseudo-effective, Conjecture 1.1, applied to the smooth projective variety  $F$ , predicts that  $\kappa(F) \geq 0$ . Choose an integer  $r \geq 1$  such that  $rK_F$  is effective. Then  $f_*\mathcal{O}_X(rK_X)$  will be nonzero, and so  $f_*\mathcal{O}_X(rK_X) \otimes \mathcal{O}_Y(\ell H)$  will have sections for  $\ell \gg 0$ . In other words, the divisor class  $rK_X + f^*(\ell H)$  is effective. By [Mor87, Prop. 1.14], we have

$$\kappa(X, rK_X + f^*(\ell H)) = \kappa(F) + \dim Y.$$

The reason is that  $rK_X + f^*(\ell H) - f^*H$  is effective once  $\ell \gg 0$ ; and that the restriction of  $rK_X + f^*(\ell H)$  to the general fiber  $F$  is equal to  $rK_F$ . But now

$$(m\ell + r)K_X - (rK_X + f^*(\ell H)) = \ell(mK_X - f^*H)$$

is of course still pseudo-effective, and so Conjecture 1.1 is also claiming that

$$\kappa(X) \geq \kappa(X, rK_X + f^*(\ell H)) = \kappa(F) + \dim Y.$$

On the other hand, one always has  $\kappa(X) \leq \kappa(F) + \dim Y$  by the easy addition formula [Mor87, Cor. 2.3], and hence (7.2).  $\square$

**8.** According to [Mor87, Prop. 1.14], the identity in (7.2) is equivalent to saying that  $mK_X - f^*H$  becomes effective for  $m$  sufficiently large and divisible. The Campana-Peternell conjecture therefore looks exactly like non-vanishing even in the general case: if  $mK_X - f^*H$  is pseudo-effective for some  $m \geq 1$ , then  $mK_X - f^*H$  should actually be effective for  $m$  sufficiently large and divisible.

**9.** Still assuming that the non-vanishing conjecture is true, we can further reduce the problem to algebraic fiber spaces with  $\kappa(F) = 0$ . The argument runs as follows. Suppose that we have an algebraic fiber space  $f: X \rightarrow Y$  with general fiber  $F$ , and an ample line bundle  $H$  on  $Y$ , such that  $mK_X - f^*H$  is pseudo-effective for some  $m \geq 1$ . As in the proof of Lemma 7.1,

$$(m\ell + r)K_X - (rK_X + f^*(\ell H))$$

is pseudo-effective for suitably chosen  $r, \ell \in \mathbb{N}$ . After another application of the procedure in §4, this time for the divisor class  $rK_X + f^*(\ell H)$ , we get a new algebraic fiber space  $f': X' \rightarrow Y'$ , with  $X'$  birational to  $X$ , such that  $\dim Y' = \kappa(F) + \dim Y$ . We also get a new ample divisor  $H'$  on  $Y'$  such that  $m'K_{X'} - f'^*H'$  is pseudo-effective for some  $m' \geq 1$ . We can now iterate this procedure, and since

$$\dim Y' = \dim Y + \kappa(F) \geq \dim Y,$$

we must eventually arrive at an algebraic fiber space  $f^{(n)}: X^{(n)} \rightarrow Y^{(n)}$  whose general fiber  $F^{(n)}$  has Kodaira dimension 0.

**10.** In this manner, we reduce the proof of the Campana-Peternell conjecture, modulo the non-vanishing conjecture, to the following special case:

**Conjecture 10.1.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space with  $\kappa(F) \geq 0$ . Let  $H$  be an ample divisor on  $Y$ . If  $mK_X - f^*H$  is pseudo-effective for some  $m \geq 1$ , then  $mK_X - f^*H$  becomes effective for  $m$  sufficiently large and divisible.*

We have seen above that Conjecture 10.1, together with the non-vanishing conjecture, is equivalent to Conjecture 1.1. Of course, I have no idea of how to prove the non-vanishing conjecture; my point is that Conjecture 10.1 is the part of the Campana-Peternell conjecture that one can hope to solve using existing methods.

