THE SERRE CONSTRUCTION IN CODIMENSION TWO

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1. Introduction

Let X be a nonsingular algebraic variety. Suppose $Z \subseteq X$ is a closed subscheme of X, with ideal sheaf $\mathscr{I}_Z$.

When Z has codimension one in X, everything is as nice as it could be: $\mathscr{I}_Z$ is a locally free sheaf, in fact a line bundle, and Z can locally be defined by a single equation. But starting in codimension two, all these pleasant things are usually false. To begin with, not every closed subscheme Z of codimension $r \geq 2$ can be defined locally by r equations. When this is possible, in other words, if the ideal sheaf $\mathscr{I}_Z$ is locally generated by r elements, the subscheme Z is called a *local complete intersection* in X (see [1, Definition on p. 185] for details). Since X is nonsingular, any such Z is automatically Cohen-Macaulay by [1, II.8.23], and as such has several useful properties.

Now let us suppose that $Z \subseteq X$ is such a local complete intersection of codimension two. We first look at the local situation near points $x \in Z$. If we let $(A, \mathfrak{m})$ be the local ring of the point x on X, the stalk of the ideal sheaf, $I = \mathscr{I}_{Z,x}$, can be generated by two elements, say $f, g \in \mathfrak{m}$. Because Z is Cohen-Macaulay, f and g form a regular sequence (see [1, II.8.21A]), and so the Koszul complex

$$0 \to A \xrightarrow{\binom{-g}{f}} A^2 \xrightarrow{(f,g)} A$$

is exact and resolves $A/I = A/(f,g)A$. Thus

$$0 \to A \xrightarrow{\binom{-g}{f}} A^2 \xrightarrow{(f,g)} I \to 0$$

is also exact. This is the local picture.

Globally (on X), we might therefore hope to find a short exact sequence of the form

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{I}_Z \to 0, \tag{1}$$

where $\mathscr{L}$ is a line bundle, and $\mathscr{E}$ a vector bundle of rank two. The purpose of this note is to investigate under which conditions such short exact sequences exist; and, should they fail to exist, whether there are good substitutes.

2. Extensions

One way to approach the above questions is to fix a line bundle $\mathscr{L}$ on X, and to ask whether or not there are *extensions* of the form (1) with $\mathscr{E}$ locally free. There is, of course, always the trivial extension $\mathscr{E} = \mathscr{L} \oplus \mathscr{I}_Z$; but it is not locally free at points of Z. Up to isomorphism, all extensions as in (1) are parametrized by the group

$$\operatorname{Ext}^1(\mathscr{I}_Z, \mathscr{L}) = \operatorname{Ext}^1_X(\mathscr{I}_Z, \mathscr{L}).$$

The reader may consult [1, Section II.6] for details about Ext-groups and $\mathscr{E}xt$-sheaves. Concerning notation, we will typically write Ext instead of Ext_X, as well as $\mathscr{E}xt$ instead of $\mathscr{E}xt_{\mathscr{O}_X}$, to make diagrams more legible. Since the ambient varietey X is always the same, this should cause no problems.

The computation of the Ext-group above is helped by the *local-to-global spectral sequence*

$$E_2^{p,q} = H^p\big(X, \mathscr{E}xt^1(\mathscr{I}_Z, \mathscr{L})\big) \Rightarrow \mathrm{Ext}^{p+q}(\mathscr{I}_Z, \mathscr{L}). \tag{2}$$

In our special case, the spectral sequence degenerates into a long exact sequence; before we can see this, however, we need to compute the $\mathscr{E}xt$-sheaves that occur in (2) and elsewhere.

