Degenerating complex variations of Hodge structure in dimension one

Claude Sabbah
Centre de Mathématiques Laurent Schwartz, École Polytechnique
claude.sabbah@polytechnique.edu

Christian Schnell
Department of Mathematics, Stony Brook University
christian.schnell@stonybrook.edu

June 16, 2022

1 Introduction

1.1 Summary of the paper

1. In this paper, we analyze the behavior of polarized complex variations of Hodge structure on the punctured unit disk. For *integral* variations of Hodge structure, this analysis was carried out by Wilfried Schmid [Sch73] in his famous article *Variation of Hodge structure: the singularities of the period mapping*, one of the central works in modern Hodge theory. The restriction to integral variations of Hodge structure is natural from the point of view of geometry; it also implies that the eigenvalues of the monodromy transformation are roots of unity, and this fact makes things technically easier. Nevertheless, there are two reasons why we want to remove this restriction now. First, we need the results for complex variations of Hodge structure to set up the theory of complex mixed Hodge modules, the topic of our "Mixed Hodge Module Project". Second, it turns out that working with complex variations clarifies many aspects of the theory that were somewhat obscured by the presence of an integral structure, especially the central role played by the Hodge metric.

2. Analytically, the behavior of a polarized variation of Hodge structure on the punctured disk turns out to be surprisingly simple. Before getting into any details, let us therefore quickly highlight the two results that are, conceptually, the most important. A complex variation of Hodge structure is a smooth vector bundle E with a flat connection d, and a decomposition into smooth subbundles $E^{p,q}$ that interacts with the connection in a certain way. The polarization defines a smooth metric on the bundle E, called the Hodge metric. In order to compare different fibers of E, we use the vector space V of multi-valued flat sections as a reference frame; together with the monodromy transformation T, it contains the same information as the flat bundle (E, d). The *first result* is that, in this reference frame, the Hodge metric degenerates in a very simple manner: near the origin, the Hodge metric of any nonzero multi-valued flat section grows or decays like a power of $|\log|t||$, where t is the coordinate on the disk; the exponents are determined by the monodromy transformation

T. This means that, up to a constant, the asymptotic behavior of the Hodge metric only depends on the underlying flat bundle.

3. The fact that the Hodge metric grows and decays at different rates of course prevents the Hodge structures from having any sort of limit as $t \to 0$. We then take the natural step of "rescaling" the Hodge metric, in order to even out these different rates. The rescaling can be done by moving the Hodge structures by elements of the symmetry group of the period domain; it involves a choice of splitting for the filtration by order of growth of the Hodge metric. The *second result* is that the rescaled Hodge structures converge to a limit, which is again a polarized Hodge structure of the same type. In particular, the asymptotic behavior of the Hodge metric also controls the asymptotic behavior of the Hodge structures in the variation. Without rescaling, one gets a "limiting" mixed Hodge structure on the vector space V, whose weight filtration is, up to a shift, the filtration by order of growth of the Hodge metric. Each subquotient of the weight filtration is again a polarized Hodge structure; the general principle at work, coming from analysis, is that one can only expect to get a meaningful limit when everything has the same order of growth.

1.2 What is new?

4. For the benefit of those readers who are already familiar with Schmid's results, we briefly summarize the new features of our approach. We mentioned already that we treat arbitrary polarized complex variations of Hodge structure on the punctured disk, without assuming that the eigenvalues of the monodromy transformation are roots of unity. In this generality, we give new – and, we think, more conceptual – proofs for all the major results in Schmid's paper, such as the estimates for the rate of growth of the Hodge norm; the existence of a limiting mixed Hodge structure; the nilpotent orbit theorem; and a simplified (but still sufficiently powerful) version of the SL(2)-orbit theorem. The results as such are of course just special cases of Mochizuki's monumental work on tame harmonic bundles [Moc07], but we think that it is worthwhile to have a self-contained and simple treatment.[1]

5. Our starting point is a direct proof for the Hodge norm estimates, based on calculations with harmonic bundles and on a comparison with certain model variations of Hodge structure. Technically, the crucial point is Simpson's "basic estimate" for the Hodge norm of the Higgs field, which is proved using Ahlfors' lemma. In Schmid's paper, the Hodge norm estimates are deduced from the orbit theorems, and therefore appear towards the end; here, they stand at the beginning. We feel that this gives a better conceptual explanation for the appearance of the monodromy weight filtration. Next, we give a different proof for the nilpotent orbit theorem, based on curvature properties of the Hodge metric and L^2-extension theorems; analytically, the key point is that one can make the curvature of the Hodge metric either positive or negative by multiplying by a suitable power of $|\log|t||$; this general idea is due to Cornalba-Griffiths [CG75] and Simpson [Sim88]. The parameter dependence of various constants, important for extending the theory to more than one dimension, is handled by using the maximum principle. We then use the Hodge norm estimates and the nilpotent orbit theorem to prove that the rescaled period mapping has a well-defined limit inside the period domain.

[1] In several places, for example [Zuc79, p. 453], it is claimed that Schmid's results carry over to polarized variations of Hodge structure whose monodromy is not quasi-unipotent; but despite considerable effort, we have not been able to adapt Schmid's proof of the nilpotent orbit theorem to this setting.

6. From the convergence of the rescaled period mapping, we deduce the existence of a limiting mixed Hodge structure; the argument is a pleasant mix of linear algebra and representation theory, and unlike in Schmid's paper, does not rely on the SL(2)-orbit theorem. (This is useful because there is a simplified proof for the SL(2)-orbit theorem by Cattani-Kaplan-Schmid [CKS86, §6], whose input is the existence of the limiting mixed Hodge structure.) We also show that, after a choice of splitting, the space of multi-valued flat sections becomes what we call a "polarized $\mathfrak{sl}_2$-Hodge structure"; this is the same kind of structure that one has on the cohomology of a compact Kähler manifold. Finally, we prove a cheap version of the SL(2)-orbit theorem that applies to an arbitrary splitting of the monodromy weight filtration (instead of the SL(2)-splitting). The degree of approximation is slightly worse than in Schmid's version, but in return, the proof is much easier. As an application, we describe the asymptotic behavior of the Hodge metric in the frame of multi-valued flat sections, something that is only implicit in Schmid's paper.

1.3 Acknowledgements

7. This paper has its immediate origins in a graduate course about variations of Hodge structure that Ch.S. taught at Stony Brook University in Fall 2019. We take this opportunity to thank all the students in the course for their interest, and for their patience while the instructor worked through the details of Schmid's paper. Ch.S. thanks Wilfried Schmid for answering some questions about the nilpotent orbit theorem; Takuro Mochizuki for a useful discussion about the definition of the limiting Hodge filtration; Dingxing Zhang for organizing a small reading seminar about Schmid's paper back in 2014; Ruijie Yang for helping with the calculations for harmonic bundles; and Bruno Klingler for several conversations about variations of Hodge structure and harmonic bundles. He also thanks Ya Deng for a very useful email exchange in 2020 about the nilpotent orbit theorem, and especially for sharing an early version of a preprint [Den22] that suggested the use of Hörmander's L^2-estimates in this context.

8. During the preparation of this paper, Ch.S. was partially supported by NSF grant DMS-1551677 and by a Simons Fellowship. He thanks the National Science Foundation and the Simons Foundation for their financial support. Large parts of this paper were written during a sabbatical stay at the Max-Planck-Institute for Mathematics and at the Kavli Institute for the Physics and Mathematics of the Universe, and Ch.S. thanks both of these institutions for providing him with excellent working conditions.

2 Overview of the results

2.1 Complex variations of Hodge structure

9. Since we are going to be working with complex Hodge structures, let us briefly recall the definition. For the purposes of this text, a **Hodge structure** of weight n on a complex vector space V is a decomposition

$$V = \bigoplus_{p+q=n} V^{p,q},$$

and a **polarization** is a hermitian pairing $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that the decomposition is orthogonal with respect to Q, and such that $(-1)^q Q$ is positive definite on the subspace

$V^{p,q}$. The **Hodge norm** of a vector $v \in V$ is

$$\|v\|^2 = \sum_{p+q=n} (-1)^q \, Q(v^{p,q}, v^{p,q}).$$

Unlike in the case of real Hodge structures, it takes *two* filtrations to describe an arbitrary complex Hodge structure: the usual Hodge filtration

$$F^p = F^p V = \bigoplus_{i \geq p} V^{i,n-i}$$

and the conjugate Hodge filtration

$$\overline{F}^q = \overline{F}^q V = \bigoplus_{j \geq q} V^{n-j,j},$$

because $V^{p,q} = F^p \cap \overline{F}^q$. When a polarization is given, the Hodge structure is still determined by the Hodge filtration $F = F^\bullet V$ alone: the reason is that

$$\overline{F}^q = \left\{ v \in V \;\middle|\; Q(v,x) = 0 \text{ for all } x \in F^{n-q+1} \right\}.$$

We are going to use the notation $\|v\|_F^2$ whenever we want to emphasize the dependence of the Hodge norm on the Hodge filtration.

Example. A Hodge structure is called **real** if $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for an $\mathbb{R}$-vector space $V_{\mathbb{R}}$, and if $\overline{F} = \sigma(F)$, where $\sigma \in \operatorname{End}_{\mathbb{R}}(V)$ is the conjugation operator $\sigma(v \otimes z) = v \otimes \overline{z}$. In the real case, one has the familiar Hodge symmetry $V^{q,p} = \sigma\big(V^{p,q}\big)$ for every $p + q = n$.

10. The paper contains a certain amount of analysis, and for that reason, it will be convenient to describe variations of Hodge structure in the language of smooth vector bundles. Let E be a smooth vector bundle on a complex manifold X. We denote by $A^k(X,E)$ the space of smooth k-forms with coefficients in E, and by $A^{i,j}(X,E)$ the space of smooth (i,j)-forms with coefficients in E. With this notation, a **variation of Hodge structure** of weight n on E is decomposition

$$E = \bigoplus_{p+q=n} E^{p,q}$$

into smooth subbundles, together with a flat connection $d \colon A^0(X,E) \to A^1(X,E)$ that maps each $A^0(X,E^{p,q})$ into the direct sum of the subspaces

$$A^{1,0}(X,E^{p,q}) \oplus A^{1,0}(X,E^{p-1,q+1}) \oplus A^{0,1}(X,E^{p,q}) \oplus A^{0,1}(X,E^{p+1,q-1}).$$

In the analysis that follows, the most important component of the connection turns out to be the **Higgs field**, which is the linear operator

$$\theta \colon A^0(X,E^{p,q}) \to A^{1,0}(X,E^{p-1,q+1}).$$

A **polarization** is a hermitian pairing

$$Q \colon A^0(X,E) \otimes_{A^0(X)} \overline{A^0(X,E)} \to A^0(X)$$

that is flat with respect to d, such that the expression

$$h(v,w) = \sum_{p+q=n} (-1)^q \, Q(v^{p,q}, w^{p,q}) \quad \text{for } v,w \in A^0(X,E)$$

defines a smooth hermitian metric on the bundle E, and the different subbundles $E^{p,q}$ are orthogonal to each other with respect to h. This metric is called the **Hodge metric** on the bundle E; for the sake of clarity, we sometimes denote it by the symbol h_E.

11. The relation with the more familiar holomorphic description comes from decomposing $d = d' + d''$ into its $(1,0)$-component $d' \colon A^0(X,E) \to A^{1,0}(X,E)$ and its $(0,1)$-component $d'' \colon A^0(X,E) \to A^{0,1}(X,E)$. Then d'' gives E the structure of a holomorphic vector bundle, which we denote by the symbol $\mathscr{E}$; and d' defines an integrable holomorphic connection $\nabla \colon \mathscr{E} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{E}$ on this bundle. The condition on d is saying that the Hodge bundles

$$F^p E = \bigoplus_{i \geq p} E^{i,n-i}$$

come from holomorphic subbundles $F^p \mathscr{E}$, and that the connection satisfies Griffiths' transversality relation $\nabla(F^p \mathscr{E}) \subseteq \Omega_X^1 \otimes_{\mathscr{O}_X} F^{p-1}\mathscr{E}$. From this point of view, the Higgs field is simply the holomorphic operator

$$\theta \colon F^p\mathscr{E}/F^{p+1}\mathscr{E} \to \Omega_X^1 \otimes_{\mathscr{O}_X} F^{p-1}\mathscr{E}/F^p\mathscr{E}$$

induced by the action of the connection ∇.

2.2 Multi-valued flat sections and monodromy

12. Our main concern in this paper is the asymptotic behavior of a polarized variation of Hodge structure on the punctured disk. Let

$$\Delta^* = \left\{ t \in \mathbb{C} \,\middle|\, 0 < |t| < 1 \right\}$$

be the punctured unit disk, and let E be a polarized variation of Hodge structure on Δ^*. In order to compare the Hodge structures on different fibers of the bundle E, we need a common reference frame; the most natural choice is the space of all multi-valued flat sections of the flat bundle (E, d).

13. Since the fundamental group of Δ^* is cyclic, a flat bundle (E,d) on the punctured disk is uniquely determined by the pair (V,T), where V is the complex vector space of all multi-valued flat sections, and $T \in \mathrm{GL}(V)$ is the monodromy transformation. To define V and T more precisely, recall that the universal covering space of Δ^* is naturally the *left* half-plane

$$\mathbb{H} = \left\{ z \in \mathbb{C} \,\middle|\, \operatorname{Re} z < 0 \right\},$$

which maps to Δ^* via the exponential function $\exp \colon \mathbb{H} \to \Delta^*$. Let V be the complex vector space of all **multi-valued flat sections** of E; concretely, V is defined to be the space of d-flat sections of the pullback bundle $\exp^* E$. The polarization on E induces a nondegenerate hermitian form

$$Q \colon V \otimes_{\mathbb{C}} \bar{V} \to \mathbb{C},$$

and translation by $2\pi i$ induces the **monodromy operator**

$$T = T_s \cdot T_u = T_s \cdot e^{2\pi i\, N} \in \mathrm{GL}(V).$$

Here T_s is diagonalizable and N is nilpotent, and one has

$$Q(T_s v, T_s w) = Q(v,w) \quad \text{and} \quad Q(Nv, w) = Q(v, Nw).$$

Note that our definition of the nilpotent operator N differs from Schmid's definition [Sch73, (4.6)] by a factor of $2\pi i$, because $\log T_u = 2\pi i N$. The advantage is that this makes N independent of the choice of $i = \sqrt{-1}$.

14. The fact that the flat bundle (E, d) is part of a polarized variation of Hodge structure puts the following restriction on the monodromy transformation (see §64 Proposition).

Proposition (Monodromy theorem)**.** *If $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $|\lambda| = 1$.*

For *integral* variations of Hodge structure, it follows (because of Kronecker's theorem) that all eigenvalues of T are roots of unity. This fact is traditionally called the "monodromy theorem"; Schmid [Sch73, Lem. 4.5] attributes it to Borel.

2.3 The Hodge norm estimates

15. Conceptually, the most important piece of the theory are the Hodge norm estimates, which describe the rate of growth of the pointwise Hodge norm for multi-valued flat sections. This result is the starting point of our treatment of the theory; in Schmid's work, it comes towards the end [Sch73, Thm. 6.6]. For a multi-valued flat section $v \in V$, the Hodge metric gives us a smooth function

$$h(v, v) = \sum_{p+q=n} (-1)^q \, Q(v^{p,q}, v^{p,q}) \colon \mathbb{H} \to [0, \infty).$$

The asymptotic behavior of this function as $|\operatorname{Re} z| \to \infty$ turns out to be surprisingly simple – in fact, it is completely controlled by the following elementary construction. The nilpotent operator $N \in \operatorname{End}(V)$ determines an increasing filtration $W_\bullet = W_\bullet V$, called the **monodromy weight filtration**. This filtration is uniquely characterized by the following two properties:

(a) For every $k \in \mathbb{Z}$, one has $N W_k \subseteq W_{k-2}$.

(b) For every $k \in \mathbb{N}$, the induced operator $N^k \colon \operatorname{gr}_k^W \to \operatorname{gr}_{-k}^W$ is an isomorphism.

An explicit formula for the monodromy weight filtration is

$$W_k = \sum_{j \in \mathbb{N}} N^j \left(\ker N^{k+2j+1} \right) \quad \text{for } k \in \mathbb{Z}.$$

The content of the Hodge norm estimates is that the monodromy weight filtration precisely captures the asymptotic behavior of the Hodge norm.

Theorem (Hodge norm estimates)**.** *For $v \in V$ any multi-valued flat section,*

$$v \in W_k \setminus W_{k-1} \quad \Longleftrightarrow \quad h(v, v) \sim |\operatorname{Re} z|^k,$$

as long as $\operatorname{Im} z$ stays in a fixed interval.

On the punctured disk, the statement is that $h(v, v) \sim |\log |t||^k$ on each angular sector.

16. The main ingredient of the proof is the following basic estimate [Sim90, §2] for the pointwise Hodge norm of the Higgs field θ, viewed as a smooth section of the induced variation of Hodge structure on the bundle $\operatorname{End}(E)$. It is analogous to the distance-decreasing property of period mappings used in Schmid's paper [Sch73, Cor. 3.17].

Theorem (Basic estimate). *One has the inequality*

$$h_{\mathrm{End}(E)}\big(\theta_{\partial/\partial t}, \theta_{\partial/\partial_t}\big) \leq \frac{C_0^2}{|t|^2(\log|t|)^2},$$

where $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$ *and* $r = \operatorname{rk} E$.

We use some elementary calculations with harmonic bundles, already done by Simpson [Sim90, §2], together with the fact that θ is nilpotent, to show that the Laplacian of the smooth function $f = h_{\mathrm{End}(E)}\big(\theta_{\partial/\partial t}, \theta_{\partial/\partial_t}\big)$ satisfies the differential inequality

$$\Delta \log f \geq 8f/C_0^2.$$

The basic estimate then follows by applying Ahlfors' lemma [Ahl38]. The value of the constant C_0 is optimal; more important than the exact value, however, is the fact that C_0 depends on nothing but the rank of the bundle E.

17. To prove the Hodge norm estimates, we first establish a special case.

Lemma. *If* $v \in V$ *is a multi-valued flat section with* $Tv = \lambda v$ *for some* $\lambda \in \mathbb{C}$, *then the function* $\varphi_v = \log h(v, v)$ *remains bounded from above as* $|\mathrm{Re}\, z| \to \infty$.

This is indeed a special case of the Hodge norm estimates, because $Tv = \lambda v$ implies $Nv = 0$, and therefore $v \in W_0$. Some further calculations with harmonic bundles, together with the basic estimate, show that:

1. The function φ_v is subharmonic on $\mathbb{H}$.
2. The first derivatives of φ_v are bounded by a constant times $|\mathrm{Re}\, z|^{-1}$.
3. One has $\varphi_v(z + 2\pi i) = \varphi_v(z)$ for every $z \in \mathbb{H}$.

By a simple calculus argument, these three properties are enough to conclude that φ_v is bounded from above as $|\mathrm{Re}\, z| \to \infty$.

18. This special case is already enough to prove that the asymptotic behavior of the Hodge norm only depends on the underlying flat bundle (E, d).

Theorem (Comparison theorem). *Let* E_1 *and* E_2 *be two polarized variations of Hodge structure on the punctured disk. If* $(E_1, d_1) \cong (E_2, d_2)$ *as flat bundles, then the Hodge metrics* h_1 *and* h_2 *are mutually bounded, up to a constant, as* $t \to 0$.

This follows by considering the induced variation of Hodge structure on the flat bundle $\mathrm{Hom}(E_1, E_2)$. To prove the Hodge norm estimates in general, we then proceed as follows. Using representation theory, we construct a "model" variation of Hodge structure on the given flat bundle (E, d), whose Hodge norm has the correct asymptotic behavior; the comparison theorem guarantees that, up to a constant, the Hodge norm on E has the same behavior. The construction of the models is done by decomposing V into irreducible representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and then explicitly constructing a model variation on each irreducible representation. The construction uses the theory of $\mathfrak{sl}_2$-Hodge structures, which we study in some depth in Chapter 5. (See also §35 in this chapter.)

Example. In the model variation of Hodge structure coming from the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$, the Hodge metric is given by the formula

$$\begin{pmatrix} |x| + y^2|x|^{-1} & -iy|x|^{-1} \\ iy|x|^{-1} & |x|^{-1} \end{pmatrix},$$

where $z = x + iy$. Since $(1,0) \in W_1$ and $(0,1) \in W_{-1}$, this is in complete agreement with the Hodge norm estimates.

2.4 Period domains and period mappings

19. Once we understand the behavior of the Hodge metric, the next question is what happens to the Hodge structures as $t \to 0$. The answer is best stated in the language of period domains and period mappings. The space of multivalued flat sections gives us a trivialization of the bundle $\exp^* E$, and so we can consider all the Hodge structures in the variation as living on the same vector space V. From our polarized variation of Hodge structure E, we then get a holomorphic **period mapping**

$$\Phi \colon \mathbb{H} \to D.$$

The **period domain** D parametrizes all Hodge structures of weight n on V, with fixed Hodge numbers $\dim V^{p,q}$, that are polarized by the pairing Q; a basic fact is that D is a homogeneous space for the real Lie group

$$G = \left\{ g \in \mathrm{GL}(V) \;\middle|\; Q(gv, gw) = Q(v,w) \text{ for all } v, w \in V \right\}.$$

The period domain is an open subset of the so-called **compact dual** $\check{D}$, which parametrizes all decreasing filtrations $F^\bullet$ on V such that $\dim F^p = \dim V^{p,q} + \dim V^{p+1,q-1} + \cdots$ for all $p \in \mathbb{Z}$. The compact dual is a projective complex manifold, and a homogeneous space for the complex Lie group $\mathrm{GL}(V)$; the complex structure on the period domain D comes from the embedding $D \subseteq \check{D}$.

20. The period mapping is holomorphic, and its differential is represented by the Higgs field θ; see §80 Lemma for the precise statement. Moreover, the monodromy transformation $T \in G$ is defined in such a way that

$$\Phi(z + 2\pi i) = T \cdot \Phi(z).$$

The eigenvalues of $T = T_s T_u$ lie on the unit circle, and so we can write $T_s = e^{2\pi i\, S}$, where $S \in \mathrm{End}(V)$ is a semisimple operator with real eigenvalues in a fixed half-open interval of length 1; then $T = e^{2\pi i (S+N)}$. One can interpret the operator $S + N$ as the residue of the connection ∇ on the canonical extension of the holomorphic vector bundle $\mathscr{E}$; see §177.

2.5 Convergence of the rescaled period mapping

21. Now we come to a crucial point in the argument. The Hodge norm estimates

$$v \in W_k \setminus W_{k-1} \quad \Longleftrightarrow \quad \|v\|^2_{\Phi(z)} \sim |\mathrm{Re}\, z|^k$$

suggest "rescaling" the period mapping, in order to even out the behavior of the Hodge norm on different parts of the weight filtration. To do this efficiently, we choose a **splitting** $H \in \mathrm{End}(V)$ for the weight filtration, with the following three properties:

1. H is semisimple, with integer eigenvalues.
2. $W_k = E_k(H) \oplus W_{k-1}$ and $[H,N] = -2N$.
3. $[H, T_s] = 0$ and $Q(Hv,w) + Q(v,Hw) = 0$ for all $v, w \in V$.

It is not hard to show that such a splitting always exists (see §139 Proposition); in general, it is far from unique. For a nonzero multi-valued flat section $v \in E_k(H)$, we then have

$$\|v\|^2_{\Phi(z)} \sim |\mathrm{Re}\, z|^k \qquad \text{and} \qquad e^{-\frac{1}{2}\log|\mathrm{Re}\, z|\, H} v = |\mathrm{Re}\, z|^{-\frac{k}{2}} v$$

Consequently, the Hodge norm of $e^{-\frac{1}{2}\log|\mathrm{Re}\, z|\, H} v$ stays bounded as $|\mathrm{Re}\, z| \to \infty$. We can achieve the same effect by moving the Hodge structures by the operator $e^{\frac{1}{2}\log|\mathrm{Re}\, z|\, H}$. Either way, the conclusion is that the **rescaled period mapping**

$$\hat{\Phi}_{S,H} \colon \mathbb{H} \to D, \quad \hat{\Phi}_{S,H}(z) = e^{\frac{1}{2}\log|\mathrm{Re}\, z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \Phi(z), \tag{21.1}$$

is invariant under the substitution $z \mapsto z + 2\pi i$, and the Hodge norm of every multi-valued flat section is uniformly bounded as $\mathrm{Re}\, z \to -\infty$. The two exponential factors are elements of the real Lie group G; this is the reason why the rescaled period mapping stays in D.

22. The second key result is that, after rescaling, the polarized Hodge structures now converge to a well-defined limit. As indicated by the notation, the limit does depend on the choice of splitting: different splittings lead to different limits.

Theorem (Convergence of the rescaled period mapping)**.** *For any choice of splitting $H \in \mathrm{End}(V)$ of the monodromy weight filtration as above, the limit*

$$e^{-N} F_H = \lim_{\mathrm{Re}\, z \to -\infty} \hat{\Phi}_{S,H}(z) \in D$$

exists in the period domain. Moreover, the filtration $F_H \in \check{D}$ satisfies

$$T_s(F_H^\bullet) \subseteq F_H^\bullet, \quad H(F_H^\bullet) \subseteq F_H^\bullet, \quad N(F_H^\bullet) \subseteq F_H^{\bullet-1}.$$

Together with the Hodge norm estimates, this is saying that a polarized variation of Hodge structure on the punctured disk behaves in a very simple way. Namely, as $t \to 0$, the Hodge metric grows (and decays) at several different rates, and this of course prevents the Hodge structures from approaching any sort of limit. But once we rescale to even out the behavior of the Hodge metric, *both* the Hodge metric and the Hodge structures converge.

2.6 The nilpotent orbit theorem

23. The most difficult step in proving the convergence of the rescaled period mapping is the **nilpotent orbit theorem**, an important result in its own right. Recall that the period mapping satisfies

$$\Phi(z + 2\pi i) = T \cdot \Phi(z),$$

and that we have $T = e^{2\pi i(S+N)}$, where $S \in \mathrm{End}(V)$ is semisimple with real eigenvalues in a fixed interval of length < 1. Consequently, the mapping

$$\mathbb{H} \to \check{D}, \quad z \mapsto e^{-z(S+N)} \Phi(z),$$

is invariant under $z \mapsto z + 2\pi i$, and therefore descends to a holomorphic mapping

$$\Psi_S \colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)} \Phi(z),$$

sometimes called the **untwisted period mapping**. The exponential factor $e^{-z(S+N)}$ is now an element of the complex Lie group $\mathrm{GL}(V)$, and so the untwisted period mapping takes values in the compact dual $\check{D}$.

24. The nilpotent orbit theorem has two parts. The first half is a convergence statement: it says that Ψ_S extends holomorphically over the origin.

Theorem (Nilpotent orbit theorem, convergence)**.** *The holomorphic mapping*

$$\Psi_S \colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)} \Phi(z),$$

extends holomorphically across the origin, and the limit filtration $\Psi_S(0) \in \check{D}$ has the property that $(S+N)\Psi_S^\bullet(0) \subseteq \Psi_S^{\bullet-1}(0)$.

The limit $\Psi_S(0)$ in the nilpotent orbit theorem depends on the choice of S, but one can show that the **limiting Hodge filtration**

$$F_{\mathrm{lim}} = \lim_{\operatorname{Re} z \to -\infty} e^{-zN} \Phi(z) = \lim_{\operatorname{Re} z \to -\infty} e^{-|\operatorname{Re} z| \, S} \Psi_S(0) \in \check{D}$$

is independent of S. It satisfies

$$T_s(F_{\mathrm{lim}}^\bullet) \subseteq F_{\mathrm{lim}}^\bullet \quad \text{and} \quad N(F_{\mathrm{lim}}^\bullet) \subseteq F_{\mathrm{lim}}^{\bullet-1}.$$

The limiting Hodge filtration does not come from a polarized Hodge structure on V (except when $N = 0$); instead, it is part of a *mixed Hodge structure*, whose weight filtration is given by the monodromy weight filtration $W_{\bullet-n}$ from above. We are going to discuss this point in more detail in Section 2.7 below.

25. We can now explain why the rescaled period mapping $\hat{\Phi}_{S,H} \colon \mathbb{H} \to D$ converges to a limit in D. The argument needs both the Hodge norm estimates (in §15 Theorem) and the nilpotent orbit theorem (in §24 Theorem). The identities $[H, N] = -2N$ and $[H, S] = 0$ can be used to write

$$\hat{\Phi}_{S,H}(z) = e^{-N} \cdot e^{\frac{1}{2}\log|\operatorname{Re} z| \, H} \, e^{-|\operatorname{Re} z| \, S} \, \Psi_S(e^z).$$

The nilpotent orbit theorem implies that $\Psi_S(e^z) \to \Psi_S(0)$ at a rate proportional to $|e^z| = e^{-|\operatorname{Re} z|}$. By analyzing the effect of the two exponential factors, and by using the fact that the eigenvalues of S lie in an interval of length < 1, one deduces the existence of the limit

$$F_H = \lim_{\operatorname{Re} z \to -\infty} e^{\frac{1}{2}\log|\operatorname{Re} z| \, H} \, e^{-|\operatorname{Re} z| \, S} \, \Psi_S(e^z) \in \check{D}.$$

The filtration $F_H \in \check{D}$ is derived from the limit $\Psi_S(0) \in \check{D}$ in the nilpotent orbit theorem in two stages:

1. The effect of the first limit

$$F_{\mathrm{lim}} = \lim_{\operatorname{Re} z \to -\infty} e^{-|\operatorname{Re} z| \, S} \Psi_S(0)$$

   is to make the filtration $\Psi_S(0)$ compatible with the eigenspace decomposition of T_s, by projecting to the subquotients of the filtration by decreasing eigenvalues of S. (Decreasing because of the minus sign in the exponent.)

2. The effect of the second limit

$$F_H = \lim_{\operatorname{Re} z \to -\infty} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} F_{\text{lim}}$$

is to make the filtration F_{lim} compatible with the eigenspace decomposition of H, by projecting to the subquotients of the weight filtration $W_\bullet$ (which is the filtration by increasing eigenvalues of H).

It follows that $\hat{\Phi}_{S,H}(z)$ converges to $e^{-N} F_H \in \check{D}$ as $\operatorname{Re} z \to -\infty$. At the same time, the Hodge norm estimates tell us that the rescaled Hodge metric is comparable to a constant metric as $\operatorname{Re} z \to -\infty$. These two facts together imply quite easily that $e^{-N} F_H \in D$.

26. The nilpotent orbit theorem also has a second half, which says that the original period mapping is very closely approximated by the **nilpotent orbit**

$$\Phi_{\text{nil}} \colon \mathbb{H} \to \check{D}, \quad \Phi_{\text{nil}}(z) = e^{zN} F_{\text{lim}}.$$

In fact, the nilpotent orbit is itself the period mapping of a polarized variation of Hodge structure (on a punctured disk of smaller radius).

Theorem (Nilpotent orbit theorem, approximation). *There are constants $C > 0$, $x_0 < 0$, and $m \in \mathbb{N}$, such that*

$$\Phi_{\text{nil}}(z) \in D \quad \text{and} \quad d_D\big(\Phi(z), \Phi_{\text{nil}}(z)\big) \le C|\operatorname{Re} z|^m e^{-\delta(T)|\operatorname{Re} z|}$$

for every $z \in \mathbb{H}$ with $\operatorname{Re} z \le x_0$. The constants C, x_0 only depend on the base point $o \in D$ and on the minimal polynomial of $T \in \operatorname{GL}(V)$; the integer m only depends on $r = \operatorname{rk} E$.

Here d_D is the G-invariant distance function on the period domain, defined by the Hodge metric (see (74.2)). The distance between the period mapping $\Phi(z)$ and the nilpotent orbit $\Phi_{\text{nil}}(z)$ is exponentially small; the constant $\delta(T)$ in the exponent is the minimal distance between consecutive eigenvalues of T (on the unit circle), divided by 2π. For developing the theory in more than one variable, it is very important that the values of the constants are essentially independent of the variation of Hodge structure.

27. When $N = 0$, the nilpotent orbit theorem is asserting that $F_{\text{lim}} \in D$; in other words, when the weight filtration is trivial, F_{lim} is the Hodge filtration of a polarized Hodge structure of weight n. When the weight filtration is not trivial, what we get instead is a limiting mixed Hodge structure (see §39 Theorem).

Note. This is not true for $\Psi_S(0)$: even when $N = 0$, the filtration $\Psi_S(0)$ is not necessarily the Hodge filtration of a polarized Hodge structure (see §190 Example).

28. The geometric meaning of the nilpotent orbit theorem is the following. Recall that the operator $S \in \operatorname{End}(V)$ depended on a choice of half-open interval I of length 1. The holomorphic vector bundle $\mathscr{E}$ has a **canonical extension** to a holomorphic vector bundle $\mathscr{E}_I$ on the disk. Up to isomorphism, it is characterized by the fact that the holomorphic connection ∇ becomes a logarithmic connection

$$\nabla \colon \mathscr{E}_I \to \Omega^1_\Delta(\log 0) \otimes_{\mathscr{O}_\Delta} \mathscr{E}_I$$

whose residue $\operatorname{Res}_0 \nabla$ has real eigenvalues in the interval I. There is a distinguished trivialization $\mathscr{E}_I \cong \mathscr{O}_\Delta \otimes_{\mathbb{C}} V$ in which

$$\nabla(v \otimes 1) = \frac{dt}{t} \otimes (S + N)v.$$

The content of §24 Theorem is that the Hodge bundles $F^p\mathscr{E}$ extends to holomorphic subbundles $F^p\mathscr{E}_I$ of the canonical extension; and that the Higgs field

$$\theta\colon F^p\mathscr{E}_I/F^{p+1}\mathscr{E}_I \to \Omega^1_\Delta(\log 0)\otimes_{\mathscr{O}_\Delta} F^{p-1}\mathscr{E}_I/F^p\mathscr{E}_I$$

has a logarithmic pole at $t=0$, with residue $S+N$. The content of §26 Theorem is that if we replace the extended Hodge bundles $F^p\mathscr{E}_I$ by trivial subbundles with fiber $\Psi^p_S(0)\subseteq V$, then we still get a polarized variation of Hodge structure of weight n, at least on a punctured disk of a fixed smaller radius.

29. Our proof of the nilpotent orbit theorem is quite different from Schmid's original argument [Sch73, §8]. It uses in a crucial way the curvature properties of the Hodge metric. A short calculation (see §50 Proposition) shows that the curvature tensor of the Hodge metric h_E on the bundle $E^{p,q}$ is given by $\Theta = -(\theta\theta^* + \theta^*\theta)$, where

$$\theta\colon A^0(\Delta^*, E^{p,q}) \to A^{1,0}(\Delta^*, E^{p-1,q+1})$$
$$\theta^*\colon A^0(\Delta^*, E^{p,q}) \to A^{0,1}(\Delta^*, E^{p+1,q-1})$$

are the Higgs field and its adjoint. (There are similar formulas for the curvature tensor on the Hodge bundles F^pE and on the quotient bundles E/F^pE.) Consequently, the expression

$$h_E\big(\Theta_{\partial/\partial t\wedge\partial/\partial\bar t}u,u\big) = h_E\big(\theta_{\partial/\partial t}u, \theta_{\partial/\partial t}u\big) - h_E\big(\theta^*_{\partial/\partial\bar t}u, \theta^*_{\partial/\partial\bar t}u\big)$$

is in general neither positive nor negative definite. However, because of the basic estimate (in §16 Theorem), any metric of the form

$$h_E\cdot|t|^a(-\log|t|)^b \quad \text{with } a\in\mathbb{R} \text{ and } b\in\mathbb{Z}$$

will have positive curvature for $b\gg 0$, and negative curvature for $b\ll 0$. This insight already appears in Simpson's work [Sim88, §10], who attributes it to the important paper by Cornalba and Griffiths [CG75, p. 29]. We use both of these properties in our proof.

30. Let us now outline the steps in our proof of the nilpotent orbit theorem. The nilpotent orbit theorem can also be deduced from Mochizuki's work on harmonic bundles [Moc02]; another analytic proof using L^2-estimates has recently been given by Deng [Den22].

1. The bundle $\operatorname{End}(E)$ inherits a polarized variation of Hodge structure of weight 0. The Higgs field $\theta_{\partial/\partial t}$ is a holomorphic section of $\operatorname{End}(E)^{-1,1}$, and the basic estimate tells us that its pointwise Hodge norm is bounded by a constant multiple of $|t|^{-2}(\log|t|)^{-2}$. The first step is to construct a lifting of $t\theta_{\partial/\partial t}$ to a holomorphic section ϑ of the Hodge bundle $F^{-1}\operatorname{End}(E)$, whose L^2-norm is controlled by an inequality of the form

$$\int_{\Delta^*} h_{\operatorname{End}(E)}(\vartheta,\vartheta)|t|^a(-\log|t|)^b\,d\mu \le C$$

where $a>-2$ and $b\gg 0$. This uses Hörmander's L^2-estimates, for which we include an elementary proof (in Section 7.1); the key point is that the metric $h_{\operatorname{End}(E)}\cdot|t|^a(-\log|t|)^b$ has positive curvature for $b\gg 0$. Using L^2-estimates to extend holomorphic bundles goes back to Cornalba-Griffiths [CG75], and in the setting of harmonic bundles, to Simpson [Sim88].

2. Pulling back to $\mathbb{H}$ gives us a holomorphic mapping $\vartheta\colon \mathbb{H} \to \operatorname{End}(V)$ that satisfies $\vartheta(z + 2\pi i) = T\vartheta(z)T^{-1}$. After untwisting, we get

$$B\colon \Delta^* \to \operatorname{End}(V), \quad B(e^z) = e^{-z(S+N)}\vartheta(z)e^{z(S+N)}.$$

Since the eigenvalues of S lie in an interval of length < 1, the L^2-estimate from the first step can be used to show that B is square-integrable on a punctured neighborhood of the origin, and therefore extends holomorphically to the entire disk.

3. We can now describe the differential of the untwisted period mapping Ψ_S using holomorphic data. The holomorphic tangent space to the complex manifold $\check{D}$ at the point $\Phi(z)$ is isomorphic to $\operatorname{End}(V)/F^0 \operatorname{End}(V)_{\Phi(z)}$, and the differential of the period mapping $\Phi\colon \mathbb{H} \to \check{D}$ is equal to

$$\theta_{\partial/\partial z} \mod F^0 \operatorname{End}(V)_{\Phi(z)}.$$

A short calculation with derivatives shows that the differential of the mapping $z \mapsto \Psi_S(e^z)$ is therefore equal to

$$\begin{aligned} &e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)} - (S+N) \\ &\qquad\equiv B(e^z) - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)}. \end{aligned}$$

The point is that the operator on the right-hand side is holomorphic.

4. Let $g\colon \mathbb{H} \to \operatorname{GL}(V)$ be the unique holomorphic solution of the ordinary differential equation

$$g'(z) = \Big(B(e^z) - (S+N)\Big)g(z),$$

subject to the initial condition $g(-1) = \mathrm{id}$. A short calculation shows that $g(z)^{-1}\Psi_S(e^z)$ must be constant, which means that

$$\Psi_S(e^z) = g(z) \cdot \Psi_S(e^{-1}).$$

The differential equation has a *regular* singular point at $t = 0$, and by the basic theory of Fuchsian differential equations, one has

$$g(z) = M(e^z) \cdot e^{zA},$$

with $M\colon \Delta^* \to \operatorname{GL}(V)$ meromorphic and $A \in \operatorname{End}(V)$ constant. Since the untwisted period mapping $\Psi_S(e^z)$ is single-valued, it follows that

$$\Psi_S(t) = M(t) \cdot \Psi_S(e^{-1}),$$

and because $\check{D}$ is a projective complex manifold, this is enough to conclude that Ψ_S extends holomorphically across the origin. (The close relationship between the nilpotent orbit theorem and regularity was already observed by Griffiths and Schmid, see [GS75, §9a] and the references cited there.)

5. The remainder of the argument consists in deriving effective estimates for the rate of convergence. Let P_λ denote the projection to the eigenspace $E_\lambda(T_s)$. The fact that

$\Psi_S(t)$ converges as $t \to 0$ can be used to show that

$$\|\theta_{\partial/\partial z} - N^{-1,1}\|^2_{\Phi(z)} + \sum_{k \leq -2} \|N^{k,-k}\|^2_{\Phi(z)} \leq C |\mathrm{Re}\, z|^{2m} e^{-2\delta(T)|\mathrm{Re}\, z|}$$
$$\sum_{k \leq -1} \|P_\lambda^{k,-k}\|^2_{\Phi(z)} \leq C |\mathrm{Re}\, z|^{2m} e^{-2\delta(T)|\mathrm{Re}\, z|}$$

for $|\mathrm{Re}\, z| \gg 0$; the exponent m only depends on $r = \mathrm{rk}\, E$, but we have no control over the value of the constant C. We then use the fact that the metric $h_{\mathrm{End}(E)} \cdot (-\log|t|)^b$ has negative curvature for $b \ll 0$, together with the maximum principle for subharmonic functions, to deduce that the above estimates actually hold on any half-plane of the form $\mathrm{Re}\, z \leq x < 0$, with a constant C that only depends on x, r, and on the minimal polynomial of $T \in \mathrm{GL}(V)$.

6. Finally, we consider the curve

$$[0, \infty) \to \check{D}, \quad x \mapsto e^{xN} \Phi(z - x),$$

joining the two points $\Phi(z)$ and $\Phi_{\mathrm{nil}}(z)$. Its derivative is controlled by the inequalities in the previous step. After integration, we obtain a bound for the distance in $\check{D}$ between these two points, and together with the Hodge norm estimates, this gives us the desired bound for their distance in D, provided that $|\mathrm{Re}\, z|$ is sufficiently large.

31. The estimates in the fifth step are saying that, modulo $F^0 \,\mathrm{End}(V)_{\Phi(z)}$, the Higgs field $\theta_{\partial/\partial z}$ converges to the nilpotent operator N at a rate that is exponential in $|\mathrm{Re}\, z|$. Moreover, the eigenspace decomposition

$$V = \bigoplus_{|\lambda|=1} E_\lambda(T_s)$$

becomes orthogonal in the limit, at a rate that is exponential in $|\mathrm{Re}\, z|$. Since the limiting Hodge filtration F_{lim} is compatible with this decomposition, both of these facts are of course also contained in the statement of the nilpotent orbit theorem itself.

2.7 The limiting mixed Hodge structure

32. Probably the most widely known result from Schmid's paper [Sch73, §6] is that, in the limit, a polarized variation of Hodge structure on the punctured disk gives rise to a mixed Hodge structure on V. The weight filtration of this "limiting" mixed Hodge structure is the shifted monodromy weight filtration $W_{\bullet - n}$, and the Hodge filtration is the limiting Hodge filtration F_{lim}. Schmid obtained this result as a consequence of his SL(2)-orbit theorem.

33. We take a somewhat different approach here, and deduce the existence of the limiting mixed Hodge structure directly from the convergence of the rescaled period mapping $\hat{\Phi}_{S,H} \colon \mathbb{H} \to D$. In fact, we prove that the vector space V is an example of a **polarized $\mathfrak{sl}_2$-Hodge structure**; this is the same kind of structure that one finds on the cohomology of a compact Kähler manifold. The proof is elementary and only uses linear algebra and some basic representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, but no further analysis.

34. Let us briefly recall how $\mathfrak{sl}_2(\mathbb{C})$ enters into the picture. Recall that this Lie algebra is 3-dimensional, and is generated by the three matrices

$$\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

subject to the relations $[\mathsf{H}, \mathsf{X}] = 2\mathsf{X}$, $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$, and $[\mathsf{X}, \mathsf{Y}] = \mathsf{H}$. In any finite-dimensional representation, the operator H acts semisimply with integer eigenvalues, and the eigenspace $E_k(\mathsf{H})$ are called the **weight spaces**. The splitting H that we chose in §21 uniquely determines a representation

$$\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}(V)$$

of this Lie algebra such that $\rho(\mathsf{H}) = H$ and $\rho(\mathsf{Y}) = -N$; the minus sign is dictated by our sign conventions for Hodge structures. It lifts to a representation of the Lie group $\mathrm{SL}_2(\mathbb{C})$; in particular, we have the **Weil element**

$$\mathsf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}),$$

which can also be written as $\mathsf{w} = e^{\mathsf{X}} e^{-\mathsf{Y}} e^{\mathsf{X}}$. The Weil element is a sort of linear algebra analogue of the Hodge $*$-operator on a compact Kähler manifold: it satisfies $\mathsf{w}^2 = (-1)^{\mathsf{H}}$, and induces isomorphisms between the two weight spaces $E_{\pm k}(\mathsf{H})$. Using a dagger to denote the adjoint of an operator with respect to the nondegenerate pairing Q, it is easy to see that $\mathsf{H}^\dagger = -\mathsf{H}$, $\mathsf{Y}^\dagger = \mathsf{Y}$, $\mathsf{X}^\dagger = \mathsf{X}$, and hence also $\mathsf{w}^\dagger = \mathsf{w}$.

35. We can now define $\mathfrak{sl}_2$-Hodge structures and their polarizations. The definition is modeled on the cohomology $H^{n+\bullet}(X, \mathbb{C})$ of an n-dimensional compact Kähler manifold, but perhaps surprisingly, it is equally relevant in our context. An **$\mathfrak{sl}_2$-Hodge structure** on a finite-dimensional complex vector space V is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on V with the following properties:

(a) Each weight space $V_k = E_k(\mathsf{H})$ has a Hodge structure of weight $n+k$; the integer n is called the **weight** of the $\mathfrak{sl}_2$-Hodge structure.

(b) The two operators

$$\mathsf{X} \colon V_k \to V_{k+2}(1) \quad \text{and} \quad \mathsf{Y} \colon V_k \to V_{k-2}(-1)$$

are morphisms of Hodge structures.

In the literature, this kind of structure is also called a "Hodge-Lefschetz structure" [Cat08] or an "$\mathbb{R}$-split mixed Hodge structure" [CKS86], but we feel that the name "$\mathfrak{sl}_2$-Hodge structure" is more descriptive.

36. The polarization on the cohomology of a compact Kähler manifold is usually discussed only for the primitive cohomology (which is the kernel of Y, in our notation). The standard definition is that, for every $k \geq 0$, the hermitian pairing

$$Q \circ \left(\mathrm{id} \otimes \mathsf{X}^k\right) \colon V_{-k} \otimes_{\mathbb{C}} \overline{V_{-k}} \to \mathbb{C}$$

should polarize the Hodge structure of weight $n-k$ on the primitive subspace $\ker \mathsf{Y} \colon V_{-k} \to V_{-(k+2)}$. Since w acts on the primitive subspace as $\frac{1}{k!}\mathsf{X}^k$, one can define polarizations more compactly with the help of the Weil element. Following Deligne [Del84], we say that a hermitian form

$$Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$$

is a **polarization** of the $\mathfrak{sl}_2$-Hodge structure V if it satisfies the four identities

$$Q \circ (\mathsf{H} \otimes \mathrm{id}) = -Q \circ (\mathrm{id} \otimes \mathsf{H}), \qquad Q \circ (\mathsf{X} \otimes \mathrm{id}) = Q \circ (\mathrm{id} \otimes \mathsf{X}),$$
$$Q \circ (\mathsf{Y} \otimes \mathrm{id}) = Q \circ (\mathrm{id} \otimes \mathsf{Y}), \qquad Q \circ (\mathsf{w} \otimes \mathrm{id}) = Q \circ (\mathrm{id} \otimes \mathsf{w}),$$

and if $Q \circ (\mathrm{id} \otimes \mathsf{w})$ polarizes the Hodge structure of weight $n+k$ on each V_k.

37. Because the Weil element w acts as $\frac{1}{k!}(-\mathsf{Y})^k$ on the "other" primitive subspace $\ker \mathsf{X}\colon V_k \to V_{k+2}$ for $k \geq 0$, an equivalent formulation is that

$$Q \circ \bigl(\mathrm{id} \otimes (-\mathsf{Y})^k\bigr)\colon V_k \otimes_{\mathbb{C}} \overline{V_k} \to \mathbb{C}$$

should polarize the Hodge structure of weight $n+k$ on $\ker \mathsf{X}$. Since $\rho(-\mathsf{Y}) = N$ in our case, this matches Schmid's definition [Sch73, Lem. 6.4].

38. We now describe how the limiting $\mathfrak{sl}_2$-Hodge structure on V comes about. The convergence of the rescaled period mapping (in §22 Theorem) gives us a filtration $F_H \in \check{D}$ with the following properties:

(a) The filtration $e^{-N} F_H \in D$ is the Hodge filtration of a Hodge structure of weight n, polarized by the pairing Q.

(b) One has $N(F_H^{\bullet}) \subseteq F_H^{\bullet-1}$, $H(F_H^{\bullet}) \subseteq F_H^{\bullet}$, and $T_s(F_H^{\bullet}) \subseteq F_H^{\bullet}$.

From these properties, we deduce by a mix of representation theory and linear algebra that F_H is the "total" Hodge filtration of an $\mathfrak{sl}_2$-Hodge structure of weight n on V.