**11.** As the discussion in §7 shows, the Campana-Peternell conjecture also has implications for Iitaka's conjecture on subadditivity of the Kodaira dimension. Let  $f: X \rightarrow Y$  be an algebraic fiber space with general fiber  $F$ . The Campana-Peternell conjecture reduces the problem of proving the inequality

$$\kappa(X) \geq \kappa(F) + \kappa(Y)$$

for arbitrary algebraic fiber spaces to the special case  $\kappa(Y) = 0$ .

**Proposition 11.1.** *Suppose that Conjecture 10.1 is true. If the inequality*

$$\kappa(X) \geq \kappa(F) + \kappa(Y)$$

*holds for every algebraic fiber space  $f: X \rightarrow Y$  with  $\kappa(Y) = 0$ , then it holds for every algebraic fiber space.*

*Proof.* We can obviously assume that  $\kappa(F) \geq 0$  and  $\kappa(Y) \geq 0$ , because the statement is vacuous otherwise. After a birational modification, we can assume that the Iitaka fibration of  $Y$  is a morphism  $p: Y \rightarrow Z$ , and that  $mK_Y - p^*H$  is effective for some  $m \geq 1$  and some ample divisor  $H$  on  $Z$ . Let  $P$  be the general fiber of  $p$ ; by construction, one has  $\kappa(P) = 0$ . Set  $g = p \circ f$ , and denote by  $G$  the general fiber of the resulting algebraic fiber space  $g: X \rightarrow Z$ .

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{p} & Z \\ & & \searrow & \nearrow & \\ & & & g & \end{array}$$

Since  $\kappa(F) \geq 0$ , the relative canonical class  $K_{X/Y}$  is pseudo-effective; consequently,

$$mK_X - g^*H \equiv mK_{X/Y} + f^*(mK_Y - p^*H)$$

is also pseudo-effective. But then (7.2) gives  $\kappa(X) = \kappa(G) + \dim Z = \kappa(G) + \kappa(Y)$ . The conclusion is that if one knew subadditivity for the algebraic fiber space  $f: G \rightarrow P$ , whose general fiber is  $F$  and whose base  $P$  satisfies  $\kappa(P) = 0$ , then one would get  $\kappa(G) \geq \kappa(F)$ , and therefore the desired subadditivity for  $f: X \rightarrow Y$ .  $\square$

### C. THE MAIN RESULT

**12.** We are going to show that Conjecture 10.1 is true under the additional (but hopefully unnecessary) assumption that the canonical divisor  $K_Y$  is pseudo-effective.

**Theorem 12.1.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space with  $\kappa(F) \geq 0$ . Let  $H$  be an ample divisor on  $Y$ . If  $mK_X - f^*H$  is pseudo-effective for some  $m \geq 1$ , then  $mK_X - f^*H$  becomes effective for  $m$  sufficiently large and divisible, provided that  $K_Y$  is pseudo-effective.*

An equivalent formulation is that, under the assumptions of the theorem,

$$\kappa(X) = \kappa(F) + \dim Y.$$

This is clear from the discussion in §8.

**13.** The strategy of the proof is as follows. Fix an integer  $r \geq 1$  such that  $rK_F$  is effective. Suppose that  $m_0K_X - f^*H$  is pseudo-effective for some  $m_0 \geq 1$ . Then  $L = (k + \ell + 1)(m_0K_X - f^*H)$  is of course also pseudo-effective for  $k, \ell \geq 1$ , and by putting things together in the right way, we get

$$(13.1) \quad f_*\mathcal{O}_X(mrK_X - f^*H) \cong f_*\mathcal{O}_X(K_X + L_n) \otimes \mathcal{O}_Y(kH) \otimes \mathcal{O}_Y(nK_Y + \ell H),$$

where  $n = mr - (k + \ell + 1)m_0 - 1$  and  $L_n = nK_{X/Y} + L$ . Since  $mrK_F$  is effective,