Lemma 1. *Let $Z \subseteq X$ be a locally complete intersection of codimension $r \geq 2$. Let $\mathscr{L}'$ be the line bundle $\det \mathscr{N}_{Z|X} \otimes i^*\mathscr{L}$ on Z. For every $q \geq 0$, we have*

$$\mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{L}) \simeq \mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{O}_X) \otimes \mathscr{L} \simeq \begin{cases} \mathscr{L}' & \text{if } q = r, \\ 0 & \text{otherwise;} \end{cases} \tag{3}$$

as well as

$$\mathscr{E}xt^q(\mathscr{I}_Z, \mathscr{L}) \simeq \mathscr{E}xt^q(\mathscr{I}_Z, \mathscr{O}_X) \otimes \mathscr{L} \simeq \begin{cases} \mathscr{L} & \text{if } q = 0, \\ \mathscr{L}' & \text{if } q = r-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Here $\mathscr{N}_{Z|X}$ is the normal bundle to Z in X; since Z is a codimension r local complete intersection, $\mathscr{N}_{Z|X} \simeq \mathscr{H}om_{\mathscr{O}_Z}(i^*\mathscr{I}_Z, \mathscr{O}_Z)$ is a vector bundle of rank r on Z. In the statement of the lemma (and elsewhere), we write $\mathscr{O}_Z$ instead of the more correct $i_*\mathscr{O}_Z$ (if $i\colon Z \to X$ is the inclusion morphism), even when $\mathscr{O}_Z$ is to be viewed as a sheaf on X. The same goes for $\mathscr{N}_{Z|X}$ and $\mathscr{L}'$.

Proof. Because tensoring with the line bundle $\mathscr{L}$ is exact, we always have the isomorphism

$$\mathscr{E}xt^q(\mathscr{F}, \mathscr{L}) \simeq \mathscr{E}xt^q(\mathscr{F}, \mathscr{O}_X) \otimes \mathscr{L}$$

for any coherent sheaf $\mathscr{F}$ on X. The line bundle $\mathscr{L}$ is therefore irrelevant for the computation we have to do.

Let us first consider (3) locally. If $x \in X$ is an arbitary point, the local ring $\mathscr{O}_{X,x}$ is regular (since X is nonsingular), and because Z is Cohen-Macaulay by [1, II.8.23], the stalk $\mathscr{I}_{Z,x}$ of its ideal sheaf can be generated by a regular sequence of length r, according to [1, II.8.21A]. On a suitable affine neighborhood $U = \operatorname{Spec} A$ of x in X, the ideal $I = \Gamma(U, \mathscr{I}_Z)$ is therefore generated by a regular sequence $f_1, \dots, f_r \in A$. By [1, III.7.10A], the Koszul complex $K_\bullet = K_\bullet(f_1, \dots, f_r; A)$ is a free resolution of $A/I = \Gamma(U, \mathscr{O}_Z)$; in other words,

$$0 \to K_r \to \dots \to K_0 \to A/I \to 0 \tag{5}$$

is exact.

Since $\mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{O}_X)\big|_U$ is the (coherent) sheaf associated to the A-module $\mathrm{Ext}_A^q(A/I, A)$, we can use the special resolution (5) to obtain

$$\Gamma(U, \mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{O}_X)) = \mathrm{Ext}_A^q(A/I, A) = H^q\big(\mathrm{Hom}(K_\bullet, A)\big).$$

Now the Koszul complex is self-dual, and so

$$H^q\big(\mathrm{Hom}(K_\bullet, A\big) \simeq H^{r-q}\big(K_\bullet\big) = \begin{cases} A/I & \text{if } q = r, \\ 0 & \text{otherwise.} \end{cases}$$

If a different regular sequence $g_1, \dots, g_r$ generating I is used, we can always write

$$(g_1, \dots, g_r) = (f_1, \dots, f_r) \cdot T$$

for a suitable $r \times r$-matrix T with entries in A. This matrix induces a map of complexes $K_\bullet(f_1, \dots, f_r; A) \to K_\bullet(g_1, \dots, g_r; A)$, and through it an isomorphism in cohomology

$$H^r\big(\mathrm{Hom}(K_\bullet(g_1, \dots, g_r; A))\big) \to H^r\big(\mathrm{Hom}(K_\bullet(f_1, \dots, f_r; A))\big)$$

which is multiplication by the unit $\det T$.