Theorem (Limiting $\mathfrak{sl}_2$-Hodge structure)**.** *The vector space V has a polarized $\mathfrak{sl}_2$-Hodge structure of weight n, whose Hodge filtration is F_H. Concretely, this means the following:*

1. *Each weight space $V_k = E_k(H)$ has a Hodge structure of weight $n+k$, whose Hodge filtration is $V_k \cap F_H$.*

2. *The operator $N\colon V_k \to V_{k-2}(-1)$ is a morphism of Hodge structures.*

3. *The Hodge structure on each primitive subspace $\ker N^{k+1}\colon V_k \to V_{-k-2}$ is polarized by the hermitian pairing $Q(-, N^k -)$.*

4. *The operator T_s is an endomorphism of the polarized $\mathfrak{sl}_2$-Hodge structure.*

39. Recall from §25 that the filtration F_H is obtained from the limiting Hodge filtration $F_{\lim}$ by the following procedure: $F_{\lim}$ induces a filtration on each subquotient W_k/W_{k-1}, and $V_k \cap F_H$ is gotten by pulling back along the isomorphism $V_k \cong W_k/W_{k-1}$. The fact that each weight space V_k has a Hodge structure of weight $n+k$ therefore means that $W_{\bullet - n}$ is the weight filtration of a mixed Hodge structure [Sch73, Thm. 6.16].

Theorem (Limiting mixed Hodge structure)**.** *The vector space V has a mixed Hodge structure, with weight filtration $W_{\bullet-n}$ and Hodge filtration $F_{\lim}$; the associated graded of this mixed Hodge structure is a polarized $\mathfrak{sl}_2$-Hodge structure of weight n with $\mathsf{Y} = -N$ and polarization Q. The two operators*

$$N\colon V \to V(-1) \quad \text{and} \quad T_s\colon V \to V$$

are morphisms of mixed Hodge structures.

The $\mathfrak{sl}_2$-Hodge structure depends on the choice of splitting H; the limiting mixed Hodge structure is independent of it. Without choosing a splitting, one gets a canonical polarized $\mathfrak{sl}_2$-Hodge structure on the associated graded $\operatorname{gr}^W_\bullet$ with respect to the monodromy weight filtration. In the terminology of [CKS86, Def. 2.26], the limiting mixed Hodge structure is therefore "polarized" by the pairing Q and the nilpotent operator N.

2.8 Asymptotic behavior of the Hodge metric

40. Finally, we describe the asymptotic behavior of the Hodge metric $h(v, v')$ where $v, v' \in V$ are two multi-valued flat sections. The Hodge norm estimates cover the case $v = v'$, but something new happens when $v \neq v'$, namely that the Hodge metric $h(v, v')$ grows more slowly than expected as $|\operatorname{Re} z| \to \infty$. This result does not appear as such in Schmid's paper, but it is implicitly contained in the statement about Siegel sets in [Sch73, Cor. 5.29].

Theorem (Asymptotic behavior of the Hodge metric)**.** *For any two multi-valued flat sections $v, v' \in V$, there is a constant $C(v, v') > 0$ such that*

$$\bigl|\langle v, v'\rangle_{\Phi(z)}\bigr| \leq C(v, v') \cdot \min\Bigl(\|v\|^2_{\Phi(z)}, \|v'\|^2_{\Phi(z)}\Bigr)$$

for all sufficiently large values of $|\operatorname{Re} z|$.

An equivalent formulation is that if $v \in V_k$ and $v' \in V_{k'}$, then

$$\langle v, v'\rangle_{\Phi(z)} = O\bigl(|\operatorname{Re} z|^{\min(k,k')}\bigr)$$

which is typically much smaller than the $|\operatorname{Re} z|^{\frac{1}{2}(k+k')}$ one would expect based on the Hodge norm estimates. We can also restate the theorem in terms of the rescaled period mapping from (21.1): the result is that if $v \in V_k$ and $v' \in V_{k'}$ belong to different weight spaces, then

$$\langle v, v'\rangle_{\hat{\Phi}_{S,H}(z)} = O\bigl(|\operatorname{Re} z|^{-\frac{1}{2}|k-k'|}\bigr). \tag{40.1}$$

Since different weight spaces are orthogonal to each other in the Hodge structure $F_\sharp = e^{-N} F_H \in D$, this is not unexpected; but not only are the weight spaces becoming orthogonal in the limit, they are doing so at a rate that depends on the difference between their weights.

41. We end the introduction with a brief sketch of the proof. Up to an error of size $e^{-\varepsilon|\operatorname{Re} z|}$, which is negligible in this context, the nilpotent orbit theorem allows us to assume that the period mapping has the form $\Phi(z) = e^{zN} F_{\lim}$. The rescaled period mapping then looks like

$$\hat{\Phi}_H(z) = e^{-N} \cdot e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} F_{\lim}.$$

This expression only depends on $|\operatorname{Re} z|$; since we are interested in its behavior as $|\operatorname{Re} z| \to \infty$, we make the substitution $u = |\operatorname{Re} z|^{-\frac{1}{2}}$, so that

$$\hat{\Phi}_H\bigl(1/u^2\bigr) = e^{-N} \cdot e^{-(\log u)H} F_{\lim}.$$

Recall from §25 that we have

$$\lim_{u \to 0} e^{-(\log u)H} F_{\lim} = F_H, \tag{41.1}$$

and hence that $\hat{\Phi}_H\bigl(1/u^2\bigr)$ converges to $F_\sharp = e^{-N} F_H$ as $u \to 0$.

42. To get a handle on the rate of convergence, we prove the following cheap version of the **SL(2)-orbit theorem**. Schmid's original result [Sch73, Thm. 5.13] has two functions: it provides a distinguished splitting H for the weight filtration, called the "SL(2)-splitting" (or "canonical splitting"); and it gives a very precise description, using power series with values in the real group G, for the rate of convergence of (41.1). Here we focus on the second aspect, using an arbitrary splitting H chosen as in §139.

Theorem (Cheap SL(2)-orbit theorem)**.** *There is a convergent power series*

$$g\colon (-\varepsilon, \varepsilon) \to G, \quad g(u) = \mathrm{id} + u g_1 + u^2 g_2 + \cdots,$$

defined for some $\varepsilon > 0$*, such that for* $0 < |u| < \varepsilon$*, one has*

$$e^{-N} \cdot e^{-(\log u)H} F_{\mathrm{lim}} = g(u)^{-1} \cdot e^{-N} F_H.$$

The coefficients $g_n \in \mathrm{End}(V)$ *of the series are* $(S_n \oplus S_{n-2} \oplus \cdots)$*-isotypical with respect to the action of* $\mathfrak{sl}_2(\mathbb{C})$ *on* $\mathrm{End}(V)$*.*

The point is that we can then write the rescaled period mapping

$$\hat{\Phi}_H\big(1/u^2\big) = g(u)^{-1} \cdot F_\sharp$$

in terms of the real group G. Now suppose that $v \in V_k$ and $v' \in V_{k'}$. Since different weight spaces are orthogonal in the Hodge structure $F_\sharp$, and since §42 Theorem gives us some control on how much the coefficients g_n can change the weight of a given vector, it is an easy matter to deduce that

$$\begin{aligned} \langle v, v' \rangle_{\hat{\Phi}_H(1/u^2)} &= \big\langle g(u) \cdot v,\, g(u) \cdot v' \big\rangle_{F_\sharp} \\ &= \sum_{m+n \geq |k-k'|} u^{m+n} \, \big\langle g_m v,\, g_n v' \big\rangle_{F_\sharp} = O\big(u^{|k-k'|}\big), \end{aligned}$$

which is equivalent to (40.1).

43. The original proof of the SL(2)-orbit theorem in [Sch73, §9] is very difficult; and even the simplified argument in [CKS86, §6] (assuming the limiting mixed Hodge structure) is quite involved. By contrast, our proof of the cheap SL(2)-orbit theorem is short and elementary. The crucial point is that

$$F_{\mathrm{lim}} = h \cdot F_H,$$

where $h = \mathrm{id} + h_1 + h_2 + \cdots \in \mathrm{GL}(V)$ has the property that

$$h_n \in E_{-n}(\operatorname{ad} H) \cap \ker(\operatorname{ad} N).$$

Such an element h can be constructed by comparing Deligne's decomposition of the limiting mixed Hodge structure (whose Hodge filtration is F_{lim}) and the Hodge decomposition of the $\mathfrak{sl}_2$-Hodge structure (whose Hodge filtration in F_H), and using the fact that, up to a Tate twist, N is a morphism in both structures. §42 Theorem follows from this by a formal argument. The crucial fact from the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is that the tensor product

$$S_k \otimes S_\ell \cong \bigoplus_{i=0}^{\min(k,\ell)} S_{k+\ell-2i}$$

of two irreducible representation decomposes in a very simple way.

3 Variations of Hodge structure and harmonic bundles

3.1 Harmonic bundles and harmonic metrics

44. Since we are going to do a certain amount of analysis, it will be convenient to describe variations of Hodge structure in terms of smooth vector bundles. Let E be a smooth vector bundle on a complex manifold X. We denote by $A^k(X,E)$ the space of smooth k-forms with coefficients in E, and by $A^{i,j}(X,E)$ the space of smooth (i,j)-forms with coefficients in E.

Definition. Let E be a smooth vector bundle on a complex manifold X. A **variation of Hodge structure** of weight n on E is a decomposition

$$E = \bigoplus_{p+q=n} E^{p,q}$$

into smooth subbundles, together with a flat connection $d\colon A^0(X,E) \to A^1(X,E)$ that maps each $A^0(X,E^{p,q})$ into the direct sum of the subspaces

$$A^{1,0}(X,E^{p,q}) \oplus A^{1,0}(X,E^{p-1,q+1}) \oplus A^{0,1}(X,E^{p,q}) \oplus A^{0,1}(X,E^{p+1,q-1}).$$

45. The definition gives us a decomposition $d = \partial + \theta + \bar{\partial} + \theta^*$ into four operators

$$\begin{aligned} \partial\colon A^0(X,E^{p,q}) &\to A^{1,0}(X,E^{p,q}) \\ \theta\colon A^0(X,E^{p,q}) &\to A^{1,0}(X,E^{p-1,q+1}) \\ \bar{\partial}\colon A^0(X,E^{p,q}) &\to A^{0,1}(X,E^{p,q}) \\ \theta^*\colon A^0(X,E^{p,q}) &\to A^{1,0}(X,E^{p+1,q-1}). \end{aligned}$$

It is easy to see that ∂ and $\bar{\partial}$ are again connections of type $(1,0)$ and $(0,1)$, whereas θ and θ^* are linear over smooth functions. As usual, we can extend all four operators to the space of all E-valued forms

$$A^*(X,E) = \bigoplus_{i,j,p,q} A^{i,j}(X,E^{p,q})$$

by enforcing the Leibniz rule. With respect to the $\mathbb{Z}^2 \otimes \mathbb{Z}^2$-grading on this space, ∂ is an operator of bidegree $(1,0) \otimes (0,0)$, because it maps $A^{i,j}(X,E^{p,q})$ into $A^{i+1,j}(X,E^{p,q})$; similarly, θ has bidegree $(1,0) \otimes (-1,1)$, $\bar{\partial}$ has bidegree $(0,1) \otimes (0,0)$, and θ^* has bidegree $(0,1) \otimes (1,-1)$. By decomposing the identity $d^2 = 0$ according to bidegree, one can deduce the following set of identities for these four operators.

Lemma. *The four operators ∂, θ, $\bar{\partial}$, and θ^* satisfy the following identities:*

$$\begin{aligned} \partial^2 = \theta^2 = \bar{\partial}^2 = (\theta^*)^2 &= 0 \\ \partial\theta + \theta\partial = \bar{\partial}\theta^* + \theta^*\bar{\partial} &= 0 \\ \bar{\partial}\theta + \theta\bar{\partial} = \partial\theta^* + \theta^*\partial &= 0 \\ \partial\bar{\partial} + \bar{\partial}\partial + \theta\theta^* + \theta^*\theta &= 0 \end{aligned}$$

46. In order to define polarizations, we consider a hermitian pairing

$$Q\colon A^0(X,E) \otimes_{A^0(X)} \overline{A^0(X,E)} \to A^0(X).$$

It takes as input two smooth sections of the bundle E, and outputs a smooth function on X. We require that Q is hermitian symmetric and $A^0(X)$-linear in its first argument, meaning that for $v, w \in A^0(X, E)$ and $f \in A^0(X)$, one has

$$Q(w, v) = \overline{Q(v, w)} \quad \text{and} \quad Q(f \cdot v, w) = f \cdot Q(v, w).$$

Of course, it follows that Q is also conjugate $A^0(X)$-linear in its second argument. We say that Q is **flat** with respect to the connection d if

$$dQ(v, w) = Q(dv, w) + Q(v, dw) \quad \text{for } v, w \in A^0(X, E).$$

Definition. Let E be a variation of Hodge structure of weight n on a complex manifold X. A **polarization** is a flat hermitian pairing

$$Q \colon A^0(X, E) \otimes_{A^0(X)} \overline{A^0(X, E)} \to A^0(X),$$

such that the formula

$$h(v, w) = \sum_{p+q=n} (-1)^q \, Q(v^{p,q}, w^{p,q}) \quad \text{for } v, w \in A^0(X, E)$$

defines a smooth hermitian metric on the bundle E, and the subbundles $E^{p,q}$ are orthogonal to each other with respect to this metric.

The hermitian metric in the definition is called the **Hodge metric**; it is the most important object in the theory. For the sake of clarity, we sometimes denote the Hodge metric on E by the symbol h_E.

47. The fact that Q is flat with respect to d gives us the following additional information about the operators θ and θ^*.

Lemma. *With respect to the Hodge metric on E, the operator θ^* is the adjoint of the operator θ, in the sense that*

$$h(\theta v, w) = h(v, \theta^* w) \quad \textit{for } v, w \in A^0(X, E).$$

Proof. Let $v \in A^0(X, E^{p,q})$ and $w \in A^0(X, E^{p-1,q+1})$ be two arbitrary sections. Then $Q(v, w) = 0$, and therefore

$$0 = dQ(v, w) = Q(dv, w) + Q(v, dw) = Q(\theta v, w) + Q(v, \theta^* w)$$

because all other terms have the wrong type. But then

$$h(\theta v, w) = (-1)^{q+1} Q(\theta v, w) = (-1)^q Q(v, \theta^* w) = h(v, \theta^* w).$$

Since the decomposition of E is orthogonal with respect to h, the result follows. $\square$

48. Let us briefly recall the relation with the (more familiar) holomorphic description of variations of Hodge structure. As usual, we decompose the flat connection into $d = d' + d''$, where $d' \colon A^0(X, E) \to A^{1,0}(X, E)$ is a connection of type $(1, 0)$, and $d'' \colon A^0(X, E) \to A^{0,1}(X, E)$ is a connection of type $(0, 1)$. The identity $d^2 = 0$ then becomes $(d')^2 = d'd'' +$

$d''d' = (d'')^2 = 0$. The operator d'' gives E the structure of a holomorphic vector bundle, which we denote by the symbol $\mathscr{E}$, and the operator d' defines a flat holomorphic connection

$$\nabla \colon \mathscr{E} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{E}.$$

Moreover, for each $p \in \mathbb{Z}$, the Hodge bundle

$$F^p E = E^{p,q} \oplus E^{p+1,q-1} \oplus E^{p+2,q-2} \oplus \cdots$$

is preserved by the operator d'', and therefore defines a holomorphic subbundle $F^p\mathscr{E}$. Since $d' = \partial + \theta$, we also get Griffiths' transversality condition

$$\nabla(F^p\mathscr{E}) \subseteq \Omega_X^1 \otimes_{\mathscr{O}_X} F^{p-1}\mathscr{E}.$$

From this perspective, we can see the other operators by looking at the quotient bundles $F^p\mathscr{E}/F^{p+1}\mathscr{E}$. The operator $\bar{\partial}$ gives each smooth bundle $E^{p,q}$ the structure of a holomorphic vector bundle $\mathscr{E}^{p,q}$, and because $d'' = \bar{\partial} + \theta^*$,

$$\mathscr{E}^{p,q} \cong F^p\mathscr{E}/F^{p+1}\mathscr{E}$$

are isomorphic as holomorphic vector bundles. Since $\bar{\partial}\theta + \theta\bar{\partial} = 0$, the operator θ defines a morphism of holomorphic vector bundles

$$\theta \colon \mathscr{E}^{p,q} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{E}^{p-1,q+1},$$

and one checks that the following diagram is commutative:

$$\begin{array}{ccc}
\mathscr{E}^{p,q} & \xrightarrow{\quad\theta\quad} & \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{E}^{p-1,q+1} \\
\downarrow{\cong} & & \downarrow{\cong} \\
F^p\mathscr{E}/F^{p+1}\mathscr{E} & \xrightarrow{\nabla} & \Omega_X^1 \otimes_{\mathscr{O}_X} F^{p-1}\mathscr{E}/F^p\mathscr{E}
\end{array}$$

The two remaining operators θ^* and ∂ in the decomposition of d have the following interpretation: θ^* is the adjoint of the Higgs field with respect to the Hodge metric (§47 Lemma), and ∂ is the Chern connection on the bundle $E^{p,q}$.

49. We are now going to do some computations with the Hodge metric; these become easier if we use the language of harmonic bundles. In fact, a polarized variation of Hodge structure is an example of a harmonic bundle, with the harmonic metric being the Hodge metric. We briefly recall the general definition. Let E be a smooth vector bundle on a complex manifold X, let $d = d' + d''$ be a flat connection on E, and let h be a hermitian metric on E. As in the construction of the Chern connection, there are unique operators

$$\delta' \colon A^0(X,E) \to A^{1,0}(X,E)$$
$$\delta'' \colon A^0(X,E) \to A^{0,1}(X,E)$$

with the property that $d' + \delta''$ and $\delta' + d''$ are metric connections on E. Then

$$\partial = \frac{1}{2}(d' + \delta') \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(d'' + \delta'')$$

are again connections of type $(1,0)$ and $(0,1)$, and

$$\theta = \frac{1}{2}(d' - \delta') \quad \text{and} \quad \theta^* = \frac{1}{2}(d'' - \delta'')$$

are linear over smooth functions. The metric h is called a **harmonic metric**, and the triple (E, d, h) is called a **harmonic bundle**, if the identity $(\overline{\partial} + \theta)^2 = 0$ holds. This is equivalent to the identities in §45 Lemma. The associated Higgs bundle is E, with the holomorphic structure defined by $\overline{\partial}$, and with Higgs field θ. As suggested by the notation, in the case of a polarized variation of Hodge structure, ∂, $\overline{\partial}$, θ, and θ^* agree with the operators that we had defined above.

Lemma. *Let E be a variation of Hodge structure with polarization Q. Then*

$$d' = \partial + \theta, \quad d'' = \overline{\partial} + \theta^*, \quad \delta' = \partial - \theta, \quad \delta'' = \overline{\partial} - \theta^*,$$

and so E is a harmonic bundle and the Hodge metric h is a harmonic metric. The associated Higgs bundle is the graded holomorphic vector bundle

$$\bigoplus_{p+q=n} \mathscr{E}^{p,q}$$

with Higgs field $\theta \colon \mathscr{E}^{p,q} \to \Omega^1_X \otimes \mathscr{E}^{p-1,q+1}$.

Proof. The claim is that the two operators $(\partial + \bar{\partial}) \pm (\theta - \theta^*)$ are metric connections. For any two sections $v, w \in A^0(X, E^{p,q})$, we have

$$dh(v,w) = (-1)^q Q(dv, w) + (-1)^q Q(v, dw) = h\big((\partial + \bar{\partial})v, w\big) + h\big(v, (\partial + \bar{\partial})w\big),$$

because all other terms have the wrong type. Therefore $\partial + \bar{\partial}$ is a metric connection on E. Since θ^* is the adjoint of θ with respect to h, it follows that $(\partial + \bar{\partial}) \pm (\theta - \theta^*)$ are also metric connections. The identities in §45 Lemma imply that $(\bar{\partial} + \theta)^2 = 0$, and so E is a harmonic bundle with harmonic metric h. The remaining assertions are clear from the discussion in the previous paragraph. □

50. The Hodge metric h gives us a hermitian metric on the holomorphic vector bundle $\mathscr{E}$ and on the subbundles $F^p\mathscr{E}$. It is not hard to derive a formula for the curvature of this metric, using the identities among the operators ∂, $\bar{\partial}$, θ, and θ^*; Schmid does this, in a more cumbersome way, in [Sch73, §7]. The curvature of the Hodge bundles will be important later, in Chapter 7, when we prove the nilpotent orbit theorem. We need one more piece of notation to state the result: we denote by

$$\pi^{p,q} \colon E \to E^{p,q}$$

the projection operator to the direct summand $E^{p,q}$; of course, $\pi^{p,q}$ is also orthogonal projection with respect to the Hodge metric.

Proposition. *Let E be a variation of Hodge structure with polarization Q.*

(a) On the holomorphic vector bundle $\mathscr{E}$, the curvature operator of the Hodge metric h is equal to $-2(\theta\theta^* + \theta^*\theta)$.

(b) On the holomorphic vector bundle $\mathscr{E}^{p,q}$, the curvature operator of the Hodge metric h is equal to $-(\theta\theta^* + \theta^*\theta)$.

(c) *On the holomorphic quotient bundle $\mathscr{E}/F^p\mathscr{E}$, the curvature operator of the induced metric is equal to $-2(\theta\theta^* + \theta^*\theta) + \theta\theta^*\pi^{p-1,q+1}$.*

Proof. Since the holomorphic structure on $\mathscr{E}$ is defined by the operator d'', the Chern connection of h is nothing but the metric connection $\delta' + d'' = \partial + \bar{\partial} - \theta + \theta^*$. A short computation shows that the curvature operator is

$$(\delta' + d'')^2 = (\partial + \bar{\partial} - \theta + \theta^*)^2 = (\partial\bar{\partial} + \bar{\partial}\partial) - (\theta\theta^* + \theta^*\theta) = -2(\theta\theta^* + \theta^*\theta),$$

using the identities in §45 Lemma. This proves (a). The proof of (b) is very similar. The holomorphic structure on $\mathscr{E}^{p,q}$ is defined by the operator $\bar{\partial}$, and so the Chern connection of h is the metric connection $\partial + \bar{\partial}$, and the curvature operator is

$$(\partial + \bar{\partial})^2 = (\partial\bar{\partial} + \bar{\partial}\partial) = -(\theta\theta^* + \theta^*\theta).$$

The derivation of (c) is slightly more complicated, because the operator $d'' = \bar{\partial} + \theta^*$ does not preserve the individual summands $E^{p,q}$. We can correct this with the help of the projection $\pi^{p,q}\colon E \to E^{p,q}$. Note that $\pi^{p,q}$ commutes with ∂ and $\bar{\partial}$, whereas

$$\pi^{p,q}\theta = \theta\pi^{p+1,q-1} \quad \text{and} \quad \pi^{p,q}\theta^* = \theta^*\pi^{p-1,q+1}.$$

Now for the proof. The Hodge decomposition is orthogonal with respect to the Hodge metric, and so, as smooth vector bundles with metric,

$$E/F^pE \cong E^{p-1,q+1} \oplus E^{p-2,q+2} \oplus E^{p-3,q+3} \oplus \cdots \tag{50.1}$$

Under this isomorphism, the holomorphic structure on E/F^pE is represented by the operator $d'' - \theta^*\pi^{p-1,q+1}$, which preserves the bundle on the right-hand side. A short computation shows that the modified operator

$$\delta' + d'' - \theta^*\pi^{p-1,q+1} = \partial + \bar{\partial} - \theta + \theta^* - \theta^*\pi^{p-1,q+1}$$

is a metric connection, and therefore equal to the Chern connection. It follows that the curvature operator of the induced metric is

$$\begin{aligned}
&\big(\delta' + d'' - \theta^*\pi^{p-1,q+1}\big)^2 \\
&\qquad = (\delta' + d'')^2 - (\partial + \bar{\partial} - \theta + \theta^*)\theta^*\pi^{p-1,q+1} - \theta^*\pi^{p-1,q+1}(\partial + \bar{\partial} - \theta + \theta^*) \\
&\qquad = -2(\theta\theta^* + \theta^*\theta) + \theta\theta^*\pi^{p-1,q+1},
\end{aligned}$$

using the identities in §45 Lemma and the fact that $\pi^{p-1,q+1}\theta = \theta\pi^{p,q} = 0$ on the vector bundle in (50.1). □

3.2 Multivalued flat sections and the Hodge metric

51. Now we are ready to start doing some computations specifically for polarized variations of Hodge structure over the punctured disk. As usual, let

$$\Delta^* = \{\, t \in \mathbb{C} \mid 0 < |t| < 1 \,\}$$

be the punctured unit disk, with coordinate t. Let E be a polarized variation of Hodge structure of weight n on Δ^*. We are going to denote by V the space of all multi-valued flat sections of (E, d). More precisely, we use the letter

$$\mathbb{H} = \{\, z \in \mathbb{C} \mid \operatorname{Re} z < 0 \,\}$$

for the *left* half-plane; the exponential function $\exp\colon \mathbb{H} \to \Delta^*$ makes it into the universal covering space of Δ^*, independently of the choice of $i = \sqrt{-1}$. To keep the notation simple, let us denote the pullbacks of E, d, and h by the same symbols; then $V \subseteq A^0(\mathbb{H}, E)$ is exactly the kernel of the operator d.

52. If $v \in V$ is a nontrivial multi-valued flat section, the pointwise Hodge norm $h(v,v)$ is a smooth function on $\mathbb{H}$ that is everywhere nonzero. The first observation is that the Higgs field $\theta_{\partial/\partial z}$ controls the first derivatives of $h(v,v)$.

Lemma. *For every nonzero $v \in V$, one has $\partial h(v,v) = -2h(\theta v, v)$.*

Proof. This is true for arbitrary harmonic bundles. By assumption, we have $d'v = d''v = 0$, and therefore $\partial v = -\theta v$. Since $\delta' + d''$ is a metric connection,

$$\partial\, h(v,v) = h(\delta' v, v) + h(v, d'' v) = h(\partial v - \theta v, v) = -2h(\theta v, v),$$

as claimed. ∎

53. Another important property of the function $h(v,v)$, discovered by Griffiths and Schmid [Sch73, Lem 7.19], is that the logarithm of $h(v,v)$ is subharmonic.

Lemma. *If $v \in V$ is nonzero, the function $\log h(v,v)$ is subharmonic on $\mathbb{H}$.*

Proof. This is again true for arbitrary harmonic bundles. From $d'v = d''v = 0$, we get $\partial v = -\theta v$ and $\bar\partial v = -\theta^* v$. For the same reason as in §52 Lemma, we have

$$\bar\partial\, h(v,v) = -2h(v, \theta v).$$

Since $\partial + \bar\partial$ is a metric connection, we then get

$$\begin{aligned}\partial\bar\partial\, h(v,v) = -2h(\partial v, \theta v) - 2h(v, \bar\partial \theta v) &= 2h(\theta v, \theta v) + 2h(v, \theta \bar\partial v)\\ &= 2h(\theta v, \theta v) - 2h(\theta^* v, \theta^* v),\end{aligned}$$

using the identity $\bar\partial \theta + \theta \bar\partial = 0$ and §47 Lemma. To shorten the next couple of formulas, we introduce the two operators

$$A = \theta_{\partial/\partial z} \quad \text{and} \quad A^* = \theta^*_{\partial/\partial \bar z},$$

which are both smooth sections of the bundle $\operatorname{End}(E)$. The notation is justified because A and A^* are adjoints under the Hodge metric h, according to §47 Lemma. We then get

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\, h(v,v) = 2\Big(h(Av, Av) + h(A^* v, A^* v)\Big).$$

We also know from §52 Lemma that

$$\frac{\partial}{\partial z}\, h(v,v) = -2h(v, Av) = -2h(A^* v, v).$$

An application of the Cauchy-Schwarz inequality now gives

$$\left|\frac{\partial}{\partial z}\, h(v,v)\right|^2 \leq \frac{1}{2}\Big(4h(v,v)h(Av,Av) + 4h(v,v)h(A^*v, A^*v)\Big) = h(v,v)\cdot \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\, h(v,v).$$

Since the Laplacian of $\varphi = \log h(v,v)$ satisfies

$$\frac{1}{4}\Delta\varphi = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\log h(v,v) = \frac{1}{h(v,v)}\cdot\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\, h(v,v) - \frac{1}{h(v,v)^2}\left|\frac{\partial}{\partial z}\, h(v,v)\right|^2$$

this inequality is what we need to conclude that $\Delta\varphi \geq 0$. ∎

54. The argument above also shows that the first derivatives of $\log h_E(v,v)$ are controlled by the operator $A = \theta_{\partial/\partial z}$, in the following sense. From §52 Lemma and the triangle inequality, we get

$$\left|\frac{\partial}{\partial z} h_E(v,v)\right| = 2\big|h_E(Av,v)\big| \leq 2h_E(v,v)^{\frac{1}{2}} h_E(Av,Av)^{\frac{1}{2}}$$
$$\leq 2h_E(v,v)\, h_{\mathrm{End}(E)}(A,A)^{\frac{1}{2}},$$

where $h_{\mathrm{End}(E)}(A,A)$ is the Hodge norm of A under the induced hermitian metric on the bundle $\mathrm{End}(E)$ (which is an upper bound for the operator norm of A with respect to the Hodge metric h_E). After dividing by $h_E(v,v)$, this says that

$$\left|\frac{\partial}{\partial z} \log h_E(v,v)\right| = \left|\frac{\partial}{\partial \bar{z}} \log h_E(v,v)\right| \leq 2h_{\mathrm{End}(E)}\big(\theta_{\partial/\partial z}, \theta_{\partial/\partial z}\big)^{\frac{1}{2}}. \tag{54.1}$$

The Hodge norm of the Higgs field therefore gives us an upper bound for the derivatives of the function $\log h_E(v,v)$.

3.3 A universal upper bound for the Higgs field

55. Clearly, our next task is to bound the quantity $h_{\mathrm{End}(E)}(\theta_{\partial/\partial z}, \theta_{\partial/\partial z})$. This is analogous to the distance-decreasing property of period mappings, and the proof naturally uses **Ahlfors' lemma**, a generalization of the Schwarz-Pick lemma from complex analysis [Ahl38]. The result is usually stated for the open unit disk; for the sake of completeness, we include a proof that covers both cases.

Lemma (Ahlfors). *Let $f\colon \mathbb{H} \to (0,+\infty)$ be a positive smooth function such that*

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log f \geq Cf$$

for a positive constant $C > 0$. Then

$$f(z) \leq \frac{(2C)^{-1}}{|\mathrm{Re}\, z|^2} \quad \textit{for all } z \in \mathbb{H}.$$

Proof. We start by proving the result on the open unit disk Δ, where the technique is easier to understand. Here the statement of Ahlfors' lemma is the following: for any positive smooth function $g\colon \Delta \to (0,+\infty)$, we have the implication

$$\frac{\partial^2}{\partial t \partial \bar{t}} \log g \geq g \quad \Longrightarrow \quad g(t) \leq \frac{2}{(1-|t|^2)^2} \quad \text{for } t \in \Delta.$$

Let Δ_R denote the open disk of radius $R > 0$. We are going to argue that

$$g(t) \leq \frac{2R^2}{(R^2-|t|^2)^2} \quad \text{for } t \in \Delta_R \tag{55.1}$$

for every $R < 1$; this is enough, because we can then let $R \to 1$ to get the result. Define an auxiliary function $u\colon \Delta_r \to (0,+\infty)$ by the formula

$$g = u \cdot \frac{2R^2}{(R^2-|t|^2)^2}.$$

We observe that u goes to zero near the boundary of Δ_R, because g is bounded on Δ_R, whereas $2R^2/(R^2 - |t|^2)^2$ goes to infinity near the boundary. Therefore u must achieve its maximum at some interior point $t_0 \in \Delta_R$. In particular, the function $\log u$ has a local maximum at t_0, and by looking at the second derivatives, we get

$$\frac{\partial^2}{\partial t \partial \bar{t}} \log u = \frac{1}{4} \left(\frac{\partial^2}{\partial x \partial x} + \frac{\partial^2}{\partial y \partial y} \right) \log u \leq 0$$

for $t = t_0$, where $t = x + iy$. It follows that

$$0 \geq \frac{\partial^2}{\partial t \partial \bar{t}} \log u = \frac{\partial^2}{\partial t \partial \bar{t}} \log g - \frac{\partial^2}{\partial t \partial \bar{t}} \left(\log \frac{2r^2}{(r^2 - |t|^2)^2} \right) \geq g - \frac{2r^2}{(r^2 - |t|^2)^2}$$

for $t = t_0$, which says exactly that $u(t_0) \leq 1$. But u had a maximum at t_0, and therefore $u \leq 1$ on the entire disk Δ_R, which is the content of (55.1).

It remains to deduce the analogous result for a smooth function $f \colon \mathbb{H} \to (0, +\infty)$. Let us write the hypothesis in the symbolic form $\partial\bar{\partial} \log f \geq f \, dz \wedge d\bar{z}$. After pulling back both 2-forms along the isomorphism

$$\Delta \to \mathbb{H}, \quad t \mapsto \frac{t+1}{t-1},$$

this becomes the condition that

$$\partial\bar{\partial} \log f \left(\frac{t+1}{t-1} \right) \geq \frac{4}{|t-1|^4} \cdot f \left(\frac{t+1}{t-1} \right) dt \wedge d\bar{t}.$$

If we now set $g(t) = 4|t-1|^{-4} \cdot f\big((t+1)/(t-1)\big)$, then we get

$$\frac{\partial^2}{\partial t \partial \bar{t}} \log g \geq g,$$

and therefore, according to the discussion above, the inequality

$$g(t) \leq \frac{2}{(1 - |t|^2)^2} \quad \text{for } t \in \Delta.$$

If we convert this back into a statement about f, the result follows. ☐

56. The following lemma establishes the fundamental inequality that we need in order to apply Ahlfors' lemma to our setting [Sim90, Thm. 1].

Lemma. *With the notation $A = \theta_{\partial/\partial z}$ and $A^* = \theta^*_{\partial/\partial \bar{z}}$, we have*

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h_{\mathrm{End}(E)}(A, A) \geq \frac{h_{\mathrm{End}(E)}\big([A^*, A], [A^*, A]\big)}{h_{\mathrm{End}(E)}(A, A)}.$$

Proof. It is best to view this as a special case of a more general result. Let (E, d, h) be a harmonic bundle on $\mathbb{H}$, and let s be a smooth section of E that satisfies $\bar{\partial} s = 0$ and $\theta s = 0$; in other words, s is a holomorphic section of the Higgs bundle E that lies in the kernel of the Higgs field. Then we claim that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h(s, s) \geq \frac{h(A^* s, A^* s)}{h(s, s)}, \tag{56.1}$$

where again $A^* = \theta^*_{\partial/\partial\bar{z}}$. The calculations are very similar to those in the proof of §53 Lemma. The identity $\partial\bar{\partial} + \bar{\partial}\partial + \theta\theta^* + \theta^*\theta = 0$ in §45 Lemma gives us $\bar{\partial}\partial s = -\theta\theta^* s$. Since $\partial + \bar{\partial}$ is a metric connection, we have

$$\bar{\partial}\, h(s,s) = h(\bar{\partial}s, s) + h(s, \partial s) = h(s, \partial s).$$

Applying the same reasoning again, we get

$$\partial\bar{\partial}\, h(s,s) = h(\partial s, \partial s) + h(s, \bar{\partial}\partial s) = h(\partial s, \partial s) - h(s, \theta\theta^* s) = h(\partial s, \partial s) - h(\theta^* s, \theta^* s).$$

As before, we can now use the Cauchy-Schwarz inequality to conclude that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log h(s,s) \geq \frac{h(A^* s, A^* s)}{h(s,s)},$$

which is of course just (56.1).

Now let us return to our specific situation. The vector bundle $\mathrm{End}(E)$, with the flat connection induced by d and the metric induced by h, is again a harmonic bundle; moreover, it is easy to see that $\theta_{\partial/\partial z}$ acts on $\mathrm{End}(E)$ as the commutator with A, and $\theta^*_{\partial/\partial\bar{z}}$ acts as the commutator with A^*. If we consider A as a section of $\mathrm{End}(E)$, it is holomorphic and trivially commutes with A, hence belongs to the kernel of both $\bar{\partial}$ and θ. As such, the calculation from above applies to it, and so

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h(A, A) \geq \frac{h\big([A^*, A], [A^*, A]\big)}{h(A, A)}.$$

This is the desired inequality. □

57. At each point of $\mathbb{H}$, the fiber of E is a vector space of dimension $r = \mathrm{rk}\, E$ with a hermitian inner product, and $\theta_{\partial/\partial z}$ is a nilpotent endomorphism. The following lemma bounds the pointwise norm of the commutator.

Lemma. *Let V be a complex vector space of finite dimension $r \geq 1$, with a hermitian inner product. If $A \in \mathrm{End}(V)$ is a nilpotent endomorphism, and if $A^* \in \mathrm{End}(V)$ denotes its adjoint with respect to the inner product, then*

$$\big\| [A^*, A] \big\|^2 \leq 2 \|A\|^4 \leq \binom{r+1}{3} \big\| [A^*, A] \big\|^2.$$

Proof. Choose an orthonormal basis $e_1, \dots, e_r \in V$ such that the matrix representing A is upper triangular; the matrix representing A^* is the conjugate transpose, hence lower triangular. Setting $a_{i,j} = \langle A e_i, e_j \rangle$, we get

$$\|A\|^2 = \sum_{i<j} |a_{i,j}|^2.$$

The k-th diagonal entry of the matrix representing $[A^*, A]$ is easily seen to be

$$d_{k,k} = \big(|a_{1,k}|^2 + \dots + |a_{k-1,k}|^2\big) - \big(|a_{k,k+1}|^2 + \dots + |a_{k,r}|^2\big),$$

and this gives us the rather simple-minded inequality

$$\big\| [A^*, A] \big\| \geq \sqrt{|d_{1,1}|^2 + \dots + |d_{r,r}|^2}.$$

Let $x \in \mathbb{R}$ be arbitrary. It is easy to see that

$$\sum_{k=1}^{r} (x+k) d_{k,k} = \sum_{i<j} \bigl((x+j) - (x+i) \bigr) |a_{i,j}|^2 = \sum_{i<j} (j-i) |a_{i,j}|^2,$$

and from the trivial lower bound $j - i \geq 1$, we therefore get

$$\sum_{i<j} |a_{i,j}|^2 \leq \sum_{k=1}^{r} (x+k) \|d_{k,k}\| \leq \left(\sum_{k=1}^{r} (x+k)^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{r} |d_{k,k}|^2 \right)^{\frac{1}{2}}.$$

The right-hand side becomes minimal for $x = -\frac{1}{2}(r+1)$, with the result that

$$\|A\|^2 \leq \sqrt{\frac{r(r+1)(r-1)}{12}} \bigl\| [A^*, A] \bigr\|.$$

This inequality is actually sharp: equality is achieved for the $r \times r$-matrix with

$$|a_{k,k+1}|^2 = k(r-k) \quad \text{for } k = 1, \dots, r-1,$$

and all other entries zero. Interestingly, this is exactly the case where V is an irreducible representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, with $\rho(\mathsf{Y}) = A$, $\rho(\mathsf{X}) = A^*$, and $\rho(\mathsf{H}) = [A^*, A]$. We leave the verification of this claim to the reader.

The proof of the other inequality is easier. Without loss of generality, we can assume that $\|A\| = \|A^*\| = 1$. The operator $[A^*, A]$ is self-adjoint; by the spectral theorem, there is an orthonormal basis $e_1', \dots, e_r'$ such that $[A^*, A] e_i' = \lambda_i e_i'$ with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Therefore

$$\bigl\| [A^*, A] \bigr\|^2 = \sum_{i=1}^{r} \lambda_i^2 = \sum_{i=1}^{r} \bigl(\|A e_i'\|^2 - \|A^* e_i'\|^2 \bigr)^2 \leq \sum_{i=1}^{r} \|A e_i'\|^4 + \sum_{i=1}^{r} \|A^* e_i'\|^4,$$

because $\lambda_i = \bigl\langle (A^*A - AA^*) e_i', e_i' \bigr\rangle = \|A e_i'\|^2 - \|A^* e_i'\|^2$. At the same time,

$$1 = \|A\|^2 = \sum_{i=1}^{r} \|A e_i'\|^2,$$

and so both sums on the right-hand side are ≤ 1. □

58. We can now apply the version of Ahlfors' lemma in §55 Lemma, and deduce the following very striking upper bound for the Hodge norm of the Higgs field.

Corollary. *Let E be a polarized variation of Hodge structure of rank $r \geq 1$ on the left half-plane $\mathbb{H}$. Then the Higgs field $\theta_{\partial/\partial z}$ satisfies the inequality*

$$h_{\mathrm{End}(E)} \bigl(\theta_{\partial/\partial z}, \theta_{\partial/\partial z} \bigr) \leq \frac{C_0^2}{(\operatorname{Re} z)^2},$$

where $C_0 = \frac{1}{2} \sqrt{\binom{r+1}{3}}$.

Proof. Since the Higgs field is trivial when $r = 1$, we can assume that $r \geq 2$. Let us again use the abbreviations $A = \theta_{\partial/\partial z}$ and $A^* = \theta_{\partial/\partial \bar{z}}$. If we put the inequalities in §56 Lemma and §57 Lemma together, we get

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h_{\mathrm{End}(E)}(A, A) \geq \frac{12}{r(r+1)(r-1)} h_{\mathrm{End}(E)}(A, A).$$

Now apply §55 Lemma to reach the desired conclusion. □

From an analytic point of view, this is really the central fact about polarized variations of Hodge structure. The amazing thing is that the constant depends on nothing but the rank of the variation of Hodge structure. This fact turns out to be especially useful for the theory in several variables.

59. The induced variation of Hodge structure on the bundle $\mathrm{End}(E)$ has rank r^2, but in that case, there is a bound on the Higgs field that is much better than simply replacing r by r^2 is the result above.

Corollary. *Let E be a polarized variation of Hodge structure of rank $r \geq 1$ on the left half-plane $\mathbb{H}$. Then the Higgs field $\theta_{\partial/\partial z}$ of the induced variation of Hodge structure on* $\mathrm{End}(E)$ *satisfies the inequality*

$$h_{\mathrm{End}(\mathrm{End}(E))}\big(\theta_{\partial/\partial z}, \theta_{\partial/\partial z}\big) \leq \frac{2rC_0^2}{(\operatorname{Re} z)^2},$$

where again $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$.

Proof. Let $A = \theta_{\partial/\partial z}$ be the Higgs field acting on V. The induced Higgs field on $\mathrm{End}(V)$ is then equal to the commutator $(\operatorname{ad} A)(X) = [A, X]$. It is therefore enough to prove that for any nilpotent matrix $A \in \mathrm{End}(V)$, one has

$$\|\operatorname{ad} A\|^2 \leq 2r\|A\|^2.$$

Let $e_1, \dots, e_r \in V$ be an orthonormal basis such that the matrix for A is strictly upper triangular. Then the matrices $E_{i,j}$, whose only nonzero entry is a 1 in the position (i, j), form an orthonormal basis for $\mathrm{End}(V)$, and a short calculation gives

$$(\operatorname{ad} A)(E_{i,j}) = \sum_{k<i} a_{k,i} E_{k,j} - \sum_{j<k} a_{j,k} E_{i,k}.$$

We therefore get

$$\begin{aligned}\|\operatorname{ad} A\|^2 &= \sum_{i,j} |(\operatorname{ad} A)(E_{i,j})|^2 \\ &= \sum_{i,j} \Big(|a_{1,i}|^2 + \dots + |a_{i-1,i}|^2 + |a_{j,j+1}|^2 + \dots + |a_{j,r}|^2\Big) \leq 2r\|A\|^2,\end{aligned}$$

as claimed. □

3.4 Proof of the monodromy theorem

60. Let us summarize the results so far. We have shown that if $v \in V$ is a nontrivial multi-valued flat section of E, then the function $\varphi = \log h(v,v)$ is smooth and subharmonic on $\mathbb{H}$, and its first derivatives are bounded by

$$\left|\frac{\partial \varphi}{\partial z}\right| = \left|\frac{\partial \varphi}{\partial \bar{z}}\right| \leq \frac{2C_0}{|\operatorname{Re} z|}, \tag{60.1}$$

where $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$ and $r = \operatorname{rk} E$. Everything up to this point, maybe except for the optimal value of the constant, was known to Simpson [Sim90, §2] back in the 1990s. Now we add a small – but crucial! – new insight, namely that the uniform bound on the first derivatives can be used directly to control the behavior of the function φ for $\operatorname{Re} z \ll 0$.

61. Before we can explain how this works, we first need to introduce the **monodromy operator** $T \in \operatorname{Aut}(V)$. Following Schmid [Sch73, (4.4)], we define this by the formula

$$(Tv)(z) = v(z - 2\pi i),$$

where $i = \sqrt{-1}$. That is to say, for any $v \in V$, the function $z \mapsto v(z - 2\pi i)$ is again a flat section of the bundle E on $\mathbb{H}$, and we define $Tv \in V$ to be this section. The flat pairing on E induces a hermitian pairing

$$Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C},$$

and one checks easily that T preserves Q, in the sense that $Q(Tv, Tw) = Q(v, w)$ for every $v, w \in V$. We write the Jordan decomposition of T in the form

$$T = T_s \cdot e^{2\pi i N},$$

with $T_s \in \operatorname{Aut}(V)$ semisimple and $N \in \operatorname{End}(V)$ nilpotent. Then

$$Q(T_s v, T_s w) = Q(v, w) \quad \text{and} \quad Q(Nv, w) = Q(v, Nw),$$

for every $v, w \in V$, because Q is conjugate-linear in the second argument. Note that T depends on the choice of $i = \sqrt{-1}$, but N is independent of it.

62. The space V of multi-valued flat sections gives us a trivialization of the vector bundle E on the left half-plane $\mathbb{H}$. We can therefore consider the polarized variation of Hodge structure as a family of Hodge structures on V; as usual, we denote by the letter $\Phi(z)$ the Hodge filtration of the Hodge structure at the point $z \in \mathbb{H}$. The definition of the monodromy operator has the following consequence.

Lemma. *For every $z \in \mathbb{H}$, we have the equality*

$$\Phi(z + 2\pi i) = T\,\Phi(z),$$

where both sides are filtrations on V.

Proof. This may be a bit confusing, so let us write out the argument. The trivialization of the pullback of the bundle E along $\exp\colon \mathbb{H} \to \Delta^*$ gives us, for every point $z \in \mathbb{H}$, an isomorphism of complex vector spaces

$$\phi_z\colon V \to (\exp^* E)\big|_z \cong E_t, \quad \phi_z(v) = v(z).$$

Here $t = e^z$. Since $v(z + 2\pi i) = (T^{-1}v)(z)$, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_{z+2\pi i}} & \\ \Big\downarrow{\scriptstyle T^{-1}} & & E_t \\ V & \xrightarrow{\phi_z} & \end{array}$$

commutes. The way the period mapping is constructed, we have

$$F^p_{\Phi(z+2\pi i)} = \phi^{-1}_{z+2\pi i}\big(F^p E_t\big) = (T^{-1})^{-1}\phi_z^{-1}\big(F^p E_t\big) = T F^p_{\Phi(z)},$$

which is what we wanted to prove. □

63. Given a multi-valued flat section $v \in V$, we can write the value of the function $h(v, v)$ at the point $z \in \mathbb{H}$ in the equivalent form

$$h(v, v)(z) = \|v\|^2_{\Phi(z)},$$

using the notation for the Hodge norm from §9. Since $T \in \mathrm{Aut}(V, Q)$, we have

$$h(v, v)(z + 2\pi i) = \|v\|^2_{T\Phi(z)} = \|T^{-1}v\|^2_{\Phi(z)} = h(T^{-1}v, T^{-1}v)(z). \tag{63.1}$$

It is this relation that connects the asymptotic behavior of the Hodge norm to the monodromy transformation T.

64. To illustrate the power of the derivative bound in (60.1), we are now going to prove the monodromy theorem [Sch73, Lem. 4.5].

Proposition. *If $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $|\lambda| = 1$.*

Proof. Let $v \in V$ be a nonzero eigenvector with $Tv = \lambda v$, and consider the smooth function $\varphi = \log h(v, v)$. The relation in (63.1) implies that

$$\varphi(z + 2\pi i) = \log|\lambda|^{-2} + \varphi(z)$$

for every $z \in \mathbb{H}$. On the other hand, we clearly have

$$\varphi(z + 2\pi i) - \varphi(z) = \int_0^{2\pi} \left(i\frac{\partial\varphi}{\partial z}(z + iy) - i\frac{\partial\varphi}{\partial \bar{z}}(z + iy) \right) dy,$$

and on account of (60.1), this leads to the inequality

$$\big|\varphi(z + 2\pi i) - \varphi(z)\big| \leq \frac{8\pi C_0}{|\mathrm{Re}\, z|},$$

where $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$. Letting $|\mathrm{Re}\, z| \to \infty$, we conclude that $\log|\lambda| = 0$, whence $|\lambda| = 1$. □

3.5 Effective bounds for the monodromy transformation

65. We can use the derivative bound in (60.1) to control the Hodge norms of the nilpotent operator N and of the projections $P_\lambda \colon V \to E_\lambda(T_s)$ to the eigenspaces of T_s. The fact that N and $\theta_{\partial/\partial z}$ are both of order $|\mathrm{Re}\, z|^{-1}$ is no accident: during the proof of the nilpotent orbit theorem, we will see that the difference between these two operators, modulo $F^0\, \mathrm{End}(V)_{\Phi(z)}$, is of order $e^{-\varepsilon|\mathrm{Re}\, z|}$. (The precise statement is in §194 Theorem.)