$$f_*\mathcal{O}_X(K_X + L_n) = f_*\mathcal{O}_X(K_X + nK_{X/Y} + L)$$

is a torsion-free sheaf on  $Y$ , of generic rank  $\dim H^0(F, mrK_F)$ . By the work of Păun and Takayama [PT18], the line bundle  $L_n$  has a naturally defined semi-positively curved singular hermitian metric  $h_n$ , and for  $m \gg 0$ , the inclusion

$$f_*(\mathcal{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n)) \subseteq f_*\mathcal{O}_X(K_X + L_n)$$

becomes generically an isomorphism. We can then use the Kollár-type vanishing theorem for the sheaf  $f_*(\mathcal{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n))$ , proved by Fujino and Matsumura [FM21], to get the desired conclusion.

**14.** Now we turn to the details. Let us first deal with the factor  $\mathcal{O}_Y(nK_Y + \ell H)$  in (13.1). Since  $K_Y$  is pseudo-effective, we can choose  $\ell \geq 1$  in such a way that the divisor  $nK_Y + \ell H$  is effective for every  $n \geq 1$ ; this kind of result is sometimes called “effective non-vanishing”, and is a simple consequence of the Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem [Laz04, 11.2.14].

**15.** To deal with the remaining factors, we need a version of effective non-vanishing for direct images of adjoint bundles. Here is the precise statement.

**Lemma 15.1.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space. Let  $H$  be an ample divisor on  $Y$ . Then there is an integer  $k \geq 1$  such that*

$$H^0(Y, f_*\mathcal{O}_X(K_X + L) \otimes \mathcal{O}_Y(kH)) \neq 0$$

for every line bundle  $(L, h_L)$  with a semi-positively curved singular hermitian metric on  $X$  such that  $f_*(\mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L)) \neq 0$ .

*Proof.* After replacing  $H$  by a sufficiently big multiple, we can assume that  $H$  is effective to begin with. To shorten the formulas, define

$$\mathcal{F} = f_*(\mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L)).$$

This is a nontrivial torsion-free coherent sheaf on  $Y$ . According to the Kollár-type vanishing theorem proved by Fujino and Matsumura [FM21, Thm. D], we have

$$H^i(Y, \mathcal{F} \otimes \mathcal{O}_Y(kH)) = 0$$

for every  $i, k \geq 1$ . Consequently,

$$\dim H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(kH)) = \chi(Y, \mathcal{F} \otimes \mathcal{O}_Y(kH))$$

is a nonzero polynomial in  $k$ , of degree at most  $\dim Y$ , hence must be nonzero for at least one value of  $k \in \{1, 2, \dots, \dim Y + 1\}$ . In fact, because  $H$  is effective, the value  $k = \dim Y + 1$  always works. Now it remains to observe that

$$H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(kH)) \subseteq H^0(Y, f_* \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_Y(kH)),$$

due to the inclusion  $\mathcal{I}(h_L) \subseteq \mathcal{O}_X$ .  $\square$

**16.** We return to our problem. Fix integers  $m_0, k, \ell \geq 1$  as above. The line bundle  $L = (k + \ell + 1)(m_0 K_X - f^* H)$  is pseudo-effective, and so it has a singular hermitian metric  $h$  with semi-positive curvature [Dem01, Prop. 6.6]. Denote by  $(L_F, h_F)$  the restriction of  $(L, h)$  to the general fiber  $F$  of the algebraic fiber space  $f: X \rightarrow Y$ .