The local computation shows two things. First, if $q \neq r$, we have $\mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{O}_X)\big|_U = 0$ for each open set U as above, and therefore $\mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{O}_X) = 0$. Secondly, we see that $\mathscr{E}xt^r(\mathscr{O}_Z, \mathscr{O}_X)\big|_U$ is isomorphic to $\mathscr{O}_U$, the isomorphism being determined through $(f_1, \dots, f_r)$. If we change to $(g_1, \dots, g_r) = (f_1, \dots, f_r) \cdot T$, the isomorphism is changed by a factor of $\det T$. Thus $\mathscr{E}xt^r(\mathscr{O}_Z, \mathscr{O}_X)$ is a certain line bundle on Z with transition functions given by $\det T$ between open sets as above. Noting that the line bundle $\det\big(i^*\mathscr{I}_Z\big)$ has transition functions $(\det T)^{-1}$ for the same open sets, we get

$$\mathscr{E}xt^r(\mathscr{O}_Z, \mathscr{O}_X) \simeq \mathscr{H}om_{\mathscr{O}_Z}(\det\big(i^*\mathscr{I}_Z\big), \mathscr{O}_Z) \simeq \det \mathscr{N}_{Z|X}$$

which leads to the isomorphism in (3).

Now (4) is a straightforward consequence of (3) and the short exact sequence $0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$. Indeed, consider the corresponding long exact sequence obtained by applying the functor $\mathscr{H}om(\_\,, \mathscr{L})$. Its first four terms are

$$\mathscr{H}om(\mathscr{O}_Z, \mathscr{L}) \to \mathscr{H}om(\mathscr{O}_X, \mathscr{L}) \to \mathscr{H}om(\mathscr{I}_Z, \mathscr{L}) \to \mathscr{E}xt^1(\mathscr{O}_Z, \mathscr{L}),$$

of which the first and last vanish by (3), while the second one is isomorphic to $\mathscr{L}$. Thus we get $\mathscr{H}om(\mathscr{I}_Z, \mathscr{L}) \simeq \mathscr{L}$ as claimed. Further on in the long exact sequence, we find (for $q > 0$)

$$\mathscr{E}xt^q(\mathscr{O}_X, \mathscr{L}) \to \mathscr{E}xt^q(\mathscr{I}_Z, \mathscr{L}) \to \mathscr{E}xt^{q+1}(\mathscr{O}_Z, \mathscr{L}) \to \mathscr{E}xt^{q+1}(\mathscr{O}_X, \mathscr{L}),$$

where the first and the fourth term vanish since $\mathscr{O}_X$ is free. Thus

$$\mathscr{E}xt^q(\mathscr{I}_Z, \mathscr{L}) \simeq \mathscr{E}xt^{q+1}(\mathscr{O}_Z, \mathscr{L}) \simeq \begin{cases} \mathscr{L}' & \text{if } q+1 = r, \\ 0 & \text{otherwise,} \end{cases}$$

and (4) is completely proved as well. □

As a consequence of this lemma (in the case $r = 2$), the spectral sequence in (2) has only two nonzero rows on the E_2-page; we therefore get a long exact sequence whose first terms are

$$\begin{aligned} 0 \to H^1(X, \mathscr{H}om(\mathscr{I}_Z, \mathscr{L})) &\to \mathrm{Ext}^1(\mathscr{I}_Z, \mathscr{L}) \\ &\to H^0(X, \mathscr{E}xt^1(\mathscr{I}_Z, \mathscr{L})) \to H^2(X, \mathscr{H}om(\mathscr{I}_Z, \mathscr{L})), \end{aligned}$$

or, using Lemma 1,

$$0 \to H^1(X, \mathscr{L}) \to \mathrm{Ext}^1(\mathscr{I}_Z, \mathscr{L}) \to H^0(Z, \mathscr{L}') \to H^2(X, \mathscr{L}). \tag{6}$$

Note that if $i\colon Z \to X$ is the inclusion morphism, we have

$$i_* \mathscr{L}' = i_*\bigl(\det \mathscr{N}_{Z|X} \otimes i^* \mathscr{L}\bigr) \simeq i_*(\det \mathscr{N}_{Z|X}) \otimes \mathscr{L}$$

by the projection formula, which allowed us to write $H^0(Z, \mathscr{L}')$ for $H^0(X, i_* \mathscr{L}')$ in (6).