Proposition. *There is a constant $C > 0$, whose value only depends on $\dim V$ and on the minimal polynomial of the monodromy transformation $T \in \mathrm{GL}(V)$, such that:*

(a) For every $v \in V$ and every $z \in \mathbb{H}$ with $\operatorname{Re} z \leq -1$, one has

$$\|Nv\|_{\Phi(z)} \leq \frac{C}{|\operatorname{Re} z|} \|v\|_{\Phi(z)}.$$

(b) For every $v \in V$ and every $z \in \mathbb{H}$ with $\operatorname{Re} z \leq -1$, one has

$$\|P_\lambda v\|_{\Phi(z)} \leq C \|v\|_{\Phi(z)}.$$

66. Given the bound on the derivative of the function $\log\|v\|^2_{\Phi(z)}$, the proof is just linear algebra. By the same argument as in the proof of §64 Proposition, we have

$$\Big|\log\|Tv\|^2_{\Phi(z)} - \log\|v\|^2_{\Phi(z)}\Big| = \Big|\log\|v\|^2_{\Phi(z-2\pi i)} - \log\|v\|^2_{\Phi(z)}\Big| \leq \frac{8\pi C_0}{|\operatorname{Re} z|};$$

after exponentiating, this becomes

$$e^{-8\pi C_0/|\operatorname{Re} z|} \leq \frac{\|Tv\|^2_{\Phi(z)}}{\|v\|^2_{\Phi(z)}} \leq e^{8\pi C_0/|\operatorname{Re} z|}.$$

If we restrict to $\operatorname{Re} z \leq -1$ for simplicity, it follows that

$$\Big|\|Tv\|^2_{\Phi(z)} - \|v\|^2_{\Phi(z)}\Big| \leq \frac{C'}{|\operatorname{Re} z|} \|v\|^2_{\Phi(z)}, \tag{66.1}$$

where $C' = (e^{8\pi C_0} - 1)/(8\pi C_0)$.

67. The following elementary lemma, applied to the inner products $\langle -, - \rangle_{\Phi(z)}$, turns the inequality in (66.1) into a bound for the operator norm of the nilpotent operator N.

Lemma. *Let V be a finite-dimensional complex vector space with an inner product $\langle -, - \rangle$. Let $T \in \mathrm{GL}(V)$ be an endomorphism whose eigenvalues have absolute value 1. Suppose that T is close to unitary, in the sense that there is a constant $\varepsilon > 0$ such that*

$$\Big|\|Tv\|^2 - \|v\|^2\Big| \leq \varepsilon \|v\|^2 \quad \text{for all } v \in V.$$

Then there are two constants $C > 0$ and $m \in \mathbb{N}$, whose values only depends on the integer $\dim V$ and on the minimal polynomial of the operator T, such that

$$\|Nv\| \leq C\varepsilon(1 + \varepsilon^m) \cdot \|v\| \quad \text{for all } v \in V,$$

where $N = (2\pi i)^{-1} \log T_u$ is the logarithm of the unipotent part of T.

Proof. Set $r = \dim V$, and choose an orthonormal basis $e_1, \dots, e_r \in V$ such that the matrix representing T is upper triangular. Write $T = U(\mathrm{id} + B)$, where U is a diagonal matrix and B is strictly upper triangular. The entries of U are the eigenvalues of T, hence of absolute value 1, and so U is unitary. Pick any two integers $1 \leq i < j \leq r$, and let $b_{i,j} = \langle Be_j, e_i \rangle$

denote the corresponding entry of the matrix representing B. By applying the hypothesis to vectors of the form $v = e_i + ze_j$ with $|z| = 1$, we get

$$2\,\mathrm{Re}(zb_{i,j}) = 2\,\mathrm{Re}\big\langle B(e_i + ze_j), e_i + ze_j\big\rangle \leq \varepsilon \|e_i + ze_j\|^2 = 2\varepsilon,$$

and by choosing z appropriately, this gives $|b_{i,j}| \leq \varepsilon$. Consequently, the L^2-norm of the operator $B \in \mathrm{End}(V)$ is bounded by $\|B\|^2 \leq \binom{r}{2}\varepsilon^2$, and therefore

$$\|Bv\| \leq r\varepsilon\|v\| \quad \text{for all } v \in V.$$

Now it is a basic fact from linear algebra that the nilpotent operator $N = (2\pi i)^{-1} \log T_u$ can be written as a polynomial in T, of the form

$$N = P(T) \cdot \prod_{\lambda} (T - \lambda\,\mathrm{id}),$$

where the product runs over the distinct eigenvalues of T; the polynomial $P(t) \in \mathbb{C}[t]$ is uniquely determined by the minimal polynomial of T. Consequently,

$$N = P(T) \cdot \prod_{\lambda} \big((U - \lambda\,\mathrm{id}) + UB\big),$$

and because the operator norm of $P(T)$ is easily bounded using the inequality $\|Tv\|^2 \leq (1+\varepsilon)\|v\|^2$, it suffices to estimate the operator norm of the product. Given our bound $r\varepsilon$ for the operator norm of B, this comes down to

$$\prod_{\lambda} (U - \lambda\,\mathrm{id}) = 0,$$

which holds because U is a diagonal matrix. □

68. The same argument also proves that the projection operators

$$P_\lambda \colon V \to E_\lambda(T_s)$$

are uniformly bounded, meaning that there is a constant $C > 0$, whose value again only depends on the integer $\dim V$ and on the minimal polynomial of $T \in \mathrm{GL}(V)$, such that

$$\|P_\lambda v\|_{\Phi(z)} \leq C\|v\|_{\Phi(z)} \quad \text{for } v \in V \text{ and } \mathrm{Re}\, z \leq -1. \tag{68.1}$$

The reason is that P_λ can again be expressed as a certain polynomial in T, whose operator norm can then be bounded using (66.1). This finishes the proof of §65 Proposition.

4 Period domains and period mappings

69. In this chapter, we briefly review the basic properties of period domains and period mappings, in the setting of *complex* Hodge structures. The theory is actually simpler than in the case of rational Hodge structures [Sch73, §3], because the Lie groups that are involved are just real forms of the general linear group $\mathrm{GL}(V)$.

4.1 The period domain and its compact dual

70. Fix a finite-dimensional complex vector space V and a hermitian pairing $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$. Let D be the **period domain** parametrizing Hodge structures of the type we are interested in: each point $o \in D$ corresponds to a Hodge structure

$$V = \bigoplus_{p+q=n} V_o^{p,q}$$

on the vector space V, with fixed Hodge numbers $\dim V_o^{p,q}$, and polarized by the hermitian pairing Q. We assume that there is at least one such Hodge structure; in particular, Q is nondegenerate and of the correct signature. In this setting, the real Lie group

$$G = \left\{ g \in \mathrm{GL}(V) \;\middle|\; Q(gv, gw) = Q(v,w) \text{ for all } v, w \in V \right\}$$

acts transitively on D, according to the rule $V_{g\cdot o}^{p,q} = g(V_o^{p,q})$. (This is easily proved by chosing an orthonormal basis adapted to the Hodge decomposition, and then mapping one such orthonormal basis to another.) The inner products at the two points o and $g \cdot o$ are related by the formula

$$\langle gv, gw \rangle_{g\cdot o} = \langle v, w \rangle_o,$$

which follows from the fact that $g \in G$.

71. The Lie algebra of G is easily seen to be

$$\mathfrak{g} = \left\{ A \in \mathrm{End}(V) \;\middle|\; Q(Av, w) + Q(v, Aw) = 0 \text{ for all } v, w \in V \right\}.$$

Note that $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}(V)$, which means that G is a non-compact real form of the complex Lie group $\mathrm{GL}(V)$. This also gives us a real structure on $\mathrm{End}(V)$; the "complex conjugate" of an endomorphism $A \in \mathrm{End}(V)$ is $-A^\dagger$, where the adjoint $A^\dagger$ with respect to the hermitian pairing Q is defined by the formula

$$Q(Av, w) = Q(v, A^\dagger w) \quad \text{for all } v, w \in V.$$

This makes sense because Q, being a polarization, is nondegenerate.

72. The stabilizer of a point $o \in D$ is the compact subgroup

$$H_o = \left\{ g \in G \;\middle|\; g(V_o^{p,q}) = V_o^{p,q} \text{ for all } p, q \right\} \subseteq G,$$

and this gives us an isomorphism $D \cong G/H_o$. We can use it to describe the real tangent space $T_o D$ in terms of Hodge structures. The Hodge structure on V induces a Hodge structure of weight 0 on $\mathrm{End}(V)$, by setting

$$\mathrm{End}(V)_o^{j,-j} = \left\{ A \in \mathrm{End}(V) \;\middle|\; A(V_o^{p,q}) \subseteq V_o^{p+j,q-j} \text{ for all } p, q \right\}.$$

It is an easy exercise to check that the decomposition

$$\mathrm{End}(V) = \bigoplus_{j \in \mathbb{Z}} \mathrm{End}(V)_o^{j,-j} \tag{72.1}$$

is an $\mathbb{R}$-Hodge structure (with respect to the real structure $\mathfrak{g}$), and that the induced polarization on $\mathrm{End}(V)$ is equal to the trace pairing

$$(A, B) \mapsto \mathrm{tr}(AB^\dagger).$$

Since the conjugate of B is $-B^\dagger$, we can view the polarization on $\mathrm{End}(V)$ as coming from the $\mathbb{R}$-valued bilinear pairing $(A, B) \mapsto -\mathrm{tr}(AB)$ on $\mathfrak{g}$.

73. Here is the proof that the trace pairing is indeed a polarization for the induced Hodge structure on $\operatorname{End}(V)$. The same argument is needed in Chapter 5, and so we give some details.

Lemma. *The pairing*

$$\operatorname{End}(V) \otimes_{\mathbb{C}} \overline{\operatorname{End}(V)} \to \mathbb{C}, \quad (A, B) \mapsto \operatorname{tr}(AB^\dagger),$$

polarizes the Hodge structure on $\operatorname{End}(V)$.

Proof. It is easy to see that $\operatorname{tr} A^\dagger = \overline{\operatorname{tr} A}$ for every $A \in \operatorname{End}(V)$. Therefore

$$\operatorname{tr}(BA^\dagger) = \operatorname{tr}\big((AB^\dagger)^\dagger\big) = \overline{\operatorname{tr}(AB^\dagger)},$$

and so the pairing is hermitian symmetric. The Hodge decomposition is orthogonal with respect to the trace pairing. Indeed, if $A \in \operatorname{End}(V)_o^{i,-i}$ and $B \in \operatorname{End}(V)_o^{j,-j}$, with $i \neq j$, then $AB^\dagger \in \operatorname{End}(V)_o^{i-j,j-i}$ is a nilpotent endomorphism, and so $\operatorname{tr}(AB^\dagger) = 0$.

Finally, we need to explain why $(-1)^j \operatorname{tr}(AA^\dagger) > 0$ if $A \in \operatorname{End}(V)_o^{j,-j}$ is nonzero. The point is that $(-1)^j A^\dagger$ is exactly the adjoint of A with respect to the inner product on V. Indeed, for $v \in V^{p,q}$ and $w \in V^{p+j,q-j}$, we have

$$\langle Av, w \rangle_o = (-1)^{q-j} Q(Av, w) = (-1)^{q-j} Q(v, A^\dagger w) = (-1)^j \langle v, A^\dagger w \rangle_o.$$

Consequently, the endomorphism $(-1)^j A^\dagger A$ is self-adjoint with respect to the inner product, and also positive definite, because

$$\big\langle (-1)^j A^\dagger A v, v \big\rangle_o = \langle Av, Av \rangle_o = \|Av\|_o^2.$$

This clearly implies that $(-1)^j \operatorname{tr}(AA^\dagger) > 0$. In fact, by choosing an orthonormal basis in V, one can easily see that this expression is nothing but the operator norm of A with respect to the inner product. ☐

74. Using the notation from (72.1), the Lie algebra of the stabilizer subgroup H_o is exactly the intersection $\mathfrak{g}_o^{0,0} = \mathfrak{g} \cap \operatorname{End}(V)_o^{0,0}$, and so we conclude that the real tangent space to the period domain at a point $o \in D$ is

$$T_o D \cong \mathfrak{g}/\mathfrak{g}_o^{0,0}. \tag{74.1}$$

From the polarized Hodge structure on $\operatorname{End}(V)$, we get a positive-definite inner product on $\mathfrak{g}$. It induces an inner product on the quotient $\mathfrak{g}/\mathfrak{g}_o^{0,0}$, hence on each tangent space $T_o D$. More precisely, define

$$\mathfrak{m}_o = \mathfrak{g} \cap \bigoplus_{k \neq 0} \operatorname{End}(V)_o^{k,-k},$$

so that $\mathfrak{g} = \mathfrak{m}_o \oplus \mathfrak{g}_o^{0,0}$. Note that $\mathfrak{m}_o$ is just a linear subspace of $\mathfrak{g}$, not a Lie subalgebra. Then $T_o D \cong \mathfrak{m}_o$, and the inner product on $T_o D$ is simply the inner product on $\mathfrak{m}_o$ coming from the Hodge structure on $\operatorname{End}(V)$. In this way, we obtain a Riemannian metric on the period domain D; one shows without trouble that this metric is G-invariant. We denote by

$$d_D \colon D \times D \to [0, +\infty] \tag{74.2}$$

the resulting G-invariant distance function on the period domain. (The value $+\infty$ is included because D does not have to be connected.)

Example. Consider Hodge structures of the form

$$V = V^{1,0} \oplus V^{0,1}$$

on the vector space $V = \mathbb{C}^2$ that are polarized by the indefinite hermitian pairing

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Writing $V^{1,0} = \mathbb{C}(1,t)$, we must have $1 - |t|^2 > 0$, hence $t \in \Delta$. The period domain is therefore the unit disk Δ; the Hodge structure corresponding to the point $t \in \Delta$ is

$$\mathbb{C}^2 = \mathbb{C}(1,t) \oplus \mathbb{C}(\bar{t},1).$$

The G-invariant Riemannian metric on the period domain is the usual Poincaré metric on Δ. Indeed, G is just the automorphism group of Δ, and so it is enough to compute the metric on the tangent space $T_0\Delta$. In the Hodge structure at the point $0 \in \Delta$, the two standard basis vectors form an orthonormal basis. A short computation shows that

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & \bar{w} \\ w & 0 \end{pmatrix} \;\middle|\; w \in \mathbb{C} \right\},$$

and that the $\mathbb{R}$-linear mapping $\mathfrak{m}_0 \to T_0\Delta$ takes the matrix $\begin{pmatrix} 0 & \bar{w} \\ w & 0 \end{pmatrix}$ to the complex number w. The inner product on the $\mathbb{R}$-vector space $\mathbb{C} = T_0\Delta$ is therefore just the usual Euclidean inner product.

75. If two points $o, p \in D$ are close to each other, then of course the corresponding inner products $\langle v, w \rangle_o$ and $\langle v, w \rangle_p$ must also be close. The following lemma makes this precise.

Lemma. *There are two constants $\delta > 0$ and $C > 0$, such that if $o, p \in D$ are two points with $d_D(p,o) < \delta$, then one has*

$$\bigl| \langle v, w \rangle_p - \langle v, w \rangle_o \bigr| \leq C \|v\|_o \|w\|_o \cdot d_D(p,o)$$

for every $v, w \in V$.

Proof. Because of the G-invariance of the metric on D, it suffices to prove this when $o \in D$ is a fixed choice of base point. The exponential mapping

$$\mathfrak{m}_o \to D, \quad A \mapsto e^A \cdot o,$$

is a real-analytic local diffeomorphism. It follows that there is a constant $\delta > 0$ such that every point $p \in D$ with $d_D(p,o) < \delta$ can be uniquely written in the form $p = e^A \cdot o$, where $A \in \mathfrak{m}_o$ satisfies $\|A\|_o \leq 2d_D(p,o)$. Clearly,

$$\bigl| \langle v, w \rangle_p - \langle v, w \rangle_o \bigr| = \bigl| \langle e^{-A} v, e^{-A} w \rangle_o - \langle v, w \rangle_o \bigr| \leq C' \|v\|_o \|w\|_o \cdot \|A\|_o$$

for some constant $C' > 0$ that is independent of $v, w \in V$. This gives the desired result (with $C = 2C'$). □

76. The period domain D is an open subset of the **compact dual** $\check{D}$, which is a closed algebraic subvariety of a product of Grassmannians, and therefore a projective complex manifold. The points of $\check{D}$ parametrize all decreasing filtrations $F = F^\bullet V$ on the vector space V, subject only to the constraint that

$$\dim F^p = \dim V^{p,q} + \dim V^{p+1,q-1} + \dim V^{p+2,q-2} + \cdots \quad \text{for all } p \in \mathbb{Z}.$$

Since we are working with complex Hodge structures, there are no further restrictions on F, unlike in Schmid's paper [Sch73, §3]. The complex Lie group $\mathrm{GL}(V)$ acts transitively on $\check{D}$, by the rule $(gF)^p = g(F^p)$. The holomorphic tangent space at a point $F \in \check{D}$ is therefore

$$T_F^{1,0}\check{D} \cong \mathrm{End}(V)/F^0\,\mathrm{End}(V), \tag{76.1}$$

where we are using the notation

$$F^0\,\mathrm{End}(V) = \left\{ A \in \mathrm{End}(V) \;\middle|\; A(F^p) \subseteq F^p \text{ for all } p \in \mathbb{Z} \right\}.$$

If we go to a point $o \in D$, and denote by F_o the Hodge filtration in the Hodge structure on V, the natural isomorphism of $\mathbb{R}$-vector spaces

$$T_o D \cong T_{F_o}^{1,0}\check{D}$$

becomes, in terms of Hodge structures, the isomorphism

$$\mathfrak{m}_o \cong \mathfrak{g}/\mathfrak{g}_o^{0,0} \cong \mathrm{End}(V)/F_o^0\,\mathrm{End}(V), \tag{76.2}$$

which is of course valid in any $\mathbb{R}$-Hodge structure of weight 0. Via the embedding $D \subseteq \check{D}$, the period domain D inherits the structure of a complex manifold.

77. We can also define a hermitian metric on the compact dual $\check{D}$, which is however not $\mathrm{GL}(V)$-invariant. For that, choose a base point $o \in D$, and denote by $\langle v, w\rangle = \langle v, w\rangle_o$ the resulting inner product on the vector space V. It induces an inner product on $\mathrm{End}(V)$, and therefore on each holomorphic tangent space

$$T_F^{1,0}\check{D} \cong \mathrm{End}(V)/F^0\,\mathrm{End}(V).$$

This gives us a hermitian metric on the compact dual $\check{D}$. Note that, unlike the G-invariant metric on the period domain D, this metric is *not* invariant under the full symmetry group $\mathrm{GL}(V)$, only under the (much smaller) unitary group $U_o \subseteq \mathrm{GL}(V)$ of the reference inner product $\langle v, w\rangle$. The resulting distance function is

$$d_{\check{D}} \colon \check{D} \times \check{D} \to [0, +\infty);$$

note that $\check{D}$ has finite diameter, because it is compact and connected. It is easy to see that the unitary group U_o acts transitively on $\check{D}$: choose a basis adapted to a given filtration, and convert it into an orthonormal basis adapted to the filtration with the help of the Gram-Schmid process.

78. We can use the exponential mapping to get local coordinates on $\check{D}$.

Lemma. *There is a constant $\delta > 0$ such that for every point $p \in \check{D}$, the set*

$$B_\delta(p) = \left\{ x \in \check{D} \;\middle|\; d_{\check{D}}(p, x) < \delta \right\}$$

is biholomorphic to an open set in Euclidean space, and the function $d_{\check{D}}(p, -)$ is comparable, on $B_\delta(p)$, to the Euclidean distance, up to a factor of 2.

Proof. Fix a base point $o \in D$. Since $\check{D}$ is a homogeneous space for the unitary group $U_o \subseteq \mathrm{GL}(V)$, and since the distance function $d_{\check{D}}$ is U_o-invariant, it suffices to prove the statement when $p = o$. From the induced Hodge structure on $\operatorname{End}(V)$, we have a direct sum decomposition $\operatorname{End}(V) = W \oplus F_o^0 \operatorname{End}(V)$, where

$$W = \bigoplus_{j<0} \operatorname{End}(V)_o^{j,-j},$$

By general theory, the exponential mapping

$$W \to \check{D}, \quad A \mapsto e^A \cdot o,$$

is a local diffeomorphism; it is also holomorphic, and therefore a local isomorphism of complex manifolds. It follows that there is a constant $\delta > 0$ such that the exponential mapping induces an isomorphism between $B_\delta(o) \subseteq \check{D}$ and an open neighborhood of the origin in the vector space W. Since the distance function $d_{\check{D}}$ is continuous, we can arrange moreover that

$$\frac{1}{2}\|A\|_o \leq d_{\check{D}}\big(e^A \cdot o, o\big) \leq 2\|A\|_o$$

on the open ball in question. □

79. We record one additional fact about the distance function $d_{\check{D}}$ that we are going to need towards the end of the proof of the nilpotent orbit theorem (in §201 Lemma). Fix an operator $g \in \mathrm{GL}(V)$, and consider the mapping

$$g \colon \check{D} \to \check{D}, \quad F \mapsto gF.$$

What is the effect of this on tangent vectors? Let $\operatorname{Ad} g \colon \operatorname{End}(V) \to \operatorname{End}(V)$ be the linear mapping $A \mapsto gAg^{-1}$. Then one checks that the diagram

$$\begin{array}{ccc} \operatorname{End}(V) & \xrightarrow{\operatorname{Ad} g} & \operatorname{End}(V) \\ \downarrow & & \downarrow \\ T_F^{1,0}\check{D} & \xrightarrow{dg} & T_{gF}^{1,0}\check{D} \end{array}$$

is commutative. In particular, left translation by g changes the length of holomorphic tangent vectors, computed using the hermitian metric on $\check{D}$, at most by the operator norm of $\operatorname{Ad} g$, relative to our fixed inner product on $\operatorname{End}(V)$. After integrating this, we arrive at the useful formula

$$d_{\check{D}}(g \cdot p, g \cdot q) \leq \max_{\|A\|=1} \|(\operatorname{Ad} g)A\| \cdot d_{\check{D}}(p, q), \tag{79.1}$$

valid for any two points $p, q \in \check{D}$.

4.2 The period mapping

80. We return to our variation of Hodge structure E on the punctured disk. At each point $z \in \mathbb{H}$, we have a polarized Hodge structure on V, whose Hodge filtration we denoted by the symbol $\Phi(z)$. This gives us the **period mapping**

$$\Phi \colon \mathbb{H} \to D;$$

we shall argue in a moment that it is holomorphic (with respect to the complex structure on the period domain induced by $D \subseteq \check{D}$.) The monodromy transformation satisfies $T \in G$, hence also $T_s \in G$ and $2\pi i\, N \in \mathfrak{g}$, and we have

$$\Phi(z + 2\pi i) = T \cdot \Phi(z).$$

From now on, we use the subscript $\Phi(z)$ to refer to the Hodge structure at the point $z \in \mathbb{H}$, to avoid confusion. So the Hodge decomposition is

$$V = \bigoplus_{p+q=n} V^{p,q}_{\Phi(z)},$$

the Hodge norm is $\|v\|_{\Phi(z)}$, and so on.

Lemma. *The derivative of the period mapping is given by*

$$d\Phi\big|_z = (\theta + \theta^*)\big|_z + \mathfrak{g}^{0,0}_{\Phi(z)},$$

where we consider both sides as $\mathbb{R}$-linear mappings from $T_z\mathbb{H}$ to the quotient $\mathfrak{g}/\mathfrak{g}^{0,0}_{\Phi(z)}$.

Proof. Fix a point $z_0 \in \mathbb{H}$. Choose an orthonormal frame for the bundle E, consisting of smooth sections $e_1, \dots, e_r \in A^0(\mathbb{H}, E)$ with the property that

$$\langle e_i, e_j \rangle_{\Phi(z)} = \delta_{i,j} \quad \text{for all } z \in \mathbb{H},$$

and such that $e_i \in A^0(\mathbb{H}, E^{p_i,q_i})$ for certain $p_i, q_i \in \mathbb{Z}$. Denote by $v_1, \dots, v_r \in V$ the unique multi-valued flat sections such that $v_i(z_0) = e_i(z_0)$. We can define a smooth function $g\colon \mathbb{H} \to \mathrm{GL}(V)$ by the condition that

$$e_j(z) = g(z) \cdot v_j \quad \text{for } j = 1, \dots, r \text{ and } z \in \mathbb{H}.$$

Clearly, $g(z_0) = \mathrm{id}$, and since

$$(-1)^{q_i} Q\Big(g(z)v_i, g(z)v_j\Big) = \langle e_i, e_j \rangle_{\Phi(z)} = \langle v_i, v_j \rangle_{\Phi(z_0)} = (-1)^{q_i} Q(v_i, v_j),$$

we have $g(z) \in G$ for all $z \in \mathbb{H}$. The function $g\colon \mathbb{H} \to G$ is a smooth lifting of the period mapping, which can now be described very concretely as

$$\Phi(z) = g(z) \cdot o \quad \text{for } z \in \mathbb{H}.$$

The derivative $d\Phi$ is easily computed from this formula. Writing

$$e_j = \sum_{i=1}^{r} g_{i,j} v_i$$

with smooth functions $g_{i,j}\colon \mathbb{H} \to \mathbb{C}$, our matrix-valued function $g\colon \mathbb{H} \to G$ is represented by the $r \times r$-matrix with entries $g_{i,j}$. At the point z_0, the derivative

$$dg\big|_{z_0} \colon T_{z_0}\mathbb{H} \to \mathfrak{g}$$

is a lifting for the derivative of the period mapping

$$d\Phi\big|_{z_0} \colon T_{z_0}\mathbb{H} \to T_{\Phi(z_0)} D.$$

It is represented by the $r \times r$-matrix with entries $dg_{i,j}$. Recall that the flat connection on E is $d = \partial + \bar{\partial} + \theta + \theta^*$. Since $v_1, \dots, v_r$ are flat sections, we get

$$(\partial + \bar{\partial} + \theta + \theta^*) e_j = de_j = \sum_{i=1}^{r} dg_{i,j} \otimes v_i.$$

Now the operator $\partial + \bar{\partial}$ preserves the subbundle $E^{p,q}$, and so this term disappears when we project into $\mathfrak{g}/\mathfrak{g}^{0,0}_{\Phi(z_0)}$. The result is that

$$dg\big|_{z_0} \equiv (\theta + \theta^*)\big|_{z_0} \mod \mathfrak{g}^{0,0}_{\Phi(z_0)},$$

and so the right-hand side indeed represents the derivative of the period mapping at the point $z_0 \in \mathbb{H}$. Note that

$$\theta\big|_{z_0} \in \operatorname{End}(V)^{-1,1}_{\Phi(z_0)}, \quad \text{and} \quad \theta^*\big|_{z_0} \in \operatorname{End}(V)^{1,-1}_{\Phi(z_0)},$$

which is the horizontality condition for the period mapping. □

81. We can now deduce quite easily that the period mapping is holomorphic.

Corollary. *The period mapping $\Phi \colon \mathbb{H} \to D$ is holomorphic.*

Proof. It is enough to prove that $\Phi \colon \mathbb{H} \to \check{D}$ is holomorphic. Recall from (76.1) that the holomorphic tangent space at $\Phi(z) \in \check{D}$ is

$$T^{1,0}_{\Phi(z)} \check{D} \cong \operatorname{End}(V)/F^0 \operatorname{End}(V)_{\Phi(z)}.$$

Using §80 Lemma, the differential of Φ is therefore equal to

$$\theta\big|_z \colon T^{1,0}_z \mathbb{H} \to \operatorname{End}(V)/F^0 \operatorname{End}(V)_{\Phi(z)},$$

and since this is $\mathbb{C}$-linear, Φ is indeed holomorphic. □

5 Results about $\mathfrak{sl}_2$-Hodge structures

82. This section contains some background on $\mathfrak{sl}_2$-Hodge structures and their polarizations. Roughly speaking, an $\mathfrak{sl}_2$-Hodge structure is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, but in the category of Hodge structures. The cohomology of a compact Kähler manifold is a typical example, but $\mathfrak{sl}_2$-Hodge structures also appear naturally in the study of degenerating variations of Hodge structure.

5.1 The Weil element and polarizations

83. Following one of several competing conventions, we will denote the three generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by the letters $\mathsf{H}, \mathsf{X}, \mathsf{Y}$. Concretely,

$$\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the relations among the three generators are

$$[\mathsf{H}, \mathsf{X}] = 2\mathsf{X}, \quad [\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}, \quad [\mathsf{X}, \mathsf{Y}] = \mathsf{H}.$$

In any finite-dimensional representation V of $\mathfrak{sl}_2(\mathbb{C})$, the element H acts semisimply with integer eigenvalues, and so the underlying vector space is the direct sum of the eigenspaces $V_k = E_k(\mathsf{H})$, for $k \in \mathbb{Z}$.

84. The following definition is of central importance in Hodge theory.

Definition. An **$\mathfrak{sl}_2$-Hodge structure** on a finite-dimensional complex vector space V is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on V with the following properties:

(a) Each weight space $V_k = E_k(\mathsf{H})$ has a Hodge structure of weight $n+k$; the integer n is called the **weight** of the $\mathfrak{sl}_2$-Hodge structure.

(b) The two operators

$$\mathsf{X}\colon V_k \to V_{k+2}(1) \quad \text{and} \quad \mathsf{Y}\colon V_k \to V_{k-2}(-1)$$

are morphisms of Hodge structures.

Example. Here are two useful examples.

1. The prototypical example of a $\mathfrak{sl}_2$-Hodge structure is the total cohomology

$$\bigoplus_{k\in\mathbb{Z}} H^{n+k}(X,\mathbb{C})$$

of an n-dimensional compact Kähler manifold (X,ω); in this case, $\mathsf{X} = 2\pi i\, L_\omega$ and $\mathsf{Y} = (2\pi i)^{-1}\Lambda_\omega$, and the weight is $n = \dim X$.

2. Any Hodge structure V of weight n may be viewed as an $\mathfrak{sl}_2$-Hodge structure of weight n by letting the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ act in a trivial way.

85. Equivalently, an $\mathfrak{sl}_2$-Hodge structure of weight n is a bigraded vector space

$$V = \bigoplus_{i,j\in\mathbb{Z}} V^{i,j}$$

that is simultaneously a representation of $\mathfrak{sl}_2(\mathbb{C})$, in a way that is compatible with the bigrading. This means that

$$\mathsf{X}\colon V^{i,j} \to V^{i+1,j+1} \quad \text{and} \quad \mathsf{Y}\colon V^{i,j} \to V^{i-1,j-1},$$

and that H acts on the subspace $V^{i,j}$ as multiplication by the integer $(i+j)-n$. This makes each of the weight spaces

$$V_k = \bigoplus_{i+j=n+k} V^{i,j}$$

into a Hodge structure of weight $n+k$. In what follows, we are always going to use the notation $V_k^{i,j}$ for the individual subspaces, to avoid confusion with the Hodge decomposition in a polarized Hodge structure on V.

86. Let V be an $\mathfrak{sl}_2$-Hodge structure. From general principles, it follows that

$$\mathsf{X}^k\colon V_{-k} \to V_k(k) \quad \text{and} \quad \mathsf{Y}^k\colon V_k \to V_{-k}(-k)$$

are isomorphisms of Hodge structures. Another way to understand this symmetry is through the action of the Lie group $\mathrm{SL}_2(\mathbb{C})$. Every finite-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ lifts to a representation of the Lie group $\mathrm{SL}_2(\mathbb{C})$, in a way that is compatible with the exponential map: for a matrix $M \in \mathfrak{sl}_2(\mathbb{C})$, the element

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!} \in \mathrm{SL}_2(\mathbb{C}),$$

acts on the representation as the exponential series $\mathrm{id} + M + \frac{1}{2}M^2 + \cdots$. Now consider the **Weil element**

$$\mathsf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}),$$

which plays a similar role as the Hodge $*$-operator in classical Hodge theory. A brief computation shows that $\mathsf{w} = \exp(\mathsf{X})\exp(-\mathsf{Y})\exp(\mathsf{X})$, and so one can use the exponential series to see how w acts on representations of $\mathfrak{sl}_2(\mathbb{C})$.

87. The point of introducing w is that it explains the symmetry between the weight spaces V_k and V_{-k}. Namely, the Lie group $\mathrm{SL}_2(\mathbb{C})$ acts on its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by conjugation, and under this action, one has

$$\mathsf{w}\mathsf{X}\mathsf{w}^{-1} = -\mathsf{Y}, \quad \mathsf{w}\mathsf{Y}\mathsf{w}^{-1} = -\mathsf{X}, \quad \mathsf{w}\mathsf{H}\mathsf{w}^{-1} = -\mathsf{H}.$$

This can be checked by direct computation: for example,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The identity $\mathsf{w}\mathsf{H}\mathsf{w}^{-1} = -\mathsf{H}$ means that the action of w interchanges the two weight spaces V_k and V_{-k}, which must therefore be of the same dimension.

Note. The square of the Weil element

$$\mathsf{w}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

acts on the weight space V_k as $(-1)^k$, and *not* as multiplication by -1. This is easiest to remember with the help of the symbolic identity $\mathsf{w}^2 = (-1)^{\mathsf{H}}$, whose right-hand side is a convenient abbreviation for the power series $e^{i\pi\mathsf{H}}$.

88. We know that $\mathsf{w}\colon V_k \to V_{-k}$ is an isomorphism for every $k \in \mathbb{Z}$. But since X and Y are only morphisms up to a Tate twist, each term in the series expansion of $\mathsf{w} = e^{\mathsf{X}}e^{-\mathsf{Y}}e^{\mathsf{X}}$ needs a different Tate twist, and so it is not immediately clear that w is a morphism of Hodge structures.

Lemma. *If V is an $\mathfrak{sl}_2$-Hodge structure, then $\mathsf{w}\colon V_k \to V_{-k}(-k)$ is an isomorphism of Hodge structures (of weight $n+k$).*

Proof. We first prove an auxiliary formula. Suppose that $b \in V_{-\ell}$ is primitive, in the sense that $\mathsf{Y}b = 0$ (and therefore $\ell \geq 0$). From $\mathsf{w}e^{-\mathsf{X}} = e^{\mathsf{X}}e^{-\mathsf{Y}}$, we get $\mathsf{w}e^{-\mathsf{X}}b = e^{\mathsf{X}}b$, and after expanding and comparing terms in degree $\ell - 2j$, also

$$\mathsf{w}\frac{\mathsf{X}^j}{j!}b = (-1)^j \frac{\mathsf{X}^{\ell-j}}{(\ell-j)!}b. \tag{88.1}$$

Now any $a \in V_k$ has a unique Lefschetz decomposition

$$a = \sum_{j \geq \max(k,0)} \frac{\mathsf{X}^j}{j!} a_j,$$

where $a_j \in V_{k-2j}$ satisfies $Y a_j = 0$. Here we only need to consider $j \geq k$ in the sum because $\mathsf{X}^{2j-k+1} a_j = 0$, which implies that $\mathsf{X}^j a_j = 0$ for $j < k$. Suppose further that $a \in V^{p,q}$, where $p + q = n + k$. Then $\mathsf{X}^i a_j \in V^{p+i,q+i}$, and by descending induction on $j \geq \max(k, 0)$, we deduce that

$$a_j \in V^{p-j,q-j}.$$

In other words, the Lefschetz decomposition holds in the category of Hodge structures. We can now check what happens when we apply w. Using (88.1),

$$\mathsf{w} a = \sum_{j \geq \max(k,0)} w \frac{\mathsf{X}^j}{j!} a_j = \sum_{j \geq \max(k,0)} (-1)^j \frac{\mathsf{X}^{j-k}}{(j-k)!} a_j \in V^{p-k,q-k},$$

and so w is a morphism of Hodge structures. Since w is bijective, it must be an isomorphism of Hodge structures, as claimed. □

89. Using the Weil element, one can give a very concise definition of polarizations for $\mathfrak{sl}_2$-Hodge structures. This idea is due to Deligne [Del84].

Definition. A **polarization** of an $\mathfrak{sl}_2$-Hodge structure V is a hermitian form

$$Q \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$$

that satisfies the four identities $H^\dagger = -H$, $X^\dagger = X$, $Y^\dagger = Y$, and $w^\dagger = w$, such that $Q \circ (\mathrm{id} \otimes \mathsf{w})$ polarizes the Hodge structure of weight $n + k$ on each V_k.

Here the dagger again means the adjoint of an operator with respect to the nondegenerate hermitian pairing Q.

90. The relation $Q \circ (\mathsf{H} \otimes \mathrm{id}) = -Q \circ (\mathrm{id} \otimes \mathsf{H})$ implies that

$$Q\big|_{V_k \otimes_{\mathbb{C}} \overline{V_\ell}} = \begin{cases} Q_k & \text{if } \ell = -k, \\ 0 & \text{otherwise.} \end{cases}$$

and so Q is really a family of sesquilinear pairings $Q_k \colon V_k \otimes_{\mathbb{C}} \overline{V_{-k}} \to \mathbb{C}$. The second condition in §89 Definition is then saying that the hermitian pairing

$$Q_k \circ (\mathrm{id} \otimes \mathsf{w}) \colon V_k \otimes_{\mathbb{C}} \overline{V_k} \to \mathbb{C}$$

polarizes the Hodge structure of weight $n + k$ on V_k. In particular, this means that the Hodge structure on the **primitive** subspace $V_{-k} \cap \ker \mathsf{Y}$ is polarized by the hermitian pairing $Q_k \circ (\mathrm{id} \otimes \mathsf{X}^k)$; this is the usual way to describe the polarization on the cohomology of a compact Kähler manifold. Deligne's definition has the advantage of giving a concise formula for the polarization on all of V, not just on the primitive subspaces.

91. With the exception of positivity, all the conditions in the definition have a nice functorial interpretation. The conjugate complex vector space $\overline{V}$ is again an $\mathfrak{sl}_2$-Hodge structure of weight n: the action of H is unchanged, but X and Y act with an extra minus sign. This sign change is dictated by the geometric case, where $\mathsf{X} = 2\pi i\, L_\omega$ and $\mathsf{Y} = (2\pi i)^{-1}\Lambda_\omega$. Likewise, if V' and V'' are $\mathfrak{sl}_2$-Hodge structures of weights n' and n'', then the tensor product $V' \otimes_{\mathbb{C}} V''$ is naturally an $\mathfrak{sl}_2$-Hodge structure of weight $n' + n''$: to be precise,

$$\big(V' \otimes_{\mathbb{C}} V''\big)_k = \bigoplus_{i+j=k} V'_i \otimes_{\mathbb{C}} V''_j,$$

and the $\mathfrak{sl}_2(\mathbb{C})$-action is given by the usual formulas

$$\begin{aligned} \mathsf{X}(v' \otimes v'') &= \mathsf{X}v' \otimes v'' + v' \otimes \mathsf{X}v'', \\ \mathsf{Y}(v' \otimes v'') &= \mathsf{Y}v' \otimes v'' + v' \otimes \mathsf{Y}v'', \\ \mathsf{H}(v' \otimes v'') &= \mathsf{H}v' \otimes v'' + v' \otimes \mathsf{H}v''. \end{aligned}$$

Lastly, we can turn $\mathbb{C}(-n)$ into an $\mathfrak{sl}_2$-Hodge structure of weight $2n$ by letting $\mathfrak{sl}_2(\mathbb{C})$ act trivially. Then all the identities in §89 Definition can be summarized in one line by saying that the hermitian form

$$Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}(-n)$$

is a morphism of $\mathfrak{sl}_2$-Hodge structures of weight $2n$.

92. From a polarization Q, we obtain a hermitian inner product on the vector space V as follows. The individual Hodge structures on the weight spaces V_k give us a direct sum decomposition

$$V = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}_{p+q-n},$$

and we denote by $v = \sum_{p,q} v^{p,q}$ the components of a given vector. The formula

$$\langle v, w\rangle = \sum_{p,q\in\mathbb{Z}} (-1)^q Q\big(v^{p,q}, \mathsf{w}(w^{p,q})\big) \tag{92.1}$$

then defines a positive definite hermitian inner product on V. By construction, the above decomposition is orthogonal with respect to this inner product. We will see in §100 Theorem how this inner product relates to polarized Hodge structures.

5.2 The irreducible representations

93. Recall that $\mathfrak{sl}_2(\mathbb{C})$ has, up to isomorphism, a unique irreducible representation of dimension $m+1$. We denote this representation by S_m. Clearly, S_0 is the trivial representation, and $S_1 = \mathbb{C}^2$ is the standard representation; for $m \geq 2$, one has $S_m = \operatorname{Sym}^m(S_1)$. Since finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ are completely reducible, one has by Schur's lemma a canonical decomposition

$$V \cong \bigoplus_{m\in\mathbb{N}} S_m \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}. \tag{93.1}$$

We can turn this into a decomposition in the category of $\mathfrak{sl}_2$-Hodge structures, by the following construction.

94. First, we argue that each irreducible representation S_m comes with a canonical $\mathfrak{sl}_2$-Hodge structure of weight m. For the trivial representation $S_0 = \mathbb{C}$, this is the trivial Hodge structure of type $(0,0)$. The **standard representation** S_1 also has a canonical $\mathfrak{sl}_2$-Hodge structure. Denote by $a = (1,0)$ and $b = (0,1)$ the two standard basis vectors, so that $\mathsf{Y}a = b$ and $\mathsf{X}b = a$. We get an $\mathfrak{sl}_2$-Hodge structure of weight 1 by considering

$$E_1(\mathsf{H}) = \mathbb{C}a \quad \text{and} \quad E_{-1}(\mathsf{H}) = \mathbb{C}b$$

as Hodge structures of type $(1,1)$ and $(0,0)$, respectively. It is polarized by the unique hermitian pairing with $Q(a,b) = 1$ and $Q(a,a) = Q(b,b) = 0$; indeed,

$$\begin{aligned}(-1)^1 Q(a, \mathsf{w}a) &= -Q(a,-b) = Q(a,b) = 1,\\ (-1)^0 Q(b, \mathsf{w}b) &= Q(b,a) = 1,\end{aligned}$$

since $\mathsf{w}a = -b$ and $\mathsf{w}b = a$. In particular, we have

$$\langle a,a\rangle = \langle b,b\rangle = 1 \quad \text{and} \quad \langle a,b\rangle = 0,$$

which means that $a, b \in S_1$ form an orthonormal basis with respect to (92.1).

95. Since everything is compatible with taking symmetric powers, the irreducible representation $S_m = \operatorname{Sym}^m(S_1)$ inherits an $\mathfrak{sl}_2$-Hodge structure of weight m. We can describe this concretely as follows. The $m+1$ elements

$$v_0 = a^m, v_1 = a^{m-1}b, \dots, v_k = a^{m-k}b^k, \dots, v_m = b^m$$

form a basis for the vector space S_m. It is easy to see that

$$\mathsf{H}v_k = (m-2k)v_k, \quad \mathsf{X}v_k = k v_{k-1}, \quad \mathsf{Y}v_k = (m-k)v_{k+1},$$

if we declare that $v_{-1} = v_{m+1} = 0$. They are morphisms up to the required Tate twists if we consider each weight space

$$E_{m-2k}(\mathsf{H}) = \mathbb{C}v_k$$

as a Hodge structure of type $(m-k, m-k)$. The induced hermitian pairing Q on S_m has the property that $Q(v_j, v_k) = 0$ if $k + j \neq m$, while

$$Q(v_k, v_{m-k}) = \frac{k!(m-k)!}{m!}.$$

Since $\mathsf{w}v_k = (\mathsf{w}a)^{m-k}(\mathsf{w}b)^k = (-1)^{m-k}v_{m-k}$, we get

$$(-1)^{m-k} Q(v_k, \mathsf{w}v_k) = Q(v_k, v_{m-k}) = \frac{k!(m-k)!}{m!} > 0,$$

as required by §89 Definition. With respect to the hermitian inner product in (92.1), the vectors $v_0, \dots, v_m \in S_m$ are again orthogonal, and

$$\langle v_k, v_k\rangle = \frac{k!(m-k)!}{m!}.$$

Note. Recall that a hermitian pairing $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ induces a hermitian pairing on the symmetric product $\operatorname{Sym}^m V$ by the formula

$$Q\bigl(v_1 \cdots v_m, w_1 \cdots w_m\bigr) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \prod_{j=1}^{m} Q\bigl(v_j, w_{\sigma(j)}\bigr),$$

where $\mathfrak{S}_m$ is the group of permutations of the set $\{1, \dots, m\}$. This comes about by viewing the symmetric product as a subspace of $V^{\otimes m}$, via the embedding

$$\operatorname{Sym}^m V \hookrightarrow V^{\otimes m}, \quad v_1 \cdots v_m \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

This explains the term $m!$ in the formulas above.

96. We continue working on the decomposition in (93.1). Next, we note that in any $\mathfrak{sl}_2$-Hodge structure V of weight n, the subspace of $\mathfrak{sl}_2(\mathbb{C})$-invariants

$$V^{\mathfrak{sl}_2(\mathbb{C})} = \bigl\{\, v \in V_0 \bigm| \mathsf{X}v = \mathsf{Y}v = 0 \,\bigr\} = \ker\bigl(\mathsf{Y}\colon V_0 \to V_{-2}(-1)\bigr)$$

is a sub-Hodge structure of V_0, and hence a Hodge structure of weight n. If V is polarized by Q, then $V^{\mathfrak{sl}_2(\mathbb{C})}$ is polarized by the restriction of Q.

97. Now let V be an $\mathfrak{sl}_2$-Hodge structure of weight n. For each $m \in \mathbb{N}$, the vector space $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$ inherits an $\mathfrak{sl}_2$-Hodge structure of weight $n - m$. According to the preceding paragraph, the subspace of $\mathfrak{sl}_2(\mathbb{C})$-invariants

$$\operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$$

is therefore a Hodge structure of weight $n - m$. If we view it as an $\mathfrak{sl}_2$-Hodge structure by letting $\mathfrak{sl}_2(\mathbb{C})$ act trivially, the evaluation mapping

$$S_m \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})} \to V$$

becomes a morphism of $\mathfrak{sl}_2$-Hodge structures of weight n, and the isomorphism in (93.1) becomes an isomorphism of $\mathfrak{sl}_2$-Hodge structures.

98. The following lemma shows that the decomposition in (93.1) is also compatible with polarizations.

Lemma. *Let V be an $\mathfrak{sl}_2$-Hodge structure of weight n. Then $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$ inherits an $\mathfrak{sl}_2$-Hodge structure of weight $n - m$, such that the evaluation morphism*

$$S_m \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})} \to V, \quad s \otimes f \mapsto f(s),$$

is a morphism of $\mathfrak{sl}_2$-Hodge structures. If V is polarized, then $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$ is also polarized, and the evaluation morphism respects the polarizations.

Proof. The first assertion is obvious, and so we focus on the second one. For clarity, let us write Q_V for the polarization on V, and Q_{S_m} for the polarization on S_m. Given $f\colon S_m \to V$, let $f^\dagger\colon V \to S_m$ be the adjoint, defined by the rule

$$Q_V\bigl(v, f(s)\bigr) = Q_{S_m}\bigl(f^\dagger(v), s\bigr).$$

A short calculation, similar to §73 Lemma, shows that the trace pairing

$$(f,g) \mapsto \frac{\operatorname{tr}(g^\dagger \circ f)}{\dim S_m}$$

polarizes the $\mathfrak{sl}_2$-Hodge structure on $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$. The induced polarization on the tensor product is therefore

$$(f \otimes s, g \otimes t) \mapsto Q_{S_m}(s,t) \cdot \frac{\operatorname{tr}(g^\dagger \circ f)}{\dim S_m}.$$

Now suppose that $f, g \in \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$. By Schur's lemma, we have

$$g^\dagger \circ f = c(f,g) \operatorname{id}_{S_m}$$

for a constant $c(f,g) \in \mathbb{C}$. Consequently,

$$Q_{S_m}(s,t) \cdot \frac{\operatorname{tr}(g^\dagger \circ f)}{\dim S_m} = c(f,g) Q_{S_m}(s,t) = Q_{S_m}\big(g^\dagger(f(s)), t\big) = Q_V\big(f(s), g(t)\big),$$

and so the induced polarization on the tensor product is indeed compatible with the given polarization on V. □

5.3 The associated Hodge structure

99. One justification for the name "$\mathfrak{sl}_2$-Hodge structure" is that every (polarized) $\mathfrak{sl}_2$-Hodge structure has canonically associated to it a (polarized) Hodge structure of the same weight. We now describe this construction. Suppose V is an $\mathfrak{sl}_2$-Hodge structure of weight n. We denote by

$$V = \bigoplus_{i,j \in \mathbb{Z}} V^{i,j}_{i+j-n}$$

the "total" Hodge decomposition, and by

$$F^p = \bigoplus_{i \geq p, j} V^{i,j}_{i+j-n} \quad \text{and} \quad \overline{F}^q = \bigoplus_{j \geq q, i} V^{i,j}_{i+j-n}$$

the total Hodge filtration and its conjugate. Clearly, $\mathsf{X}(F^p) \subseteq F^{p+1}$ and $\mathsf{Y}(F^p) \subseteq F^{p-1}$, and likewise for the other filtration $\overline{F}$. If Q is a polarization of the $\mathfrak{sl}_2$-Hodge structure, one can recover the conjugate Hodge filtration $\overline{F}$ from the Hodge filtration F because

$$\overline{F}^q = \big\{ v \in V \;\big|\; Q(v,x) = 0 \text{ for all } x \in F^{n-q+1} \big\}. \tag{99.1}$$

Note. In fact, a polarized $\mathfrak{sl}_2$-Hodge structure can be described completely by specifying the representation V, the weight n, the Hodge filtration F, and the hermitian pairing Q. This data determines the conjugate Hodge filtration $\overline{F}$ according to (99.1), and therefore the subspaces

$$V^{i,j}_k = E_k(\mathsf{H}) \cap F^i \cap \overline{F}^j$$

in the Hodge decomposition of each weight space. Note that the weight is *not* determined by F and Q alone.