**Lemma 16.1.** *For  $n \gg 0$ , we have  $\mathcal{I}(h_F^{1/n}) = \mathcal{O}_F$ .*

*Proof.* Since  $F$  is compact, we can cover  $F$  by finitely many open sets on which  $L_F$  is trivial. In any trivialization, the singular hermitian metric  $h_F$  can be written in the form  $e^{-\varphi}$ , with  $\varphi$  plurisubharmonic. This reduces the problem to the following local statement: Let  $\varphi$  be a plurisubharmonic function on a neighborhood of  $0 \in \mathbb{C}^d$ , not identically equal to  $-\infty$ ; then the function  $e^{-\varphi/n}$  is locally integrable for  $n \gg 0$ . This is straightforward. The Lelong number  $\nu(\varphi, 0)$  is finite by construction; see for example [Dem01, Thm. 2.8]. But by a result of Skoda [Dem01, Lem. 5.6], the function  $e^{-\varphi/n}$  becomes locally integrable once  $2n > \nu(\varphi, 0)$ .  $\square$

**17.** Choose  $m$  sufficiently large so that  $\mathcal{I}(h_F^{1/n}) = \mathcal{O}_F$ ; recall from the construction above that  $n = rm - (k + \ell + 1)m_0 - 1$ . Since we know that

$$f_* \mathcal{O}_X(K_X + L_n) = f_* \mathcal{O}_X(K_X + nK_{X/Y} + L)$$

is nontrivial, we can now apply [PT18, Thm. 5.1.2]. During the proof of this result, Păun and Takayama show that the twisted Narasimhan-Simha metric on the fibers of  $f: X \rightarrow Y$  induces a semi-positively curved singular hermitian metric  $h_n$  on the line bundle  $L_n = nK_{X/Y} + L$ , and that the inclusion

$$f_*(\mathcal{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n)) \subseteq f_* \mathcal{O}_X(K_X + L_n)$$

is generically an isomorphism; this is a consequence of the fact that  $\mathcal{I}(h_F^{1/n}) = \mathcal{O}_F$ . Together with Lemma 15.1, these two facts imply that

$$H^0(Y, f_* \mathcal{O}_X(K_X + L_n) \otimes \mathcal{O}_Y(kH)) \neq 0.$$

Since  $nK_Y + \ell H$  is effective, we conclude from (13.1) that  $f_* \mathcal{O}_X(mrK_X - f^* H)$  has nontrivial global sections for  $m \gg 0$ , and hence that  $mrK_X - f^* H$  becomes effective for  $m \gg 0$ . This is what we wanted to show.

**18.** Here is an example where this gives a new result.

*Example 18.1.* Let  $f: X \rightarrow A$  be an algebraic fiber space over an abelian variety, with  $\kappa(F) \geq 0$ . If  $mK_X - f^* H$  is pseudo-effective for some  $m \geq 1$  and some ample divisor  $H$  on the abelian variety, then  $mK_X - f^* H$  is effective for  $m \gg 0$ , and therefore  $\kappa(X) = \kappa(F) + \dim A$ . According to [PS14, Thm. 2.1], it also follows that the pullback of every holomorphic one-form on  $A$  has a non-empty zero locus on  $X$ , but now under the much weaker assumption that  $mK_X - f^* H$  is pseudo-effective (instead of effective).

**19.** What about the case when  $K_Y$  is not pseudo-effective? By [BDPP13, Cor. 0.3], this is equivalent to  $Y$  being covered by rational curves. Using the MRC fibration and Theorem 12.1, one can easily reduce the proof of Conjecture 10.1 to the case where  $Y$  is rationally connected. The argument is very similar to the proof of Proposition 11.1, and so we only sketch it. After a birational modification, we can assume that the MRC fibration of  $Y$  is represented by an algebraic fiber space  $p: Y \rightarrow Z$ . If we denote by  $P$  the general fiber of  $p$ , then  $P$  is rationally connected [Kol96, IV.5]. On the other hand,  $Z$  is not uniruled [GHS03, Cor. 1.4], and so  $K_Z$  is pseudo-effective. Set  $g = p \circ f$ , and again denote by  $G$  the general fiber of the resulting algebraic fiber space  $g: X \rightarrow Z$ .