Our extension problem thus leads us to sections of the line bundle $\mathscr{L}' = \det \mathscr{N}_{Z|X} \otimes i^* \mathscr{L}$ on Z; however, only those sections are of interest that lie in the kernel of the map

$$c \underset{\text{def}}{=} d_2^{0,1}\colon H^0(Z, \mathscr{L}') \to H^2(X, \mathscr{L}).$$

We shall see later what exactly c does.

Global sections of $\mathscr{L}$. For the time being, we investigate more closely the map in the sequence (6) that associates to an extension $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{I}_Z \to 0$ a global section of the line bundle $\mathscr{L}'$ on Z. This map is actually an edge homomorphism of the spectral sequence, namely the map

$$\operatorname{Ext}^1(\mathscr{I}_Z, \mathscr{L}) \to H^0(X, \mathscr{E}xt^1(\mathscr{I}_Z, \mathscr{L}))$$

and as such easy to describe explicitly.

Consider the long exact sequence obtained by applying $\mathscr{H}om(\_\,, \mathscr{L})$ to our short exact sequence; the first six terms are

$$\begin{aligned} 0 \to \mathscr{H}om(\mathscr{I}_Z, \mathscr{L}) \to \mathscr{H}om(\mathscr{E}, \mathscr{L}) \to \mathscr{H}om(\mathscr{L}, \mathscr{L}) \\ \to \mathscr{E}xt^1(\mathscr{I}_Z, \mathscr{L}) \to \mathscr{E}xt^1(\mathscr{E}, \mathscr{L}) \to \mathscr{E}xt^1(\mathscr{L}, \mathscr{L}) \end{aligned}$$

The line bundle $\mathscr{L}$ satisfies $\mathscr{H}om(\mathscr{L}, \mathscr{L}) \simeq \mathscr{O}_X$, with the identity corresponding to the constant section 1; with this, and the results of Lemma 1, we simplify the six-term sequence to

$$0 \to \mathscr{L} \to \mathscr{H}om(\mathscr{E}, \mathscr{L}) \to \mathscr{O}_X \xrightarrow{\delta} \mathscr{L}' \to \mathscr{E}xt^1(\mathscr{E}, \mathscr{L}) \to 0. \tag{7}$$

Thus we get a distinguished section $\delta_X(1)$ of $i_* \mathscr{L}'$; as before, we view this as a global section of $\mathscr{L}'$ on Z. This is the image of our extension under the edge homomorphism. (The proof of this assertion is a pleasant exercise.)

A different approach. The sequence in (6) can also be obtained in a different, more direct way (which is useful for questions of functoriality). In this second approach, we make the short exact sequence $0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$ our point of departure. The corresponding long exact sequence for Ext-groups is, in part,

$$\begin{aligned} \operatorname{Ext}^1(\mathscr{O}_Z, \mathscr{L}) \to \operatorname{Ext}^1(\mathscr{O}_X, \mathscr{L}) \\ \to \operatorname{Ext}^1(\mathscr{I}_Z, \mathscr{L}) \to \operatorname{Ext}^2(\mathscr{O}_Z, \mathscr{L}) \to \operatorname{Ext}^2(\mathscr{O}_X, \mathscr{L}). \end{aligned} \tag{8}$$

Using the local-to-global spectral sequence, most of these groups can be computed. Indeed, we have on the one hand the spectral sequence

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathscr{O}_X, \mathscr{L})) \Rightarrow \operatorname{Ext}^{p+q}(\mathscr{O}_X, \mathscr{L});$$

since $\mathscr{E}xt^q(\mathscr{O}_X, \mathscr{L}) = 0$ unless $q = 0$, the edge homomorphism

$$\operatorname{Ext}^q(\mathscr{O}_X, \mathscr{L}) \to H^0(X, \mathscr{H}om(\mathscr{O}_X, \mathscr{L}) = H^0(X, \mathscr{L})$$

is an isomorphism. On the other hand, the spectral sequence

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathscr{O}_Z, \mathscr{L})) \Rightarrow \operatorname{Ext}^{p+q}(\mathscr{O}_Z, \mathscr{L}),$$

together with the isomorphisms in Lemma 1, gives

$$\mathrm{Ext}^q(\mathscr{O}_Z, \mathscr{L}) \simeq H^{q-2}(X, \mathscr{E}xt^2(\mathscr{O}_Z, \mathscr{L})) \simeq H^{q-2}(Z, \mathscr{L}').$$

If we substitute into (8) above, we retrieve our previous sequence (6); indeed, since only Lemma 1 and local-to-global spectral sequences were used in both cases (and since the latter is functorial), it is easy to believe that the two really are the same.