100. The following theorem explains how to pass from a (polarized) $\mathfrak{sl}_2$-Hodge structure to a (polarized) Hodge structure of the same weight. Later in this chapter, we are going to prove a sort of converse to this result, which is the key to the existence of limiting mixed Hodge structures.

Theorem. *Let V be an $\mathfrak{sl}_2$-Hodge structure of weight n, and let Q be a polarization. Then the following is true:*

(a) $e^{\mathsf{Y}}F$ and $e^{-\mathsf{Y}}\overline{F}$ are the two Hodge filtrations of a polarized Hodge structure of weight n, with polarization Q.

(b) The inner product determined by this polarized Hodge structure is equal to the one determined by the $\mathfrak{sl}_2$-Hodge structure in (92.1).

(c) We have $\mathsf{H}^* = \mathsf{H}$ and $\mathsf{Y}^* = \mathsf{X}$, where $*$ denotes the adjoint with respect to the inner product determined by the polarized Hodge structure.

Proof. The decomposition in (93.1) implies that it is enough to prove this for the irreducible representations S_m. Furthermore, everything is compatible with taking symmetric powers, and so we only need to consider the standard representation S_1. Here $F^1 = \overline{F}^1 = \mathbb{C}a$, and consequently

$$e^{\mathsf{Y}}F^1 = \mathbb{C}(a+b) \quad \text{and} \quad e^{-\mathsf{Y}}\overline{F}^1 = \mathbb{C}(a-b).$$

Since $Q(a+b, a-b) = 0$, we obtain a Hodge structure $V = V^{1,0} \oplus V^{0,1}$ with

$$V^{1,0} = \mathbb{C}(a+b) \quad \text{and} \quad V^{0,1} = \mathbb{C}(a-b).$$

Also, Q is a polarization because $Q(a+b, a+b) = 2 = -Q(a-b, a-b)$. The Hodge decompositions of the two standard basis vectors are

$$a = \frac{1}{2}(a+b) + \frac{1}{2}(a-b) \quad \text{and} \quad b = \frac{1}{2}(a+b) - \frac{1}{2}(a-b),$$

from which it follows that a and b form an orthonormal basis with respect to the inner product coming from the Hodge structure. Since $\mathsf{H}a = a$ and $\mathsf{H}b = -b$, we see that H is self-adjoint with respect to the inner product; similarly, $\mathsf{Y}a = b$ and $\mathsf{X}b = a$ imply that $\mathsf{Y}^* = \mathsf{X}$. Recalling the definition of the inner product in (92.1), we obviously have

$$\langle a, a \rangle = \langle b, b \rangle = 1 \quad \text{and} \quad \langle a, b \rangle = 0,$$

and so the two inner products agree. □

101. The analogy with the cohomology of a compact Kähler manifold is striking: V has a hermitian inner product; $\mathsf{Y} = \mathsf{X}^*$ is the adjoint of the operator X with respect to this inner product; $\mathsf{H} = [\mathsf{X}, \mathsf{X}^*]$ is the commutator of X and its adjoint.

102. The next result relates the Weil operator in the Hodge structure $e^{\mathsf{Y}}F$ to representation theory. Using the total Hodge decomposition

$$V = \bigoplus_{i,j} V^{i,j}_{i+j-n},$$

we define an involution $C \in \mathrm{GL}(V)$ by the rule

$$Cv = (-1)^j v \quad \text{for } v \in V^{i,j}_{i+j-n}.$$

This is the analogue of the Weil operator for an $\mathfrak{sl}_2$-Hodge structure.

Proposition. *Let $C_\sharp \in \mathrm{End}(V)$ be the Weil operator of the Hodge structure whose Hodge filtration is $F_\sharp = e^{\mathsf{Y}} F$. Then $C_\sharp = \mathsf{w} \circ C$.*

Proof. As before, it suffices to prove this in the case of the standard $\mathfrak{sl}_2$-Hodge structure on S_1. If we again denote the two basis vectors by a and b, then

$$Ca = -a, \quad Cb = b, \quad \mathsf{w}a = -b, \quad \mathsf{w}b = a, \quad C_\sharp a = b, \quad C_\sharp b = a,$$

from which it follows that $C_\sharp = \mathsf{w} \circ C$. □

103. For later use, let us also record the Hodge decompositions of the three operators $\mathsf{H}, \mathsf{X}, \mathsf{Y} \in \mathrm{End}(V)$ that constitute the $\mathfrak{sl}_2(\mathbb{C})$-representation. Let

$$\mathrm{End}(V) = \bigoplus_{j \in \mathbb{Z}} \mathrm{End}(V)^{j,-j}$$

denote the Hodge structure of weight 0 on $\mathrm{End}(V)$, induced by the Hodge structure $e^{\mathsf{Y}} F$ on V. For any $A \in \mathrm{End}(V)$, we write the Hodge decomposition in the form

$$A = \sum_{j \in \mathbb{Z}} A_j,$$

with $A_j \in \mathrm{End}(V)^{j,-j}$.

Lemma. *The Hodge decompositions of $\mathsf{Y}, \mathsf{X}, \mathsf{H} \in \mathrm{End}(V)$ look like*

$$\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1, \quad \mathsf{X} = -\mathsf{Y}_{-1} + \mathsf{Y}_0 - \mathsf{Y}_1, \quad \mathsf{H} = -2\mathsf{Y}_{-1} + 2\mathsf{Y}_1,$$

and conversely, we have

$$4\mathsf{Y}_{-1} = \mathsf{Y} - \mathsf{H} - \mathsf{X}, \quad 2\mathsf{Y}_0 = \mathsf{Y} + \mathsf{X}, \quad 4\mathsf{Y}_1 = \mathsf{Y} + \mathsf{H} - \mathsf{X}$$

Proof. It is again enough to prove this for the standard $\mathfrak{sl}_2$-Hodge structure on S_1. Denote the two basis vectors by a and b; then the Hodge decomposition is

$$S_1 = \mathbb{C}(a+b) \oplus \mathbb{C}(a-b).$$

With respect to the basis a, b, we have

$$\mathsf{Y} - \mathsf{H} - \mathsf{X} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathsf{Y} + \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{Y} + \mathsf{H} - \mathsf{X} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

and it is easy to see that these three matrices have Hodge type $(-1,1)$, $(0,0)$, and $(1,-1)$, respectively. This suffices to conclude the proof. □

104. One nice consequence of these identities is that the isotypical components of the $\mathfrak{sl}_2(\mathbb{C})$-representation are actually sub-Hodge structures. This is most easily proved with the help of the **Casimir operator**

$$\Omega = 2(\mathsf{XY} + \mathsf{YX}) + \mathsf{H}^2 + \mathrm{id} = 4\mathsf{XY} + (\mathsf{H} + \mathrm{id})^2 \in \mathrm{End}(V).$$

By general theory, Ω commutes with $\mathsf{X}, \mathsf{H}, \mathsf{Y}$, and acts on the m-th summand in the decomposition (93.1) as multiplication by $(m+1)^2$. A brief computation using the identities in §103 Lemma gives

$$\begin{aligned}\Omega &= 2(-\mathsf{Y}_{-1} + \mathsf{Y}_0 - \mathsf{Y}_1)(\mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1) \\ &\qquad + 2(\mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1)(-\mathsf{Y}_{-1} + \mathsf{Y}_0 - \mathsf{Y}_1) + 4(\mathsf{Y}_1 - \mathsf{Y}_{-1})^2 + \mathrm{id} \\ &= (2\mathsf{Y}_0 + \mathrm{id})^2 - 16\mathsf{Y}_1\mathsf{Y}_{-1}.\end{aligned}$$

This means that Ω is an endomorphism of the Hodge structure on V, and therefore that each isotypical component $S_m \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$ in the decomposition (93.1) is automatically a sub-Hodge structure.

5.4 The associated variation of Hodge structure

105. In fact, a polarized $\mathfrak{sl}_2$-Hodge structure of weight n determines a polarized variation of Hodge structure of weight n on the punctured disk Δ^*. These "model variations" are important, both as examples of the general theory, and because they are used to prove the Hodge norm estimates.

106. Let V be an $\mathfrak{sl}_2$-Hodge structure of weight n, and let Q be a polarization. From the previous section, we keep the notation

$$F^p = \bigoplus_{i \geq p, j} V^{i,j}_{i+j-n}$$

for the total Hodge filtration. By construction, $\mathsf{H}(F^p) \subseteq F^p$, $\mathsf{Y}(F^p) \subseteq F^{p-1}$, and $\mathsf{X}(F^p) \subseteq F^{p+1}$. We now define a holomorphic mapping

$$\Phi \colon \mathbb{H} \to \check{D}, \quad \Phi(z) = e^{-z\mathsf{Y}}F.$$

At the point $z = -1$, the filtration $e^{\mathsf{Y}}F$ is the Hodge filtration of a polarized Hodge structure of weight n (by §100 Theorem), and therefore $\Phi(-1) \in D$. Now write $z = x+iy$, with $x < 0$. From the relation $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$, we get

$$e^{-z\mathsf{Y}} = e^{-iy\mathsf{Y}} e^{|x|\mathsf{Y}} = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} e^{\mathsf{Y}} e^{\frac{1}{2}\log|x|\,\mathsf{H}},$$

and therefore

$$\Phi(z) = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} \cdot e^{\mathsf{Y}}F = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} \cdot \Phi(-1). \tag{106.1}$$

Both exponential factors are elements of the real group $G = \mathrm{Aut}(V, Q)$, because $\mathsf{H}^\dagger = -\mathsf{H}$ and $\mathsf{Y}^\dagger = \mathsf{Y}$, and so we conclude that $\Phi(z) \in D$ for every $z \in \mathbb{H}$.

107. We are going to show that Φ is the period mapping of a polarized variation of Hodge structure. Let $E = \mathbb{H} \times V$ be the trivial smooth vector bundle with fiber V, and let $d \colon A^0(\mathbb{H}, E) \to A^1(\mathbb{H}, E)$ be the flat connection induced by differentiation. The Hodge structures $\Phi(z)$ give us a decomposition

$$E = \bigoplus_{p+q=n} E^{p,q}$$

into smooth subbundles. This can be described concretely as follows. Let

$$V = \bigoplus_{p+q=n} V^{p,q}$$

denote the Hodge decomposition in the Hodge structure $\Phi(-1)$. Due to (106.1), the Hodge decomposition in the Hodge structure $\Phi(z)$ is then equal to

$$V = \bigoplus_{p+q=n} V^{p,q}_{\Phi(z)} = \bigoplus_{p+q=n} e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} V^{p,q},$$

and so the subbundle $E^{p,q} \subseteq E$ is the image of the smooth mapping

$$\mathbb{H} \times V^{p,q} \to \mathbb{H} \times V, \quad (z, v) \mapsto e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} v.$$

108. It is now a simple matter to verify the conditions in §44 Definition. Every smooth section in $A^0(\mathbb{H}, E^{p,q})$ can be written in the form

$$e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} f,$$

where $f \colon \mathbb{H} \to V^{p,q}$ is a smooth function. We have

$$\begin{aligned} &d\Big(e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} f\Big) \\ &\qquad = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} \left(-ie^{\frac{1}{2}\log|x|\,\mathsf{H}} \mathsf{Y} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} f \otimes dy + \frac{1}{2|x|} \mathsf{H} f \otimes dx + df \right) \\ &\qquad = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} \left(-\frac{i}{|x|} \mathsf{Y} f \otimes dy + \frac{1}{2|x|} \mathsf{H} f \otimes dx + df \right) \end{aligned}$$

Remembering the identities (from §103 Lemma)

$$\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1 \quad \text{and} \quad \mathsf{H} = -2\mathsf{Y}_{-1} + 2\mathsf{Y}_1,$$

we can rewrite the factor in parentheses as

$$-\frac{i}{|x|} \mathsf{Y} f \otimes dy + \frac{1}{2|x|} \mathsf{H} f \otimes dx + df = df - \frac{1}{|x|} \mathsf{Y}_{-1} f \otimes dz + \frac{1}{|x|} \mathsf{Y}_1 f \otimes d\bar{z} - \frac{1}{2|x|} \mathsf{Y}_0 f \otimes (dz - d\bar{z})$$

It follows that $d = \partial + \theta + \bar{\partial} + \theta^*$, just as in §44 Definition, with

$$\begin{aligned} g^{-1} \partial g &= \frac{\partial}{\partial z} - \frac{1}{2|x|} \mathsf{Y}_0 \otimes dz, & g^{-1} \theta g &= -\frac{1}{|x|} \mathsf{Y}_{-1} \otimes dz, \\ g^{-1} \bar{\partial} g &= \frac{\partial}{\partial \bar{z}} + \frac{1}{2|x|} \mathsf{Y}_0 \otimes d\bar{z}, & g^{-1} \theta^* g &= \frac{1}{|x|} \mathsf{Y}_1 \otimes dz, \end{aligned}$$

where $g = e^{-iy\mathsf{Y}} e^{-\frac{1}{2}\log|x|\,\mathsf{H}} \in G$. Evidently, these operators have the required bidegrees $(1,0) \otimes (0,0)$, $(0,1) \otimes (0,0)$, $(1,0) \otimes (-1,1)$, and $(0,1) \otimes (1,-1)$, respectively. Therefore Φ is indeed the period mapping of a polarized variation of Hodge structure on $\mathbb{H}$. From the identity

$$\Phi(z + 2\pi i) = e^{-2\pi i \mathsf{Y}} \Phi(z),$$

we see that the variation of Hodge structure is pulled back from the punctured disk Δ^*; the monodromy operator is the unipotent operator

$$T = e^{-2\pi i \mathsf{Y}} \in G,$$

and in our usual notation, we therefore have $N = -\mathsf{Y}$.

109. In fact, we can easily turn this into an example with non-unipotent monodromy as well. Let $S \in \operatorname{End}(V)$ be an operator with the property that $S^\dagger = S$, and assume that S is an endomorphism of the $\mathfrak{sl}_2$-Hodge structure on V. Concretely, this means that $[\mathsf{H}, S] = [\mathsf{Y}, S] = 0$ and that $S(F^p) \subseteq F^p$ for every $p \in \mathbb{Z}$. It follows that S is self-adjoint with respect to the inner product in (92.1), and so S is automatically semisimple with real eigenvalues. We still have

$$\Phi(z + 2\pi i) = e^{2\pi i (S - \mathsf{Y})} \Phi(z),$$

and so we can also consider Φ as the period mapping of a polarized variation of Hodge structure on Δ^* with monodromy operator

$$T = T_s T_u = e^{2\pi i S} e^{-2\pi i \mathsf{Y}} \in G.$$

This is the Jordan decomposition of T.

110. As an aside, let us verify the result in §58 Corollary about the pointwise Hodge norm of the Higgs field in this example. Since $\Phi(z) = g \cdot \Phi(-1)$, we have

$$\left\| \theta_{\partial/\partial z} \right\|^2_{\Phi(z)} = \left\| g^{-1} \theta_{\partial/\partial z} g \right\|^2_{\Phi(-1)} = \frac{1}{|x|^2} \left\| Y_{-1} \right\|^2_{\Phi(-1)},$$

and so the pointwise Hodge norm of the Higgs field is exactly a constant multiple of $|x|^{-1}$ in this case. (It is a pleasant exercise to check that when $V = S_m$ is the irreducible representation of dimension $m+1$, and hence $\operatorname{rk} E = m+1$, one has $\left\| Y_{-1} \right\|^2_{\Phi(-1)} = \frac{1}{4}\binom{m+2}{3}$, and so equality is achieved in §58 Corollary; this explains what we found during the proof of §57 Lemma.)

111. We are now going to derive a simple formula for the Hodge metric in our "model" variation of Hodge structure. To simplify the notation, let us write $\|v\|$ and $\langle v, w \rangle$ for the inner product in (92.1); according to §100 Theorem, we have

$$\|v\| = \|v\|_{\Phi(-1)} \quad \text{and} \quad \langle v, w \rangle = \langle v, w \rangle_{\Phi(-1)},$$

and so this agrees with our usual notation. From (106.1), we get

$$\left\langle v, w \right\rangle_{\Phi(z)} = \left\langle e^{\frac{1}{2} \log |x| \, \mathsf{H}} e^{iy\mathsf{Y}} v, \, e^{\frac{1}{2} \log |x| \, \mathsf{H}} e^{iy\mathsf{Y}} w \right\rangle$$

and since $\mathsf{H}^* = \mathsf{H}$ and $\mathsf{Y}^* = \mathsf{X}$ by §100 Theorem, we can rewrite this as

$$\left\langle v, w \right\rangle_{\Phi(z)} = \left\langle e^{\log |x| \, \mathsf{H}} e^{iy\mathsf{Y}} v, \, e^{iy\mathsf{Y}} w \right\rangle = \left\langle e^{-iy\mathsf{X}} e^{\log |x| \, \mathsf{H}} e^{iy\mathsf{Y}} v, \, w \right\rangle.$$

112. The conclusion that we can draw from this is that the weight filtration

$$W_\ell = E_\ell(\mathsf{H}) \oplus E_{\ell-1}(\mathsf{H}) \oplus E_{\ell-2}(\mathsf{H}) \oplus \cdots$$

controls the rate of growth of the Hodge norm in these examples. Indeed, suppose that we have a vector $v \in V$ with $\mathsf{H}v = \ell v$. Then

$$e^{-iy\mathsf{X}} e^{\log|x| \, \mathsf{H}} e^{iy\mathsf{Y}} v = e^{-iy\mathsf{X}} \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} |x|^{\ell - 2k} \mathsf{Y}^k v = \sum_{j,k=0}^{\infty} (-1)^j \frac{(iy)^{j+k}}{j! k!} |x|^{\ell - 2k} \mathsf{X}^j \mathsf{Y}^k v,$$

and since different weight spaces are orthogonal under the inner product, we get

$$\|v\|^2_{\Phi(z)} = \sum_{k=0}^{\infty} (-1)^k \frac{(iy)^{2k}}{(k!)^2} |x|^{\ell-2k} \langle \mathsf{X}^k \mathsf{Y}^k v, v\rangle = \sum_{k=0}^{\infty} \frac{y^{2k}}{(k!)^2} |x|^{\ell-2k} \|\mathsf{Y}^k v\|^2.$$

As long as $y = \operatorname{Im} z$ remains bounded, this expression grows like $|x|^\ell = |\operatorname{Re} z|^\ell$; in fact, the leading term is exactly $|\operatorname{Re} z|^\ell \|v\|^2$.

5.5 Recognizing polarized $\mathfrak{sl}_2$-Hodge structures

113. We end our discussion of $\mathfrak{sl}_2$-Hodge structures by proving a converse to §100 Theorem. The result below gives us a way to recognize polarized $\mathfrak{sl}_2$-Hodge structures, and is going to be the key step in the construction of the limiting mixed Hodge structure. Let V be a finite-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and let $Q \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ be a nondegenerate hermitian pairing such that $\mathsf{X}^\dagger = \mathsf{X}$, $\mathsf{Y}^\dagger = \mathsf{Y}$, and $\mathsf{H}^\dagger = -\mathsf{H}$, where the dagger always means the adjoint of an operator with respect to Q.

114. Now suppose that we have a decreasing filtration F on the vector space V, with the following two properties:

(a) One has $\mathsf{Y}(F^\bullet) \subseteq F^{\bullet-1}$ and $\mathsf{H}(F^\bullet) = F^\bullet$.

(b) The filtration $e^{\mathsf{Y}} F$ is the Hodge filtration of a polarized Hodge structure of weight n on V, polarized by the pairing Q.

It will be convenient to have a name for this kind of object, so let us agree to call such a filtration F an **$\mathfrak{sl}_2$-Hodge filtration** (of weight n).

Theorem. *If F is an $\mathfrak{sl}_2$-Hodge filtration on the vector space V, of weight n, then V has a unique $\mathfrak{sl}_2$-Hodge structure of weight n such that*

$$F^p = \bigoplus_{i \geq p, j} V^{i,j}_{i+j-n} \quad \textit{for all } p \in \mathbb{Z},$$

and this $\mathfrak{sl}_2$-Hodge structure is polarized by the pairing Q.

115. The proof is basically just linear algebra, with a little bit of representation theory mixed in. We note the following useful corollary.

Corollary. *If $S \in \operatorname{End}(V)$ satisfies $[\mathsf{H}, S] = [\mathsf{Y}, S] = 0$ and $S^\dagger = S$, and if $S(F^\bullet) \subseteq F^\bullet$, then S is an endomorphism of the $\mathfrak{sl}_2$-Hodge structure that also respects the polarization.*

116. The proof of §114 Theorem uses the decomposition

$$V \cong \bigoplus_{m \in \mathbb{N}} S_m \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$$

into isotypical components, coming from Schur's lemma. Recall that each S_m has a canonical polarized $\mathfrak{sl}_2$-Hodge structure of weight m. According to §100 Theorem, the total Hodge filtration $F^\bullet S_m$ is an $\mathfrak{sl}_2$-Hodge filtration in the above sense. We are going to argue that the $\mathfrak{sl}_2$-Hodge filtrations on S_m and V give each vector space $\operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$ a polarized Hodge structure of weight $n-m$, in a way that is compatible with the above decomposition. The result we want will then follow from the fact that each irreducible representation S_m is a polarized $\mathfrak{sl}_2$-Hodge structure of weight m.

117. The first thing to do is to analyze the subspace of $\mathfrak{sl}_2(\mathbb{C})$-invariants; everything else is going to follow from this case by functoriality.

Proposition. *Let F be an $\mathfrak{sl}_2$-Hodge filtration. Then*

$$V^{\mathfrak{sl}_2(\mathbb{C})} = \left\{ v \in V \;\middle|\; \mathsf{H}v = \mathsf{Y}v = 0 \right\}$$

has a Hodge structure of weight n, polarized by the restriction of Q, whose Hodge filtration is $F \cap V^{\mathfrak{sl}_2(\mathbb{C})}$. This Hodge structure is compatible with the polarized Hodge structure on V defined by the filtration $e^{\mathsf{Y}}F$.

118. The idea of the proof is to analyze the Hodge decompositions of the two operators $\mathsf{H}, \mathsf{Y} \in \operatorname{End}(V)$. Note that H is "real", in the sense that it belongs to the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$; the other two operators $\mathsf{X}, \mathsf{Y} \in i\mathfrak{g}$ are "purely imaginary", due to the fact that the pairing Q is hermitian. (The same thing happens for the $\mathfrak{sl}_2(\mathbb{C})$-representation on the cohomology of a compact Kähler manifold (X, ω), because $\mathsf{X} = 2\pi i\, L_\omega$ and $\mathsf{Y} = (2\pi i)^{-1} \Lambda_\omega$ in that case.)

119. To simplify the notation, let us from now on denote by

$$V = \bigoplus_{p+q=n} V^{p,q}$$

the Hodge structure of weight n whose Hodge filtration is $e^{\mathsf{Y}}F$. Since Q is a polarization, it is easy to see that the conjugate Hodge filtration is $e^{-\mathsf{Y}}\overline{F}$, where

$$\overline{F}_q = \left\{ v \in V \;\middle|\; Q(v, x) = 0 \text{ for all } x \in F^{n-q+1} \right\}.$$

The Hodge structure on V induces a Hodge structure of weight 0 on the vector space $\operatorname{End}(V)$. Recall that the Lie algebra

$$\mathfrak{g} = \left\{ A \in \operatorname{End}(V) \;\middle|\; A^\dagger = -A \right\}$$

defines a real structure; the induced polarization is $(A, B) \mapsto \operatorname{tr}(A \circ B^\dagger)$, where $B^\dagger$ means the adjoint of B relative to the nondegenerate hermitian pairing Q. For an operator $A \in \operatorname{End}(V)$, we denote by

$$A = \sum_{j \in \mathbb{Z}} A_j$$

its Hodge decomposition in the Hodge structure on $\operatorname{End}(V)$, with $A_j \in \operatorname{End}(V)^{j,-j}$. Concretely, this means that $A_j(V^{p,q}) \subseteq V^{p+j,q-j}$ for all $p, q \in \mathbb{Z}$.

120. If we write the Hodge decompositions of $\mathsf{Y}, \mathsf{H} \in \operatorname{End}(V)$ as

$$\mathsf{Y} = \sum_{j \in \mathbb{Z}} \mathsf{Y}_j \quad \text{and} \quad \mathsf{H} = \sum_{j \in \mathbb{Z}} \mathsf{H}_j,$$

then our assumptions on Y and H can be expressed as follows.

Lemma. *We have*

$$\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1 \quad \textit{and} \quad \mathsf{H} = -2\mathsf{Y}_{-1} + \mathsf{H}_0 + 2\mathsf{Y}_1,$$

and these operators satisfy $\mathsf{Y}_{-1}^\dagger = \mathsf{Y}_1$, $\mathsf{Y}_0^\dagger = \mathsf{Y}_0$, and $\mathsf{H}_0^\dagger = -\mathsf{H}_0$.

Proof. From the fact that $\mathsf{Y}^\dagger = \mathsf{Y}$, we get $(\mathsf{Y}_j)^\dagger = \mathsf{Y}_{-j}$ for every $j \in \mathbb{Z}$. The condition $\mathsf{Y}(F^p) \subseteq F^{p-1}$ implies that $\mathsf{Y}(e^{\mathsf{Y}} F^p) \subseteq e^{\mathsf{Y}} F^{p-1}$, which means that $\mathsf{Y}_j = 0$ for $j \leq -2$. But then also $\mathsf{Y}_j = 0$ for $j \geq 2$, and so actually

$$\mathsf{Y} = \mathsf{Y}_{-1} + \mathsf{Y}_0 + \mathsf{Y}_1.$$

Similarly, we have $\mathsf{H}^\dagger = -\mathsf{H}$, hence $(\mathsf{H}_j)^\dagger = -\mathsf{H}_{-j}$ for every $j \in \mathbb{Z}$. The condition $\mathsf{H}(F^p) \subseteq F^p$ implies that

$$(\mathsf{H} + 2\mathsf{Y}) e^{\mathsf{Y}} F^p = e^{\mathsf{Y}} \mathsf{H} e^{-\mathsf{Y}} \cdot e^{\mathsf{Y}} F^p = e^{\mathsf{Y}} \mathsf{H}(F^p) \subseteq e^{\mathsf{Y}} F^p,$$

and so $\mathsf{H}_j + 2\mathsf{Y}_j = 0$ for $j \leq -1$. In particular, $\mathsf{H}_{-1} = -2\mathsf{Y}_{-1}$ and $\mathsf{H}_j = 0$ for all $j \leq -2$. But then $\mathsf{H}_j = 0$ for $j \geq 2$, and $\mathsf{H}_1 = -(\mathsf{H}_{-1})^\dagger = 2(\mathsf{Y}_{-1})^\dagger = 2\mathsf{Y}_1$. □

121. The relation $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$ gives us the following additional identity.

Lemma. *We have* $2\mathsf{Y}_0 = 2[\mathsf{Y}_{-1}, \mathsf{Y}_1] + [\mathsf{Y}_0, \mathsf{H}_0]$.

Proof. Consider the component of $2\mathsf{Y} = [\mathsf{Y}, \mathsf{H}]$ in the subspace $\operatorname{End}(V)^{0,0}$. From the preceding lemma, we get

$$2\mathsf{Y}_0 = [\mathsf{Y}_1, \mathsf{H}_{-1}] + [\mathsf{Y}_0, \mathsf{H}_0] + [\mathsf{Y}_{-1}, \mathsf{H}_1] = -2[\mathsf{Y}_1, \mathsf{Y}_{-1}] + [\mathsf{Y}_0, \mathsf{H}_0] + 2[\mathsf{Y}_{-1}, \mathsf{Y}_1],$$

which simplifies to the desired identity. □

122. We can now prove §117 Proposition. On V, we have a polarized Hodge structure of weight n with Hodge filtration $e^{\mathsf{Y}} F$. It is enough to show that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is a sub-Hodge structure; the remaining assertions then follow because $e^{\mathsf{Y}} F$ and F induce the same filtration on $V^{\mathfrak{sl}_2(\mathbb{C})}$, due to the fact that Y acts trivially.

123. To say that $V^{\mathfrak{sl}_2(\mathbb{C})}$ is a sub-Hodge structure means that whenever we take a vector $v \in V^{\mathfrak{sl}_2(\mathbb{C})}$, and write its Hodge decomposition as

$$v = \sum_p v_p,$$

with $v_p \in V^{p,q}$, then each $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$. This is trivially satisfied if $v = 0$. If $v \neq 0$, let $p \in \mathbb{Z}$ be the least integer such that $v_p \neq 0$. It is clearly enough to prove that $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$, because we can then repeat the same argument for $v - v_p$. Our goal is therefore to show that $\mathsf{Y} v_p = \mathsf{H} v_p = 0$.

124. From the fact that $\mathsf{Y} v = 0$ and the Hodge decomposition, we deduce that

$$\mathsf{Y}_{-1} v_p = 0 \quad \text{and} \quad \mathsf{Y}_0 v_p + \mathsf{Y}_{-1} v_{p+1} = 0.$$

From the fact that $\mathsf{H} v = 0$, we get the additional piece of information that

$$\mathsf{H}_0 v_p - 2\mathsf{Y}_{-1} v_{p+1} = 0.$$

In particular, $\mathsf{H}_0 v_p = -2\mathsf{Y}_0 v_p$, which is interesting, because Y_0 and H_0 behave very differently under taking adjoints. We can use this different behavior to show that $Q(\mathsf{Y}_0 v_p, v_p) = 0$. Since $(\mathsf{Y}_0)^\dagger = \mathsf{Y}_0$ and $(\mathsf{H}_0)^\dagger = -\mathsf{H}_0$, we have

$$Q(2\mathsf{Y}_0 v_p, v_p) = -Q(\mathsf{H}_0 v_p, v_p) = Q(v_p, \mathsf{H}_0 v_p) = -Q(v_p, 2\mathsf{Y}_0 v_p) = -Q(2\mathsf{Y}_0 v_p, v_p),$$

and therefore $Q(2\mathsf{Y}_0 v_p, v_p) = 0$. Now we combine this with the identity in §121 Lemma. Because we already know that $\mathsf{Y}_{-1} v_p = 0$, this gives

$$\begin{aligned} 0 = Q(2\mathsf{Y}_0 v_p, v_p) &= Q(2\mathsf{Y}_{-1}\mathsf{Y}_1 v_p, v_p) + Q(\mathsf{Y}_0\mathsf{H}_0 v_p, v_p) - Q(\mathsf{H}_0\mathsf{Y}_0 v_p, v_p) \\ &= 2Q(\mathsf{Y}_1 v_p, \mathsf{Y}_1 v_p) + Q(\mathsf{H}_0 v_p, \mathsf{Y}_0 v_p) + Q(\mathsf{Y}_0 v_p, \mathsf{H}_0 v_p) \\ &= 2Q(\mathsf{Y}_1 v_p, \mathsf{Y}_1 v_p) - 4Q(\mathsf{Y}_0 v_p, \mathsf{Y}_0 v_p). \end{aligned}$$

Here we used the fact that $(\mathsf{Y}_{-1})^\dagger = \mathsf{Y}_1$ and $(\mathsf{H}_0)^\dagger = -\mathsf{H}_0$, and also $\mathsf{H}_0 v_p = -2\mathsf{Y}_0 v_p$. Now $\mathsf{Y}_0 v_p \in V^{p,q}$, and since Q is a polarization,

$$Q(\mathsf{Y}_0 v_p, \mathsf{Y}_0 v_p) = (-1)^q \|\mathsf{Y}_0 v_p\|^2,$$

where $\|-\|$ means the Hodge norm in the polarized Hodge structure $e^{\mathsf{Y}} F$. Likewise, $\mathsf{Y}_1 v_p \in V^{p+1,q-1}$, and so

$$Q(\mathsf{Y}_1 v_p, \mathsf{Y}_1 v_p) = (-1)^{q-1} \|\mathsf{Y}_1 v_p\|^2.$$

Putting everything together, we find that

$$0 = \|\mathsf{Y}_1 v_p\|^2 + 2\|\mathsf{Y}_0 v_p\|^2,$$

which clearly implies that $\mathsf{Y}_1 v_p = 0$ and $\mathsf{Y}_0 v_p = 0$. But then also $\mathsf{H}_0 v_p = 0$, and so we have proved that $\mathsf{Y} v_p = \mathsf{H} v_p = 0$, hence $v_p \in V^{\mathfrak{sl}_2(\mathbb{C})}$. This finishes the proof of §117 Proposition.

125. The next step is to establish some basic functoriality. For the sake of clarity, let us denote the $\mathfrak{sl}_2$-Hodge filtrations on S_m and V by the symbols $F^\bullet S_m$ and $F^\bullet V$; the resulting Hodge structures have weight m and n. We get an induced filtration on $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$ according to the rule

$$F^k \operatorname{Hom}_{\mathbb{C}}(S_m, V) = \big\{ f \colon S_m \to V \;\big|\; f(F^p S_m) \subseteq F^{p+k} V \text{ for all } p \in \mathbb{Z} \big\}.$$

The induced $\mathfrak{sl}_2(\mathbb{C})$-representation is easy to describe: for $f \colon S_m \to V$, one has

$$(\mathsf{H} f)(v) = \mathsf{H} f(v) - f(\mathsf{H} v), \quad (\mathsf{X} f)(v) = \mathsf{X} f(v) - f(\mathsf{X} v), \quad (\mathsf{Y} f)(v) = \mathsf{Y} f(v) - f(\mathsf{Y} v).$$

Observe that $\mathfrak{sl}_2(\mathbb{C})$ acts trivially on a linear mapping $f \colon S_m \to V$ exactly when f is a morphism of $\mathfrak{sl}_2(\mathbb{C})$-representations.

126. Since it is important, let us check very carefully that $F^\bullet \operatorname{Hom}_{\mathbb{C}}(S_m, V)$ is indeed an $\mathfrak{sl}_2$-Hodge filtration (of weight $n - m$).

Lemma. *Suppose that $F^\bullet V$ is an $\mathfrak{sl}_2$-Hodge filtration of weight n. Then the filtration $F^\bullet \operatorname{Hom}_{\mathbb{C}}(S_m, V)$ is an $\mathfrak{sl}_2$-Hodge filtration of weight $n - m$.*

Proof. We need to check that the filtration on $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$ satisfies the two conditions in the definition. We can again use the modified trace pairing on $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$ as a polarization. Given a linear mapping $f \colon S_m \to V$, we denote by $f^\dagger \colon V \to S_m$ the adjoint with respect to the nondegenerate pairings Q_{S_m} and Q_V; to be precise,

$$Q_V\big(f(v), s\big) = Q_{S_m}\big(v, f^\dagger(s)\big) \quad \text{for all } v \in V \text{ and } s \in S_m.$$

On $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$, we have the hermitian pairing

$$\operatorname{Hom}_{\mathbb{C}}(S_m, V) \otimes_{\mathbb{C}} \overline{\operatorname{Hom}_{\mathbb{C}}(S_m, V)} \to \mathbb{C}, \quad (f, g) \mapsto \frac{\operatorname{tr}(g^\dagger \circ f)}{\dim S_m}. \tag{126.1}$$

Let $f \in F^k \operatorname{Hom}_{\mathbb{C}}(S_m, V)$ be arbitrary. For any $s \in F^p S_m$, we have $f(s) \in F^{p+k}V$, and therefore

$$(\mathsf{H}f)(s) = \mathsf{H}f(s) - f(\mathsf{H}s) \in \mathsf{H}(F^{p+k}V) + f(F^p S_m) \subseteq F^{p+k}V,$$

which proves that $\mathsf{H}f \in F^k \operatorname{Hom}_{\mathbb{C}}(S_m, V)$. Similarly, $\mathsf{Y}f \in F^{k-1} \operatorname{Hom}_{\mathbb{C}}(S_m, V)$.

It remains to show that the filtration $e^{\mathsf{Y}} F^{\bullet} \operatorname{Hom}_{\mathbb{C}}(S_m, V)$ defines a Hodge structure of weight $n - m$ on $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$, polarized by the pairing in (126.1). Since

$$(e^{\mathsf{Y}} f)(v) = e^{\mathsf{Y}} f\big(e^{-\mathsf{Y}} v\big),$$

it is not hard to see that

$$e^{\mathsf{Y}} F^k \operatorname{Hom}_{\mathbb{C}}(S_m, V) = \big\{ f \colon S_m \to V \ \big|\ f\big(e^{\mathsf{Y}} F^p S_m\big) \subseteq e^{\mathsf{Y}} F^{p+k} V \text{ for all } p \in \mathbb{Z} \big\};$$

but the right-hand side is exactly the Hodge filtration of the induced Hodge structure of weight $n - m$ on $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$. The proof that the pairing in (126.1) polarizes this Hodge structure is similar to the proof of §73 Lemma. ☐

127. We can now prove §114 Theorem. Let $F^{\bullet}V$ be an $\mathfrak{sl}_2$-Hodge filtration on V, of weight n. Our starting point is the decomposition in (93.1). Fix some $m \in \mathbb{N}$. According to §126 Lemma, the induced filtration

$$F^k \operatorname{Hom}_{\mathbb{C}}(S_m, V) = \big\{ f \colon S_m \to V \ \big|\ f(F^p S_m) \subseteq F^{p+k} V \text{ for all } p \in \mathbb{Z} \big\}$$

is an $\mathfrak{sl}_2$-Hodge filtration of weight $n - m$, and the modified trace pairing

$$\operatorname{Hom}_{\mathbb{C}}(S_m, V) \otimes_{\mathbb{C}} \overline{\operatorname{Hom}_{\mathbb{C}}(S_m, V)} \to \mathbb{C}, \quad (f, g) \mapsto \frac{\operatorname{tr}(g^{\dagger} \circ f)}{\dim S_m}$$

is a polarization. §117 Proposition tells us that the subspace

$$H_m = \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})} \subseteq \operatorname{Hom}_{\mathbb{C}}(S_m, V)$$

has a Hodge structure of weight $n - m$, with Hodge filtration

$$F^k H_m = F^k \operatorname{Hom}_{\mathbb{C}}(S_m, V) \cap H_m.$$

Moreover, we know that this Hodge structure is polarized by the restriction of the modified trace pairing. But for $f, g \in H_m$, the composition $g^{\dagger} \circ f$ is an endomorphism of S_m as an $\mathfrak{sl}_2(\mathbb{C})$-representation, hence (by Schur's lemma) a multiple of the identity. Thus $g^{\dagger} \circ f = c(f, g) \operatorname{id}$ for some constant $c(f, g) \in \mathbb{C}$, and because of how we defined the modified trace pairing, this says exactly that

$$c \colon H_m \otimes_{\mathbb{C}} \overline{H_m} \to \mathbb{C}$$

polarizes the Hodge structure on $H_m = \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$.

Lemma. *The evaluation morphism*

$$\bigoplus_{m \in \mathbb{N}} S_m \otimes_{\mathbb{C}} H_m \to V$$

induces an isomorphism between the $\mathfrak{sl}_2$-Hodge filtrations on both sides.

Proof. We know that the mapping is an isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-representations. By the same argument as in §98 Lemma, one shows that the isomorphism is compatible with the hermitian pairings on both sides. Let

$$S_m = \bigoplus_{p+q=m} S_m^{p,q} \quad \text{and} \quad H_m = \bigoplus_{p+q=n-m} H_m^{p,q}$$

denote the Hodge decompositions in the Hodge structures on S_m and W. By construction, H_m is a sub-Hodge structure of $\operatorname{Hom}_{\mathbb{C}}(S_m, V)$, and so

$$H_m^{p,q} = \left\{ f \colon S_m \to V \;\middle|\; f(S_m^{j,m-j}) \subseteq V^{j+p,m-j+q} \text{ for all } j \in \mathbb{Z} \right\}.$$

We are going to argue that the evaluation morphism is a morphism of Hodge structures of weight n. Let $p \in \mathbb{Z}$ be an integer. As with any tensor product, the $(p, n-p)$-subspace in the Hodge decomposition of the left-hand side is

$$\bigoplus_{m \in \mathbb{N}} \bigoplus_{j \in \mathbb{Z}} S_m^{j,m-j} \otimes_{\mathbb{C}} H_m^{p-j,n-m-p+j};$$

the evaluation morphism takes this into the subspace $V^{p,n-p}$, and is therefore a morphism of Hodge structures of weight n. Since morphisms of Hodge structures strictly respect the Hodge filtration, we get isomorphisms

$$\bigoplus_{m \in \mathbb{N}} \bigoplus_{j \in \mathbb{Z}} e^{\mathsf{Y}} F^j S_m \otimes_{\mathbb{C}} F^{p-j} H_m \cong e^{\mathsf{Y}} F^p V$$

between the Hodge filtrations on both sides. After multiplying by $e^{-\mathsf{Y}}$, this gives us the desired isomorphisms

$$\bigoplus_{m \in \mathbb{N}} \bigoplus_{j \in \mathbb{Z}} F^j S_m \otimes_{\mathbb{C}} F^{p-j} H_m \cong F^p V. \qquad \square$$

128. By construction, the $\mathfrak{sl}_2$-Hodge filtration on each S_m comes from a polarized $\mathfrak{sl}_2$-Hodge structure of weight m. If we let $\mathfrak{sl}_2(\mathbb{C})$ act trivially on the Hodge structure $\operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$, the direct sum

$$\bigoplus_{m \in \mathbb{N}} S_m \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(S_m, V)^{\mathfrak{sl}_2(\mathbb{C})}$$

inherits a polarized $\mathfrak{sl}_2$-Hodge structure of weight $m + (n-m) = n$. This puts an $\mathfrak{sl}_2$-Hodge structure of weight n on V, and the lemma above says that the $\mathfrak{sl}_2$-Hodge filtration $F^\bullet V$ agrees with the total Hodge filtration of this $\mathfrak{sl}_2$-Hodge structure, and that the polarizations are compatible. §114 Theorem is therefore proved in full.

Note. A posteriori, we can deduce from §103 Lemma that $\mathsf{H}_0 = 0$; but this fact does not play a role during the proof.

129. In fact, one can improve §114 Theorem by taking out the assumption that V is a representation of $\mathfrak{sl}_2(\mathbb{C})$. This gives a minimal set of conditions by which one can recognize a polarized $\mathfrak{sl}_2$-Hodge structure. Interestingly, a version of this result is used in the proof of the decomposition theorem [dCM05], in order to establish the relative Hard Lefschetz theorem. This is another place where the global theory (cohomology with coefficients in a polarized variation of Hodge structure) and the local theory (degenerating variations of Hodge structure) meet rather unexpectedly.

130. Here is the minimal set of conditions. This time, we only assume that

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

is a finite-dimensional graded complex vector space; we let $\mathsf{H} \in \operatorname{End}(V)$ be the semisimple operator that acts as multiplication by k on the subspace V_k. Let $\mathsf{Y} \in \operatorname{End}(V)$ be a nilpotent operator such that $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$; note that we are *not* assuming that $\mathsf{Y}^k \colon V_k \to V_{-k}$ is an isomorphism. Further, let $Q \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ be a hermitian pairing such that $\mathsf{Y}^\dagger = \mathsf{Y}$ and $\mathsf{H}^\dagger = -\mathsf{H}$, where the dagger means the adjoint with respect to Q. Lastly, suppose that we have a decreasing filtration F on the vector space V, with the following two properties:

(a) One has $\mathsf{Y}(F^\bullet) \subseteq F^{\bullet-1}$ and $\mathsf{H}(F^\bullet) = F^\bullet$.

(b) The filtration $e^{\mathsf{Y}}F$ is the Hodge filtration of a polarized Hodge structure of weight n on V, polarized by the pairing Q (which is therefore nondegenerate).

Corollary. *Under these assumptions, F is an $\mathfrak{sl}_2$-Hodge structure of weight n; consequently, V is an $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the hermitian pairing Q.*

Proof. We are going to argue that $\mathsf{Y}^k \colon V_k \to V_{-k}$ is an isomorphism for every $k \in \mathbb{N}$. This implies that H and Y determine a representation of $\mathfrak{sl}_2(\mathbb{C})$ on V, and together with the other conditions above, is enough for applying §114 Theorem. An equivalent formulation is that the increasing filtration

$$W_k = V_k \oplus V_{k-1} \oplus V_{k-2} \oplus \cdots$$

is the monodromy weight filtration of the nilpotent operator Y. By assumption, $e^{\mathsf{Y}}F \in D$. Let us show that $e^{-z\mathsf{Y}}F \in D$ for every $z \in \mathbb{H}$. The two operators $e^{-\frac{1}{2}\log|\operatorname{Re} z|\,\mathsf{H}}$ and $e^{i \operatorname{Im} z\,\mathsf{Y}}$ belong to the real Lie group $G = \operatorname{Aut}(V, Q)$, due to the fact that $\mathsf{H}^\dagger = -\mathsf{H}$ and $\mathsf{Y}^\dagger = \mathsf{Y}$. From the relation $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$ and the fact that $\mathsf{H}(F^\bullet) \subseteq F^\bullet$, we thus get

$$e^{-z\mathsf{Y}}F = e^{-i\operatorname{Im} z\mathsf{Y}} e^{|\operatorname{Re} z|\mathsf{Y}}F = e^{-i\operatorname{Im} z\,\mathsf{Y}} e^{-\frac{1}{2}\log|\operatorname{Re} z|\,\mathsf{H}} \cdot e^{\mathsf{Y}}F \in D.$$

Because $\mathsf{Y}(F^\bullet) \subseteq F^{\bullet-1}$, the mapping

$$\Phi \colon \mathbb{H} \to D, \quad \Phi(z) = e^{-z\mathsf{Y}}F,$$

is the period mapping of a polarized variation of Hodge structure of weight n, with monodromy operator $T = e^{-2\pi i\mathsf{Y}}$. In our usual notation, we therefore have $N = -\mathsf{Y}$.

Now comes the crucial point: because of the Hodge norm estimates, we can recognize the monodromy weight filtration of N – and hence that of Y – by the order of growth of the Hodge norm. Let $v \in V_k$ be any nonzero vector. As long as $\operatorname{Im} z$ remains bounded, we have

$$\|v\|^2_{\Phi(z)} = \left\| e^{\frac{1}{2}\log|\operatorname{Re} z|\,\mathsf{H}} e^{i\operatorname{Im} z\,\mathsf{Y}} v \right\|^2_{e^{\mathsf{Y}}F} \sim |\operatorname{Re} z|^k \cdot \|v\|^2_{e^{\mathsf{Y}}F},$$

and so according to §15 Theorem, the filtration W we defined above must be the monodromy weight filtration of Y. By construction, $\operatorname{gr}^W_k \cong V_k$, and so $\mathsf{Y}^k \colon V_k \to V_{-k}$ is an isomorphism for every $k \in \mathbb{N}$. This concludes the proof. □

6 The Hodge norm estimates

131. In this chapter, we use the results about harmonic bundles that we derived in Chapter 3 to prove the Hodge norm estimates. The point of these estimates is to control the order of growth of the Hodge norm of a multi-valued flat section in terms of the monodromy weight filtration of the nilpotent operator N.

Note. In fact, exactly the same argument proves the Hodge norm estimates for an arbitrary harmonic bundle on the punctured disk whose Higgs field $\theta_{\partial/\partial t}$ is nilpotent.

6.1 Boundedness of the Hodge norm for flat sections

132. Let us start by showing that, because of the derivative bound in (60.1), the Hodge norm of any multi-valued flat section can grow at most like a power of $|\operatorname{Re} z|$.

Proposition. *Let $v \in V$ be a multi-valued flat section. Then*

$$\|v\|_{\Phi(z)} \leq e^{2C_0\pi} \max\Big(|\operatorname{Re} z|^{2C_0}, |\operatorname{Re} z|^{-2C_0}\Big) \|v\|_{\Phi(-1)}$$

$$\|v\|_{\Phi(-1)} \leq e^{2C_0\pi} \max\Big(|\operatorname{Re} z|^{2C_0}, |\operatorname{Re} z|^{-2C_0}\Big) \|v\|_{\Phi(z)}$$

on the horizontal strip $|\operatorname{Im} z| \leq \pi$, where $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$ and $r = \operatorname{rk} E$.