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{p} & Z \\ & & \searrow & \nearrow & \\ & & & g & \end{array}$$

We can assume that the ample line bundle has the form  $L = L' + p^*L''$ , with  $L'$  relatively ample over  $Z$ , and  $L''$  ample on  $Z$ . Now  $f: G \rightarrow P$  is an algebraic fiber space over a rationally connected base, with general fiber  $F$ . Since  $mK_X - f^*L$  is pseudo-effective,  $mK_G - f^*(L'|_P)$  is also pseudo-effective; if Conjecture 10.1 is true for the algebraic fiber space  $f: G \rightarrow P$ , then we get

$$\kappa(G) = \kappa(F) + \dim P.$$

At the same time, Theorem 12.1 applies to the algebraic fiber space  $g: X \rightarrow Z$ , because  $K_Z$  is pseudo-effective; together with the identity above, this gives

$$\kappa(X) = \kappa(G) + \dim Z = \kappa(F) + \dim Y.$$

As explained in §8, this identity is equivalent to  $mK_X - f^*L$  being effective for  $m$  sufficiently large and divisible. In this way, Conjecture 10.1 is reduced to the case where  $Y$  is rationally connected.

**20.** Unfortunately, I am not able to say anything even in the case  $Y = \mathbb{P}^1$ .

*Example 20.1.* Let  $f: X \rightarrow \mathbb{P}^1$  be an algebraic fiber space with  $\kappa(F) \geq 0$ . If the divisor class  $mK_X - f^*\mathcal{O}(1)$  is pseudo-effective for some  $m \geq 1$ , then is it true that  $mK_X - f^*\mathcal{O}(1)$  must be effective for  $m \gg 0$ ? The argument from above breaks down because the canonical bundle of the base is no longer pseudo-effective.

**21.** One can imagine a proof of Conjecture 10.1 along the following lines. Choose  $k, m \geq 1$  such that  $mK_X - f^*(kH)$  is pseudo-effective and  $H^0(F, K_F + mK_F) \neq 0$ . Suppose that one could find a singular hermitian metric  $h$  on the line bundle  $mK_X - f^*(kH)$ , with semi-positive curvature, such that all sections in  $H^0(F, K_F + mK_F)$  are square-integrable relative to the induced metric  $h_F$  on  $mK_F$ . (In fact, just one nontrivial section would be enough.) Then the direct image sheaf

$$\mathcal{F} = f_* \left( \mathcal{O}_X(K_X + mK_X - f^*(kH)) \otimes \mathcal{I}(h) \right)$$

would be nonzero, and one could get the desired result by adjusting the coefficients  $k$  and  $m$ , and repeating the argument from above. The construction in §17 actually produces such a metric when  $K_Y$  is pseudo-effective. Does the same kind of metric still exist when  $K_Y$  is not pseudo-effective?

## REFERENCES

- [BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248. MR 3019449
- [CP11] Frédéric Campana and Thomas Peternell, *Geometric stability of the cotangent bundle and the universal cover of a projective manifold*, Bull. Soc. Math. France **139** (2011), no. 1, 41–74, with an appendix by Matei Toma. MR 2815027
- [Dem01] Jean-Pierre Demailly, *Multiplier ideal sheaves and analytic methods in algebraic geometry*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 1–148. MR 1919457
- [FM21] Osamu Fujino and Shin-ichi Matsumura, *Injectivity theorem for pseudo-effective line bundles and its applications*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 849–884. MR 4324359
- [GHS03] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67. MR 1937199
- [Has18] Kenta Hashizume, *On the non-vanishing conjecture and existence of log minimal models*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 1, 89–104. MR 3749345
- [Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180
- [Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals. MR 2095472
- [Mor87] Shigefumi Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331. MR 927961
- [PS14] Mihnea Popa and Christian Schnell, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. (2) **179** (2014), no. 3, 1109–1120. MR 3171760
- [PS17] ———, *Viehweg’s hyperbolicity conjecture for families with maximal variation*, Invent. Math. **208** (2017), no. 3, 677–713.
- [PT18] Mihai Păun and Shigeharu Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, J. Algebraic Geom. **27** (2018), no. 2, 211–272. MR 3764276

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