Thus one possible description of the map c is through the following commutative diagram,

$$\begin{array}{ccccc}
\mathrm{Ext}^2(\mathscr{O}_Z, \mathscr{L}) & \xrightarrow{\simeq} & H^0(X, \mathscr{E}xt^2(\mathscr{O}_Z, \mathscr{L})) & \xrightarrow{\simeq} & H^0(Z, \mathscr{L}') \\
\downarrow & & & & \downarrow c \\
\mathrm{Ext}^2(\mathscr{O}_X, \mathscr{L}) & \xrightarrow{\simeq} & H^2(X, \mathscr{H}om(\mathscr{O}_X, \mathscr{L})) & \xrightarrow{\simeq} & H^2(X, \mathscr{L})
\end{array}$$

in which the first vertical arrow is the map induced from $\mathscr{O}_X \to \mathscr{O}_Z$.

3. Local freeness of $\mathscr{E}$

Now we return to the main question, namely whether or not in an extension

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{I}_Z \to 0$$

the coherent sheaf $\mathscr{E}$ is locally free; or perhaps, more generally, how far $\mathscr{E}$ is from being locally free. Since $\mathscr{I}_Z$ is the same as $\mathscr{O}_X$ outside of Z, this is really a question about points of Z only. Luckily, there is a simple device, called *Serre's criterion*, to decide this question for us.

As before, we first consider the local situation, where Serre's criterion is the following algebraic statement.

Lemma 2. *Let $(A, \mathfrak{m})$ be a regular local ring, and let M be a finitely generated A-module. The following three statements are equivalent.*

1. *M is projective (and hence free, A being a local ring).*
2. *$\mathrm{Ext}^q_A(M, N) = 0$ for every finitely generated A-module N, and all $q > 0$.*
3. *$\mathrm{Ext}^q_A(M, A) = 0$ for all $q > 0$.*

Proof. The equivalence of (1) and (2) is a basic fact in homological algebra. Since (2) clearly implies (3), we only have to show that (3) is strong enough to give us (2) back.

Let N be a finitely generated A-module. To show that $\mathrm{Ext}^q_A(M, N) = 0$, we take a finitely generated free module A^k mapping onto N, and let N_1 be the kernel,

$$0 \to N_1 \to A^k \to N \to 0.$$

For all $q > 0$, the associated long exact sequence gives

$$\mathrm{Ext}^q_A(M, N) \simeq \mathrm{Ext}^{q+1}_A(M, N_1).$$

Applying the same procedure to N_1, and continuing in like manner, we find that for all $t \geq 0$,

$$\mathrm{Ext}^q_A(M, N) \simeq \mathrm{Ext}^{q+t}_A(M, N_t)$$

for suitably chosen finitely generated modules N_t. But A is a regular local ring and so, by Serre's theorem, of finite global projective dimension (equal to $\dim A$). In other words, all Ext-groups past $\dim A$ vanish; if we chooose t sufficiently large, we see that $\operatorname{Ext}_A^q(M,N) = 0$, proving (2). □

If we apply this lemma to all local rings on X (they are without exception regular because X is nonsingular), we get the following global form of Lemma 2.

Lemma 3. *Let $\mathscr{F}$ be a coherent sheaf on a nonsingular algebraic variety X. Then $\mathscr{F}$ is locally free at a point $x \in X$ if, and only if, the stalk $\mathscr{E}xt^q(\mathscr{F}, \mathscr{O}_X)_x = 0$ for all $q > 0$.*

As a consequence, $\mathscr{F}$ is locally free exactly on the open set

$$\bigcap_{q>0} \bigl(X - \operatorname{Supp} \mathscr{E}xt^q(\mathscr{F}, \mathscr{O}_X)\bigr)$$

(since the $\mathscr{E}xt$-sheaves are zero for large q, the intersection is a finite one).