Proof. Assuming that $v \neq 0$, we consider the function $\varphi = \log\|v\|^2_{\Phi(z)}$. Set $z = x + iy$. Then

$$\big|\varphi(x+iy) - \varphi(-1)\big| \leq \big|\varphi(x+iy) - \varphi(-1+iy)\big| + \big|\varphi(-1+iy) - \varphi(-1)\big|.$$

The derivative bound in (60.1) implies that

$$\left|\frac{\partial\varphi}{\partial x}\right| = \left|\frac{\partial\varphi}{\partial z} + \frac{\partial\varphi}{\partial \bar{z}}\right| \leq \frac{4C_0}{|\operatorname{Re} z|},$$

with a similar inequality for $\partial\varphi/\partial y$. Therefore

$$\big|\varphi(x+iy) - \varphi(-1+iy)\big| \leq \left|\int_{-1}^{x} \frac{4C_0}{|t|}\, dt\right| = 4C_0\big|\log|x|\big|;$$

in the same manner, we have for $|y| \leq \pi$ the inequality

$$\big|\varphi(-1+iy) - \varphi(-1)\big| \leq 4C_0|y| \leq 4C_0\pi.$$

Putting everything together, we get

$$-4C_0\pi - 4C_0\big|\log|\operatorname{Re} z|\big| \leq \log\|v\|^2_{\Phi(z)} - \log\|v\|^2_{\Phi(-1)} \leq 4C_0\pi + 4C_0\big|\log|\operatorname{Re} z|\big|,$$

from which the desired inequalities follow by exponentiation. □

133. The following result is a first step towards more precise estimates for Hodge norms of multi-valued flat sections. The proof is disarmingly simple.

Proposition. *Let $v \in V$ be a multi-valued flat section such that $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$. Then for every $x_0 < 0$, we have the inequality*

$$\|v\|_{\Phi(z)} \leq e^{4C_0/|x_0|} \|v\|_{\Phi(x_0)}$$

for all $z \in \mathbb{H}$ with $\operatorname{Re} z \leq x_0$; here $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$ and $r = \operatorname{rk} E$.

Using the Jordan decomposition $T = T_s e^{2\pi i N}$, the condition $Tv = \lambda v$ is equivalent to $T_s v = \lambda v$ and $Nv = 0$. The inequality in the proposition is therefore a special case of the Hodge norm estimates for multi-valued flat sections. Indeed, $Nv = 0$ implies that $v \in W_0$, and so we expect the Hodge norm of v to remain bounded as $\operatorname{Re} z \to -\infty$. It turns out that this special case is all that is needed to prove the Hodge norm estimates in general: the power of the statement comes from the fact that it applies to *all* polarized variations of Hodge structure on Δ^*. The universal character of the inequality also makes it very useful for studying variations of Hodge structure in several variables, as we plan to show in a sequel to this paper.

134. Now for the proof of the proposition. The semisimple part of the monodromy transformation $T = T_s e^{2\pi i N}$ give rise to a decomposition

$$V = \bigoplus_{|\lambda|=1} E_\lambda(T_s),$$

and each eigenspace $E_\lambda(T_s)$ is preserved by N. Let us consider a fixed nonzero vector $v \in E_\lambda(T_s) \cap \ker N$. As in the proof of the monodromy theorem (in §64 Proposition), these conditions imply that the function

$$\varphi = \log h(v, v)$$

is invariant under the substitution $z \mapsto z + 2\pi i$. In addition, we have already proved that φ is smooth and subharmonic (§53 Lemma), and that

$$\left|\frac{\partial \varphi}{\partial z}\right| = \left|\frac{\partial \varphi}{\partial \bar{z}}\right| \leq \frac{2C_0}{|\operatorname{Re} z|},$$

Recall that we derived this inequality in (60.1); here $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$ and $r = \operatorname{rk} E$.

Example. To get a feeling for these conditions, let us consider a toy example: a smooth function $f\colon (-\infty, 0) \to \mathbb{R}$ with the property that $f'' \geq 0$ and $|f'(x)| \leq C|x|^{-1}$. Since $f'' \geq 0$, the function f' must be increasing, and therefore

$$0 = \lim_{x \to -\infty} f'(x) \leq f'.$$

But this means that f is itself increasing, and so we get

$$f(x) \leq f(x_0)$$

for every $x \leq x_0 < 0$. In particular, f is bounded above as $x \to -\infty$. A similar convexity argument also shows up in Mochizuki's work on the asymptotic behavior of tame harmonic bundles [Moc07, Lem. 2.23].

135. In order to apply the same reasoning as in the example, we need a function that only depends on $x = \operatorname{Re} z$. We therefore consider the vertical averages

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(x+iy)\,dy.$$

Differentiation under the integral sign gives

$$f''(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_{x,x}(x+iy)\,dy \geq -\frac{1}{2\pi} \int_0^{2\pi} \varphi_{y,y}(x+iy)\,dy,$$

since $\Delta\varphi = \varphi_{x,x} + \varphi_{y,y} \geq 0$. The integral evaluates to

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_{y,y}(x+iy)\,dy = \frac{1}{2\pi}\Big(\varphi_y(x+2\pi i) - \varphi_y(x)\Big) = 0,$$

due to the fact that $\varphi(x+iy)$ is periodic in y of period 2π. Therefore $f(x)$ is a convex function of x. Moreover, we know from (60.1) that

$$|\varphi_x(x+iy)| \leq \frac{4C_0}{|x|},$$

where $C_0 = \frac{1}{2}\sqrt{\binom{n+1}{3}}$. Plugging this into the integral from above gives

$$|f'(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} |\varphi_x(x+iy)|\,dy \leq \frac{4C_0}{|x|}.$$

As in the example, it follows that $f(x) \leq f(x_0)$ for every $x \leq x_0 < 0$.

136. All that is left is to relate the behavior of $\varphi(x+iy)$ to that of $f(x)$. We have

$$-\frac{4C_0}{|x|} \leq \varphi_y(x+iy) \leq \frac{4C_0}{|x|},$$

which can be integrated to give

$$-\frac{4C_0}{|x|}u \leq \varphi(x+iy+iu) - \varphi(x+iy) \leq \frac{4C_0}{|x|}u$$

for all $u \geq 0$. If we now average over $u \in [0, 2\pi]$, we get

$$-\frac{4C_0}{|x|} \leq f(x) - \varphi(x+iy) \leq \frac{4C_0}{|x|}.$$

For $x \leq x_0 < 0$, we have $f(x) \leq f(x_0)$, and therefore

$$\varphi(x+iy) \leq f(x) + \frac{4C_0}{|x|} \leq f(x_0) + \frac{4C_0}{|x|} \leq \varphi(x_0) + \frac{8C_0}{|x_0|}.$$

This gives the desired upper bound for $\|v\|_{\Phi(z)}$ after exponentiation.

6.2 The comparison theorem

137. One striking consequence of §133 Proposition is that the behavior of the Hodge norm as $t \to 0$ only depends on the underlying flat vector bundle (E, d). Simpson proves a similar result for tame harmonic bundles, but with conditions that appear to be much more restrictive [Sim90, Cor. 4.3].

Theorem. *Let E_1, E_2 be polarized variations of Hodge structure over Δ^*. If (E_1, d_1) and (E_2, d_2) are isomorphic as smooth vector bundles with connection, then for every $0 < r < 1$, there is a constant $C_r > 0$ such that*

$$C_r^{-1} \cdot h_{E_1} \leq h_{E_2} \leq C_r \cdot h_{E_1}$$

at all points $t \in \Delta^*$ with $|t| \leq r$.

Proof. Let $f\colon E_1 \to E_2$ be an isomorphism between the underlying smooth vector bundles that preserves the connections. We consider f as a single-valued flat section of the bundle $\mathrm{Hom}(E_1, E_2)$. Since this bundle underlies a polarized variation of Hodge structure, §133 Proposition applies to it. Taking $x_0 = \log r$ in the statement, we get a bound for the pointwise Hodge norm of f, of the form

$$h_{\mathrm{Hom}(E_1,E_2)}(f, f) \leq C_r,$$

valid for every $t \in \Delta^*$ with $|t| \leq r$. Here the constant C_r equals $e^{8C_0/|\log r|}$ times the value of the Hodge norm at the point $t = r$. Since the Hodge norm is an upper bound for the pointwise operator norm of f, this gives us the inequality

$$h_{E_2} \leq C_r \cdot h_{E_1}.$$

The reverse inequality follows by applying the same reasoning to the inverse isomorphism $f^{-1}\colon E_2 \to E_1$. □

6.3 Order of growth and the weight filtration

138. Let $W = W_\bullet V$ be the monodromy weight filtration of the nilpotent operator N. Our next goal is to prove that the weight filtration controls the behavior of the function $h(v, v)$ as $\mathrm{Re}\, z \to -\infty$, at least on horizontal strips of bounded height. The idea is to construct another variation of Hodge structure on the same underlying flat vector bundle (E, d), using representation theory. §137 Theorem guarantees that the Hodge metrics of the two variations are equivalent up to a constant, and this implies the Hodge norm estimates.

139. To get started, we need to upgrade the nilpotent operator $N \in \mathrm{End}(V)$ into a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let us denote the standard basis elements by $\mathsf{H}, \mathsf{X}, \mathsf{Y} \in \mathfrak{sl}_2(\mathbb{C})$; then

$$[\mathsf{H}, \mathsf{X}] = 2\mathsf{X}, \quad [\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}, \quad [\mathsf{X}, \mathsf{Y}] = \mathsf{H}.$$

As $N \in \mathrm{End}(V)$ is nilpotent, one can find a semisimple endomorphism $H \in \mathrm{End}(V)$ with integral eigenvalues, such that $[H, N] = -2N$. In fact, one can make a more careful choice of H; the additional properties are going to be useful for us later on.

Proposition. *One can find $H \in \mathrm{End}(V)$ with the following properties:*

(a) H is semisimple with integral eigenvalues.

(b) One has $[H, N] = -2N$ and $W_k = E_k(H) \oplus W_{k-1}$ for every $k \in \mathbb{Z}$.

(c) One has $Q(Hv, w) + Q(v, Hw) = 0$ for every $v, w \in V$.

(d) H commutes with T_s.

Proof. The proof is easy, and so we only give a sketch. Consider an arbitrary nilpotent endomorphism $N \in \operatorname{End}(V)$ of a finite-dimensional vector space V. Denote by $\Sigma(N) \subseteq \operatorname{End}(V)$ the set of all semisimple endomorphisms H with integer eigenvalues, such that $[H, N] = -2N$ and $W_j = E_j(H) \oplus W_{j-1}$ for every $j \in \mathbb{Z}$. It is easy to see that $\Sigma(N) \neq \emptyset$, for example by choosing a basis that puts N into Jordan canonical form. This already shows that splittings satisfying (a) and (b) exist, a special case of the Jacobson-Morozov theorem.

Let us now compare two arbitrary elements $H, H' \in \Sigma(N)$. The difference $H' - H$ commutes with N and satisfies $(H' - H)(W_\bullet) \subseteq W_{\bullet-1}$. Denote by $\operatorname{ad} H$ the semisimple operator on $\operatorname{End}(V)$, defined as $(\operatorname{ad} H)(A) = [H, A]$; then $H' - H$ belongs to the direct sum of the eigenspaces $E_k(\operatorname{ad} H)$ with $k \leq -1$. From this, one deduces that

$$H' = e^B H e^{-B}$$

for a unique endomorphism $B \in \operatorname{End}(V)$ such that $[B, N] = 0$ and $B(W_\bullet) \subseteq W_{\bullet-1}$. Conversely, for any $B \in \operatorname{End}(V)$ with these two properties, one has $e^B H e^{-B} \in \Sigma(N)$.

Now we describe how to adjust a given splitting $H_0 \in \Sigma(N)$ so that (c) holds. Suppose that V comes with a nondegenerate hermitian pairing $Q \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that $Q(Nv, w) = Q(v, Nw)$ for all $v, w \in V$. For $A \in \operatorname{End}(V)$, denote by $A^\dagger \in \operatorname{End}(V)$ the adjoint with respect to Q; thus $N^\dagger = N$. It is easy to see that $-H_0^\dagger \in \Sigma(N)$, and so by the above, one has

$$-(H_0)^\dagger = e^B H_0 e^{-B}$$

for a unique $B \in \operatorname{End}(V)$ with $[N, B] = 0$ and $B(W_\bullet) \subseteq W_{\bullet-1}$; by uniqueness, $B^\dagger = B$. Consequently, the new splitting

$$H = e^{\frac{1}{2}B} H_0 e^{-\frac{1}{2}B} \in \Sigma(N)$$

satisfies $H = -H^\dagger$, which is just a different way of writing (c). If we suppose in addition that $S \in \operatorname{End}(V)$ is semisimple, commutes with N, and satisfies $S^\dagger = S$, then we can easily arrange that moreover $[H, S] = 0$ (by considering each eigenspace of S separately). This shows that splittings with all four properties exist. □

140. We now define a representation

$$\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}(V), \quad \rho(\mathsf{H}) = H,\ \rho(\mathsf{Y}) = -N.$$

The reason for using $-N$ (instead of the seemingly more natural N) has to do with the sign conventions for $\mathfrak{sl}_2$-Hodge structures. With this choice, each eigenspace $E_\lambda(T_s)$ is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and

$$V = \bigoplus_{|\lambda|=1} E_\lambda(T_s).$$

Moreover, the eigenspaces $E_k(H)$ and $E_\ell(H)$ are orthogonal with respect to Q unless $k = -\ell$.

6.4 Proof of the Hodge norm estimates

141. We can now prove the Hodge norm estimates in general.

Theorem. *Let E be a polarized variation of Hodge structure on Δ^*. If $v \in V$ is a multi-valued flat section such that $v \in W_k$ and $v \notin W_{k-1}$, then the function*

$$|\mathrm{Re}\, z|^{-k} \cdot \|v\|^2_{\Phi(z)}$$

is uniformly bounded on every region of the form $\mathrm{Re}\, z \leq x_0 < 0$ *and* $|\mathrm{Im}\, z| \leq y_0$.

Proof. As in §140, we choose a representation of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space

$$V = \bigoplus_{|\lambda|=1} E_\lambda(T_s)$$

that commutes with T_s, is compatible with the pairing Q, and such that $\mathsf{Y} \in \mathfrak{sl}_2(\mathbb{C})$ acts as the nilpotent operator $-N$. The representation is completely reducible, hence

$$V \cong \bigoplus_{|\lambda|=1} \bigoplus_{m \in \mathbb{N}} S_m \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}\big(S_m, E_\lambda(T_s)\big)^{\mathfrak{sl}_2(\mathbb{C})}.$$

Each S_m has a canonical $\mathfrak{sl}_2$-Hodge structure of weight m, and by putting suitable Hodge structures on the vector spaces in the above decomposition, V becomes an $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the pairing Q. By the construction in §109, we can arrange that the monodromy transformation of the resulting variation of Hodge structure on Δ^* is equal to $T = T_s e^{2\pi i N}$. In this way, we obtain another polarized variation of Hodge structure on Δ^* whose underlying flat vector bundle is isomorphic to (E, d). §137 Theorem shows that the two Hodge metrics are mutually bounded up to a constant. Since we already know the Hodge norm estimates for variations of Hodge structure coming from $\mathfrak{sl}_2$-Hodge structures (by §112), this finishes the proof of the Hodge norm estimates in general. ☐

142. One can sharpen the Hodge norm estimates by using the decomposition

$$V = \bigoplus_{k \in \mathbb{Z}} E_k(H)$$

coming from the $\mathfrak{sl}_2(\mathbb{C})$-representation. If $v \in E_k(H)$ is a nonzero in the k-th weight space, then the Hodge norm estimates show that

$$\|v\|^2_{\Phi(z)} \cdot |\mathrm{Re}\, z|^{-k}$$

is bounded from above and below by positive constants (as long as $\mathrm{Im}\, z$ lies in a bounded interval). This suggests rescaling by the operator $e^{-\frac{1}{2}\log|\mathrm{Re}\, z|\, H}$, because

$$e^{-\frac{1}{2}\log|\mathrm{Re}\, z|\, H} v = |\mathrm{Re}\, z|^{-\frac{k}{2}} v,$$

The point of insisting that $H^\dagger = -H$ is that $e^{-\frac{1}{2}\log|\mathrm{Re}\, z|\, H} \in G$. Furthermore, we can remove the restriction on the imaginary part by choosing a logarithm of T_s: a semisimple operator $S \in \mathrm{End}(V)$, with real eigenvalues in a fixed half-open interval of length 1, such that $T_s = e^{2\pi i S}$ and $[S, N] = 0$. The relation

$$\|v\|^2_{\Phi(z+2\pi i)} = \|T^{-1} v\|^2_{\Phi(z)}$$

means that the rescaled expression

$$\left\| e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v \right\|_{\Phi(z)}^2 \tag{142.1}$$

now depends only on $t = e^z$, and therefore descends to a smooth function on Δ^* that is bounded from above and below by a positive constant as $t \to 0$. In Chapter 8, we are going to argue that (142.1) actually converges to a positive definite hermitian inner product on V (as a consequence of the nilpotent orbit theorem).

6.5 A more direct proof

143. We end this chapter by sketching another proof, without representation theory, for the fact that the monodromy weight filtration governs the rate of growth of the Hodge norm. In this section only, let us denote the monodromy weight filtration of the nilpotent operator $N \in \operatorname{End}(V)$ by the symbol $M_\bullet$. Let us also define the growth order filtration

$$W_k = \left\{ v \in V \;\middle|\; \|v\|_{\Phi(z)}^2 = O\big(|\operatorname{Re} z|^k\big) \text{ as } |\operatorname{Re} z| \to \infty \right\},$$

again assuming that $\operatorname{Im} z$ stays in a bounded interval. We are going to prove directly that $W_k = M_k$ for every $k \in \mathbb{Z}$.

144. We know from §132 Proposition that $W_k = 0$ for $k \leq -2C_0$, and $W_k = V$ for $k \geq 2C_0$. This means that the growth order filtration has finite length, just like the monodromy weight filtration. Another simple observation is that

$$Q(W_k, W_\ell) = 0 \quad \text{if } k + \ell \leq -1. \tag{144.1}$$

Indeed, for $v \in W_k$ and $w \in W_\ell$, we have

$$|Q(v,w)|^2 \leq \|v\|_{\Phi(z)}^2 \|w\|_{\Phi(z)}^2 = O\big(|\operatorname{Re} z|^{k+\ell}\big).$$

As long as $k + \ell \leq -1$, the right-hand side is going to zero as $|\operatorname{Re} z| \to \infty$; because $Q(v,w)$ is constant, it follows that $Q(v,w) = 0$.

145. We can use the basic estimate (in §58 Corollary) to show that $N(W_\bullet) \subseteq W_{\bullet-2}$; recall that this is one of the two conditions that characterize the monodromy weight filtration.

Lemma. *We have $N(W_k) \subseteq W_{k-2}$ for all $k \in \mathbb{Z}$.*

Proof. This is an immediate consequence of §65 Proposition. In fact, we proved that there is a constant $C > 0$ such that

$$\|Nv\|_{\Phi(z)} \leq \frac{C}{|\operatorname{Re} z|} \|v\|_{\Phi(z)} \quad \text{for } v \in V \text{ and } \operatorname{Re} z \leq -1,$$

and so $v \in W_k$ implies that $Nv \in W_{k-2}$. □

146. The monodromy weight filtration has the property that $\ker N^{k+1} \subseteq M_k$. So far, we only know (from §133 Proposition) that $\ker N \subseteq W_0$. The following lemma generalizes this to arbitrary powers of N.

Lemma. *We have $\ker N^{k+1} \subseteq W_k$ for $k \geq 0$.*

Proof. Together with the triangle inequality, §133 Proposition implies that $\ker N \subseteq W_0$. Now consider any vector $v \in \ker N^{k+1}$. As in §95, let S_k denote the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with $\dim S_k = k+1$. Define a linear mapping $f \colon S_k \to V$ by setting

$$f(v_j) = \frac{j!(k-j)!}{k!}(-N)^j v \quad \text{for } j = 0, 1, \dots, k;$$

then $f(v_0) = v$ and $f \circ \mathsf{Y} = -N \circ f$. According to Section 5.4, S_k is the space of multi-valued flat sections of a polarized variation of Hodge structure of weight k, whose monodromy operator is $T = e^{-2\pi i \mathsf{Y}}$; we also know from §112 that the Hodge norm of any vector in S_k can grow at most like $|\operatorname{Re} z|^k$. Consequently, $\operatorname{Hom}_{\mathbb{C}}(S_k, V)$ is the space of multi-valued flat sections of a polarized variation of Hodge structure of weight $n-k$, and $f \in \operatorname{Hom}_{\mathbb{C}}(S_k, V)$ satisfies $Nf = 0$. By §133 Proposition, the Hodge norm of f is bounded; since the Hodge norm is an upper bound for the operator norm, it follows that $\|v\|^2_{\Phi(z)} = O\big(|\operatorname{Re} z|^k\big)$. □

147. We recall two facts about the monodromy weight filtration; as we are only sketching the proof, we omit the details. First,

$$M_k = \sum_{j \in \mathbb{N}} N^j \big(\ker N^{k+2j+1}\big) \quad \text{for } k \in \mathbb{Z}.$$

Second, since $N^\dagger = N$ is self-adjoint with respect to the hermitian pairing Q, one has

$$M_k^\perp = \big\{\, v \in V \;\big|\; Q(v, x) = 0 \text{ for all } x \in M_k \,\big\} = M_{-k-1} \quad \text{for } k \in \mathbb{Z}.$$

We can use this to prove that $W_k = M_k$. By §146 Lemma, we have

$$M_k = \sum_{j \in \mathbb{N}} N^j \big(\ker N^{k+2j+1}\big) \subseteq \sum_{j \in \mathbb{N}} N^j \big(W_{k+2j}\big) \subseteq W_k,$$

where the last inclusion is due to §145 Lemma. Dually, we have

$$W_k \subseteq W_{-k-1}^\perp \subseteq M_{-k-1}^\perp = M_k,$$

where we used (144.1) for the first inclusion. It follows that $W_k = M_k$ for all $k \in \mathbb{Z}$, and so the filtration by order of growth of the Hodge metric is indeed equal to the monodromy weight filtration.

7 The nilpotent orbit theorem

148. This chapter is devoted to the proof of the nilpotent orbit theorem. The theorem has several parts, and it is going to take us some time to prove all of them. Our first goal is to prove that the "untwisted period mapping"

$$\Psi_S \colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)} \Phi(z),$$

extends holomorphically to the entire disk Δ. The general idea is to construct a holomorphic extension of the Higgs field $\theta_{\partial/\partial t}$ with the help of Hörmander's L^2-estimates, and then to use this extension to prove that Ψ_S is meromorphic at $t = 0$. This is enough because $\check{D}$ is a projective complex manifold.

7.1 Hörmander's L^2-estimates

149. We need only the most basic version of Hörmander's L^2-estimates for vector bundles, in one complex dimension. The technique, in the special case of the trivial bundle, is explained very nicely in [Ber10, Lecture 1]. We include an elementary proof for arbitrary vector bundles here, both in order to keep the paper self-contained, and to show the reader that the L^2-estimates in one dimension are not difficult – in fact, most of the argument is just integration by parts.

150. Let $X \subseteq \mathbb{C}$ be a domain in $\mathbb{C}$, with the usual Euclidean metric $\omega = \frac{i}{2} dz \wedge d\bar{z}$. Let E be a smooth vector bundle on X with a hermitian metric h, and suppose that E has the structure of a holomorphic vector bundle; this corresponds to a connection $d'' \colon A^0(X,E) \to A^{0,1}(X,E)$ of type $(0,1)$; the integrability condition $(d'')^2 = 0$ is automatically satisfied in dimension one. One technique for constructing holomorphic sections of E is to solve the $\bar{\partial}$-equation

$$d''u = f d\bar{z}$$

for a given section $f \in A^0(X,E)$. The general idea is that this can be done, provided h has positive curvature. Let us write the Chern connection in the form $\delta' + d''$, where $\delta' \colon A^0(X,E) \to A^{1,0}(X,E)$ is a connection of type $(1,0)$. Let

$$\Theta = (\delta' + d'')^2 \in A^{1,1}\big(X, \operatorname{End}(E)\big)$$

be the curvature operator of the metric h. The main result is as follows [Dem12, VIII.§6].

Theorem (Hörmander)**.** *Suppose that there is a positive function $\rho \colon X \to \mathbb{R}$ such that*

$$\int_X h\big(\Theta_{\partial/\partial z \wedge \partial/\partial \bar{z}}\, \alpha, \alpha\big)\, d\mu \geq \int_X \rho^2 h(\alpha, \alpha)\, d\mu \quad \textit{for all } \alpha \in A_c^0(X,E).$$

Let $f \in A^0(X,E)$ be an arbitrary smooth section. Then the equation $d''u = f d\bar{z}$ has a solution $u \in A^0(X,E)$ that moreover satisfies the L^2-estimate

$$\int_X h(u,u)\, d\mu \leq \int_X \frac{1}{\rho^2} h(f,f)\, d\mu,$$

provided that the integral on the right-hand side is finite.

151. The proof uses one nontrivial (but well-known) fact, namely the regularity of the $\bar{\partial}$-equation in one complex dimension. Let $f \colon X \to \mathbb{C}$ be a smooth function. Suppose that a measurable function $u \colon X \to \mathbb{C}$ is a **weak solution** to the equation

$$\frac{\partial u}{\partial \bar{z}} = f, \tag{151.1}$$

meaning that for every compactly supported smooth function $\varphi \in A_c^0(X)$, one has

$$\int_X f \varphi \, d\mu = - \int_X u \frac{\partial \varphi}{\partial \bar{z}}\, d\mu.$$

Then after modifying u on a set of measure zero, if necessary, the function u is smooth and solves (151.1) in the usual sense.

152. Now we turn to the proof of §150 Theorem. We start by using integration by parts to relate the two operators $d''_{\partial/\partial\bar{z}}$ and $\delta'_{\partial/\partial z}$.

Lemma. *Let $u \in A^0(X,E)$ be a smooth section, and let $\alpha \in A^0_c(X,E)$ be a smooth section with compact support. Then*

$$\int_X h\big(d''_{\partial/\partial\bar{z}} u, \alpha\big)\, d\mu = -\int_X h\big(u, \delta'_{\partial/\partial z}\alpha\big)\, d\mu.$$

Proof. Since $\delta' + d''$ is a metric connection, we have

$$\frac{\partial}{\partial\bar{z}} h(u,\alpha) = h\big(d''_{\partial/\partial\bar{z}} u, \alpha\big) + h\big(u, \delta'_{\partial/\partial z}\alpha\big).$$

Now integrate over X and use Stokes' theorem to get the result. □

153. The general idea, which should be familiar from the proof of Hodge's theorem about harmonic forms, is to construct a "weak solution" to the equation by Hilbert space techniques. We let $L^2(X,E)$ be the Hilbert space of all measurable sections of the bundle E with finite L^2-norm

$$\|u\|_h^2 = \int_X h(u,u)\, d\mu.$$

We say that $u \in L^2(X,E)$ is a **weak solution** of the equation $d''u = f d\bar{z}$ if

$$\int_X h(f,\alpha)\, d\mu = -\int_X h(u, \delta'_{\partial/\partial z}\alpha)\, d\mu$$

for every smooth section $\alpha \in A^0_c(X,E)$ with compact support. The regularity theory of the $\bar{\partial}$-operator implies that weak solutions are actually smooth.

Proposition. *Let $f \in A^0(X,E)$. If $u \in L^2(X,E)$ is a weak solution to the equation $d''u = f d\bar{z}$, then $u \in A^0(X,E)$ and $d''u = f d\bar{z}$ in the usual sense.*

Proof. This is a local problem; after replacing X by an open neighborhood of a given point, we may assume that E is a trivial holomorphic bundle of rank r. Let $s_1, \dots, s_r \in A^0(X,E)$ be a holomorphic frame, and define the smooth functions

$$h_{i,j} = h(s_i, s_j).$$

The $r \times r$-matrix with entries $h_{i,j}$ is hermitian and positive definite; let $h^{i,j}$ be the entries of the inverse matrix. A simple calculation shows that

$$\delta'_{\partial/\partial z} s_i = \sum_{j,k} \frac{\partial h_{i,j}}{\partial z} h^{j,k} s_k$$

Now let $\alpha = \sum_i \alpha_i s_i \in A^0_c(X,E)$ be an arbitrary smooth section of E with compact support. Since δ' is a connection of type $(1,0)$, we have

$$\begin{aligned}\delta'_{\partial/\partial z}\alpha &= \sum_i \frac{\partial \alpha_i}{\partial z} s_i + \sum_i \alpha_i \delta'_{\partial/\partial z} s_i = \sum_{i,j,k} \Big(\frac{\partial \alpha_i}{\partial z} h_{i,j} h^{j,k} + \alpha_i \frac{\partial h_{i,j}}{\partial z} h^{j,k} \Big) s_k \\ &= \sum_{i,j,k} \frac{\partial(\alpha_i h_{i,j})}{\partial z} h^{j,k} s_k.\end{aligned}$$

Write the given section $u \in L^2(X, E)$ as $u = \sum_i u_i s_i$, with measurable functions $u_i \colon X \to \mathbb{C}$. Since h is conjugate-linear in the second argument, we get

$$h(u, \delta'_{\partial/\partial z}\alpha) = \sum_{i,j,k,\ell} u_\ell h_{\ell,k} \frac{\partial(\overline{\alpha}_i h_{j,i})}{\partial \bar{z}} h^{k,j} = \sum_{i,j} u_j \frac{\partial(\overline{\alpha}_i h_{j,i})}{\partial \bar{z}}.$$

Because u is a weak solution to $d''u = f d\bar{z}$, we therefore have

$$\int_X \sum_{i,j} h_{j,i} f_j \overline{\alpha}_i \, d\mu = \int_X h(f, \alpha) \, d\mu$$
$$= -\int_X h\bigl(u, \delta'_{\partial/\partial z}\alpha\bigr) \, d\mu = -\int_X \sum_{i,j} u_j \frac{\partial(\overline{\alpha}_i h_{j,i})}{\partial \bar{z}} \, d\mu,$$

where $f = \sum_j f_j s_j$. Now let $\varphi_1, \dots, \varphi_r \colon X \to \mathbb{C}$ be arbitrary compactly supported smooth functions, and set

$$\alpha = \sum_i \alpha_i s_i = \sum_{i,j} \overline{\varphi}_j h^{j,i} s_i;$$

with this choice, we have $\varphi_j = \sum_i h_{j,i} \overline{\alpha}_i$, and so the identity from above becomes

$$\int_X \sum_j f_j \varphi_j \, d\mu = -\int_X \sum_j u_j \frac{\partial \varphi_j}{\partial \bar{z}} \, d\mu.$$

This shows that each coefficient function $u_j \colon X \to \mathbb{C}$ by itself is a weak solution of the ordinary $\bar{\partial}$-equation $\partial u_j / \partial \bar{z} = f_j$. By standard regularity theory, we can modify each u_j on a set of measure zero and make it smooth; then $u \in A^0(X, E)$. We can now use integration by parts, as in §152 Lemma, and deduce that

$$\int_X h(f, \alpha) \, d\mu = -\int_X h\bigl(u, \delta'_{\partial/\partial z}\alpha\bigr) \, d\mu = \int_X h\bigl(d''_{\partial/\partial \bar{z}} u, \alpha\bigr) \, d\mu$$

for every $\alpha \in A^0_c(X, E)$. But this means exactly that $d''u = f d\bar{z}$. □

154. To find the desired weak solution, we use the Riesz representation theorem. The image of the linear mapping

$$\delta'_{\partial/\partial z} \colon A^0_c(X, E) \to L^2(X, E) \tag{154.1}$$

is a linear subspace of $L^2(X, E)$, in general not closed. On this subspace, we can define a conjugate-linear functional by the formula

$$\delta'_{\partial/\partial z}\alpha \mapsto -\int_X h(f, \alpha) \, d\mu.$$

The following lemma shows that this is well-defined and bounded.

Lemma. *For every $\alpha \in A^0_c(X, E)$, we have*

$$\left| \int_X h(f, \alpha) \, d\mu \right| \leq \left(\int_X \frac{1}{\rho^2} h(f, f) \, d\mu \right)^{1/2} \left(\int_X h\bigl(\delta'_{\partial/\partial z}\alpha, \delta'_{\partial/\partial z}\alpha\bigr) \, d\mu \right)^{1/2},$$

where $\rho \colon X \to \mathbb{R}$ is the positive function from §150 Theorem.

Proof. The Cauchy-Schwarz inequality gives

$$\left| \int_X h(f,\alpha)\, d\mu \right| \leq \left(\int_X \frac{1}{\rho^2} h(f,f)\, d\mu \right)^{1/2} \left(\int_X \rho^2 h(\alpha,\alpha)\, d\mu \right)^{1/2},$$

and in view of the inequality in §150 Theorem, it is therefore enough to prove that

$$\int_X h\big(\Theta_{\partial/\partial z \wedge \partial/\partial \bar{z}}\, \alpha, \alpha\big)\, d\mu \leq \int_X h\big(\delta'_{\partial/\partial z}\alpha, \delta'_{\partial/\partial z}\alpha\big)\, d\mu.$$

The curvature operator is $\Theta = \delta' d'' + d'' \delta'$, and so

$$h\big(\Theta_{\partial/\partial z \wedge \partial/\partial \bar{z}}\, \alpha, \alpha\big) = h\big(\delta'_{\partial/\partial z} d''_{\partial/\partial \bar{z}}\alpha, \alpha\big) - h\big(d''_{\partial/\partial \bar{z}} \delta'_{\partial/\partial z}\alpha, \alpha\big).$$

If we integrate over X and apply §152 Lemma, we get

$$\int_X h\big(\Theta_{\partial/\partial z \wedge \partial/\partial \bar{z}}\, \alpha, \alpha\big)\, d\mu = \int_X h\big(\delta'_{\partial/\partial z}\alpha, \delta'_{\partial/\partial z}\alpha\big)\, d\mu - \int_X h\big(d''_{\partial/\partial \bar{z}}\alpha, d''_{\partial/\partial \bar{z}}\alpha\big)\, d\mu.$$

Since the second term is negative, this gives us the inequality we need. □

155. The inequality in §154 Lemma says that the conjugate-linear functional

$$\delta'_{\partial/\partial z}\alpha \mapsto -\int_X h(f,\alpha)\, d\mu$$

is well-defined, bounded, and of norm at most $\|f/\rho\|_h$. By continuity, it extends uniquely to a conjugate-linear functional

$$L^2(X,E) \to \mathbb{C}$$

that is identically zero on the orthogonal complement of the image of (154.1); its norm is of course still at most $\|f/\rho\|_h$. By the Riesz representation theorem, this conjugate-linear functional is in turn represented by a unique element $u \in L^2(X,E)$. By construction, we have

$$\int_X h\big(u, \delta'_{\partial/\partial z}\alpha\big)\, d\mu = -\int_X h(f,\alpha)\, d\mu$$

for every $\alpha \in A^0_c(X,E)$, and so u is a weak solution to the equation $d''u = f d\bar{z}$. This weak solution also satisfies the L^2-estimate because

$$\int_X h(u,u)\, d\mu = \|u\|_h^2 \leq \|f/\rho\|_h^2 = \int_X \frac{1}{\rho^2} h(f,f)\, d\mu.$$

Now an application of §153 Proposition finishes the proof of §150 Theorem.

156. If $v \in L^2(X,E)$ is orthogonal to the image of (154.1), then

$$\int_X h(d''_{\partial/\partial \bar{z}} v, \alpha)\, d\mu = -\int_X h(v, \delta'_{\partial/\partial z}\alpha)\, d\mu = 0$$

for every $\alpha \in A^0_c(X,E)$, and so $v \in A^0(X,E)$ is smooth and satisfies $d''v = 0$, hence is holomorphic. This means that the solution $u \in A^0(X,E)$ constructed above is orthogonal to the space of holomorphic sections of E, and so it is the (unique) solution of the equation $d''u = f d\bar{z}$ with minimal L^2-norm.

7.2 Hodge bundles and metrics with positive curvature

157. Now let E be a polarized variation of Hodge structure on Δ^*. In order to apply Hörmander's theory to the Hodge bundles, we need a metric with positive curvature. According to §50 Proposition, the curvature operator of the Hodge metric h on the holomorphic vector bundle $\mathscr{E}^{p,q}$ is

$$\Theta = -(\theta\theta^* + \theta^*\theta).$$

For any smooth section $u \in A^0(\Delta^*, E^{p,q})$, we therefore have

$$h\big(\Theta_{\partial/\partial t \wedge \partial/\partial \bar{t}}\, u, u\big) = h\big(\theta_{\partial/\partial t} u, \theta_{\partial/\partial t} u\big) - h\big(\theta^*_{\partial/\partial \bar{t}} u, \theta^*_{\partial/\partial \bar{t}} u\big).$$

The right-hand side is unfortunately not positive, but at least we know that the troublesome second term can be no bigger than

$$h\big(\theta^*_{\partial/\partial \bar{t}} u, \theta^*_{\partial/\partial \bar{t}} u\big) \leq h\big(\theta^*_{\partial/\partial \bar{t}}, \theta^*_{\partial/\partial \bar{t}}\big) h(u,u) \leq \frac{1}{4}\binom{r+1}{3} \frac{1}{|t|^2 (\log|t|)^2} h(u,u),$$

using the universal bound for the Higgs field in §58 Corollary. (Here $r = \operatorname{rk} E$.)

158. We can try to fix this problem by multiplying the Hodge metric by a factor of the form $e^{-\varphi}$, where $\varphi\colon \Delta^* \to \mathbb{R}$ is a suitable weight function. Let us denote the new hermitian metric temporarily by the symbol $h^\varphi = he^{-\varphi}$. Since

$$\begin{aligned} \bar{\partial} h^\varphi(v,w) &= \bar{\partial} h(v,w) \cdot e^{-\varphi} + h(v,w) e^{-\varphi}(-\bar{\partial}\varphi) \\ &= h^\varphi(d''v, w) + h^\varphi(u, \delta' v) - h^\varphi(u, \partial\varphi), \end{aligned}$$

the new Chern connection is $\delta' + d'' - \partial\varphi$, and so the new curvature operator is

$$\Theta^\varphi = (\delta' + d'' - \partial\varphi)^2 = \Theta + \partial\bar{\partial}\varphi.$$

Our inequality from above now changes into

$$h^\varphi\big(\Theta^\varphi_{\partial/\partial t \wedge \partial/\partial \bar{t}}\, u, u\big) = h^\varphi\big(\theta_{\partial/\partial t} u, \theta_{\partial/\partial t} u\big) - h^\varphi\big(\theta^*_{\partial/\partial \bar{t}} u, \theta^*_{\partial/\partial \bar{t}} u\big) + \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \cdot h^\varphi(u,u).$$

The right-hand side is greater or equal to

$$h^\varphi(u,u) \cdot \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} - \frac{1}{4}\binom{r+1}{3} \frac{1}{|t|^2 (\log|t|)^2} \right),$$

and as long as the term in parentheses is positive, the metric h^φ has positive curvature. The following lemma gives us a family of suitable weight functions.

Lemma. *For $a \in \mathbb{R}$ and $b \in \mathbb{Z}$, the function $e^{-\varphi} = |t|^a (-\log|t|)^b$ satisfies*

$$\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} = \frac{b}{4|t|^2 (\log|t|)^2}$$

Proof. A small calculation, which can be made easier by pulling back to $\mathbb{H}$. □

159. This gives us a nice family of hermitian metrics on the Hodge bundles to which we can apply Hörmander's L^2-estimates.

Corollary. *Let $a \in \mathbb{R}$ and $b \in \mathbb{N}$, subject to the condition $b \geq \binom{r+1}{3} + 1$, where $r = \operatorname{rk} E$. Let $f \in A^0(\Delta^*, E^{p,q})$ be a smooth section with the property that*

$$\int_{\Delta^*} h(f,f)\, |t|^{a+2} (-\log|t|)^{b+2}\, d\mu < +\infty.$$

Then there is a smooth section $u \in A^0(\Delta^*, E^{p,q})$ that solves the $\bar\partial$-equation $\bar\partial u = f d\bar t$ and also satisfies the L^2-estimate

$$\int_{\Delta^*} h(u,u)\, |t|^{a} (-\log|t|)^{b}\, d\mu \leq 4 \int_{\Delta^*} h(f,f)\, |t|^{a+2} (-\log|t|)^{b+2}\, d\mu.$$

Proof. We apply §150 Theorem to the Hodge bundle $E^{p,q}$, with the holomorphic structure defined by the operator $\bar\partial$ and with the hermitian metric

$$h^{\varphi} = h \cdot |t|^{a} (-\log|t|)^{b}$$

For $b \geq \binom{r+1}{3} + 1$, this metric has positive curvature; in fact, we now have

$$h^{\varphi}\Big(\Theta^{\varphi}_{\partial/\partial t \wedge \partial/\partial \bar t} u, u\Big) \geq h^{\varphi}(u,u) \cdot \frac{1}{4|t|^2 (\log|t|)^2} \tag{159.1}$$

for every smooth section $u \in A^0(\Delta^*, E^{p,q})$. The assumptions of the theorem are therefore met with $1/\rho^2 = 4|t|^2(\log|t|)^2$. □

160. In what follows, we are actually going to apply the L^2-estimates to the induced variation of Hodge structure on the bundle $\operatorname{End}(E)$. Because of the better bound in §59 Corollary, the result takes the following form.

Corollary. *Let $a \in \mathbb{R}$ and $b \in \mathbb{N}$, subject to the condition $b \geq 2r\binom{r+1}{3} + 1$. Let $f \in A^0\big(\Delta^*, \operatorname{End}(E)^{p,q}\big)$ be a smooth section with the property that*

$$\int_{\Delta^*} h_{\operatorname{End}(E)}(f,f)\, |t|^{a+2} (-\log|t|)^{b+2}\, d\mu < +\infty.$$

Then there is a smooth section $u \in A^0\big(\Delta^*, \operatorname{End}(E)^{p,q}\big)$ that solves the equation $\bar\partial u = f d\bar t$ and also satisfies the L^2-estimate

$$\int_{\Delta^*} h_{\operatorname{End}(E)}(u,u)\, |t|^{a} (-\log|t|)^{b}\, d\mu \leq 4 \int_{\Delta^*} h_{\operatorname{End}(E)}(f,f)\, |t|^{a+2} (-\log|t|)^{b+2}\, d\mu.$$

7.3 Constructing a holomorphic lifting of the Higgs field

161. After these preliminaries, we now turn to the actual proof of the nilpotent orbit theorem. Fix a half-open interval $I \subseteq \mathbb{R}$ of length 1, and let $S \in \operatorname{End}(V)$ be the unique semisimple operator such that $T_s = e^{2\pi i S}$ and such that the eigenvalues of S are real and contained in I. If we denote by $P_\lambda \in \operatorname{End}(V)$ the projection to the eigenspace $E_\lambda(T_s)$, we can write S compactly as

$$S = \sum_{\alpha \in I} \alpha P_{e^{2\pi i \alpha}}.$$

The differential of the holomorphic mapping $z \mapsto \Psi_S(e^z) = e^{-z(S+N)}\Phi(z)$ at the point $z \in \mathbb{H}$ is equal to

$$e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)} - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)}.$$

The differential is holomorphic, modulo the indicated subspace, but the operator on the left-hand side is *not* holomorphic as a section of the bundle $\operatorname{End}(V)$. The idea of the proof is to lift this operator to a holomorphic section of the bundle $\operatorname{End}(V)$ with the help of Hörmander's L^2-estimates, and then to use the properties of this lifting to show that Ψ_S extends.

162. Let us begin by looking at the Higgs field itself. By construction, the operator $\theta_{\partial/\partial z} \colon \mathbb{H} \to \operatorname{End}(V)$ is the pullback of $t\theta_{\partial/\partial t}$, and as such, it satisfies

$$\theta_{\partial/\partial z}(z + 2\pi i) = T\,\theta_{\partial/\partial z}(z)\,T^{-1}.$$

Recall that we have an induced variation of Hodge structure of weight 0 on the bundle $\operatorname{End}(E)$. The Higgs field $t\theta_{\partial/\partial t}$ is a smooth section of the Hodge bundle $F^{-1}\operatorname{End}(E)$ on Δ^*. Now the Hodge bundle is a holomorphic vector bundle, whose holomorphic structure is defined by the operator $d''_{\operatorname{End}(E)} = [d'', -]$, but the Higgs field is not a holomorphic section. Instead, because of the identity $\bar{\partial}\theta + \theta\bar{\partial} = 0$, only its projection to the quotient bundle

$$\operatorname{End}(E)^{-1,1} \cong F^{-1}\operatorname{End}(E)/F^0\operatorname{End}(E)$$

is holomorphic, for the holomorphic structure defined by $\bar{\partial}_{\operatorname{End}(E)} = [\bar{\partial}, -]$.

163. The L^2-estimates allow us to lift $t\theta_{\partial/\partial t}$ to a holomorphic section of the bundle $F^{-1}\operatorname{End}(E)$ in a controlled manner. We will make a specific choice of the two parameters $a \in \mathbb{R}$ and $b \in \mathbb{Z}$ a little later.

Proposition. *Set $r = \operatorname{rk} E$. For every $a > -2$ and every $b \geq 2r\binom{r+1}{3} + 1$, there is a holomorphic section ϑ of the Hodge bundle $F^{-1}\operatorname{End}(E)$ such that*

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0\operatorname{End}(E),$$

and which also satisfies the L^2-estimate

$$\int_{\Delta^*} h_{\operatorname{End}(E)}(\vartheta, \vartheta)\,|t|^a(-\log|t|)^b\,d\mu \leq C,$$

with a constant $C > 0$ whose exact value only depends on r, a, and b.

Proof. Since the Higgs field is trivial when $r = 1$, we may assume without loss of generality that $r \geq 2$. Using the identities in §45 Lemma, we compute that

$$d''_{\operatorname{End}(E)}\big(t\theta_{\partial/\partial t}\big) = \bar{\partial}_{\operatorname{End}(E)}\big(t\theta_{\partial/\partial t}\big) + \theta^*_{\operatorname{End}(E)}\big(t\theta_{\partial/\partial t}\big) = t[\theta^*_{\partial/\partial \bar{t}}, \theta_{\partial/\partial t}]d\bar{t}.$$

Temporarily define $f_0 = t[\theta^*_{\partial/\partial \bar{t}}, \theta_{\partial/\partial t}]$; this is a smooth section of the Hodge bundle $\operatorname{End}(E)^{0,0}$. The universal bound on the Higgs field in §58 Corollary gives

$$h(f_0, f_0) \leq 2|t|^2 \cdot h\big(\theta_{\partial/\partial t}, \theta_{\partial/\partial t}\big)^2 \leq \frac{2C_0^4}{|t|^2(\log|t|)^4}.$$

Provided that $a > -2$ and $b \geq 2$, the integral

$$\int_{\Delta^*} h(f_0, f_0)\, |t|^{a+2} (-\log|t|)^{b+2}\, d\mu \leq 2C_0^4 \int_{\Delta^*} |t|^a (\log|t|)^{b-2}\, d\mu$$

is finite. Once $b \geq 2r\binom{r+1}{3} + 1$, the version of the L^2-estimates in §159 Corollary gives us a solution $u_0 \in A^0\bigl(\Delta^*, \operatorname{End}(E)^{0,0}\bigr)$ to the equation $\bar{\partial}_{\operatorname{End}(E)} u_0 + f_0 d\bar{t} = 0$ that satisfies the L^2-estimate

$$\int_{\Delta^*} h(u_0, u_0)\, |t|^a (-\log|t|)^b\, d\mu \leq 8C_0^4 \int_{\Delta^*} |t|^a (-\log|t|)^{b-2}\, d\mu$$

Continuing in this way, we produce smooth sections $u_k, f_k \in A^0\bigl(\Delta^*, \operatorname{End}(E)^{k,-k}\bigr)$, indexed by $k \in \mathbb{N}$, such that

$$f_{k+1} = [\theta^*_{\partial/\partial\bar{t}}, u_k] \quad \text{and} \quad \bar{\partial}_{\operatorname{End}(E)} u_{k+1} + f_{k+1} d\bar{t} = 0,$$

subject to the L^2-estimate

$$\begin{aligned}\int_{\Delta^*} h(u_{k+1}, u_{k+1})\, |t|^a (-\log|t|)^b\, d\mu &\leq 4 \int_{\Delta^*} h(f_{k+1}, f_{k+1})\, |t|^{a+2} (-\log|t|)^{b+2}\, d\mu \\ &\leq 8C_0^2 \int_{\Delta^*} h(u_k, u_k) |t|^a (-\log|t|)^b\, d\mu\end{aligned}$$

Let $p \in \mathbb{N}$ be the biggest integer such that $\operatorname{End}(E)^{p,-p} \neq 0$. By construction,

$$\vartheta = t\theta_{\partial/\partial t} + \sum_{k=0}^{p} u_k \in A^0\bigl(\Delta^*, F^{-1} \operatorname{End}(E)\bigr)$$

satisfies $d''_{\operatorname{End}(E)}(\vartheta) = 0$, hence is a holomorphic section of the Hodge bundle $F^{-1}\operatorname{End}(E)$, with the same image in $\operatorname{End}(E)^{-1,1} \cong F^{-1}\operatorname{End}(E)/F^0\operatorname{End}(E)$ as the Higgs field $t\theta_{\partial/\partial t}$. We also get another valuable piece of information about ϑ from the L^2-estimates for the individual solutions u_k. On the one hand,

$$\int_{\Delta^*} h(u_k, u_k)\, |t|^a (-\log|t|)^b\, d\mu \leq (8C_0^2)^{k+1} \cdot C_0^2 \int_{\Delta^*} |t|^a (-\log|t|)^{b-2}\, d\mu$$

by combining the inequalities above. On the other hand,

$$h\bigl(t\theta_{\partial/\partial t}, t\theta_{\partial/\partial t}\bigr) \leq \frac{C_0^2}{(\log|t|)^2},$$

and so we have a similar inequality

$$\int_{\Delta^*} h\bigl(t\theta_{\partial/\partial t}, t\theta_{\partial/\partial t}\bigr)\, |t|^a (-\log|t|)^b\, d\mu \leq C_0^2 \int_{\Delta^*} |t|^a (-\log|t|)^{b-2}\, d\mu.$$

If we put everything together and remember that the Hodge decomposition on $\operatorname{End}(E)$ is orthogonal with respect to the Hodge metric, the result is that

$$\begin{aligned}\int_{\Delta^*} h(\vartheta, \vartheta)\, |t|^a (-\log|t|)^b\, d\mu &\leq \sum_{k=0}^{p+1} (8C_0^2)^k \cdot C_0^2 \int_{\Delta^*} |t|^a (-\log|t|)^{b-2}\, d\mu \\ &\leq C_0^2 \sum_{k=0}^{p+1} (8C_0^2)^k \cdot \frac{2\pi (b-2)!}{(a+2)^{b-1}},\end{aligned}$$

after evaluating the integral. □

164. Recall that the pullback of the flat bundle (E, d) along $\exp\colon \mathbb{H} \to \Delta^*$ has a canonical trivialization by the space V of multi-valued flat sections. Let us use the same letter

$$\vartheta\colon \mathbb{H} \to \operatorname{End}(V)$$

also for the pullback of $\vartheta \in A^0\big(\Delta^*, F^{-1}\operatorname{End}(E)\big)$ to the half-plane $\mathbb{H}$; this is now a holomorphic mapping with the property that

$$\vartheta(z) \in F^{-1}\operatorname{End}(V)_{\Phi(z)} \quad \text{and} \quad \vartheta(z) - \theta_{\partial/\partial z} \in F^0\operatorname{End}(V)_{\Phi(z)}$$

for every $z \in \mathbb{H}$. As a pullback, it transforms according to the rule

$$\vartheta(z + 2\pi i) = T\vartheta(z)T^{-1}.$$

Recall that $T = e^{2\pi i(S+N)}$, where $S \in \operatorname{End}(V)$ is semisimple with real eigenvalues contained in a half-open interval of length 1. The expression $e^{-z(S+N)}\vartheta(z)e^{z(S+N)}$ is again invariant under the substitution $z \mapsto z + 2\pi i$, and therefore

$$e^{-z(S+N)}\,\vartheta(z)\,e^{z(S+N)} = B(e^z)$$

for a well-defined holomorphic mapping $B\colon \Delta^* \to \operatorname{End}(V)$.

165. We can use our integral estimate for ϑ to prove that B extends holomorphically across the origin. To do that, we first need to know how much conjugation by the operator $e^{-z(S+N)}$ affects the norm of an endomorphism. Define

$$\delta(T) = \min\big\{\, \delta > 0 \;\big|\; T \text{ has two eigenvalues of the form } \lambda \text{ and } \lambda \cdot e^{2\pi i\delta} \,\big\};$$

concretely, this is the minimal distance between consecutive eigenvalues of T (on the unit circle), divided by 2π. In the extreme case where all eigenvalues of T are equal to each other, we have $\delta(T) = 1$. Also define

$$m(N) = \min\big\{\, m \in \mathbb{N} \;\big|\; N^m \neq 0 \,\big\}$$

as the index of nilpotency of the operator $N = \log T_u$.

Lemma. *There is a constant $C > 0$ such that*

$$\begin{aligned}\big\|e^{-z(S+N)}Ae^{z(S+N)}\big\|_{\Phi(-1)} \\ \leq Ce^{(1-\delta(T))|\operatorname{Re} z|}\max\Big(|\operatorname{Re} z|^{m(N)+2\sqrt{2r}C_0}, |\operatorname{Re} z|^{-2\sqrt{2r}C_0}\Big)\|A\|_{\Phi(z)}\end{aligned}$$

for every $A \in \operatorname{End}(V)$, uniformly on the horizontal strip

$$\big\{\, z \in \mathbb{H} \;\big|\; |\operatorname{Im} z| \leq \pi \,\big\}.$$

Here $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$, and the exact value of the constant C only depends on $r = \operatorname{rk} E$ and on the minimal polynomial of $T \in \operatorname{GL}(V)$.

Proof. In order to simplify the notation, we denote by $\|v\| = \|v\|_{\Phi(-1)}$ the Hodge norm at the point $-1 \in \mathbb{H}$. Let us first consider the dominant term e^{-zS}. Let $P_\lambda\colon V \to E_\lambda(T_s)$ denote the projection to the λ-eigenspace of T_s. Then

$$S = \sum_{\alpha \in I} \alpha P_{e^{2\pi i\alpha}}.$$

Let $A \in \operatorname{End}(V)$ be an arbitrary endomorphism. We have

$$e^{-zS} A e^{zS} = \sum_{\alpha,\beta} e^{(\alpha-\beta)|\operatorname{Re} z|} e^{-i(\alpha-\beta)\operatorname{Im} z} P_{e^{2\pi i\alpha}} A P_{e^{2\pi i\beta}}$$

and so the triangle inequality and submultiplicativity of the L^2-norm give

$$\left\| e^{-zS} A e^{zS} \right\| \leq e^{(1-\delta)|\operatorname{Re} z|} \left(\sum_\lambda \|P_\lambda\| \right)^2 \|A\|, \tag{165.1}$$

where $\delta = \delta(T)$, due to the fact that $\alpha_{\max} - \alpha_{\min} \leq 1 - \delta$. The term e^{-zN} is easier to analyze because N is nilpotent. Set $m = m(N)$. Then

$$e^{-zN} A e^{zN} = \sum_{i,j=0}^{m} \frac{(-1)^i z^{i+j}}{i!\, j!} N^i A N^j$$

and therefore

$$\left\| e^{-zN} A e^{zN} \right\| \leq \sum_{i,j=0}^{m} \frac{|z|^{i+i}}{i!\, j!} \|N\|^{i+j} \|A\| = \sum_{k=0}^{m} \frac{|2z|^k}{k!} \|N\|^k \|A\|.$$

Now we consider $A \in \operatorname{End}(V)$ as a multi-valued flat section of the induced variation of Hodge structure on $\operatorname{End}(E)$. Remembering the improved bound for the Higgs field in §59 Corollary, we get from §132 Proposition the inequality

$$\|A\| \leq e^{2\sqrt{2r}C_0\pi} \max\Big(|\operatorname{Re} z|^{2\sqrt{2r}C_0}, |\operatorname{Re} z|^{-2\sqrt{2r}C_0} \Big) \|A\|_{\Phi(z)}$$

Putting everything together, we arrive at

$$\left\| e^{-z(S+N)} A e^{z(S+N)} \right\| \leq C e^{(1-\delta)|\operatorname{Re} z|} \max\Big(|\operatorname{Re} z|^{m+2\sqrt{2r}C_0}, |\operatorname{Re} z|^{-2\sqrt{2r}C_0} \Big) \|A\|_{\Phi(z)}$$

with a constant $C > 0$ that only depends on the two integers $r = \operatorname{rk} E$ and $m = m(N)$, and on the Hodge norms $\|P_\lambda\|_{\Phi(-1)}$ and $\|N\|_{\Phi(-1)}$. The latter quantities are in turn bounded, in §65 Proposition, by a constant whose value only depends on r and on the minimal polynomial of $T \in \operatorname{GL}(V)$. This gives the desired result. $\square$

166. We are now ready to show that $B \colon \Delta^* \to \operatorname{End}(V)$ extends holomorphically across the origin. We choose

$$a = -2 + 2\delta(T) > -2 \quad \text{and} \quad b = 2r \binom{r+1}{3} + 1,$$

and then construct ϑ and B according to the procedure described above. Since $b = 8rC_0^2 + 1 \geq m(N) + 2\sqrt{2r}C_0$, §165 Lemma gives

$$\left\| B(e^z) \right\|_{\Phi(-1)}^2 \leq C^2 e^{-a|\operatorname{Re} z|} \max\Big(|\operatorname{Re} z|^b, |\operatorname{Re} z|^{-b} \Big) \|\vartheta\|_{\Phi(z)}^2;$$

the restriction on $|\operatorname{Im} z|$ becomes irrelevant here, because $B(e^z)$ is of course invariant under the substitution $z \mapsto z + 2\pi i$. Now $|t| = e^{-|\operatorname{Re} z|}$ and $-\log|t| = |\operatorname{Re} z|$, and so we can rewrite this inequality in the form

$$\min\Big(1, (\log|t|)^{2b}\Big) \cdot \|B(t)\|_{\Phi(-1)}^2 \leq C^2 h(\vartheta, \vartheta) |t|^a (-\log|t|)^b.$$

The L^2-estimate for ϑ in §163 Proposition now gives us

$$\int_{\Delta^*} \min\Big(1, (\log|t|)^{2b}\Big) \cdot \|B(t)\|^2_{\Phi(-1)} \, d\mu \leq C', \tag{166.1}$$

with a constant $C' > 0$ whose value has the same dependence on parameters as in §165 Lemma. Since $B\colon \Delta^* \to \operatorname{End}(V)$ is holomorphic, this is enough to conclude that B extends holomorphically across the origin.

167. As an aside, we note that the bound on the integral also gives us some control over the pointwise norm of $B(t)$. To simplify the calculations, we use the inequality

$$\min\Big(1, (\log|t|)^{2b}\Big) \geq (1 - |t|)^{2b},$$

which follows from the fact that $|\log x| \geq 1 - x$ for $0 < x \leq 1$. The following result is a simple modification of the formula for the Bergman kernel on the unit disk.

Lemma. *For any holomorphic function $f\colon \Delta \to \mathbb{C}$, one has*

$$|f(t)|^2 \leq \frac{2^b(b+1)}{(1-|t|^2)^{b+2}} \cdot \frac{1}{\pi} \int_\Delta (1-|t|)^b |f|^2 \, d\mu.$$

Proof. Consider the power series expansion

$$f = \sum_{n=0}^\infty a_n t^n.$$

An easy computation shows that

$$\begin{aligned}
\frac{1}{\pi} \int_\Delta (1-|t|)^b |f|^2 \, d\mu &= \sum_{n=0}^\infty |a_n|^2 \frac{1}{\pi} \int_\Delta (1-|t|)^b |t|^{2n} \, d\mu \\
&= \sum_{n=0}^\infty |a_n|^2 \int_0^1 2(1-r)^b r^{2n+1} \, dr = \sum_{n=0}^\infty |a_n|^2 \frac{2(2n+1)!b!}{(2n+2+b)!}.
\end{aligned}$$

We can now apply the Cauchy-Schwarz inequality to get

$$|f(t)|^2 \leq \sum_{n=0}^\infty |t|^{2n} \frac{(2n+2+b)!}{2(2n+1)!b!} \cdot \sum_{n=0}^\infty |a_n|^2 \frac{2(2n+1)!b!}{(2n+2+b)!}.$$

To estimate the first series, we set $x = |t|^2$ and compute as follows:

$$\begin{aligned}
\sum_{n=0}^\infty |t|^{2n} \frac{(2n+2+b)!}{2(2n+1)!b!} &= \frac{1}{2b!} \sum_{n=0}^\infty (2n+2)(2n+3)\cdots(2n+2+b) x^n \\
&\leq \frac{1}{2b!} \sum_{n=0}^\infty (2n+2)(2n+4)\cdots(2n+2+2b) x^n \\
&= \frac{2^b}{b!} \sum_{n=0}^\infty (n+1)\cdots(n+1+b) x^n \\
&= \frac{2^b}{b!} \left(\frac{d}{dx}\right)^{b+1} \sum_{n=0}^\infty x^n = \frac{2^b(b+1)}{(1-x)^{b+2}} = \frac{2^b(b+1)}{(1-|t|^2)^{b+2}}.
\end{aligned}$$

The desired result follows. □

168. If we apply this result to the integral bound for $B(t)$ in (166.1), for example by choosing an orthonormal basis of $\operatorname{End}(V)$ relative to the inner product coming from the Hodge structure $\Phi(-1)$, and then writing $B(t)$ as a matrix, we get

$$\|B(t)\|_{\Phi(-1)} \leq \frac{C}{(1-|t|^2)^{2b+2}}, \tag{168.1}$$

where $b = 2r\binom{r+1}{3} + 1$, and where $C > 0$ is a constant that again depends only on $r = \operatorname{rk} E$ and on the minimal polynomial of $T \in \mathrm{GL}(V)$. The interesting thing is that $B(t)$ still behaves rather nicely near the outer boundary of the unit disk, even though all we have is a bound on its L^2-norm.

7.4 Holomorphic extension of the untwisted period mapping

169. We return to our analysis of the untwisted period mapping

$$\Psi_S(e^z) = e^{-z(S+N)}\Phi(z).$$

Recall that the differential of the mapping $z \mapsto \Psi_S(e^z)$ is given by

$$e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)} - (S+N) \equiv B(e^z) - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)}$$

Here we are using the fact that $B(e^z) - e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)}$ lies in $F^0 \operatorname{End}(V)_{\Psi_S(e^z)}$. The point is that the operator on the right-hand side is holomorphic.

170. Now suppose that we have a holomorphic mapping

$$g\colon \mathbb{H} \to \mathrm{GL}(V)$$

that solves the ordinary differential equation

$$g'(z) = \Big(B(e^z) - (S+N)\Big)g(z), \tag{170.1}$$

subject to the initial condition $g(-1) = \mathrm{id}$. A short calculation with derivatives shows that the mapping $g(z)^{-1}\Psi_S(e^z)$ is then constant, which means that

$$\Psi_S(e^z) = g(z)\cdot\Psi_S(e^{-1}) = g(z)\cdot e^{(S+N)}\Phi(-1).$$

The key to the remainder of the proof is that the differential equation in (170.1) has a **regular singular point** at $t = 0$.

171. Let us first prove that (170.1) has a solution with the desired properties. For every $v \in V$, the differential equation

$$f'(z) = \Big(B(e^z) - (S+N)\Big)f(z)$$

has a unique solution $f\colon \mathbb{H} \to V$ that is holomorphic and satisfies the initial condition $f(-1) = v$; this is Cauchy's theorem. Let $v_1, \dots, v_r \in V$ be a basis, and let $f_1, \dots, f_r \colon \mathbb{H} \to V$ be the corresponding solutions. By uniqueness, the values $f_1(z), \dots, f_r(z) \in V$ must be linearly independent for every $z \in \mathbb{H}$. We can therefore define a holomorphic mapping

$$g\colon \mathbb{H} \to \mathrm{GL}(V)$$

by requiring that $g(z)v_i = f_i(z)$ for $i = 1, \dots, r$. It follows that $g(-1) = \mathrm{id}$, and so g is the desired solution to (170.1).

172. Clearly, $g(z+2\pi i)g(-1+2\pi i)^{-1}$ is also a solution, and so by uniqueness,

$$g(z+2\pi i) = g(z)g(-1+2\pi i).$$

Let $A \in \operatorname{End}(V)$ be the unique operator such that $g(-1+2\pi i) = e^{2\pi i A}$ and such that all eigenvalues of A have their real part contained in the interval $[0,1)$. Then $g(z)e^{-zA}$ is invariant under $z \mapsto z+2\pi i$, and so

$$g(z) = M(e^z)e^{zA}$$

for a unique holomorphic mapping $M\colon \Delta^* \to \mathrm{GL}(V)$. Since (170.1) has a regular singular point at $t=0$, the basic theory of Fuchsian differential equations implies that $M(t)$ is meromorphic at $t=0$.

173. Let us briefly recall how this is proved. We use the shorthand $\|v\| = \|v\|_{\Phi(-1)}$ for the Hodge norm in the Hodge structure $\Phi(-1)$. Since $B\colon \Delta \to \operatorname{End}(V)$ is holomorphic, there is a constant $C>0$ such that

$$\big\|B(e^z) - (S+N)\big\| \leq C \quad \text{for } \operatorname{Re} z \leq -1.$$

Let $z \in \mathbb{H}$ be any point with $\operatorname{Re} z \leq -1$. Consider the auxiliary function

$$\varphi\colon [0,1] \to \mathbb{R}, \quad \varphi(x) = \big\|g\big(-1+x(z+1)\big)\big\|^2.$$

The derivative of φ is given by the formula

$$\varphi'(x) = 2\operatorname{Re}\Big\langle (z+1)g'\big(-1+x(z+1)\big),\, g\big(-1+x(z+1)\big)\Big\rangle,$$

and so the differential equation in (170.1), together with the Cauchy-Schwarz inequality and the bound on $B(e^z)-(S+N)$, leads to the simple inequality

$$|\varphi'(x)| \leq 2C|z+1|\cdot\varphi(x).$$

This is easily integrated, with the result that

$$\|g(z)\|^2 \leq \|g(-1)\|^2 e^{2C|z+1|} \leq r e^{2C|z+1|},$$

remembering that $g(-1) = \mathrm{id}$ and $r = \dim V$. Since the L^2-norm is submultiplicative, we then get

$$\|M(e^z)\| \leq \|g(z)\| \cdot \|e^{-zA}\| \leq \sqrt{r} e^{C|z+1|} \cdot e^{|z|\|A\|} \leq C' e^{d|\operatorname{Re} z|}$$

for some $C'>0$ and some $d \in \mathbb{N}$; note that we may assume that $|\operatorname{Im} z| \leq \pi$. But this says exactly that $M(t)$ has a pole of order at most d at $t=0$.

174. We can now finish the proof of the first half of the nilpotent orbit theorem, still under the assumption that the eigenvalues of S lie in a fixed half-open interval I of length 1. By construction, we have

$$\Psi_S(e^z) = g(z)\cdot e^{S+N}\Phi(-1) = M(e^z)\cdot e^{zA}\cdot e^{S+N}\Phi(-1)$$

for every $z \in \mathbb{H}$. The left-hand side is invariant under the substitution $z \mapsto z+2\pi i$, and so the operator $e^{2\pi i A} \in \mathrm{GL}(V)$ has to leave the point $e^{S+N}\Phi(-1) \in \check{D}$ fixed. It follows that A itself must leave the point fixed, and so we can erase the factor e^{zA} from the identity above. The result is that

$$\Psi_S(t) = M(t)\cdot e^{S+N}\Phi(-1)$$

for $t \in \Delta^*$. Because $M(t)$ is meromorphic at $t=0$, and $\check{D}$ is projective, this is enough to conclude that Ψ_S extends holomorphically across the origin.

175. Removing the restriction on the eigenvalues of S is now an easy matter. Suppose that $S \in \operatorname{End}(V)$ is *any* operator such that $T_s = e^{2\pi i S}$; of course, S is then automatically semisimple with real eigenvalues. Let $S(I) \in \operatorname{End}(V)$ be the choice we were using above, with eigenvalues contained in the given interval $I \subseteq \mathbb{R}$. Then $S(I) - S$ has integer eigenvalues, and so

$$\Psi_S(t) = e^{z(S(I)-S)}\Psi_{S(I)}(t) = t^{S(I)-S}\Psi_{S(I)}(t)$$

is obtained from the holomorphic maping $\Psi_{S(I)}(t)$ by acting by a semisimple operator whose eigenvalues are powers of t. It follows that $\Psi_S(t)$ is also meromorphic, and therefore extends holomorphically to Δ for the same reason as before.

176. In particular, we now have a well-defined limit $\Psi_S(0) \in \check{D}$. Let us write $F^\bullet \operatorname{End}(V)_{\Psi_S(0)}$ for the induced filtration on $\operatorname{End}(V)$. It remains to show that

$$S + N \in F^{-1}\operatorname{End}(V)_{\Psi_S(0)}.$$

Recall that the derivative of the holomorphic mapping $\Psi_S(e^z)$ is equal to

$$B(e^z) - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)},$$

and that the operator $B(e^z)$ is holomorphic and belongs to $F^{-1}\operatorname{End}(V)_{\Psi_S(e^z)}$. Since $\Psi_S \colon \Delta \to \check{D}$ is holomorphic, the chain rule shows that the derivative of $\Psi_S(e^z)$ goes to zero like $|t| = e^{-|\operatorname{Re} z|}$. Passing to the limit, we get

$$B(0) - (S+N) \in F^0 \operatorname{End}(V)_{\Psi_S(0)},$$

and therefore $S + N \in F^{-1}\operatorname{End}(V)_{\Psi_S(0)}$, as claimed. This completes the proof of §24 Theorem, which contained the convergence assertions in the nilpotent orbit theorem.

177. The nilpotent orbit theorem has the following more geometric interpretation, in terms of certain bundles on the unit disk. Recall that the operator $S \in \operatorname{End}(V)$ in $T = e^{2\pi i(S+N)}$ depended on a choice of half-open interval I of length 1. Let $\mathscr{E}_S$ denote the canonical extension [Del70, Prop. 5.2] of the holomorphic vector bundle $\mathscr{E}$, such that the connection extends to a logarithmic connection $\nabla \colon \mathscr{E}_S \to \Omega^1_\Delta(\log 0) \otimes \mathscr{E}_S$ whose residue $\operatorname{Res}_0 \nabla$ has eigenvalues in the same interval I. The bundle $\mathscr{E}_S$ has a unique trivialization

$$V \otimes_{\mathbb{C}} \mathscr{O}_\Delta \cong \mathscr{E}_S$$

such that the logarithmic connection becomes $\nabla(v \otimes 1) = (S+N)v \otimes dt$; therefore $S + N$ is exactly the Jordan decomposition of the residue $\operatorname{Res}_0 \nabla$. Under this trivialization, the Hodge bundle $F^p\mathscr{E}$ goes to the subbundle

$$\left\{ \left(t, \Psi_S^p(t)\right) \,\middle|\, t \in \Delta^* \right\} \subseteq \Delta^* \times V$$

of the trivial bundle; §24 Theorem is therefore saying exactly that the Hodge bundles extend to holomorphic subbundles $F^p\mathscr{E}_S$ of the canonical extension.

178. Now if we introduce the quotient bundles

$$\mathscr{E}_S^{p,q} = F^p\mathscr{E}_S / F^{p+1}\mathscr{E}_S,$$

then the operator $B(t)$ that we constructed above gives a holomorphic extension of the Higgs field to a morphism of holomorphic vector bundles

$$\mathscr{E}_S^{p,q} \to \Omega^1_\Delta(\log 0) \otimes \mathscr{E}_S^{p-1,q+1}$$

whose residue at $t = 0$ is exactly the operator $S + N \in \operatorname{End}(V)$. This gives us a sort of canonical extension of the Higgs bundle $\bigoplus \mathscr{E}^{p,q}$ to a logarithmic Higgs bundle $\bigoplus \mathscr{E}_S^{p,q}$, on which the residue $\operatorname{Res}_0 \theta_{\partial/\partial t}$ of the Higgs field has eigenvalues in the interval I. The fiber at $t = 0$ is isomorphic to the graded vector space

$$\bigoplus_{p \in \mathbb{Z}} \Psi_S^p(0)/\Psi_S^{p+1}(0),$$

and we have $\operatorname{Res}_0 \theta_{\partial/\partial t} = S + N$.

7.5 The limiting Hodge filtration

179. From the untwisted period mapping, we get a sort of "limit" filtration $\Psi_S(0) \in \check{D}$, but it depends on the choice of $S \in \operatorname{End}(V)$, and therefore has little intrinsic meaning. In this section, we introduce an additional filtration $F_{\text{lim}} \in \check{D}$, called the "limiting Hodge filtration", that only depends on the period mapping itself. The limiting Hodge filtration is important in two ways: it allows one to approximate the original period mapping by a nilpotent orbit; and it shows up in the limiting mixed Hodge structure.

180. For technical reasons, we again have to restrict to the case where the eigenvalues of $S \in \operatorname{End}(V)$ are contained in a half-open interval of length 1.

Proposition. *As long as* $|\operatorname{Im} z|$ *remains bounded, the limit*

$$F_{\text{lim}} = \lim_{|\operatorname{Re} z| \to \infty} e^{-zN} \Phi(z) = \lim_{x \to \infty} e^{-xS} \Psi_S(0) \in \check{D}$$

exists and is called the ***limiting Hodge filtration****. It satisfies*

$$N(F_{\text{lim}}^{\bullet}) \subseteq F_{\text{lim}}^{\bullet - 1} \quad \textit{and} \quad T_s(F_{\text{lim}}^{\bullet}) \subseteq F_{\text{lim}}^{\bullet}.$$

181. From the formula for the untwisted period mapping, it is clear that

$$e^{-zN} \Phi(z) = e^{zS} \Psi_S(e^z).$$

Since $\Psi_S \colon \Delta \to \check{D}$ is holomorphic, there is a constant $C > 0$ such that

$$d_{\check{D}}\big(\Psi_S(t), \Psi_S(0)\big) \leq C|t| \tag{181.1}$$

for small values of $|t|$. According to (79.1), translation by the operator $e^{zS} \in \operatorname{GL}(V)$ distorts distances in the compact dual $\check{D}$ by at most the operator norm of $\operatorname{Ad} e^{zS}$. We can estimate this using the following variant of §165 Lemma.

Lemma. *Let* $S \in \operatorname{End}(V)$ *be a semisimple with real eigenvalues. Then*

$$\|\operatorname{Ad} e^{zS}\| \leq C e^{(\alpha_{\max} - \alpha_{\min})|\operatorname{Re} z|},$$

where $\alpha_{\min}$ *and* $\alpha_{\max}$ *are the smallest and largest eigenvalue of* S*.*

Proof. Let $A \in \operatorname{End}(V)$ be an arbitrary endomorphism. We have

$$e^{zS} A e^{-zS} = \sum_{\alpha, \beta} e^{(\beta - \alpha)|\operatorname{Re} z|} e^{i \operatorname{Im} z (\alpha - \beta)} P_\alpha A P_\beta,$$

where $P_\alpha \colon V \to E_\alpha(S)$ are the projections to the eigenspaces of S. Since the L^2-norm is submultiplicative, we get

$$\|e^{zS} A e^{-zS}\| \leq e^{(\alpha_{\max}-\alpha_{\min})|\operatorname{Re} z|} \left(\sum_\alpha \|P_\alpha\| \right)^2 \|A\|,$$

and therefore the desired upper bound on the operator norm of $\operatorname{Ad} e^{zS}$. $\square$

182. Applying this lemma to (181.1), we find that

$$d_{\check{D}}\bigl(e^{-zN}\Phi(z), e^{zS}\Psi_S(0)\bigr) \leq C' e^{-(1-\rho)|\operatorname{Re} z|} \tag{182.1}$$

for some constant $C' > 0$, where $\rho = \alpha_{\max} - \alpha_{\min} < 1$ is the difference between the largest and smallest eigenvalue of S. It remains to understand the effect of the exponential factor $e^{-|\operatorname{Re} z| S}$ on the filtration $\Psi_S(0)$.

183. The following lemma does this in slightly greater generality; we will use it again (in §209) when we prove the convergence of the rescaled period mapping.

Lemma. *Let $S \in \operatorname{End}(V)$ be a semisimple operator with real eigenvalues. For any filtration $F \in \check{D}$, the limit*

$$\hat{F} = \lim_{x \to \infty} e^{xS} F$$

exists in $\check{D}$, and the filtration $\hat{F}$ is compatible with the eigenspace decomposition $V = \bigoplus_\alpha E_\alpha(S)$. Moreover, there is a constant $C \geq 0$ such that

$$d_{\check{D}}\bigl(\hat{F}, e^{xS} F\bigr) \leq C e^{-\delta x},$$

where $\delta > 0$ is the smallest distance between consecutive eigenvalues of S.

Proof. Since a filtration is just a collection of subspaces, it suffices to prove that for any subspace $W \subseteq V$ of dimension d, the limit

$$\hat{W} = \lim_{x \to \infty} e^{xS} W$$

exists (in the Grassmannian of d-dimensional subspaces of V), and satisfies

$$\hat{W} = \bigoplus_{\alpha \in \mathbb{R}} E_\alpha(S) \cap \hat{W}.$$

To make it clear what is going on, let us first do the case where $W = \mathbb{C}v$ is one-dimensional. Write $v = \sum_\alpha v_\alpha$, where $S v_\alpha = \alpha v_\alpha$. Then

$$e^{xS} v = \sum_\alpha e^{\alpha x} v_\alpha.$$

Let $\beta \in \mathbb{R}$ be the largest number such that $v_\beta \neq 0$. From

$$e^{xS}(\mathbb{C}v) = \mathbb{C}\left(v_\beta + \sum_{\alpha < \beta} e^{-(\beta - \alpha)x} v_\alpha \right),$$

we see that $\lim_{x\to\infty} e^{xS}(\mathbb{C}v)$ exists and equals $\mathbb{C}v_\beta$. So the effect of the limit is to extract the component of v for the largest possible eigenvalue. Moreover, the rate of convergence is evidently $e^{-(\beta-\beta')x}$, where $\beta' < \beta$ is the next largest eigenvalue of S.

To deal with the general case, let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be the eigenvalues of S in increasing order, and consider the filtration $G_\bullet$ by increasing eigenvalues, with terms

$$G_j = E_{\alpha_1}(S) \oplus \cdots \oplus E_{\alpha_j}(S) \subseteq V.$$

Set $W_1 = G_1 \cap W$; choose a subspace $W_2 \subseteq W$ such that $W_1 \oplus W_2 = G_2 \cap W$; and so on. Continuing in this way, we obtain a collection of subspaces $W_1, \dots, W_n \subseteq W$, possibly zero-dimensional, with the property that

$$G_j \cap W = W_1 \oplus \cdots \oplus W_j.$$

By construction, any nonzero vector $v \in W_j$ must have a nontrivial component $v_{\alpha_j} \in E_{\alpha_j}(S)$, and as we have seen above, $e^{xS}(\mathbb{C}v)$ therefore converges to $\mathbb{C}v_{\alpha_j}$ at a rate of $e^{-(\alpha_j-\alpha_{j-1})x}$. It follows that the subspace $e^{xS}W_j$ converges, at the same rate, to a subspace $\hat{W}_j \subseteq E_{\alpha_j}(S)$, which consists of the $E_{\alpha_j}(S)$-components of all the vectors in W_j. Putting everything together, we find that

$$\lim_{x\to\infty} e^{xS}W = \bigoplus_{j=1}^{n} \hat{W}_j.$$

The rate of convergence is evidently $e^{-\delta x}$, where $\delta > 0$ is the smallest distance between consecutive eigenvalues of S. □

184. Here is an equivalent way for describing the limit filtration $\hat{F}$ in terms of the filtration $G_\bullet$ by increasing eigenvalues of S. Projecting $W \subseteq V$ to the subquotient G_j/G_{j-1} yields the subspace

$$(G_j \cap W + G_{j-1})/G_{j-1} \subseteq G_j/G_{j-1}.$$

Since $G_j \cap W + G_{j-1} = W_j + G_{j-1}$, the subspace $\hat{W}_j \subseteq E_\alpha(S)$ that we used during the proof is exactly the preimage of the above subspace under the isomorphism $E_{\alpha_j}(S) \cong G_j/G_{j-1}$. So the effect of the limit

$$\hat{F} = \lim_{x\to\infty} e^{xS}F$$

is to make the filtration F compatible with the eigenspace decomposition of S, by projecting to the subquotients of the filtration by increasing eigenvalues.

185. The lemma shows that the limit

$$F_{\mathrm{lim}} = \lim_{x\to\infty} e^{-xS}\Psi_S(0) \in \check{D}$$

exists, and that the resulting filtration satisfies $S(F^\bullet_{\mathrm{lim}}) \subseteq F^\bullet_{\mathrm{lim}}$. On account of (182.1), it follows that the limit

$$\lim_{|\mathrm{Re}\, z|\to\infty} e^{-zN}\Phi(z) = \lim_{|\mathrm{Re}\, z|\to\infty} e^{i\,\mathrm{Im}\, z\, S}e^{-|\mathrm{Re}\, z|S}\Phi(z) = F_{\mathrm{lim}}$$

also exists, provided that $|\mathrm{Im}\, z|$ remains bounded in the process. In fact, the limiting Hodge filtration F_{lim} is obtained from $\Psi_S(0)$ by projecting to the subquotients of the filtration by *decreasing* eigenvalues of S. This is due to the minus sign in the exponential factor $e^{-|\mathrm{Re}\, z|S}$.

186. We already know that $T_s(F^\bullet_{\lim}) \subseteq F^\bullet_{\lim}$. To complete the proof of §180 Proposition, we have to analyze how the nilpotent operator N acts on the limiting Hodge filtration $F_{\lim}$. Recall that we have

$$(S+N)\Psi^p_S(0) \subseteq \Psi^{p-1}_S(0) \quad \text{for all } p \in \mathbb{Z}.$$

From the definition of $F_{\lim}$, we now get

$$NF^p_{\lim} = (S+N)F^p_{\lim} = \lim_{x\to\infty} e^{-xS}(S+N)\Psi^p_S(0) \subseteq \lim_{x\to\infty} e^{-xS}\Psi^{p-1}_S(0) = F^{p-1}_{\lim},$$

as desired.

187. It turns out that there is another way to relate the limiting Hodge filtration to untwisted period mappings. This is useful for the proof of the nilpotent orbit theorem in higher dimensions, which we plan to discuss in a sequel to this paper. The idea, which already occurs in [Sch73, §8], is to go to a finite covering of Δ^* of sufficiently high degree.

188. For any $m \geq 1$, we can pull back E along the finite covering space

$$\Delta^* \to \Delta^*, \quad t \mapsto t^m.$$

The result is another variation of Hodge structure on Δ^*. It has the same space of multivalued flat sections, but the monodromy operator changes to T^m, and the period mapping changes to $\Phi(mz)$. This is convenient, because it means that we do not have to introduce any new notation. Let $I \subseteq \mathbb{R}$ be any half-open interval of length 1. Let $S_m \in \operatorname{End}(V)$ be the unique semisimple operator with

$$T^m_s = e^{2\pi i m S_m},$$

such that mS_m has eigenvalues in I; the eigenvalues of S_m itself are contained in $\frac{1}{m}I$. Because $T_s = e^{2\pi i S}$, the eigenvalues of the operator $e^{2\pi i(S-S_m)}$ are m-th roots of unity.

Lemma. *On each eigenspace of T_s, the operator $e^{2\pi i(S-S_m)}$ acts as multiplication by an m-th root of unity; these roots of unity are distinct if $m\delta(T) \geq 1$.*

Proof. Suppose that $I = [\alpha, \alpha+1)$; the remaining case is similar. List the eigenvalues of S in increasing order as

$$\alpha \leq \alpha_1 < \cdots < \alpha_n < \alpha + 1;$$

the distance between adjacent eigenvalues is at least $\delta(T)$. Write $m\alpha_j = k_j + \beta_j$, where $k_j \in \mathbb{Z}$ and $\beta_j \in I$; then the eigenvalues of mS_m are exactly $\beta_1, \dots, \beta_n$. Now

$$k_j = \lfloor m\alpha_j - a \rfloor,$$

and since $m\alpha_{j+1} - m\alpha_j \geq m\delta(T) \geq 1$, it follows that $k_{j+1} > k_j$; similarly, the fact that $m\alpha_r - m\alpha_1 < m$ implies that $k_n - k_1 < m$. The eigenvalues of $e^{2\pi i(S-S_m)}$ are therefore the complex numbers $e^{2\pi i k_j/m}$, which are distinct m-th roots of unity. □

189. Now we consider a variant of the untwisted period mapping. The expression

$$e^{-mz(S_m+N)}\Phi(mz)$$

is invariant under the substitution $z \mapsto z + 2\pi i$, and so it again descends to a holomorphic mapping

$$\Psi_m \colon \Delta^* \to \check{D}, \quad \Psi_m(e^z) = e^{-mz(S_m+N)}\Phi(mz).$$

This in of course just the untwisted period mapping Ψ_{mS_m} for the pullback of E along $t \mapsto t^m$. A brief calculation shows that

$$\Psi_m(e^{2\pi i/m} \cdot t) = e^{2\pi i(S-S_m)} \cdot \Psi_m(t).$$

This identity has the following nice consequence.

Proposition. *Let $m \in \mathbb{N}$ be such that $m\delta(T) \geq 1$, and choose $S_m \in \operatorname{End}(V)$ as above. The holomorphic mapping*

$$\Psi_m \colon \Delta^* \to \check{D}, \quad \Psi_m(e^z) = e^{-mz(S_m+N)}\Phi(mz),$$

extends holomorphically across the origin, and $\Psi_m(0) = F_{\lim}$.

Proof. We already know from the proof of the nilpotent orbit theorem that Ψ_m extends holomorphically across the origin. It remains to show that the filtration $\Psi_m(0) \in \check{D}$ agrees with the limiting Hodge filtration $F_{\lim}$. From the identity

$$\Psi_m(e^{2\pi i/m} \cdot t) = e^{2\pi i(S-S_m)} \cdot \Psi_m(0),$$

we conclude that the filtration $\Psi_m(0) \in \check{D}$ is preserved by the operator $e^{2\pi i(S-S_m)}$. By §188 Lemma, the eigenvalues of this operator are m-th roots of unity, with a different root of unity ζ_λ on each eigenspace $E_\lambda(T_s)$. Consequently,

$$P_\lambda = \frac{1}{m}\sum_{k=0}^{m-1}\bigl(\zeta_\lambda^{-1}e^{2\pi i(S-S_m)}\bigr)^k,$$

and one deduces easily that $T_s \cdot \Psi_m(0) = \Psi_m(0)$. For $|t|$ sufficiently small, we have

$$d_{\check{D}}\bigl(\Psi_m(t), \Psi_m(0)\bigr) \leq C|t|,$$

which translates into

$$d_{\check{D}}\Bigl(e^{-z(S_m+N)}\Phi(z), \Psi_m(0)\Bigr) \leq Ce^{-\frac{1}{m}|\operatorname{Re} z|}.$$

After applying (79.1) and (165.1) to control the distortion caused by translating by the operator $e^{zS_m} \in \operatorname{GL}(V)$, we get

$$d_{\check{D}}\Bigl(e^{-zN}\Phi(z), e^{zS_m}\Psi_m(0)\Bigr) \leq C\left(\sum_\lambda \|P_\lambda\|_{\Phi(-1)}\right)^2 e^{-\frac{1-\rho}{m}|\operatorname{Re} z|}.$$

Since $e^{zS_m}\Psi_m(0) = \Psi_m(0)$, we conclude that

$$\Psi_m(0) = \lim_{|\operatorname{Re} z| \to \infty} e^{-zN}\Phi(z) = F_{\lim}$$

is indeed equal to the limiting Hodge filtration. ∎

190. Here is a small example that shows the difference between the filration $\Psi_S(0)$ and the limiting Hodge filtration $F_{\lim}$.

Example. Let $\alpha > 0$ be a real number. Using the notation from §74 Example, consider the polarized variation of Hodge structure $\Phi\colon \mathbb{H} \to D$ with

$$V^{1,0}_{\Phi(z)} = \mathbb{C}\big(1, e^{\alpha z}\big) \quad \text{and} \quad V^{0,1}_{\Phi(z)} = \mathbb{C}\big(e^{\alpha \bar{z}}, 1\big).$$

Here $N = 0$ and so $F^1_{\lim} = \mathbb{C}(1,0)$, which is the Hodge filtration of a polarized Hodge structure. On the other hand, if we use

$$S = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix},$$

then $\Psi^1_S(0) = \mathbb{C}(1,1)$, and this is *not* the Hodge filtration of a polarized Hodge structure.

7.6 Effective estimates for the rate of convergence

191. What is missing from the above proof of the nilpotent orbit theorem are good estimates for the rate of convergence of the untwisted period mapping. The purpose of this section is to obtain such estimates, by using the curvature properties of the Hodge metric and the maximum principle.

192. Let us first see what kind of estimates we can derive from the fact that the untwisted period mapping Ψ_S extends across the origin. Recall that the differential of the holomorphic mapping $\Psi_S(e^z)$ is equal to

$$e^{-z(S+N)} \theta_{\partial/\partial z} e^{z(S+N)} - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)}.$$

At the same time, for $\varepsilon > 0$ sufficiently small, we can certainly find a holomorphic mapping $g\colon \Delta_r \to \mathrm{GL}(V)$ with $g(0) = \mathrm{id}$ such that $\Psi_S(t) = g(t) \cdot \Psi_S(0)$. By the chain rule, the differential of $\Psi_S(e^z)$ is therefore also equal to

$$e^z \cdot g'(e^z) g(e^z)^{-1} \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)},$$

provided that $\operatorname{Re} z < \log \varepsilon$. Putting both things together, we get

$$\theta_{\partial/\partial z} - (S+N) \equiv e^z \cdot e^{z(S+N)} g'(e^z) g(e^z)^{-1} e^{-z(S+N)} \mod F^0 \operatorname{End}(V)_{\Phi(z)},$$

at least when $\operatorname{Re} z < \log \varepsilon$. Arguing as in the proof of §165 Lemma to estimate the effect of conjugating by $e^{z(S+N)}$, the Hodge norm of the operator on the right-hand side is bounded, for $|\operatorname{Re} z| \gg 0$, by a constant multiple of

$$|\operatorname{Re} z|^m e^{-\delta(T)|\operatorname{Re} z|},$$

where $m = m(N) + 2\sqrt{2r} C_0$; the exact value of the constant depends of course on $g(t)$ and hence on $\Psi_S(t)$. We remind the reader that $0 < \delta(T) \leq 1$ is the minimal distance between consecutive eigenvalues of T on the unit circle, divided by 2π. If we denote by

$$N = \sum_{k \in \mathbb{Z}} N^{k,-k} \quad \text{and} \quad S = \sum_{k \in \mathbb{Z}} S^{k,-k} \quad \text{and} \quad P_\lambda = \sum_{k \in \mathbb{Z}} P_\lambda^{k,-k}$$

the Hodge decompositions, then it follows that the quantity

$$\|\theta_{\partial/\partial z} - N^{-1,1} - S^{-1,1}\|^2_{\Phi(z)} + \sum_{k \leq -2} \|N^{k,-k} + S^{k,-k}\|^2_{\Phi(z)}$$

is bounded, for $|\operatorname{Re} z| \gg 0$, by a constant multiple of $|\operatorname{Re} z|^{2m} e^{-2\delta(T)|\operatorname{Re} z|}$.

193. We can improve the estimates from above by changing the interval containing the eigenvalues of S. Recall that $T_s = e^{2\pi i S}$, and that the eigenvalues of S are supposed to lie in a half-open interval of length 1. Write the eigenvalues of T_s in the form $\lambda_j = e^{2\pi i \alpha_j}$, say with $0 \leq \alpha_1 < \cdots < \alpha_n < 1$. If we gradually slide the interval $[0,1)$ over to the right, we obtain the following n choices for the operator S, namely

$$S = \sum_{j=1}^{n} \alpha_j P_{\lambda_j} + (P_{\lambda_1} + \cdots + P_{\lambda_k}),$$

for $k = 0, 1, \dots, n-1$. If we apply the argument in the preceding paragraph to each choice of S, and then take a suitable linear combination of the resulting inequalities, we find that there are two constants $C > 0$ and $b \in \mathbb{N}$, such that

$$\begin{aligned} \|\theta_{\partial/\partial z} - N^{-1,1}\|^2_{\Phi(z)} + \sum_{k \leq -2} \|N^{k,-k}\|^2_{\Phi(z)} &\leq C |\operatorname{Re} z|^{2m} e^{-2\delta(T)|\operatorname{Re} z|} \\ \sum_{k \leq -1} \|P_\lambda^{k,-k}\|^2_{\Phi(z)} &\leq C |\operatorname{Re} z|^{2m} e^{-2\delta(T)|\operatorname{Re} z|} \end{aligned}$$

for $|\operatorname{Re} z| \gg 0$. Here $m = m(N) + 2\sqrt{2r} C_0$, but we cannot say anything about the value of the constant C, or about how big $|\operatorname{Re} z|$ has to be for the inequality to hold; these are going to depend, in some unspecified way, on the variation of Hodge structure E. The problem is that these estimates are not *effective*.

Note. In fact, the argument above gives a somewhat better estimate for P_λ. Instead of $\delta(T)$, the optimal exponent is the minimal distance, on the unit circle, from λ to the two immediately adjacent eigenvalues of T.

194. We can make the above estimates effective – and independent of the specific variation of Hodge structure! – by exploiting once more the curvature properties of the Hodge metric. The precise result we are going to prove is the following.

Theorem. *Given $x < 0$, there are constants $C > 0$ and $m \in \mathbb{N}$, such that*

$$\begin{aligned} \|\theta_{\partial/\partial z} - N^{-1,1}\|^2_{\Phi(z)} + \sum_{k \leq -2} \|N^{k,-k}\|^2_{\Phi(z)} &\leq C |\operatorname{Re} z|^{2b} e^{-2\delta(T)|\operatorname{Re} z|} \\ \sum_{k \leq -1} \|P_\lambda^{k,-k}\|^2_{\Phi(z)} &\leq C |\operatorname{Re} z|^{2b} e^{-2\delta(T)|\operatorname{Re} z|} \end{aligned}$$

for every $z \in \mathbb{H}$ with $\operatorname{Re} z \leq x$. The exact value of C only depends on x, on $r = \operatorname{rk} E$, and on the minimal polynomial of $T \in \operatorname{GL}(V)$; the exact value of b only depends on r.

195. We need another basic fact about metrics with negative curvature on holomorphic vector bundles. Let E be a smooth vector bundle with a hermitian metric h, defined on a domain $X \subseteq \mathbb{C}$, and suppose that E has the structure of a holomorphic vector bundle, given by a connection $d'' \colon A^0(X,E) \to A^{0,1}(X,E)$ of type $(0,1)$. As before, we write the Chern connection as $\delta' + d''$, where $\delta' \colon A^0(X,E) \to A^{1,0}(X,E)$, and denote by

$$\Theta = (\delta' + d'')^2 \in A^{1,1}\big(X, \operatorname{End}(E)\big)$$

the curvature operator of the metric.

Lemma. *Suppose that the hermitian metric h has semi-negative curvature, in the sense that for every $u \in A^0(X, E)$, one has*

$$h\big(\Theta_{\partial/\partial z \wedge \partial/\partial \overline{z}}\, u, u\big) \leq 0.$$

Then for every nontrivial holomorphic section $u \in A^0(X, E)$ with $d''u = 0$, the function $\log h(u, u)$ is subharmonic on X.

Proof. Since $\delta' + d''$ is a metric connection, we have $\bar{\partial} h(u, u) = h(u, \delta' u)$, hence

$$\partial\bar{\partial} h(u, u) = h(\delta' u, \delta' u) + h(u, d''\delta' u) = h(\delta' u, \delta' u) + h(u, \Theta u).$$

After evaluating this on $\partial/\partial z \wedge \partial/\partial\overline{z}$, we get

$$\frac{\partial^2}{\partial z \partial \overline{z}} h(u, u) = h\big(\delta'_{\partial/\partial z} u, \delta'_{\partial/\partial z} u\big) - h\big(\Theta_{\partial/\partial z \wedge \partial/\partial \overline{z}}\, u, u\big) \geq h\big(\delta'_{\partial/\partial z} u, \delta'_{\partial/\partial z} u\big).$$

At all points of X where $h(u, u) > 0$, we now get $\Delta \log h(u, u) \geq 0$ from the Cauchy-Schwarz inequality, by the same argument as in the proof of §53 Lemma. Since $\log h(u, u)$ is locally bounded from above, it follows that $\log h(u, u)$ is a well-defined subharmonic function with values in $[-\infty, \infty)$. □

196. We return to our usual setting where E is a polarized variation of Hodge structure on Δ^*. The result above has the following implication for $\operatorname{End}(E)$.

Proposition. *Let $u \in A^0\big(\Delta^*, \operatorname{End}(E)\big)$ be a smooth section with the property that $d''u \in A^0\big(\Delta^*, F^0 \operatorname{End}(E)\big)$. Write the Hodge decomposition of u as*

$$u = \sum_{k \in \mathbb{Z}} u^{k,-k},$$

where $u^{k,-k} \in A^0\big(\Delta^*, \operatorname{End}(E)^{k,-k}\big)$. Then for every $b \geq 4r\binom{r+1}{3}$, the function

$$\log\left((-\log t)^{-b} \sum_{k \leq -1} h\big(u^{k,-k}, u^{k,-k}\big)\right)$$

is subharmonic on Δ^*.