In this form, the criterion fits our situation like a glove. We already obtained, in (7) above, the sequence

$$0 \to \mathscr{L} \to \mathscr{H}om(\mathscr{E}, \mathscr{L}) \to \mathscr{O}_X \xrightarrow{\delta} \mathscr{L}' \to \mathscr{E}xt^1(\mathscr{E}, \mathscr{L}) \to 0. \tag{7}$$

by applying the functor $\mathscr{E}xt^q(\_\,, \mathscr{L})$ to the extension $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{I}_Z \to 0$. But we also get

$$\mathscr{E}xt^q(\mathscr{E}, \mathscr{O}_X) \simeq \mathscr{E}xt^q(\mathscr{I}_Z, \mathscr{O}_X) = 0$$

for all $q \geq 2$. By the criterion in Lemma 3, then, $\mathscr{E}$ fails to be locally free exactly on the closed subset $\operatorname{Supp} \mathscr{E}xt^1(\mathscr{E}, \mathscr{O}_X) = \operatorname{Supp} \mathscr{E}xt^1(\mathscr{E}, \mathscr{L})$ of Z; because of (7), this is where the section $s = \delta_X(1)$ of the line bundle $\mathscr{L}'$ on Z vanishes. Turned around, $\mathscr{E}$ is locally free at all those points $x \in Z$ where $s(x) \neq 0$ (and, of course, on all of $X - Z$).

Reflexiveness. It should now be apparent that we cannot expect $\mathscr{E}$ to be locally free in general. However, in most cases $\mathscr{E}$ will be at least *reflexive*. If we write $\mathscr{F}^\vee = \mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$ for the dual of a coherent sheaf $\mathscr{F}$, this means that the natural map $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ is an isomorphism. Let us investigate under which conditions this is the case.

We first write our extension in the form

$$0 \to \mathscr{L} \xrightarrow{i} \mathscr{E} \xrightarrow{p} \mathscr{O}_X \to \mathscr{O}_Z \to 0;$$

in (7), we had a similar sequence, namely

$$0 \to \mathscr{L} \xrightarrow{p^*} \mathscr{H}om(\mathscr{E}, \mathscr{L}) \xrightarrow{i^*} \mathscr{O}_X \xrightarrow{\delta} \mathscr{L}' \to \mathscr{E}xt^1(\mathscr{E}, \mathscr{L}) \to 0.$$

To compare the two, we define a morphism of sheaves $\phi \colon \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L})$. If $U \subseteq X$ is any open set, we have an exact sequence

$$0 \to \mathscr{L}(U) \xrightarrow{i_U} \mathscr{E}(U) \xrightarrow{p_U} \mathscr{O}_X(U).$$

For a section $e \in \mathscr{E}(U)$, let $\phi_U(e) \colon \mathscr{E}\big|_U \to \mathscr{L}\big|_U$ be given by the following rule: If $V \subseteq U$ is an open subset, then a section $e' \in \mathscr{E}(V)$ is mapped to the unique section $t \in \mathscr{L}(V)$ such that

$$i_V(t) = p_V(e\big|_V) \cdot e' - p_V(e') \cdot e\big|_V.$$

One verifies that this prescription makes ϕ a well-defined morphism of sheaves.

Lemma 4. *The following diagram commutes.*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathscr{L} & \xrightarrow{\ i\ } & \mathscr{E} & \xrightarrow{\ p\ } & \mathscr{O}_X & \longrightarrow & \mathscr{O}_Z & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle -\,\mathrm{id}} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & \mathscr{L} & \xrightarrow{\ p^*\ } & \mathscr{H}om(\mathscr{E}, \mathscr{L}) & \xrightarrow{\ i^*\ } & \mathscr{O}_X & \xrightarrow{\ \delta\ } & \mathscr{L}' & &
\end{array}$$