Proof. We apply the result from above to the quotient bundle $\operatorname{End}(E)/F^0 \operatorname{End}(E)$, with the hermitian metric h induced by the Hodge metric on $\operatorname{End}(E)$. According to §50 Proposition, the curvature tensor of this metric satisfies

$$h\big(\Theta_{\partial/\partial t \wedge \partial/\partial \bar{t}}\, u, u\big) \leq 2h\big(\theta_{\partial/\partial t} u, \theta_{\partial/\partial t} u\big) \leq r\binom{r+1}{3} \frac{1}{|t|^2 (\log|t|)^2} h(u, u),$$

using the improved bound for the Higgs field of $\operatorname{End}(E)$ in §59 Corollary. Just as in §158, we now consider the modified hermitian metric

$$h^\varphi = h \cdot e^{-\varphi} = h \cdot (-\log|t|)^{-b},$$

for $b \in \mathbb{N}$. Its curvature tensor satisfies

$$h^\varphi\big(\Theta^\varphi_{\partial/\partial t \wedge \partial/\partial \bar{t}}\, u, u\big) \leq \left(r\binom{r+1}{3} - \frac{b}{4}\right) \cdot \frac{1}{|t|^2 (\log|t|)^2} h^\varphi(u, u) \leq 0,$$

provided we choose $b \geq 4r\binom{r+1}{3}$. By assumption, u gives a holomorphic section of the quotient bundle $\operatorname{End}(E)/F^0 \operatorname{End}(E)$, and so the result now follows from §195 Lemma. □

197. We can now use the **maximum principle** to make the estimates in §192 effective. Since the operator $P_\lambda \in \operatorname{End}(V)$ commutes with T, its defines a flat section of the bundle $\operatorname{End}(E)$ on Δ^*. Define

$$f = \sum_{k \leq -1} h\bigl(P_\lambda^{k,-k}, P_\lambda^{k,-k}\bigr),$$

which is a smooth function on Δ^*. According to §196 Proposition, the function

$$\varphi = \log\bigl(f \cdot (-\log|t|)^{-b}\bigr)$$

is subharmonic on Δ^*, where $b = 4r\binom{r+1}{3}$. We also know from §192 that, in a small neighborhood of the origin, f is bounded from above by

$$C'(-\log|t|)^{2m(N)+4\sqrt{2r}C_0}|t|^{2\delta(T)}.$$

Since $b = 16rC_0^2 \geq 2m(N) + 4\sqrt{2r}C_0$, it follows that

$$\varphi - 2\delta(T)\log|t| \leq \log C'$$

for $|t|$ sufficiently small. In particular, this means that the subharmonic function $\varphi - 2\delta(T)\log|t|$ is bounded from above on every compact subset of Δ. Now the maximum principle for subharmonic functions implies that

$$\varphi - 2\delta(T)\log|t| \leq \max_{|t|=R}\Bigl(\varphi(t) - 2\delta(T)\log R\Bigr)$$

for every $0 < R < 1$. Since $P_\lambda \in \operatorname{End}(V)$ is a flat section, we can estimate the right-hand side with the help of §132 Proposition. After exponentiating, the conclusion is that

$$\sum_{k \leq -1} h\bigl(P_\lambda^{k,-k}, P_\lambda^{k,-k}\bigr) \leq C(-\log|t|)^{2b}|t|^{2\delta(T)} \quad \text{for } |t| \leq R,$$

where $b = 4r\binom{r+1}{3}$ and, assuming without loss of generality that $R > e^{-1}$,

$$C = R^{-2\delta(T)}e^{4\sqrt{2r}C_0\pi}(-\log R)^{-2(b+2\sqrt{2r}C_0)}\|P_\lambda\|^2_{\Phi(-1)}.$$

Since §65 Proposition contains an upper bound on $\|P_\lambda\|_{\Phi(-1)}$, with a constant that only depends on $r = \dim V$ and on the minimal polynomial of $T \in \mathrm{GL}(V)$, this gives us the first inequality in §194 Theorem, just with different notation.

198. To prove the other inequality, we apply the same argument to $t\theta_{\partial/\partial t} - N$, viewed as a holomorphic section of the quotient bundle $\operatorname{End}(E)/F^0\operatorname{End}(E)$ on Δ^*. Everything works out because the Hodge norm of $t\theta_{\partial/\partial t}$ is bounded from above by $C_0(-\log|t|)^{-1}$, according to §58 Corollary, and because $\|N\|_{\Phi(-1)}$ is bounded by virtue of §65 Proposition. This finishes the proof of the effective estimates in §194 Theorem.

7.7 Approximation by nilpotent orbits

199. In this section, we prove the second half of the nilpotent orbit theorem. Roughly speaking, the result is that the original period mapping $\Phi(z)$ can be approximated very well by the nilpotent orbit $\Phi_{\mathrm{nil}}(z) = e^{zN}F_{\mathrm{lim}}$. We fix an arbitrary base point $o \in D$, and for simplicity, we denote by $\|v\| = \|v\|_o$ the resulting norm on the vector space V. We use this norm to define the metric, and hence the distance function $d_{\check{D}}$, on the compact dual $\check{D}$.

Theorem. *There are constants $C > 0$, $x_0 < 0$, and $m \in \mathbb{N}$ such that*

$$\Phi_{\mathrm{nil}}(z) = e^{zN} F_{\lim} \in D \quad \textit{and} \quad d_D\big(\Phi(z), \Phi_{\mathrm{nil}}(z)\big) \leq C|\operatorname{Re} z|^m e^{-\delta(T)|\operatorname{Re} z|}$$

for every $z \in \mathbb{H}$ with $\operatorname{Re} z \leq x_0$. The constants C, x_0 only depend on the base point $o \in D$ and on the minimal polynomial of $T \in \mathrm{GL}(V)$; the integer m only depends on $r = \operatorname{rk} E$.

200. We first note that, due to the G-invariance of the metric on D, it is enough to prove the theorem under the additional assumption that $\Phi(-1) = o$. Here is why. Choose an element $g \in G$ such that $g \cdot \Phi(-1) = o$, and consider the modified period mapping $\Phi'(z) = g \cdot \Phi(z)$. The monodromy transformation changes to $T' = gTg^{-1}$, which has the same minimal polynomial; the limiting Hodge filtration changes to $F'_{\lim} = gF_{\lim}$. If the theorem is known for the modified period mapping Φ', then

$$g \cdot e^{zN} F_{\lim} = e^{zN'} F'_{\lim} \in D \quad \text{for } \operatorname{Re} z \leq x_0,$$

with a constant x_0 that depends on $o \in D$ and on the minimal polynomial of T'; but then clearly $e^{zN} F_{\lim} \in D$ in the same range. In the same way, the distance estimate for Φ' implies that for Φ itself.

201. We assume from now on that $\Phi(-1) = o$. The following lemma tells us how close a point of $\check{D}$ has to be to $\Phi(z)$ in order to belong to D.

Lemma. *There is a constant $\varepsilon > 0$ such that*

$$d_{\check{D}}\big(p, \Phi(z)\big) \leq \varepsilon \cdot |\operatorname{Re} z|^{-4C_0}$$

implies both that $p \in D$ and that

$$d_D\big(p, \Phi(z)\big) \leq 2n e^{4C_0\pi} |\operatorname{Re} z|^{4C_0} \cdot d_{\check{D}}\big(p, \Phi(z)\big).$$

The exact value of ε only depends on the choice of base point $o \in D$.

Proof. Since D is open in $\check{D}$, there is a constant $\delta > 0$ such that

$$\big\{\, p \in \check{D} \;\big|\; d_{\check{D}}(p, o) < \delta \,\big\} \subseteq D;$$

because d_D and $d_{\check{D}}$ are continuous, we can arrange moreover that

$$d_D(p, o) \leq 2 d_{\check{D}}(p, o)$$

on the open ball in question. Since we are using the Hodge norm at the base point $o \in D$ to define the metric on $\check{D}$, this second condition means that δ depends on the choice of $o \in D$. Fix a point $z \in \mathbb{H}$ with $\operatorname{Re} z \leq -1$, and choose $g \in G$ such that $\Phi(z) = g \cdot \Phi(-1)$. For any $v \in V$, we have

$$\|v\|_{\Phi(z)} = \|v\|_{g \cdot \Phi(-1)} = \|g^{-1} v\|_{\Phi(-1)}.$$

According to §132 Proposition,

$$e^{-2C_0\pi} |\operatorname{Re} z|^{-2C_0} \|v\|_{\Phi(-1)} \leq \|v\|_{\Phi(z)} \leq e^{2C_0\pi} |\operatorname{Re} z|^{2C_0} \|v\|_{\Phi(-1)},$$

where $C_0 = \frac{1}{2}\sqrt{\binom{r+1}{3}}$ and $r = \operatorname{rk} E$. Putting both things together, we get an upper bound on the operator norms of g and g^{-1}, and therefore

$$\max\big(\|g\|_{\Phi(-1)}, \|g^{-1}\|_{\Phi(-1)}\big) \leq \sqrt{r} e^{2C_0\pi} |\operatorname{Re} z|^{2C_0}$$

This gives us a bound on the operator norm of $\operatorname{Ad} g \colon \operatorname{End}(V) \to \operatorname{End}(V)$; indeed,

$$\|gAg^{-1}\|_{\Phi(-1)} \leq \|g\|_{\Phi(-1)} \|A\|_{\Phi(-1)} \|g^{-1}\|_{\Phi(-1)} \leq re^{4C_0\pi} |\operatorname{Re} z|^{4C_0} \|A\|_{\Phi(-1)}.$$

According to the discussion in §79, for any $p \in \check{D}$, we have

$$d_{\check{D}}\big(g^{-1}p, o\big) \leq re^{4C_0\pi} |\operatorname{Re} z|^{4C_0} \cdot d_{\check{D}}\big(p, \Phi(z)\big)$$

Define $\varepsilon = e^{-4C_0\pi} \cdot \delta/r$. As long as $d_{\check{D}}\big(p, \Phi(z)\big) < \varepsilon \cdot |\operatorname{Re} z|^{-4C_0}$, we can conclude from this that $g^{-1}p \in D$ and, therefore, $p \in D$; we also get the distance estimate

$$d_D\big(p, \Phi(z)\big) = d_D\big(g^{-1}p, o\big) \leq 2d_{\check{D}}\big(g^{-1}p, o\big) \leq 2re^{4C_0\pi} |\operatorname{Re} z|^{4C_0} \cdot d_{\check{D}}\big(p, \Phi(z)\big).$$

This is what we wanted to show. □

202. Fix a point $z \in \mathbb{H}$, and consider the curve

$$[0, \infty) \to \check{D}, \quad x \mapsto e^{xN} \Phi(z - x).$$

At $x = 0$, this starts at the point $\Phi(z) \in D$; as $x \to \infty$, it converges to the point

$$\lim_{x \to \infty} e^{xN} \Phi(z - x) = e^{zN} F_{\lim} \in \check{D}.$$

The derivative at a given point $x > 0$ is easily seen to be

$$N - e^{xN} \theta_{\partial/\partial z}(z - x) e^{-xN} \mod F^0 \operatorname{End}(V)_{e^{xN}\Phi(z-x)}.$$

We know from the effective estimates in §194 Theorem that

$$\|\theta_{\partial/\partial z} - N^{-1,1}\|^2_{\Phi(z-x)} + \sum_{k \leq -2} \|N^{k,-k}\|^2_{\Phi(z-x)} \leq C^2 \big(|\operatorname{Re} z| + x\big)^{2b} e^{-2\delta(|\operatorname{Re} z| + x)}.$$

where $\delta = \delta(T)$, and where the value of C depends on $r = \operatorname{rk} E$ and on the minimal polynomial of $T \in \mathrm{GL}(V)$. Arguing as in the proof of §165 Lemma, it follows that the length of the derivative, measured using the metric on $\check{D}$, is bounded from above by

$$C' \big(|\operatorname{Re} z| + x\big)^{b + 2\sqrt{2r} C_0} e^{-\delta(|\operatorname{Re} z| + x)} \sum_{k=0}^{m(N)} \frac{(2x)^k}{k!} \|N\|^k_{\Phi(-1)}.$$

After integrating this expression over the interval $[0, \infty)$, we get a bound for the distance between $\Phi(z)$ and $e^{zN} F_{\lim}$ that looks like

$$d_{\check{D}}\big(\Phi(z), e^{zN} F_{\lim}\big) \leq C'' |\operatorname{Re} z|^{b + 2\sqrt{2r} C_0} e^{-\delta|\operatorname{Re} z|},$$

with a constant $C'' > 0$ whose exact value is somewhat complicated, but depends only on the two integers $m(N)$ and $r = \operatorname{rk} E$ and on the minimal polynomial of $T \in \mathrm{GL}(V)$.

203. Now let us choose $x_0 < 0$ in such a way that $\operatorname{Re} z \leq x_0$ implies

$$C'' |\operatorname{Re} z|^{b + 2\sqrt{2r} C_0} e^{-\delta|\operatorname{Re} z|} \leq \varepsilon \cdot |\operatorname{Re} z|^{-4C_0},$$

where ε is the constant from §201 Lemma. We can then conclude that $e^{zN} F_{\lim} \in D$; we also get the distance estimate

$$d_D\big(\Phi(z), e^{zN} F_{\lim}\big) \leq 2re^{4C_0\pi} |\operatorname{Re} z|^{4C_0} \cdot C'' |\operatorname{Re} z|^{b + 2\sqrt{2r} C_0} e^{-\delta|\operatorname{Re} z|}.$$

Since all the constants on the right-hand side have the correct dependence on parameters, this completes the proof of the second half of the nilpotent orbit theorem.

204. We close this chapter with two remarks about the nilpotent orbit theorem:

1. The nilpotent orbit theorem guarantees that $e^{zN} F_{\lim} \in D$ for $\operatorname{Re} z \leq x_0$, where the constant $x_0 < 0$ only depends on $\operatorname{rk} E$ and on the minimal polynomial of T. One may wonder whether this actually holds for every $z \in \mathbb{H}$. We do not know the answer to this question, and we were not able to find any relevant examples in the literature.

2. One can easily prove a variant of the nilpotent orbit theorem for the filtration $\Psi_S(0)$. The statement is that there are constants $C > 0$ and $m \in \mathbb{N}$ such that

$$e^{z(S+N)} \Psi_S(0) \in D \quad \text{and} \quad d_D\bigl(\Phi(z), e^{z(S+N)} \Psi_S(0)\bigr) \leq C |\operatorname{Re} z|^m e^{-\delta(T)|\operatorname{Re} z|},$$

provided that $|\operatorname{Re} z| \gg 0$. In fact, §183 Lemma gives us the distance estimate

$$d_{\check{D}}\bigl(e^{-|\operatorname{Re} z|\, S} \Psi_S(0), F_{\lim}\bigr) \leq C e^{-\delta(T)|\operatorname{Re} z|},$$

and so the desired result follows from §199 Theorem and its proof.

8 Convergence of the rescaled period mapping

205. In this chapter, we use the nilpotent orbit theorem and its consequences to show that the **rescaled period mapping**

$$\hat{\Phi}_{S,H} \colon \mathbb{H} \to D, \quad \hat{\Phi}_{S,H}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \Phi(z),$$

has a well-defined limit in D as $|\operatorname{Re} z| \to \infty$. Recall from §139 that the operator $H \in \operatorname{End}(V)$ is a splitting for the monodromy weight filtration $W_\bullet$.

Theorem. *Let $H \in \operatorname{End}(V)$ be as in §139 Proposition. Then the limit*

$$e^{-N} F_H = \lim_{\operatorname{Re} z \to -\infty} \hat{\Phi}_{S,H}(z) \in D$$

exists in the period domain. The resulting filtration $F_H \in \check{D}$ satisfies

$$T_s(F_H^\bullet) \subseteq F_H^\bullet, \quad H(F_H^\bullet) \subseteq F_H^\bullet, \quad N(F_H^\bullet) \subseteq F_H^{\bullet-1}.$$

We remind the reader that the two operators H and $i(S+N)$ both belong to the Lie algebra $\mathfrak{g}$ of the real group $G = \operatorname{Aut}(V, Q)$, which means that

$$e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \in G.$$

This is why the rescaled period mapping stays in D. Also note that the rescaled period mapping depends both on S and on H; we will see during the proof that the filtration $F_H \in \check{D}$ only depends on H, justifying the notation.

8.1 Proof of convergence in the compact dual

206. Let us first rewrite everything in terms of Ψ_S. We have

$$\begin{aligned} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \Phi(z) &= e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} e^{z(S+N)} \Psi_S(e^z) \\ &= e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z|(S+N)} \Psi_S(e^z). \end{aligned}$$

The relation $[H, N] = -2N$ implies that

$$e^{\frac{1}{2}\log x\, H} N e^{-\frac{1}{2}\log x\, H} = e^{-\log x} \cdot N = \frac{N}{x}, \tag{206.1}$$

and therefore gives us the useful identity

$$e^{\frac{1}{2}\log x\, H} e^{-xN} e^{-\frac{1}{2}\log x\, H} = e^{-N}.$$

Putting everything together, we arrive at

$$e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \Phi(z) = e^{-N} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z|\, S} \Psi_S(e^z), \tag{206.2}$$

due to the fact that $[S, N] = [S, H] = 0$ (by §139 Proposition).

207. We know from the nilpotent orbit theorem that $\Psi_S(e^z)$ converges to $\Psi_S(0)$ at a rate proportional to $|e^z| = e^{-\operatorname{Re} z}$. The effect of the two exponential factors is controlled by §183 Lemma. For the exponential factor $e^{\frac{1}{2}\log|\operatorname{Re} z|\, H}$, the filtration by increasing eigenvalues of H is exactly the monodromy weight filtration $W_\bullet$; moreover, the rate of convergence is $e^{-\frac{1}{2}\log|\operatorname{Re} z|} = |\operatorname{Re} z|^{-\frac{1}{2}}$, since the eigenvalues of H will generally be consecutive integers. For the other exponential factor $e^{-|\operatorname{Re} z|\, S} = e^{|\operatorname{Re} z|(-S)}$, the relevant filtration is by *decreasing* eigenvalues of S (because of the minus sign); the rate of convergence is $e^{-\delta|\operatorname{Re} z|}$, where $\delta > 0$ is the minimal distance among consecutive eigenvalues of S.

208. Let us now prove the convergence of the rescaled period mapping. Since $\Psi_S \colon \Delta \to \check{D}$ is holomorphic, there is a constant $C > 0$ such that

$$d_{\check{D}}\big(\Psi_S(e^z), \Psi_S(0)\big) \leq C e^{-|\operatorname{Re} z|} \quad \text{for } |\operatorname{Re} z| \gg 0.$$

According to §181 Lemma, the operator norm of $\operatorname{Ad} e^{-|\operatorname{Re} z|\, S}$ is bounded by a constant multiple of $e^{(\alpha_{\max}-\alpha_{\min})|\operatorname{Re} z|}$. From (79.1), we therefore get

$$d_{\check{D}}\Big(e^{-|\operatorname{Re} z|\, S}\Psi_S(e^z),\, e^{-|\operatorname{Re} z|\, S}\Psi_S(0)\Big) \leq C' e^{-\delta(T)|\operatorname{Re} z|},$$

using the fact that $(1 + \alpha_{\min}) - \alpha_{\max} \geq \delta(T)$ is greater or equal to the minimal distance between adjacent eigenvalues of T.

209. Recall that the limiting Hodge filtration $F_{\lim} \in \check{D}$ satisfies

$$F_{\lim} = \lim_{x \to \infty} e^{-xS} \Psi_S(0)$$

and that §183 Lemma gives us a distance estimate of the form

$$d_{\check{D}}\Big(e^{-|\operatorname{Re} z|\, S}\Psi_S(0), F_{\lim}\Big) \leq C'' e^{-\delta(T)|\operatorname{Re} z|}.$$

Because of the triangle inequality, we then get

$$d_{\check{D}}\Big(e^{-|\operatorname{Re} z|\, S}\Psi_S(e^z), F_{\lim}\Big) \leq (C' + C'') e^{-\delta(T)|\operatorname{Re} z|}. \tag{209.1}$$

210. Next, we have to analyze the effect of the second exponential factor $e^{\frac{1}{2}\log|\operatorname{Re} z|\, H}$. On the one hand, we have

$$d_{\check{D}}\Big(e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z|\, S}\Psi_S(t),\, e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} F_{\lim}\Big) \leq C'''|\operatorname{Re} z|^{2m(N)} e^{-\delta(T)|\operatorname{Re} z|},$$

due to the fact that the operator norm of $\operatorname{Ad} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H}$ is bounded by a constant multiple of $|\operatorname{Re} z|^{2m}$ by §181 Lemma; here $m(N)$ is the largest integer such that $N^m \neq 0$. On the other hand, the limit

$$F_H = \lim_{|\operatorname{Re} z|\to\infty} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} F_{\lim} \in \check{D}$$

exists by §183 Lemma. Putting everything together, we find that the limit

$$\lim_{|\operatorname{Re} z|\to\infty} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z|\, S}\Psi_S(e^z) = F_H$$

exists in $\check{D}$. It follows that the Hodge filtrations of the rescaled period mapping

$$\hat{\Phi}_{S,H}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)}\Phi(z) \in D$$

converge, in the compact dual $\check{D}$, to the filtration $e^{-N}F_H$.

211. The filtration F_H can be described concretely as follows. First, we take the filtration $\Psi_S(0)$ from the nilpotent orbit theorem and make it compatible with the eigenspace decomposition of S, by projecting to the subquotients of the filtration by *decreasing* eigenvalues of S. By construction,

$$T_s \cdot F_{\lim}^{\bullet} = F_{\lim}^{\bullet}.$$

Similarly, the filtration F_H is obtained by starting from $F_{\lim}$, and making it compatible with the eigenspace decomposition of H by projecting to the subquotients of the monodromy weight filtration $W_\bullet$ (which is the filtration by *increasing* eigenvalues of H). This gives us the two relations

$$T_s \cdot F_H^{\bullet} = F_H^{\bullet} \quad \text{and} \quad H \cdot F_H^{\bullet} = F_H^{\bullet}.$$

We already know from the nilpotent orbit theorem that $N \cdot F_{\lim}^{\bullet} \subseteq F_{\lim}^{\bullet-1}$. From this, we can easily deduce that

$$N \cdot F_H^{\bullet} \subseteq F_H^{\bullet-1}.$$

Indeed, the filtration $F_H \in \check{D}$ was defined in such a way that

$$F_H^p = \lim_{x\to\infty} e^{xH} F_{\lim}^p.$$

From (206.1), we have $e^{-xH} N e^{xH} = e^{2x} N$, which gives

$$N \cdot F_H^p = \lim_{x\to\infty} N e^{xH} F_{\lim}^p = \lim_{x\to\infty} e^{xH} N F_{\lim}^p \subseteq \lim_{x\to\infty} e^{xH} F_{\lim}^{p-1} = F_H^{p-1}.$$

This is what we wanted to prove.

8.2 Proof that the limit belongs to the period domain

212. It remains to argue that $e^{-N} F_H \in D$, and hence that

$$\hat{\Phi}_{S,H}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \Phi(z)$$

actually converges to a polarized Hodge structure on V. From the Hodge norm estimates, we know that the corresponding inner products

$$\langle v, w \rangle_{\hat{\Phi}_{S,H}(z)} = \left\langle e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v,\; e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} w \right\rangle_{\Phi(z)}$$

remain *bounded* as $|\operatorname{Re} z| \to \infty$, for every $v, w \in V$. Equivalently, if we fix a norm $\|-\|$ on the vector space V, we have an upper bound

$$\|v\|^2_{\hat{\Phi}_{S,H}(z)} = \left\| e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v \right\|^2_{\Phi(z)} \leq C\|v\|^2$$

for some constant $C > 0$. To prove the convergence, we need a lower bound. This can easily be obtained with the help of the following trick.

213. Fix a basis $v_1, \ldots, v_r \in V$, and consider the $r \times r$-matrix $M(z)$ with entries

$$\left\langle e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v_i,\; e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v_j \right\rangle^2_{\Phi(z)}$$

We need to bound the inverse matrix $M(z)^{-1}$, and since we know that all entries of $M(z)$ stay bounded as $|\operatorname{Re} z| \to \infty$, we only need to control the function $\det M(z)$. But the determinant can be computed in a different way. The wedge product

$$v_1 \wedge \cdots \wedge v_r \in \det V$$

is a multi-valued flat section of the variation of Hodge structure on $\det E$. Since $\operatorname{rk}(\det E) = 1$, we have, with the appropriate notation,

$$H(v_1 \wedge \cdots \wedge v_r) = N(v_1 \wedge \cdots \wedge v_r) = 0,$$
$$S(v_1 \wedge \cdots \wedge v_r) = \alpha \cdot v_1 \wedge \cdots \wedge v_r)$$

for some $\alpha \in \mathbb{R}$. Therefore the exponential factors act trivially and

$$\det M(z) = \|v_1 \wedge \cdots \wedge v_r\|^2_{\Phi(z)}.$$

But now the variation of Hodge structure on $\det E$ can only have a single Hodge type (p, q), with $p + q = kr$. Therefore the Hodge norm of $v_1 \wedge \cdots \wedge v_r$ is equal to

$$\|v_1 \wedge \cdots \wedge v_r\|^2_{\Phi(z)} = (-1)^q Q(v_1 \wedge \cdots \wedge v_r, v_1 \wedge \cdots \wedge v_r),$$

which is $(-1)^q$ times the determinant of the $r \times r$-matrix with entries $Q(v_i, v_j)$. Anyway, the conclusion is that $\det M(z)$ is a nonzero constant. This implies that the inverse matrix $M(z)^{-1}$ is also bounded, and hence that

$$\frac{1}{C}\|v\|^2 \leq \left\| e^{\frac{1}{2}(z-\bar{z})(S+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v \right\|^2_{\Phi(z)} \leq C\|v\|^2$$

for a suitable constant $C > 0$ and all $v \in V$.

214. Now we get the result that we want from the following lemma. Pretty much the same argument also appears in [Kas85, Lem 4.2.1].

Lemma. *Let $f\colon \mathbb{N} \to D$ be a sequence of points of D such that:*

1. *The limit $\lim_{m\to\infty} f(m)$ exists in $\check{D}$.*
2. *There is a constant $C > 0$ such that*

$$\frac{1}{C}\|v\|^2 \leq \|v\|^2_{f(m)} \leq C\|v\|^2$$

for all vectors $v \in V$.

Then $\lim_{m\to\infty} f(m) \in D$.

Proof. For each $m \in \mathbb{N}$, we have the Hodge decomposition

$$V = \bigoplus_{p+q=n} V^{p,q}_{f(m)}.$$

After passing to a subsequence, we can assume that each limit

$$W^{p,q} = \lim_{m\to\infty} V^{p,q}_{f(m)}$$

exists (by compactness of the Grassmannian). We have to prove that

$$V = \bigoplus_{p+q=n} W^{p,q},$$

and that this Hodge structure of weight n is polarized by the pairing Q. Since the different subspaces $W^{p,q}$ are obviously orthogonal to each other under Q, it suffices to show that $(-1)^q Q$ is positive definite on $W^{p,q}$. By hypothesis,

$$(-1)^q Q(v,v) = \|v\|^2_{f(m)} \geq \frac{1}{C}\|v\|^2$$

for all $v \in V^{p,q}_{f(m)}$. After passing to the limit, the same is then true for $v \in W^{p,q}$, which means that $(-1)^q Q$ is positive definite. It is easy to deduce from this that

$$W^{p,q} \cap W^{p',q'} = \{0\}$$

whenever $(p,q) \neq (p',q')$, and so we do get a polarized Hodge structure. □

215. We can use the effective estimates from the proof of the nilpotent orbit theorem to get a better bound for the distance between

$$\hat{\Phi}_{S,H}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S+N)}\Phi(z) \in D$$

and its limit $e^{-N}F_H \in D$. According to §199 Theorem, we have

$$d_D\big(\Phi(z), e^{zN}F_{\lim}\big) \leq C|\operatorname{Re} z|^m e^{-\delta(T)|\operatorname{Re} z|},$$

for every $z \in \mathbb{H}$ with $\operatorname{Re} z \leq x_0$, with constants $C > 0$ and $x_0 < 0$ that are basically independent of the period mapping in question. Since the exponential factors in the definition of $\hat{\Phi}_{S,H}(z)$ belong to the real Lie group G, it follows that

$$d_D\Big(\hat{\Phi}_{S,H}(z),\, e^{-N} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} F_{\mathrm{lim}}\Big) \leq C|\operatorname{Re} z|^m e^{-\delta(T)|\operatorname{Re} z|}.$$

Here we used (206.1) to simplify the second argument. Because H has integer eigenvalues, $e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} F_{\mathrm{lim}}$ converges to F_H at a rate proportional to $|\operatorname{Re} z|^{-\frac{1}{2}}$. This means that there is a constant $C' > 0$, whose value depends on F_{lim}, such that

$$d_D\big(\hat{\Phi}_{S,H}(z), e^{-N} F_H\big) \leq C'|\operatorname{Re} z|^{-\frac{1}{2}}.$$

Schmid's SL(2)-orbit theorem gives much more precise information about the convergence; we are going to prove a weak form of this result later, in Chapter 11.

9 Results about mixed Hodge structures

216. In this chapter, we review some basic facts about complex mixed Hodge structures. We also discuss Deligne's functorial splitting for the weight filtration; we are going to need it during the proof of the cheap SL(2)-orbit theorem in Chapter 11. Everything in this chapter is just (somewhat tedious) linear algebra, but the results are important for the study of degenerating variations of Hodge structure, especially in higher dimensions.

9.1 Complex mixed Hodge structures

217. Unlike in the real case, we need *three* filtrations to describe a mixed Hodge structure, because the Hodge decomposition in a complex Hodge structure is not determined by the Hodge filtration alone.

Definition. A **mixed Hodge structure** on a finite-dimensional complex vector space V consists of an increasing filtration $W_\bullet$ with $W_n = 0$ for $n \ll 0$ and $W_n = V$ for $n \gg 0$, and two decreasing filtrations $F^\bullet$ and $\overline{F}^\bullet$, such that each subquotient

$$\operatorname{gr}_n^W = W_n / W_{n-1}$$

has a Hodge structure of weight n, given by the two induced filtrations

$$F^\bullet \operatorname{gr}_n^W = (F^\bullet \cap W_n + W_{n-1})/W_{n-1}$$
$$\overline{F}^\bullet \operatorname{gr}_n^W = (\overline{F}^\bullet \cap W_n + W_{n-1})/W_{n-1}.$$

The filtration $W_\bullet$ is called the **weight filtration**.

218. To save space, we are going to use the shorthand notation

$$F^p W_n = F^p \cap W_n \quad \text{and} \quad \overline{F}^q W_n = \overline{F}^q \cap W_n$$

from now on. The (p,q)-subspace in the Hodge decomposition of gr_n^W is

$$F^p \operatorname{gr}_n^W \cap \overline{F}^q \operatorname{gr}_n^W = \frac{(F^p W_n + W_{n-1}) \cap (\overline{F}^q W_n + W_{n-1})}{W_{n-1}}.$$

In order to have a mixed Hodge structure on V, the direct sum of these subspaces (over $p+q=n$) must equal gr_n^W, which means concretely that

$$W_n = \sum_{p+q=n} (F^p W_n + W_{n-1}) \cap (\overline{F}^q W_n + W_{n-1})$$

and that, whenever $p+q>n$, one has

$$(F^p W_n + W_{n-1}) \cap (\overline{F}^q W_n + W_{n-1}) = W_{n-1}.$$

From this, one deduces, by induction on the length of the weight filtration, that

$$F^p W_n \cap \Big(\overline{F}^q W_n + \overline{F}^{q-1} W_{n-1} + \overline{F}^{q-2} W_{n-2} + \cdots\Big) = 0 \tag{218.1}$$

whenever $p+q>n$. Either way, this is a fairly complicated set of conditions.

Example. A mixed Hodge structure is called **real** if $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for an $\mathbb{R}$-vector space $V_{\mathbb{R}}$, and if $\overline{F} = \sigma(F)$ and $W = \sigma(W)$, where $\sigma \in \operatorname{End}_{\mathbb{R}}(V)$ is the conjugation operator $\sigma(v \otimes z) = v \otimes \bar{z}$. Since the weight filtration is defined over $\mathbb{R}$, each subquotient gr_n^W is a real Hodge structure of weight n.

219. Here are some general operations on mixed Hodge structures:

1. Let V be a mixed Hodge structure. For any $k \in \mathbb{Z}$, we obtain a new mixed Hodge structure $V(k)$ on the same underlying vector space by setting

   $$W_n V(k) = W_{n+2k} V, \quad F^p V(k) = F^{p-k} V, \quad \overline{F}^q V(k) = \overline{F}^{q-k} V$$

   for $n,p,q \in \mathbb{Z}$. This operation is called the **Tate twist**.

2. Let V be a mixed Hodge structure. The **dual** vector space $\operatorname{Hom}(V,\mathbb{C})$ inherits a mixed Hodge structure, with

   $$\begin{aligned} W_n \operatorname{Hom}(V,\mathbb{C}) &= \big\{\, f\colon V \to \mathbb{C} \;\big|\; f(W_{-n-1}) = 0 \,\big\}, \\ F^p \operatorname{Hom}(V,\mathbb{C}) &= \big\{\, f\colon V \to \mathbb{C} \;\big|\; f(F^{-p+1}) = 0 \,\big\}, \\ \overline{F}^q \operatorname{Hom}(V,\mathbb{C}) &= \big\{\, f\colon V \to \mathbb{C} \;\big|\; f(\overline{F}^{-q+1}) = 0 \,\big\}. \end{aligned}$$

   It is easy to see that $\operatorname{gr}_n^W \operatorname{Hom}(V,\mathbb{C}) \cong \operatorname{Hom}\big(\operatorname{gr}_{-n}^W V, \mathbb{C}\big)$ are isomorphic Hodge structures of weight n.

3. Let V_1 and V_2 be mixed Hodge structures. Their **tensor product** $V_1 \otimes V_2$ is again a mixed Hodge structure, with

   $$\begin{aligned} W_n(V_1 \otimes V_2) &= \sum_{n'+n''=n} W_{n'} V_1 \otimes W_{n''} V_2, \\ F^p(V_1 \otimes V_2) &= \sum_{p'+p''=p} F^{p'} V_1 \otimes F^{p''} V_2, \\ \overline{F}^q(V_1 \otimes V_2) &= \sum_{q'+q''=q} \overline{F}^{q'} V_1 \otimes \overline{F}^{q''} V_2. \end{aligned}$$

   As expected, one has an isomorphism of Hodge structures

   $$\operatorname{gr}_n^W (V_1 \otimes V_2) \cong \bigoplus_{n'+n''=n} \operatorname{gr}_{n'}^W V_1 \otimes \operatorname{gr}_{n''}^W V_2.$$

4. An important special case is the **endomorphism algebra** $\operatorname{End}(V)$ of the vector space V underlying a mixed Hodge structure. It has a mixed Hodge structure with

$$\begin{aligned} W_n \operatorname{End}(V) &= \{\, f\colon V \to V \mid f(W_\bullet) \subseteq W_{\bullet+n} \,\}, \\ F^p \operatorname{End}(V) &= \{\, f\colon V \to V \mid f(F^\bullet) \subseteq F^{\bullet+p} \,\}, \\ \overline{F}^q \operatorname{End}(V) &= \{\, f\colon V \to V \mid f(\overline{F}^\bullet) \subseteq \overline{F}^{\bullet+q} \,\}. \end{aligned}$$

With these definitions, the multiplication map $\operatorname{End}(V) \otimes \operatorname{End}(V) \to \operatorname{End}(V)$ becomes a morphism of mixed Hodge structures.

9.2 Deligne's splitting construction

220. A mixed Hodge structure is called **split** if it is a direct sum of Hodge structures of different weights, with the obvious weight filtration. For example, any $\mathfrak{sl}_2$-Hodge structure is a split mixed Hodge structure. Being split is equivalent to having a decomposition

$$V = \bigoplus_{i,j \in \mathbb{Z}} V^{i,j}$$

with the property that

$$W_n = \bigoplus_{i+j \leq n} V^{i,j}, \quad F^p = \bigoplus_{i \geq p, j} V^{i,j}, \quad \overline{F}^q = \bigoplus_{j \geq q, i} V^{i,j}.$$

If V is a mixed Hodge structure, then the associated graded

$$\operatorname{gr}_\bullet^W = \bigoplus_{n \in \mathbb{Z}} \operatorname{gr}_n^W$$

becomes in a natural way a split mixed Hodge structure.

221. A general mixed Hodge structure is of course not split. Nevertheless, Deligne [Del71, Lem. 1.2.11] showed that there is a functorial decomposition

$$V = \bigoplus_{i,j \in \mathbb{Z}} I^{i,j}$$

into subspaces, with the property that

$$W_n = \bigoplus_{i+j \leq n} I^{i,j} \qquad \text{and} \qquad F^p = \bigoplus_{i \geq p, j} I^{i,j}, \tag{221.1}$$

and such that $I^{i,j}$ maps isomorphically to the (i,j)-subspace in the Hodge decomposition of $\operatorname{gr}_{i+j}^W$, under the projection from W_{i+j} to $\operatorname{gr}_{i+j}^W$. The subspaces in question are defined by the formula

$$I^{i,j} = F^i \cap W_{i+j} \cap \left(\overline{F}^j \cap W_{i+j} + \overline{F}^{j-1} \cap W_{i+j-2} + \overline{F}^{j-2} \cap W_{i+j-3} + \cdots \right). \tag{221.2}$$

The decomposition is functorial in the following sense. Suppose that $f\colon V_1 \to V_2$ is a morphism of mixed Hodge structures, which means that

$$f(W_n V_1) \subseteq W_n V_2, \quad f(F^p V_1) \subseteq F^p V_2, \quad f(\overline{F}^q V_1) \subseteq \overline{F}^q V_2$$

for all $p, q, n \in \mathbb{Z}$. Then with the obvious notation, we have $f(I_1^{i,j}) \subseteq I_2^{i,j}$ for all $i, j \in \mathbb{Z}$. This is an immediate consequence of the formula for $I^{i,j}$.

Example. When the mixed Hodge structure is split, one has $I^{i,j} = F^i \cap W_{i+j} \cap \overline{F}^j$ and

$$\overline{F}^q = \bigoplus_{j \geq q, i} I^{i,j}.$$

In the case of a non-split mixed Hodge structure, there is no simple formula for $\overline{F}$ in terms of the subspaces $I^{i,j}$.

222. Let us try to understand Deligne's construction from a slightly different point of view. To get a decomposition of V, all we need is a **splitting** $H \in \operatorname{End}(V)$ for the weight filtration; by this we mean that H is semisimple with integer eigenvalues, and that $W_n = E_n(H) \oplus W_{n-1}$ for all $n \in \mathbb{Z}$. Such a splitting induces an isomorphism

$$V = \bigoplus_{n \in \mathbb{Z}} E_n(H) \cong \bigoplus_{n \in \mathbb{Z}} \operatorname{gr}_n^W,$$

and the desired decomposition of V is then simply the image of the Hodge decomposition in the direct sum of Hodge structures on the right-hand side.

223. Any splitting preserves the weight filtration, and acts on gr_n^W as multiplication by n. This condition actually characterizes splittings, as the following lemma shows.

Lemma. *Let V be a mixed Hodge structure.*

(a) Suppose that $H \in W_0 \operatorname{End}(V)$ acts on each subquotient gr_n^W as multiplication by n. Then H is a splitting of the weight filtration.

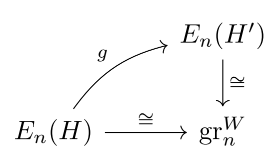
(b) If H and H' are two splittings of the weight filtration, then there is a unique element $g \in \operatorname{GL}(V)$ such that $g - \operatorname{id} \in W_{-1} \operatorname{End}(V)$ and $H' = gHg^{-1}$.

Proof. We first prove (a). For any $n \in \mathbb{Z}$, we have $(H - n\,\mathrm{id})(W_n) \subseteq W_{n-1}$ because of the assumptions on H. If $W_{n_0 - 1} = 0$ and $W_{n_1} = V$, we get

$$\prod_{n=n_0}^{n_1} (H - n\,\mathrm{id}) = 0,$$

and so H is semisimple with integer eigenvalues. It also follows that $W_n = E_n(H) \oplus W_{n-1}$, and so H is indeed a splitting.

Next, we prove (b). There is a unique element $g \in \operatorname{GL}(V)$ that maps each eigenspace $E_n(H)$ into the eigenspace $E_n(H')$ and makes all the following diagrams commute:



Clearly then $H' = gHg^{-1}$, and also $g - \operatorname{id} \in W_{-1} \operatorname{End}(V)$, as claimed. □

224. Now we return to Deligne's construction. Using the mixed Hodge structure on $\operatorname{End}(V)$, we can express the condition in (221.1) as

$$H \in F^0 W_0 \operatorname{End}(V) = F^0 \operatorname{End}(V) \cap W_0 \operatorname{End}(V).$$

Deligne's result then takes the following form.

Proposition. *Let V be a mixed Hodge structure. There is a unique splitting $H \in \operatorname{End}(V)$ for the weight filtration with the property that*

$$H \in F^0 W_0 \operatorname{End}(V) \cap \Big(\overline{F}^0 W_0 \operatorname{End}(V) + \overline{F}^{-1} W_{-2} \operatorname{End}(V) + \overline{F}^{-2} W_{-3} \operatorname{End}(V) + \cdots \Big).$$

Moreover, if $v \in F^p \cap W_{p+q} \cap \overline{F}^q$, then one has $Hv = (p+q)v$.

Proof. We first prove existence. Consider the endomorphism h of

$$\operatorname{gr}_\bullet^W = \bigoplus_{n \in \mathbb{Z}} \operatorname{gr}_n^W$$

that acts as multiplication by n on the summand gr_n^W. In terms of the mixed Hodge structure on $\operatorname{End}(V)$, this endomorphism is an element of $\operatorname{gr}_0^W \operatorname{End}(V)$, of Hodge type $(0,0)$. We start by lifting h in an arbitrary way to an element $H \in F^0 W_0 \operatorname{End}(V)$; according to §223 Lemma, this is a splitting of the weight filtration. We are now going to adjust this initial choice.

We can also lift h to an element $\bar{H}_0 \in \overline{F}^0 W_0 \operatorname{End}(V)$. Then $H - \bar{H}_0 \in W_{-1} \operatorname{End}(V)$, and we consider its image in $\operatorname{gr}_{-1}^W \operatorname{End}(V)$, which has a Hodge structure of weight -1. As such,

$$\operatorname{gr}_{-1}^W \operatorname{End}(V) = F^0 \operatorname{gr}_{-1}^W \operatorname{End}(V) \oplus \overline{F}^0 \operatorname{gr}_{-1}^W \operatorname{End}(V),$$

and so we can find $H_{-1} \in F^0 W_{-1} \operatorname{End}(V)$ and $\bar{H}_{-1} \in \overline{F}^0 W_{-1} \operatorname{End}(V)$ such that $H - \bar{H}_0$ and $H_{-1} + \bar{H}_{-1}$ have the same image in $\operatorname{gr}_{-1}^W \operatorname{End}(V)$. If we replace H by $H - H_{-1}$, and $\bar{H}_0$ by $\bar{H}_0 + \bar{H}_{-1}$, we can arrange that $H - \bar{H}_0 \in W_{-2} \operatorname{End}(V)$. Next, we project into

$$\operatorname{gr}_{-2}^W \operatorname{End}(V) = F^0 \operatorname{gr}_{-2}^W \operatorname{End}(V) \oplus \overline{F}^{-1} \operatorname{gr}_{-2}^W \operatorname{End}(V).$$

As before, we can find $H_{-2} \in F^0 W_{-2} \operatorname{End}(V)$ and $\bar{H}_{-2} \in \overline{F}^{-1} W_{-2} \operatorname{End}(V)$ such that $H - \bar{H}_0$ and $H_{-2} + \bar{H}_{-2}$ have the same image in $\operatorname{gr}_{-2}^W \operatorname{End}(V)$; after replacing H by $H - \bar{H}_0$, we get $H - \bar{H}_0 - \bar{H}_{-2} \in W_{-3} \operatorname{End}(V)$. Continuing in this manner, we obtain

$$H = \bar{H}_0 + \bar{H}_{-2} + \bar{H}_{-3} + \cdots,$$

where $\bar{H}_k \in \overline{F}^{k+1} W_k \operatorname{End}(V)$ for $k \leq -2$. This is the desired splitting. It is clear from this argument that the splitting H is optimal among all splittings that preserve F, in the sense that the subspace of $\operatorname{End}(V)$ in the definition is as small as possible.

Uniqueness of H follows in the same way. If we have two splittings $H, H' \in \operatorname{End}(V)$ with the stated properties, then their difference $H - H'$ is an element of

$$F^0 W_{-1} \operatorname{End}(V) \cap \Big(\overline{F}^0 W_0 \operatorname{End}(V) + \overline{F}^{-1} W_{-2} \operatorname{End}(V) + \overline{F}^{-2} W_{-3} \operatorname{End}(V) + \cdots \Big).$$

But since $\operatorname{End}(V)$ is a mixed Hodge structure, this intersection is trivial by (218.1).

Finally, let us suppose that we have a vector $v \in F^p \cap \overline{F}^q \cap W_n$, where $n = p + q$. Then

$$Hv - nv \in F^p \cap W_{n-1} \cap \Big(\overline{F}^q W_{n-1} + \overline{F}^{q-1} W_{n-2} + \overline{F}^{q-2} W_{n-3} + \cdots \Big),$$

and as the intersection on the right-hand side is again trivial, it follows that $Hv = nv$. ∎

225. We are going to denote the splitting in §224 Proposition by the symbol $H(W, F, \overline{F})$ if we want to emphasize its dependence on the mixed Hodge structure.

226. It is now an easy matter to deduce Deligne's formula for the subspaces $I^{i,j}$ in (221.2). Let $n = p + q$, and suppose that we have a vector of Hodge type (p, q) in gr_n^W. We can lift it to $v \in F^p W_n$ and also to $\bar{v} \in \overline{F}^q W_n$. Let $P_n \colon V \to E_n(H)$ denote the projection to the n-eigenspace of H; since H is semisimple with integer eigenvalues, this equals

$$P_n = \prod_{k \neq n} \frac{1}{n-k}(H - k \operatorname{id}),$$

where the product runs over all eigenvalues of H different from n. As $v - \bar{v} \in W_{n-1}$, we have $P_n v = P_n \bar{v}$. Now $v \in F^p \cap W_n$, and therefore

$$P_n v \in F^p \cap W_n.$$

On the other hand, we have $\bar{v} \in \overline{F}^q \cap W_n$, and therefore

$$P_n \bar{v} \in \overline{F}^q \cap W_n + \overline{F}^{q-1} \cap W_{n-2} + \overline{F}^{q-2} \cap W_{n-3} + \cdots,$$

because of the defining properties of H in §224 Proposition. This proves that $P_n v \in I^{p,q}$. The conclusion is that, in the decomposition of V induced by the splitting H, each subspace is contained in some $I^{p,q}$.

227. Conversely, let us show that H acts as multiplication by $i + j$ on the subspace $I^{i,j}$. Set $n = i + j$. For any vector $v \in I^{i,j}$, we have $v \in F^i \cap W_n$, hence

$$Hv - nv \in F^i \cap W_{n-1}.$$

At the same time, the formula in (221.2) and the properties of H imply that

$$Hv - nv \in \overline{F}^j W_{n-1} + \overline{F}^{j-1} W_{n-2} + \overline{F}^{j-2} W_{n-3} + \cdots$$

Since the intersection between these two subspaces is trivial by (218.1), we get $Hv = nv$, as claimed. This is enough to conclude that, under the isomorphism

$$V = \bigoplus_{n \in \mathbb{Z}} E_n(H) \cong \bigoplus_{n \in \mathbb{Z}} \operatorname{gr}_n^W,$$

the subspace of type (i, j) in the Hodge decomposition on the right-hand side corresponds exactly to the subspace $I^{i,j}$.

228. Let us also check that Deligne's construction respects the basic operations on mixed Hodge structures; this point is not covered very well in the literature [Mor78, Prop. 1.9].

Proposition. *The splitting in §224 Proposition is functorial; it is also compatible with duals and tensor products.*

Proof. Let us first consider tensor products. Suppose that V_1 and V_2 are two mixed Hodge structures, and set $V = V_1 \otimes V_2$. Denote the three splittings coming from §224 Proposition by $H_1 \in \operatorname{End}(V_1)$, $H_2 \in \operatorname{End}(V_2)$, and $H \in \operatorname{End}(V)$. Then the claim is that $H = H_1 \otimes \operatorname{id} + \operatorname{id} \otimes H_2$. Now the operator $H_1 \otimes \operatorname{id} + \operatorname{id} \otimes H_2 \in \operatorname{End}(V)$ acts on

$$\operatorname{gr}_n^W V \cong \bigoplus_{n' + n'' = n} \operatorname{gr}_{n'}^W V_1 \otimes \operatorname{gr}_{n''}^W V_2$$

as multiplication by n; since it is also semisimple with integer eigenvalues, it is a splitting of the weight filtration on V. From the definition of the mixed Hodge structure on $V = V_1 \otimes V_2$, it is easy to see that both $H_1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes H_2$ belong to

$$F^0 W_0 \operatorname{End}(V) \cap \Big(\overline{F}^0 W_0 \operatorname{End}(V) + \overline{F}^{-1} W_{-2} \operatorname{End}(V) + \overline{F}^{-2} W_{-3} \operatorname{End}(V) + \cdots \Big).$$

The result we want now follows from the uniqueness statement in §224 Proposition.

The same kind of argument proves that if V is a mixed Hodge structure, and if $H \in \operatorname{End}(V)$ denotes the splitting in §224 Proposition, then the splitting of the mixed Hodge structure on $\operatorname{Hom}(V, \mathbb{C})$ is given by the formula $f \mapsto -f \circ H$.

Lastly, let us prove the functoriality of the splitting. Suppose that $f \colon V_1 \to V_2$ is a morphism of mixed Hodge structures. Let $H_1 \in \operatorname{End}(V_1)$ and $H_2 \in \operatorname{End}(V_2)$ denote the two splittings. By the above, the splitting of the mixed Hodge structure on $\operatorname{Hom}(V_1, V_2)$ is then given by $Hf = H_2 \circ f - f \circ H_1$. Since $f \in \operatorname{Hom}(V_1, V_2)$ is a morphism, it satisfies

$$f \in F^0 \operatorname{Hom}(V_1, V_2) \cap \overline{F}^0 \operatorname{Hom}(V_1, V_2) \cap W_0 \operatorname{Hom}(V_1, V_2).$$

By §224 Proposition, it follows that $Hf = 0$, which translates into $H_2 \circ f = f \circ H_1$. □

229. What about complex conjugation? If we swap the role of the two filtrations F and $\overline{F}$, we obtain another splitting $\bar{H} = H(W, \overline{F}, F)$, whose characteristic property is that

$$\bar{H} \in \overline{F}^0 W_0 \operatorname{End}(V) \cap \Big(F^0 W_0 \operatorname{End}(V) + F^{-1} W_{-2} \operatorname{End}(V) + F^{-2} W_{-3} \operatorname{End}(V) + \cdots \Big).$$

The corresponding subspaces in the decomposition of V are then

$$\bar{I}^{i,j} = \overline{F}^j \cap W_{i+j} \cap \big(F^i \cap W_{i+j} + F^{i-1} \cap W_{i+j-2} + F^{i-2} \cap W_{i+j-3} + \cdots \big).$$

We have $H = \bar{H}$ if and only if $H \in \overline{F}^0 \operatorname{End}(V)$ if and only if the mixed Hodge structure is split; we leave the proof of this assertion as an easy exercise.

230. One can also understand the result in §224 Proposition by comparing the two splittings H and $\bar{H}$. We have $\bar{H} - H \in W_{-2} \operatorname{End}(V)$, and therefore

$$\begin{aligned} \bar{H} - H \in & \big(F^{-1} W_{-2} \operatorname{End}(V) + F^{-2} W_{-3} \operatorname{End}(V) + \cdots \big) \cap \\ & \big(\overline{F}^{-1} W_{-2} \operatorname{End}(V) + \overline{F}^{-2} W_{-3} \operatorname{End}(V) + \cdots \big). \end{aligned}$$

This element functions as a sort of "extension class" for the mixed Hodge structure (because V is split if and only if $\bar{H} - H = 0$).

231. Given a mixed Hodge structure, let us consider the subspace

$$R(W_{-2}) = \big(F^{-1} W_{-2} + F^{-2} W_{-3} + \cdots \big) \cap \big(\overline{F}^{-1} W_{-2} + \overline{F}^{-2} W_{-3} + \cdots \big).$$

The next proposition describes its properties.

Proposition. *In any mixed Hodge structure, one has a decomposition*

$$W_{-1} = F^0 W_{-1} \oplus \overline{F}^0 W_{-1} \oplus R(W_{-2}).$$

The subspace $R(W_{-2})$ is a sub-mixed Hodge structure, and

$$R(W_{-2}) = \bigoplus_{i,j \le -1} I^{i,j} = \bigoplus_{i,j \le -1} \bar{I}^{i,j}.$$

Proof. The formula in (221.2) shows that $I^{i,j} \subseteq R(W_{-2})$ for $i, j \leq -1$; by symmetry, it follows that $\bar{I}^{i,j} \subseteq R(W_{-2})$ as well. Now remember that $I^{i,j}$ maps isomorphically to the subspace of type (i, j) in the Hodge decomposition of $\operatorname{gr}^W_{i+j}$. It follows readily that

$$W_n = F^0 W_n + \overline{F}^0 W_n + W_n \cap R(W_{-2})$$

for every $n \leq -1$. Indeed, the subspace on the right-hand side maps onto gr^W_n under the projection $W_n \to \operatorname{gr}^W_n$, because its image contains the subspace of type (i, j) in the Hodge decomposition as long as either $i \geq 0$, or $j \geq 0$, or $i, j \leq -1$. At the same time, it is easy to see that the three subspaces are linearly independent. Indeed, if we have $v + \bar{v} + w = 0$ with $v \in F^0 W_{-1}$, $\bar{v} \in F^0 W_{-1}$, and $w \in R(W_{-2})$, then

$$v = -(\bar{v} + w) \in F^0 W_{-1} \cap \left(\overline{F}^0 W_{-1} + \overline{F}^{-1} W_{-2} + \overline{F}^{-2} W_{-3} + \cdots\right),$$

and this intersection is trivial by (218.1). By symmetry, we get $v = \bar{v} = w = 0$. Since

$$W_{-1} = \bigoplus_{i+j \leq -1} I^{i,j} = F^0 W_{-1} \oplus \bigoplus_{\substack{i+j \leq -1 \\ i \leq -1}} I^{i,j} = \overline{F}^0 W_{-1} \oplus \bigoplus_{\substack{i+j \leq -1 \\ i \leq -1}} \bar{I}^{i,j},$$

this also proves the nice identity

$$R(W_{-2}) = \bigoplus_{i,j \leq -1} I^{i,j} = \bigoplus_{i,j \leq -1} \bar{I}^{i,j}.$$

Because of the formulas for F and W in (221.1), and a similar formula for $\overline{F}$ in terms of the subspaces $\bar{I}^{i,j}$, this is enough to conclude that $R(W_{-2})$ is a sub-mixed Hodge structure. □

232. With this notation, we have $\bar{H} - H \in R\big(W_{-2} \operatorname{End}(V)\big)$. According to §223 Lemma, there is a unique element $g \in \mathrm{GL}(V)$ such that

$$\bar{H} = gHg^{-1} \quad \text{and} \quad g - \mathrm{id} \in R\big(W_{-2} \operatorname{End}(V)\big). \tag{232.1}$$

This gives another way to think about the characteristic property of Deligne's splitting H.

233. In a Hodge structure of weight n, one has $V = F^0 \oplus \overline{F}^{n+1}$. It turns out that a mixed Hodge structure still determines a canonical complement for the subspace F^0.

Proposition. *In any mixed Hodge structure, one has*

$$V = F^0 \oplus \sum_{n \in \mathbb{Z}} \overline{F}^{n+1} W_n.$$

In terms of Deligne's decomposition, one has

$$\sum_{n \in \mathbb{Z}} \overline{F}^{n+1} W_n = \bigoplus_{i \leq -1, j} I^{i,j}.$$

Proof. From the formula for $I^{i,j}$ in (221.2), it is clear that $I^{i,j} \subseteq \sum_n \overline{F}^{n+1} W_n$ as long as $i \leq -1$. Since we already know from (221.1) that

$$V = F^0 \oplus \bigoplus_{i \leq -1, j} I^{i,j},$$

it is therefore enough to prove that $F^0 \cap \sum_n \overline{F}^{n+1} W_n = 0$. This is easily checked by projecting to the subquotients gr^W_n in descending order. □

9.3 Mixed Hodge structures and hermitian pairings

234. Let us briefly discuss what happens when the vector space V comes equipped with a nondegenerate hermitian pairing $Q\colon V \otimes \overline{V} \to \mathbb{C}$. The pairing induces an isomorphism

$$Q\colon V \to \operatorname{Hom}(\overline{V}, \mathbb{C}), \quad v \mapsto Q(v, -),$$

between V and the conjugate dual vector space. We make the assumption that

$$Q\colon V \to \operatorname{Hom}(\overline{V}, \mathbb{C})(-n)$$

is actually an isomorphism of mixed Hodge structures (for a certain integer $n \in \mathbb{Z}$). Since the role of the two filtrations F and $\overline{F}$ get swapped in the mixed Hodge structure on $\overline{V}$, this amounts concretely to the following two conditions:

1. We have $W_k = \left\{ v \in V \,\middle|\, Q(v, x) = 0 \text{ for all } x \in W_{2n-k-1} \right\}$.
2. We have $\overline{F}^q = \left\{ v \in V \,\middle|\, Q(v, x) = 0 \text{ for all } x \in F^{n-q+1} \right\}$.

They imply that, for every $k \in \mathbb{Z}$, the hermitian pairing Q induces an isomorphism of Hodge structures between gr_k^W and the conjugate dual of $\operatorname{gr}_{2n-k}^W$ (with a Tate twist by n).

235. Just as in §71, the presence of the pairing turns $\operatorname{End}(V)$ into a real mixed Hodge structure. The real structure is given by

$$\mathfrak{g} = \left\{ A \in \operatorname{End}(V) \,\middle|\, A^\dagger = -A \right\},$$

the Lie algebra of the real Lie group G. The "complex conjugate" of an endomorphism $A \in \operatorname{End}(V)$ is again $\sigma(A) = -A^\dagger$, where the dagger means the adjoint with respect to Q. The two conditions in the previous paragraph are saying that $\sigma\bigl(W_k \operatorname{End}(V)\bigr) = W_k \operatorname{End}(V)$ and that $\overline{F}^q \operatorname{End}(V) = \sigma\bigl(F^q \operatorname{End}(V)\bigr)$, and so $\operatorname{End}(V)$ is indeed a real mixed Hodge structure.

236. From §224 Proposition, we get a splitting $H \in \operatorname{End}(V)$ of the mixed Hodge structure. By swapping the role of the two filtrations F and $\overline{F}$, we get another splitting $\bar{H} \in \operatorname{End}(V)$.

Lemma. *We have $\bar{H} = 2n \operatorname{id} - H^\dagger$, where the dagger means the adjoint with respect to Q.*

Proof. Since the splitting in §224 Proposition is functorial, this follows from the fact that $Q\colon V \to \operatorname{Hom}(\overline{V}, \mathbb{C})(-n)$ is an isomorphism of mixed Hodge structures. □

237. By averaging the two splittings H and $\bar{H} = 2n \operatorname{id} - H^\dagger$, we can create a splitting that is "real", meaning compatible with the hermitian pairing Q.

Proposition. *The operator $H_\mathbb{R} = \frac{1}{2}(H - H^\dagger) \in \mathfrak{g}$ is the unique real splitting of the shifted weight filtration $W_{n+\bullet}$ with the property that*

$$H_\mathbb{R} - (H - n \operatorname{id}) \in R\bigl(W_{-2} \operatorname{End}(V)\bigr).$$

Proof. The operator $\frac{1}{2}(H + \bar{H}) - n \operatorname{id} = \frac{1}{2}(H - H^\dagger) \in \mathfrak{g}$ preserves the weight filtration, and acts on $\operatorname{gr}_{n+k}^W$ as multiplication by k; it is therefore a splitting of the shifted weight filtration $W_{n+\bullet}$. Recall from §231 Proposition that $\bar{H} - H \in R\bigl(W_{-2} \operatorname{End}(V)\bigr)$. The proof of uniqueness is the same as in §224 Proposition. □

10 The limiting mixed Hodge structure

238. The purpose of this chapter is show that the vector space V of multi-valued flat sections has a mixed Hodge structure on it; following Schmid, we shall call this the **limiting mixed Hodge structure**. Before stating the result, we briefly recall the definition of a (complex) mixed Hodge structure.

10.1 The limiting $\mathfrak{sl}_2$-Hodge structure

239. We are going to derive the existence of the limiting mixed Hodge structure from a more precise result. Recall that, after having chosen a splitting $H \in \operatorname{End}(V)$ for the monodromy weight filtration, we get a unique representation

$$\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}(V)$$

with the property that $\rho(\mathsf{H}) = H$ and $\rho(\mathsf{Y}) = -N$. It turns out that this representation is part of a polarized $\mathfrak{sl}_2$-Hodge structure. The "total" Hodge filtration of this $\mathfrak{sl}_2$-Hodge structure is obtained from the rescaled period mapping

$$\hat{\Phi}_{S,H} \colon \mathbb{H} \to D, \quad \hat{\Phi}_{S,H}(z) = e^{\frac{1}{2}\log|\operatorname{Re} z| H} e^{-\frac{1}{2}(z-\bar{z})(S+N)} \Phi(z).$$

In Chapter 8, we showed that $\hat{\Phi}_{S,H}$ converges to a well-defined limit

$$e^{-N} F_H = \lim_{|\operatorname{Re} z| \to \infty} \hat{\Phi}_{S,H}(z) \in D,$$

which is the Hodge filtration of a polarized Hodge structure of weight n on the vector space V (with polarization Q). The filtration F_H has the property that

$$H F_H^\bullet \subseteq F_H^\bullet, \qquad T_s F_H^\bullet \subseteq F_H^\bullet, \qquad N F_H^\bullet \subseteq F_H^{\bullet - 1}. \tag{239.1}$$

Since $e^{\mathsf{Y}} F_H$ is the Hodge filtration of a polarized Hodge structure of weight n, the criterion in §114 Theorem shows that F_H is the total Hodge filtration of a polarized $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the hermitian pairing Q. Moreover, by §115 Corollary, the semisimple part T_s is necessarily an endomorphism of this polarized $\mathfrak{sl}_2$-Hodge structure.

240. We observed earlier that the conjugate Hodge filtration is $e^{-\mathsf{Y}} \overline{F}_H$, where

$$\overline{F}_H^q = \left\{ v \in V \;\middle|\; Q(v,x) = 0 \text{ for all } x \in F^{n-q+1} \right\}.$$

It follows that the two "total" Hodge filtrations of the $\mathfrak{sl}_2$-Hodge structure are F_H and $\overline{F}_H$, and so the Hodge structure on the weight space V_k is given by

$$V_k^{i,j} = V_k \cap F_H^i \cap \overline{F}_H^j.$$

In particular, this says that $\mathsf{Y}(V_k^{i,j}) \subseteq V_{k-2}^{i-1,j-1}$ and $\mathsf{X}(V_k^{i,j}) \subseteq V_{k+2}^{i+1,j+1}$; especially the second inclusion is not at all obvious from the conditions in (239.1).

10.2 The limiting mixed Hodge structure

241. In terms of the limiting Hodge filtrations $F_{\lim}$ and $\overline{F}_{\lim}$ coming from the nilpotent orbit theorem, one can describe F_H and its conjugate as

$$F_H = \lim_{x \to \infty} e^{xH} F_{\lim} \quad \text{and} \quad \overline{F}_H = \lim_{x \to \infty} e^{xH} \overline{F}_{\lim}.$$

The effect of the limit is that, under the isomorphism $E_k(H) \cong W_k/W_{k-1}$, one has

$$F_H^p \cap E_k(H) \cong \big(F_{\lim}^p \cap W_k + W_{k-1}\big)/W_{k-1},$$

and similarly for the conjugate filtration $\overline{F}_H$. This means that the passage from $(F_{\lim}, \overline{F}_{\lim})$ to $(F_H, \overline{F}_H)$ is exactly the passage from a mixed Hodge structure to its associated graded.

Corollary. *The three filtrations $W_{\bullet-n}(N)$, $F_{\lim}$ and $\overline{F}_{\lim}$ determine a mixed Hodge structure on the vector space V. The associated graded of this mixed Hodge structure is a polarized $\mathfrak{sl}_2$-Hodge structure of weight n. The operators*

$$N\colon V \to V(-1) \quad \textit{and} \quad T_s\colon V \to V$$

are morphisms of mixed Hodge structures.

Proof. Set $W_k = W_k(N)$. The first assertion is clear because each weight space

$$V_k = E_k(H) \cong W_k/W_{k-1}$$

has a Hodge structure of weight $n+k$. The second assertion follows because we have $N(W_k) \subseteq W_{k-2}$ and $T_s(W_k) \subseteq W_k$ for all $k \in \mathbb{Z}$, and also $N(F_{\lim}^p) \subseteq F_{\lim}^{p-1}$ and $T_s(F_{\lim}^p) \subseteq F_{\lim}^p$ for all $p \in \mathbb{Z}$. □

10.3 A formula for the period mapping

242. It is sometimes useful to have a more explicit description of the period mapping. At least on a small neighborhood of the origin, we can write down a fairly concrete formula for $\Phi(z)$ with the help of the limiting mixed Hodge structure. For variations of Hodge structure with unipotent monodromy, this result can be found in [CK89, (2.5)].

243. The nilpotent orbit theorem (in §24 Theorem) tell us that

$$\Psi_S(e^z) = e^{-z(S+N)}\Phi(z)$$

converges to $\Psi_S(0) \in \check{D}$ as $|\operatorname{Re} z| \to \infty$. Since $\Psi_S(t)$ is holomorphic, we should therefore be able to express $\Psi_S(t)$ in terms of $\Psi_S(0)$, using functions that are holomorphic on a neighborhood of the origin. To do this, recall the formula for the holomorphic tangent space

$$T^{1,0}_{\Psi_S(0)}\check{D} \cong \operatorname{End}(V)/F^0 \operatorname{End}(V)_{\Psi_S(0)}.$$

We can use the limiting mixed Hodge structure to find a subalgebra $\mathfrak{q} \subseteq \operatorname{End}(V)$ such that

$$\operatorname{End}(V) = \mathfrak{q} \oplus F^0 \operatorname{End}(V)_{\Psi_S(0)}. \tag{243.1}$$

Recall that V has a mixed Hodge structure with Hodge filtration $F_{\lim}$ and weight filtration $W_{\bullet-n}$, and that the semisimple operator T_s is an endomorphism of this mixed Hodge structure. Consider the induced mixed Hodge structure on $\operatorname{End}(V)$. According to the general result in §233 Proposition, the subspace

$$\mathfrak{q} = \sum_{n\in\mathbb{Z}} \overline{F}_{\lim}^{n+1} W_n \operatorname{End}(V)$$

is a vector space complement of $F^0_{\lim} \operatorname{End}(V)$, and hence

$$\operatorname{End}(V) = \mathfrak{q} \oplus F^0_{\lim} \operatorname{End}(V). \tag{243.2}$$

It is also clearly a subalgebra, and we have $N \in \mathfrak{q}$ because $N \colon V \to V(-1)$ is a morphism of mixed Hodge structures. Moreover, since S is an endomorphism of the mixed Hodge structure, the operator $\operatorname{ad} S$ preserves $\mathfrak{q}$; this gives us a decomposition

$$\mathfrak{q} = \bigoplus_{\alpha \in \mathbb{R}} \mathfrak{q}_\alpha,$$

where $\mathfrak{q}_\alpha = \mathfrak{q} \cap E_\alpha(\operatorname{ad} S)$. This decomposition is compatible with the Lie algebra structure on $\operatorname{End}(V)$, in the sense that $[\mathfrak{q}_\alpha, \mathfrak{q}_\beta] \subseteq \mathfrak{q}_{\alpha+\beta}$ by the Jacobi identity.

244. Now (243.1) follows from (243.2) by a formal argument. Let $\alpha_1 > \cdots > \alpha_m$ be the distinct eigenvalues of $\operatorname{ad} S$ on $\operatorname{End}(V)$, in decreasing order. Setting $\lambda_k = e^{2\pi i \alpha_k}$, we have

$$\frac{E_{\alpha_1}(\operatorname{ad} S) \oplus \cdots \oplus E_{\alpha_{k-1}}(\operatorname{ad} S) \oplus E_{\alpha_k}(\operatorname{ad} S)}{E_{\alpha_1}(\operatorname{ad} S) \oplus \cdots \oplus E_{\alpha_{k-1}}(\operatorname{ad} S)} \cong E_{\lambda_k}(\operatorname{Ad} T_s),$$

and under this isomorphism, the filtration induced by $\Psi_S(0)$ on the left-hand side goes to the limiting Hodge filtration $F_{\lim}$. We can then derive (243.1) by induction on $k = 1, \dots, m$.

245. Since $T^{1,0}_{\Psi_S(0)} \check{D} \cong \mathfrak{q}$, the exponential mapping

$$\exp \colon \mathfrak{q} \to \check{D}, \quad A \mapsto e^A \cdot \Psi_S(0),$$

restricts to a biholomorphism between a small neighborhood of $0 \in \mathfrak{q}$ and a small neighborhood of the point $\Psi_S(0) \in \check{D}$. Consequently, there is a constant $\varepsilon > 0$, whose exact value depends on the period mapping under consideration, and a unique holomorphic mapping

$$\tilde{\Gamma} \colon \Delta_\varepsilon \to \mathfrak{q}$$

such that $\tilde{\Gamma}(0) = 0$ and $\Psi_S(t) = e^{\tilde{\Gamma}(t)} \cdot \Psi_S(0)$ for $|t| < \varepsilon$. As long as $\operatorname{Re} z < \log \varepsilon$, this gives us a (preliminary) formula for the period mapping:

$$\Phi(z) = e^{z(S+N)} e^{\tilde{\Gamma}(e^z)} \cdot \Psi_S(0) \tag{245.1}$$

Note that, unlike in [CK89, (2.5)], the relevant filtration here is *not* the limiting Hodge filtration $F_{\lim}$, but the filtration $\Psi_S(0)$ coming from the nilpotent orbit theorem.

246. Now we would like to derive from (245.1) a formula for $\Phi(z)$ that only involves the limiting Hodge filtration $F_{\lim}$. The key is the following lemma.

Lemma. *One has $\Psi_S(0) = e^A F_{\lim}$ for a unique element $A \in \mathfrak{q}$. Decomposing $A = \sum_\alpha A_\alpha$, with $A_\alpha \in \mathfrak{q}_\alpha$, one has $A_\alpha = 0$ for $\alpha \leq 0$.*

Proof. As before, the exponential mapping $A \mapsto e^A \cdot F_{\lim}$ gives a biholomorphism between a neighborhood of $0 \in \mathfrak{q}$ and a neighborhood of the point $F_{\lim} \in \check{D}$. Recall that $F_{\lim} = \lim_{x \to \infty} e^{-xS} \Psi_S(0)$. For $x \gg 0$, it follows that there is an element $B(x) \in \mathfrak{q}$ such that $e^{-xS} \Psi_S(0) = e^{B(x)} F_{\lim}$. Since S preserves the filtration $F_{\lim}$, we can rewrite this as

$$\Psi_S(0) = e^{xS} e^{B(x)} e^{-xS} \cdot F_{\lim} = e^A \cdot F_{\lim},$$

where $A = e^{x \operatorname{ad}(S)} B(x) \in \mathfrak{q}$ is necessarily constant. Now we decompose this constant endomorphism as $A = \sum_\alpha A_\alpha$, with $A_\alpha \in \mathfrak{q} \cap E_\alpha(\operatorname{ad} S)$. Then

$$e^{-xS} \Psi_S(0) = e^{-xS} e^A e^{xS} \cdot F_{\mathrm{lim}} = \exp\left(\sum_\alpha e^{-\alpha x} A_\alpha \right) \cdot F_{\mathrm{lim}}.$$

As $x \to \infty$, this can only converge to F_{lim} if $A_\alpha = 0$ for $\alpha \leq 0$. □

247. Putting the lemma and (245.1) together, we find that

$$\Phi(z) = e^{z(S+N)} e^{\tilde{\Gamma}(e^z)} e^A \cdot F_{\mathrm{lim}} = e^{z(S+N)} e^{\Gamma(e^z)} \cdot F_{\mathrm{lim}},$$

where $\Gamma \colon \Delta_\varepsilon \to \mathfrak{q}$ is holomorphic and satisfies $\Gamma(0) = A$. We can do even better by decomposing $\Gamma(t) = \sum_\alpha \Gamma_\alpha(t)$; each individual function $\Gamma_\alpha \colon \Delta_\varepsilon \to \mathfrak{q}_\alpha$ is holomorphic, and $\Gamma_\alpha(0) = 0$ for $\alpha \leq 0$. Since S preserves the filtration F_{lim}, we finally get

$$\Phi(z) = e^{zN} \cdot e^{zS} e^{\Gamma(e^z)} e^{-zS} \cdot F_{\mathrm{lim}} = e^{zN} \cdot \exp\left(\sum_\alpha e^{\alpha z} \, \Gamma_\alpha(e^z) \right) \cdot F_{\mathrm{lim}}.$$

To summarize, we have proved the following result.

Proposition. *There is a small positive number $\varepsilon > 0$, and a collection of holomorphic functions $\Gamma_\alpha \colon \Delta_\varepsilon \to \mathfrak{q}_\alpha$ with $\Gamma_\alpha(0) = 0$ for $\alpha \leq 0$, such that the formula*

$$\Phi(z) = e^{zN} \cdot \exp\left(\sum_\alpha e^{\alpha z} \, \Gamma_\alpha(e^z) \right) \cdot F_{\mathrm{lim}}$$

describes the period mapping on the half-plane $\operatorname{Re} z < \log \varepsilon$.

Note that each term in the sum vanishes as $|\operatorname{Re} z| \to \infty$. The rate at which this happens is controlled by the smallest positive eigenvalue of $\operatorname{ad} S$; but this is the smallest distance among consecutive eigenvalues of S, which we had earlier denoted by $\delta(T)$. This shows one more time that $e^{-zN}\Phi(z)$ approaches its limit F_{lim} at a rate proportional to $e^{-\delta(T)|\operatorname{Re} z|}$.

11 Asymptotic behavior of the Hodge metric and the SL(2)-orbit theorem

248. In this chapter, we study the rate of convergence of the rescaled period mapping, and prove a cheap version of Schmid's famous SL(2)-orbit theorem [Sch73, Thm. 5.13].

249. On the vector space V, we have the limiting Hodge filtration F_{lim}, as well as the conjugate filtration $\overline{F}_{\mathrm{lim}}$, defined by

$$\overline{F}^q_{\mathrm{lim}} = \left\{ v \in V \;\middle|\; Q(v, w) = 0 \text{ for all } w \in F^{n+1-q}_{\mathrm{lim}} \right\}.$$

These put a mixed Hodge structure on V, with weight filtration $W_{\bullet - n} = W_{\bullet - n}(N)$; moreover, $N \colon V \to V(-1)$ is a morphism of mixed Hodge structures. At the same time, V also has a polarized $\mathfrak{sl}_2$-Hodge structure of weight n, whose two Hodge filtrations are F_H and $\overline{F}_H$, where

$$\overline{F}^q_H = \left\{ v \in V \;\middle|\; Q(v, w) = 0 \text{ for all } w \in F^{n+1-q}_H \right\}.$$

250. We begin by comparing the two filtrations F_{lim} and F_H.

Lemma. *There is an element $h \in \mathrm{GL}(V)$ with $h - \mathrm{id} \in W_{-1} \operatorname{End}(V)$ and $hNh^{-1} = N$, such that $F_{\mathrm{lim}} = hF_H$.*

Proof. Let $H(W_{\bullet-n}, F_{\mathrm{lim}}, \overline{F}_{\mathrm{lim}})$ denote Deligne's splitting of the limiting mixed Hodge structure (as in §224 Proposition). The corresponding subspaces

$$I_{\mathrm{lim}}^{i,j} = F_{\mathrm{lim}}^{i} \cap W_{i+j-n} \cap \left(\overline{F}_{\mathrm{lim}}^{j} \cap W_{i+j-n} + \overline{F}_{\mathrm{lim}}^{j-1} \cap W_{i+j-2-n} + \overline{F}_{\mathrm{lim}}^{j-2} \cap W_{i+j-3-n} + \cdots \right)$$

decompose V into a direct sum, in such a way that

$$W_k = \bigoplus_{i+j \leq n+k} I_{\mathrm{lim}}^{i,j} \qquad \text{and} \qquad F_{\mathrm{lim}}^{p} = \bigoplus_{i \geq p, j} I_{\mathrm{lim}}^{i,j};$$

moreover, one has $N(I_{\mathrm{lim}}^{i,j}) \subseteq I_{\mathrm{lim}}^{i-1,j-1}$, due to the fact that $N \colon V \to V(-1)$ is a morphism of mixed Hodge structures. In terms of the splitting, $[H(W_{\bullet-n}, F_{\mathrm{lim}}, \overline{F}_{\mathrm{lim}}), N] = -2N$.

Recall that in an $\mathfrak{sl}_2$-Hodge structure of weight n, one has

$$V_k^{i,j} = F_H^i \cap V_k \cap \overline{F}_H^j,$$

whenever $i+j = n+k$. Under the projection from W_{i+j-n} to gr_{i+j-n}^W, the two subspaces $I_{\mathrm{lim}}^{i,j}$ and $V_{i+j-n}^{i,j}$ both map isomorphically to the (i,j)-subspace in the Hodge decomposition of gr_{i+j-n}^W, and so there is a unique isomorphism $h^{i,j}$ making the following diagram commute:

$$\begin{array}{ccc} & & V_{i+j-n}^{i,j} \\ & \swarrow^{h^{i,j}} & \downarrow \cong \\ I_{\mathrm{lim}}^{i,j} & \xrightarrow{\cong} & \left(\mathrm{gr}_{i+j-n}^W\right)^{i,j} \end{array}$$

Consequently, there is a unique automorphism $h \in \mathrm{GL}(V)$ such that $h\big|_{V_{i+j-n}^{i,j}} = h^{i,j}$ for all $i, j \in \mathbb{Z}$. It follows that

$$h(F_H^p) = \bigoplus_{i \geq p} h\left(V_{i+j-n}^{i,j}\right) = \bigoplus_{i \geq p} I_{\mathrm{lim}}^{i,j} = F_{\mathrm{lim}}^p.$$

By construction, h preserves the weight filtration W and acts as the identity on each gr_k^W; therefore $h - \mathrm{id} \in W_{-1} \operatorname{End}(V)$. Moreover, the diagram

$$\begin{array}{ccc} V_{i+j-n}^{i,j} & \xrightarrow{h^{i,j}} & I_{\mathrm{lim}}^{i,j} \\ \downarrow N & & \downarrow N \\ V_{i+j-2-n}^{i-1,j-1} & \xrightarrow{h^{i-1,j-1}} & I_{\mathrm{lim}}^{i-1,j-1} \end{array}$$

is commutative, and this implies that $hN = Nh$. In terms of the two splittings, we have

$$H(W_{\bullet-n}, \overline{F}, \overline{F}_{\mathrm{lim}}) = h\left(H + n\,\mathrm{id}\right)h^{-1},$$

which is consistent with §223 Lemma. ☐

251. Note that, in the lemma above, the splitting H was arbitrary, subject only to the conditions in §139 Proposition. If we pick a splitting that is adapted to the limiting mixed Hodge structure, we can get a much better result. Indeed, suppose we use instead the splitting $H_{\mathbb{R}}(W_{\bullet -n}, F_{\lim}, \overline{F}_{\lim}) \in \mathfrak{g}$ for the filtration $W_\bullet$, constructed in §237 Proposition, which we got by averaging the two splittings $H(W_{\bullet -n}, F_{\lim}, \overline{F}_{\lim})$ and $H(W_{\bullet -n}, \overline{F}_{\lim}, F_{\lim})$. This also has all the properties required by §139 Proposition. Then

$$H_{\mathbb{R}}(W_{\bullet -n}, F_{\lim}, \overline{F}_{\lim}) - \big(H(W_{\bullet -n}, F_{\lim}, \overline{F}_{\lim}) - n\,\mathrm{id}\big) \in R\big(W_{-2}\operatorname{End}(V)\big),$$

and so the element $h \in \mathrm{GL}(V)$ in §250 Lemma even satisfies $h - \mathrm{id} \in W_{-2}\operatorname{End}(V)$.

252. We return to our investigation of the two filtrations $F_{\lim}$ and F_H. Let us write the element $h \in \mathrm{GL}(V)$ as a finite sum

$$h = \mathrm{id} + h_{-1} + h_{-2} + \cdots,$$

where each $h_{-k} \in E_{-k}(\operatorname{ad} H) \cap \ker(\operatorname{ad} N)$. Recall from §210 that

$$F_H = \lim_{x\to\infty} e^{\frac{1}{2}\log x\, H} F_{\lim}.$$

We can analyze the convergence more precisely as follows. For $x > 0$, one has

$$e^{\frac{1}{2}\log x\, H} F_{\lim} = e^{\frac{1}{2}\log x\, H} h F_H = e^{\frac{1}{2}\log x\, H} h e^{-\frac{1}{2}\log x\, H} F_H,$$

due to the fact that H preserves the filtration F_H. Now

$$e^{\frac{1}{2}\log x\, H} h e^{-\frac{1}{2}\log x\, H} = \mathrm{id} + \sum_{k=1}^{\infty} x^{-k/2} h_{-k},$$

and since $[N, h_{-k}] = 0$, we finally obtain

$$e^{\frac{1}{2}\log x\, H} e^{-xN} F_{\lim} = e^{-N} e^{\frac{1}{2}\log x\, H} F_{\lim} = \left(\mathrm{id} + \sum_{k=1}^{\infty} x^{-k/2} h_{-k}\right) \cdot F_\sharp,$$

where $F_\sharp = e^{-N} F_H \in D$. Note that the filtration on the right-hand side belongs to D when $x \gg 0$. With the substitution $u = x^{-1/2}$, we can rewrite this as

$$\left(\mathrm{id} + \sum_{k=1}^{\infty} u^k h_{-k}\right) \cdot F_\sharp,$$

as long as u is sufficiently close to 0.

253. To understand the behavior of the Hodge metric, we have to express this in terms of the real group G. For simplicity, let us denote by

$$V = \bigoplus_{p+q=n} V^{p,q}$$

the Hodge structure with Hodge filtration $F_\sharp = e^{-N} F_H$, and by

$$\operatorname{End}(V) = \bigoplus_{j\in\mathbb{Z}} \operatorname{End}(V)^{j,-j}$$

the induced Hodge structure of weight 0 on $\operatorname{End}(V)$. Recall that this is an $\mathbb{R}$-Hodge structure; the Lie algebra $\mathfrak{g}$ of the real group G gives the real structure. Setting

$$\mathfrak{m} = \mathfrak{g} \cap \bigoplus_{j \neq 0} \operatorname{End}(V)^{j,-j}$$

we then have $\operatorname{End}(V) = \mathfrak{m} \oplus F^0 \operatorname{End}(V)$; compare (76.2). In concrete terms, this decomposition works as follows. For an endomorphism $A \in \operatorname{End}(V)$, denote by $A_j \in \operatorname{End}(V)^{j,-j}$ the components in the Hodge decomposition; then

$$\sum_{j<0} \bigl(A_j - (A_j)^\dagger\bigr) \in \mathfrak{m} \quad \text{and} \quad \sum_{j \geq 0} A_j + \sum_{j<0} (A_j)^\dagger \in F^0 \operatorname{End}(V)$$

254. Recall that the subspace $\mathfrak{m} \subseteq \operatorname{End}(V)$ maps isomorphically to the tangent space of D at the point $F_\sharp$. The exponential mapping

$$\operatorname{End}(V) \to D, \quad A \mapsto e^A \cdot F_\sharp,$$

is real analytic, and therefore restricts to a real-analytic diffeomorphism between a neighborhood of the origin in $\mathfrak{m}$ and a neighborhood of the point $F_\sharp \in D$, it follows that there is a small positive number $\varepsilon > 0$ and two unique real analytic functions

$$B \colon (-\varepsilon, \varepsilon) \to \mathfrak{m} \quad \text{and} \quad C \colon (-\varepsilon, \varepsilon) \to F^0 \operatorname{End}(V)$$

with the property that $B(0) = C(0) = 0$ and

$$\left(\operatorname{id} + \sum_{k=1}^{\infty} u^k h_{-k} \right) e^{C(u)} = e^{B(u)}.$$

Let us expand B and C into convergent power series of the form

$$B(u) = \sum_{k=1}^{\infty} u^k B_{-k} \quad \text{and} \quad C(u) = \sum_{k=1}^{\infty} u^k C_{-k},$$

where $B_{-k} \in \mathfrak{m}$ and $C_{-k} \in F^0 \operatorname{End}(V)$.

255. From the power series expansion of

$$e^{B(u)} = \operatorname{id} + uB_{-1} + u^2 \Bigl(B_{-2} + \frac{1}{2} B_{-1}^2 \Bigr) + \cdots,$$

we see that there are universal non-commutative polynomials P_{-k} in the variables $B_{-1}, B_{-2}, \dots,$ such that

$$e^{B(u)} = \operatorname{id} + \sum_{k=1}^{\infty} u^k P_{-k}(B_{-1}, \dots, B_{-k}).$$

The first few terms are easily computed to be

$$\begin{aligned} P_{-1}(B_{-1}) &= B_{-1}, \\ P_{-2}(B_{-1}, B_{-2}) &= B_{-2} + \frac{1}{2} B_{-1}^2 \\ P_{-3}(B_{-1}, B_{-2}, B_{-3}) &= B_{-3} + \frac{1}{2} B_{-2} B_{-1} + \frac{1}{2} B_{-1} B_{-2} + \frac{1}{6} B_{-1}^3. \end{aligned}$$

In general, the coefficients of $k! P_{-k}$ are positive integers; moreover, P_{-k} is homogeneous of degree k, if we declare that the degree of B_{-j} is j. It is easy to see that the variable B_{-k} always appears in P_{-k} with coefficient 1.

256. From the $\mathfrak{sl}_2$-Hodge structure on V, the vector space $\operatorname{End}(V)$ inherits an $\mathfrak{sl}_2$-Hodge structure of weight 0. In particular, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts on $\operatorname{End}(V)$, and therefore induces a decomposition

$$\operatorname{End}(V) \cong \bigoplus_{m\in\mathbb{N}} S_m \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}\bigl(S_m, \operatorname{End}(V)\bigr)^{\mathfrak{sl}_2(\mathbb{C})}.$$

The summand with S_m is called the ***S_m*-isotypical component**. According to the discussion in §104, the isotypical components are sub-Hodge structures of the Hodge structure on $\operatorname{End}(V)$. They are also stable under taking adjoints; therefore the decomposition

$$\operatorname{End}(V) = \mathfrak{m} \oplus F^0 \operatorname{End}(V)$$

respects the isotypical components.

257. Since each operator $h_{-k} \in E_{-k}(\operatorname{ad} H) \cap \ker(\operatorname{ad} N)$ is primitive, it obviously belongs to the S_k-isotypical component of the representation. This has the following consequence for the operators $B_{-k}, C_{-k} \in \operatorname{End}(V)$.

Proposition. *For each $k \geq 1$, both B_{-k} and C_{-k} belong to the sum of the S_j-isotypical components with $j = k, k-2, k-4, \dots$.*

Proof. Expanding the relation

$$\left(\mathrm{id} + \sum_{k=1}^{\infty} u^k P_{-k}(B_{-1}, \dots, B_{-k})\right) = \left(\mathrm{id} + \sum_{k=1}^{\infty} u^k h_{-k}\right)\left(\mathrm{id} + \sum_{k=1}^{\infty} u^k P_{-k}(C_{-1}, \dots, C_{-k})\right)$$

from above now gives us the following series of identities:

$$P_{-k}(B_{-1}, \dots, B_{-k}) = h_{-k} + P_{-k}(C_{-1}, \dots, C_{-k}) + \sum_{j=1}^{k-1} h_{-k+j} \cdot P_{-j}(C_{-1}, \dots, C_{-j})$$

For $k = 1$, this reads $B_{-1} = h_{-1} + C_{-1}$; here $B_{-1} \in \mathfrak{m}$ and $C_{-1} \in F^0 \operatorname{End}(V)$. Remembering that $\operatorname{End}(V) = \mathfrak{m} \oplus F^0 \operatorname{End}(V)$, we see again that this equation uniquely determines B_{-1} and C_{-1}. Since h_{-1} belongs to the S_1-isotypical component of the representation, and since this is a sub-Hodge structure, it follows that B_{-1} and C_{-1} also belong to the S_1-isotypical component.

We can now prove the assertion by induction on $k \geq 1$. Rearranging the identity from above, we obtain an identity of the form

$$B_{-k} - C_{-k} = P(B_{-1}, \dots, B_{-k+1}, C_{-1}, \dots, C_{-k+1}, h_{-1}, \dots, h_{-k})$$

for a certain non-commutative polynomial P, homogeneous of degree k. Now each term on the right-hand side is a product of operators of the form B_{-j}, C_{-j} or h_{-j}, and by induction, each such operator is in the direct sum of the S_{j-2i}-isotypical components for $i \geq 0$. Since

$$S_k \otimes S_\ell \cong \bigoplus_{i=0}^{\min(k,\ell)} S_{k+\ell-2i},$$

and since the degree of each term is $-k$, this gives the result. □

258. This has the following nice consequence for the nilpotent orbit $e^{zN} F_{\lim}$. The result is of course less precise than the SL(2)-orbit theorem [Sch73, Thm. 5.13], but in return, the proof is *much* easier, and we believe that the version below is actually sufficient for all practical purposes.

Theorem. *There is a constant $\varepsilon > 0$, whose value depends on $F_{\lim}$ and N, and a real-analytic function $B\colon (-\varepsilon, \varepsilon) \to \mathfrak{g}$, such that*

$$\Phi_{\mathrm{nil}}(x+iy) = e^{(x+iy)N} F_{\lim} = e^{iyN} e^{-\frac{1}{2}\log|x|\, H} e^{B(|x|^{-1/2})} \cdot e^{-N} F_H$$

as long as $|x| > 1/\varepsilon^2$. In the power series expansion

$$B(u) = \sum_{k=1}^{\infty} u^k B_{-k},$$

each coefficient $B_{-k} \in \mathfrak{g}$ belongs to the direct sum of the S_m-isotypical components of the $\mathfrak{sl}_2(\mathbb{C})$-representation with $m = k, k-2, k-4, \dots$.

The point is that all the operators on the right-hand side of this formula belong to the real group G. This makes it possible to describe the asymptotic behavior of the Hodge metric or the Weil operator in any nilpotent orbit.

Note. Instead of an arbitrary splitting $H \in \operatorname{End}(V)$, we can use the splitting

$$H_{\mathbb{R}}(W_{\bullet - n}, F_{\lim}, \overline{F}_{\lim}) \in \mathfrak{g}$$

given by §237 Proposition. Then $h - \mathrm{id} \in W_{-2} \operatorname{End}(V)$, and an analysis of the proof shows that $B_{-1} = 0$. With this choice, the statement of §258 Theorem becomes closer to the full SL(2)-orbit theorem in [Sch73, Thm. 5.13].

259. We can use §258 Theorem to get more precise information about the asymptotic behavior of the Hodge metric. The Hodge norm estimates tell us the exact rate of growth of $\|v\|^2_{\Phi(z)}$ for any $v \in V$, but so far, we have not said anything about the behavior of the inner products $\langle v, w\rangle_{\Phi(z)}$ when $v, w \in V$ are two different multi-valued flat sections.

260. Let us denote by $\langle v, w\rangle$ the inner product coming from the polarized Hodge structure $F_\sharp = e^{-N} F_H$. Set $z = x + iy$. According to §199 Theorem, we have

$$d_D\big(\Phi(x+iy), e^{(x+iy)N} F_{\lim}\big) \leq C|x|^m e^{-\delta(T)|x|}$$

as long as $x \leq x_0$, with constants $C > 0$ and $x_0 < 0$ that are basically independent of the period mapping in question. Since the exponential factors in the definition of $\hat{\Phi}_{S,H}$ belong to the real Lie group G, it follows that

$$d_D\Big(\hat{\Phi}_{S,H}(x+iy),\, e^{\frac{1}{2}\log|x|\, H} e^{xN} F_{\lim}\Big) \leq C|x|^m e^{-\delta(T)|x|}.$$

We now restrict our attention to $|x| > 1/\varepsilon^2$, and set $u = |x|^{-\frac{1}{2}}$. Then

$$e^{\frac{1}{2}\log|x|\, H} e^{xN} F_{\lim} = e^{B(u)} F_\sharp,$$

and so our distance estimate becomes

$$d_D\Big(\hat{\Phi}_{S,H}(x+iy),\, e^{B(u)} F_\sharp\Big) \leq C|x|^m e^{-\delta(T)|x|}.$$

§75 Lemma allows us to conclude that, as long as $|x|$ is sufficiently large,

$$\left|\langle v, w\rangle_{\hat{\Phi}_{S,H}(x+iy)} - \langle v, w\rangle_{e^{B(u)}F_\sharp}\right| \leq C' \|v\|_{e^{B(u)}F_\sharp} \|w\|_{e^{B(u)}F_\sharp} \cdot |x|^m e^{-\delta(T)|x|}$$

for some constant $C' > 0$ that is independent of $v, w \in V$.

261. Now suppose that $v \in E_k(H)$ and $w \in E_\ell(H)$ belong to possibly different weight spaces, say with $k \leq \ell$. The behavior of the inner product

$$\langle v, w\rangle_{e^{B(u)}F_\sharp} = \langle e^{-B(u)}v, e^{-B(u)}w\rangle$$

is very easy to understand, given what we know about the coefficients of the series $B(u)$. The whole expression is a convergent power series in u. Since different eigenspaces of H are orthogonal under the inner product $\langle v, w\rangle$, the leading term in this series is divisible by $u^{\ell-k}$. It follows that there is a constant $C > 0$ such that for $|\operatorname{Re} z| \gg 0$, one has

$$\left|\langle v, w\rangle_{\hat{\Phi}_{S,H}(z)}\right| \leq C\|v\|\|w\| \cdot |\operatorname{Re} z|^{-(\ell-k)/2}. \tag{261.1}$$

From the definition of the rescaled period mapping in (21.1), we get

$$\big\langle v,\, w\big\rangle_{\hat{\Phi}_{S,H}(z)} = |\operatorname{Re} z|^{-(k+\ell)/2} \cdot \big\langle e^{\frac{1}{2}(z-\overline{z})(S+N)}v,\, e^{\frac{1}{2}(z-\overline{z})(S+N)}w\big\rangle_{\Phi(z)},$$

and as long as $\operatorname{Im} z$ remains bounded, it follows that

$$\left|\langle v, w\rangle_{\Phi(z)}\right| \leq C'\|v\|\|w\| \cdot |\operatorname{Re} z|^k \tag{261.2}$$

for a different constant $C' > 0$ and $|\operatorname{Re} z| \gg 0$.

262. We know from the Hodge norm estimates that $\|v\|^2_{\Phi(z)}$ and $\|w\|^2_{\Phi(z)}$ are of order $|\operatorname{Re} z|^k$ and $|\operatorname{Re} z|^\ell$, respectively. Consequently, (261.2) is equivalent to the statement that, for $|\operatorname{Re} z| \gg 0$, one has

$$\left|\langle v, w\rangle_{\Phi(z)}\right| \leq C'' \min\big(\|v\|^2_{\Phi(z)}, \|w\|^2_{\Phi(z)}\big).$$

Because of the triangle inequality, we deduce that for any two multi-valued flat sections $v, w \in V$, there is a constant $C(v, w) > 0$ such that

$$\left|\langle v, w\rangle_{\Phi(z)}\right| \leq C(v, w) \cdot \min\Big(\|v\|^2_{\Phi(z)}, \|w\|^2_{\Phi(z)}\Big)$$

for $|\operatorname{Re} z| \gg 0$. This finishes the proof of §40 Theorem from the introduction.

References

[Ahl38] Lars V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), no. 3, 359–364. MR 1501949

[Ber10] Bo Berndtsson, *An introduction to things $\overline{\partial}$*, Analytic and algebraic geometry, IAS/Park City Math. Ser., vol. 17, Amer. Math. Soc., Providence, RI, 2010, pp. 7–76. MR 2743815

[Cat08] Eduardo Cattani, *Mixed Lefschetz theorems and Hodge-Riemann bilinear relations*, Int. Math. Res. Not. IMRN (2008), no. 10, Art. ID rnn025, 20. MR 2429243

[CG75] Maurizio Cornalba and Phillip Griffiths, *Analytic cycles and vector bundles on non-compact algebraic varieties*, Invent. Math. **28** (1975), 1–106. MR 367263

[CK89] Eduardo Cattani and Aroldo Kaplan, *Degenerating variations of Hodge structure*, Astérisque (1989), no. 179-180, 9, 67–96, Actes du Colloque de Théorie de Hodge (Luminy, 1987). MR 1042802 (91k:32019)

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457–535. MR 840721 (88a:32029)

[dCM05] Mark de Cataldo and Luca Migliorini, *The Hodge theory of algebraic maps*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 5, 693–750. MR 2195257 (2007a:14016)

[Del70] Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970. MR 0417174 (54 #5232)

[Del71] ———, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 0498551 (58 #16653a)

[Del84] ———, *Positivité: signes*, Personal note, dated Feb. 11, 1984.

[Dem12] Jean-Pierre Demailly, *Complex analytic and differential geometry*, 2012.

[Den22] Ya Deng, *On the nilpotent orbit theorem of complex variation of Hodge structures*, 2022, preprint `arXiv:2203.04266`.

[GS75] Phillip Griffiths and Wilfried Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, Discrete subgroups of Lie groups and applicatons to moduli (Internat. Colloq., Bombay, 1973), 1975, pp. 31–127. MR 0419850

[Kas85] Masaki Kashiwara, *The asymptotic behavior of a variation of polarized Hodge structure*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 4, 853–875. MR 817170 (87h:32049)

[Moc02] Takuro Mochizuki, *Asymptotic behaviour of tame nilpotent harmonic bundles with trivial parabolic structure*, J. Differential Geom. **62** (2002), no. 3, 351–559. MR 2005295

[Moc07] ———, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. I*, Mem. Amer. Math. Soc. **185** (2007), no. 869, xii+324. MR 2281877

[Mor78] John W. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 137–204. MR 516917

[Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. MR 0382272 (52 #3157)

[Sim88] Carlos T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918. MR 944577

[Sim90] ———, *Harmonic bundles on noncompact curves*, J. Amer. Math. Soc. **3** (1990), no. 3, 713–770. MR 1040197

[Zuc79] Steven Zucker, *Hodge theory with degenerating coefficients. L_2 cohomology in the Poincaré metric*, Ann. of Math. (2) **109** (1979), no. 3, 415–476. MR 534758 (81a:14002)