Proof. Concerning the first square in the diagram, we show that for every open set $U \subseteq X$,

$$\begin{array}{ccc}
\mathscr{L}(U) & \xrightarrow{\ i_U\ } & \mathscr{E}(U) \\
\big\downarrow{\scriptstyle -\,\mathrm{id}_U} & & \big\downarrow{\scriptstyle \phi_U} \\
\mathscr{L}(U) & \xrightarrow{\ p_U^*\ } & \mathscr{H}om(\mathscr{E}, \mathscr{L})(U)
\end{array}$$

is commutative. Take any section $r \in \mathscr{L}(U)$. Then $p_U^*(-r)$ is the morphism $\mathscr{E}\big|_U \to \mathscr{L}\big|_U$ that maps any $e' \in \mathscr{E}(V)$ to $-p_V(e') \cdot r\big|_V$. On the other hand, by definition of ϕ, the morphism $\phi_U(i_u(r))$ sends $e' \in \mathscr{E}(V)$ to the unique $t \in \mathscr{L}(V)$ with

$$i_V(t) = -p_V(e') \cdot i_U(r)\big|_V + p_V(i_U(r)\big|_V) \cdot e' = -i_V(p_V(e') \cdot r\big|_V),$$

in other words, to $t = -p_V(e') \cdot r\big|_V$. Thus the square commutes.

For the second square, we similarly show that

$$\begin{array}{ccc}
\mathscr{E}(U) & \xrightarrow{\ p_U\ } & \mathscr{O}_X(U) \\
\big\downarrow{\scriptstyle \phi_U} & & \big\downarrow{\scriptstyle \mathrm{id}_U} \\
\mathscr{H}om(\mathscr{E}, \mathscr{L})(U) & \xrightarrow{\ i_U^*\ } & \mathscr{O}_X(U)
\end{array}$$

is a commutative diagram. If $e \in \mathscr{E}(U)$, then $\phi_U \circ i_U \colon \mathscr{L}\big|_U \to \mathscr{L}\big|_U$ sends any $r \in \mathscr{L}(V)$ to the unique $t \in \mathscr{L}(V)$ for which

$$i_V(t) = p_V(e\big|_V) \cdot i_V(r) - p_V(i_V(r)) \cdot e\big|_V = i_V(p_V(e\big|_V) \cdot r),$$

and thus to $t = -p_V(e\big|_V) \cdot r$. Thus the map $\phi_U \circ i_U$ is multiplication by $p_U(e) \in \mathscr{O}_X(U)$, showing that the second square is also commutative.

As for the third one, it commutes by definition of the section s. □

Now if the section s is such that the corresponding map $s \colon \mathscr{O}_Z \to \mathscr{L}' = \det \mathscr{N}_{Z|X} \otimes i^*\mathscr{L}$ is injective, a bit of diagram chasing in the above diagram shows that $\phi \colon \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L}) \simeq \mathscr{E}^\vee \otimes \mathscr{L}$ is an isomorphism. As a consequence,

$$\mathscr{E}^{\vee\vee} \simeq \left(\mathscr{E}^\vee \otimes \mathscr{L}\right)^\vee \simeq \mathscr{E}^\vee \otimes \mathscr{L} \simeq \mathscr{E},$$

and so $\mathscr{E}$ is reflexive.

Summary. Here is a summary of the results obtained so far. As usual, we write $\mathscr{L}' = \det \mathscr{N}_{Z|X} \otimes i^* \mathscr{L}$.

(1) For every extension $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{I}_Z \to 0$, there is a distinguished global section $s \in H^0(Z, \mathscr{L}')$, and s is in the kernel of the map $c\colon H^0(Z, \mathscr{L}') \to H^2(X, \mathscr{L})$. Conversely, for every section in the kernel of c, there is at least one extension, unique up to the group $H^1(X, \mathscr{L})$.

(2) The coherent sheaf $\mathscr{E}$ is locally free precisely on the open set $(X - Z) \cup \operatorname{Supp} s$.

(3) If $s\colon \mathscr{O}_Z \to \mathscr{L}'$ is injective, then $\mathscr{E} \simeq \mathscr{E}^\vee \otimes \mathscr{L}$, and $\mathscr{E}$ is reflexive.

References

[1] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.