A CONDITION FOR AN IDEAL IN A POWER SERIES RING TO BE GENERATED BY CONSTANTS

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Let A be a Noetherian ring in which every nonzero integer is invertible, meaning that $\mathbb{Q} \subseteq A$. The following lemma gives a necessary and sufficient condition for an ideal in the ring of power series $A[\![x]\!]$ to be generated by elements of A. This answers a question by Yu-Han Liu.

Lemma. *Let $I \subseteq A[\![x]\!]$ be an arbitrary ideal in the ring of power series over A. Then I is generated by elements of A if, and only if, it is closed under differentiation. In other words, the condition is that $f'(x) \in I$ for every power series $f(x)$ in the ideal I.*

Proof. The condition in the lemma is clearly necessary. To prove that it is also sufficient, let $I \subseteq A[\![x]\!]$ be an ideal containing the derivatives of all its members. Since the ring $A[\![x]\!]$ is Noetherian, we can choose finitely many generators $f_1(x), f_2(x), \dots, f_n(x)$ for the ideal I. Let us write

$$\phi(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)^\dagger$$

for the column vector determined by these n power series; similarly, we let

$$\phi'(x) = \big(f_1'(x), f_2'(x), \dots, f_n'(x)\big)^\dagger$$

be the vector of derivatives. By assumption, all components of $\phi'(x)$ are again contained in I; we can therefore find an $n \times n$-matrix $B(x)$, with coefficients in the ring $A[\![x]\!]$, such that

$$\phi'(x) = B(x)\phi(x). \tag{1}$$

We write out the the quantities in this equation in the form

$$B(x) = B_0 + xB_1 + x^2 B_2 + \cdots \quad \text{and} \quad \phi(x) = \phi_0 + x\phi_1 + x^2\phi_2 + \cdots,$$

where each B_i (resp., each ϕ_i) is a matrix (resp., a vector) with entries in the ring A. The identity in (1) can be used to express $\phi_1, \phi_2, \dots$ in terms of ϕ_0 and the matrices B_i. Indeed, by equating coefficients at x^k, we find that

$$k \cdot \phi_k = B_0\phi_{k-1} + B_1\phi_{k-2} + \cdots + B_{k-1}\phi_0.$$

When applied recursively, this gives

$$\phi_1 = B_0\phi_0, \quad \phi_2 = \frac{1}{2}\big(B_0^2 + B_1\big)\phi_0, \quad \phi_3 = \frac{1}{6}\big(B_0^3 + 2B_1B_0 + B_0B_1 + 2B_2\big)\phi_0,$$

and so forth. What we find then is that

$$\phi(x) = \phi_0 + xB_0\phi_0 + \frac{1}{2}x^2\big(B_0^2 + B_1\big)\phi_0 + \cdots = C(x)\phi_0,$$

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for a certain $n \times n$-matrix $C(x)$ whose entries are again power series. The coefficients of $C(x)$ are universal expressions in the matrices B_i; their exact shape, of course, does not matter for our purposes.

To finish the proof, we observe that the matrix $C(x)$ is invertible over $A[\![x]\!]$, because $C(0) = \mathrm{id}$. We thus have both $\phi(x) = C(x)\phi_0$ and $\phi_0 = C(x)^{-1}\phi(x)$. Since the components of the vector $\phi(x)$ generate the ideal I, it follows that the components of $\phi_0 = \big(f_1(0), f_2(0), \dots, f_n(0)\big)$ are also generators. We conclude that I is generated by elements of A, which is what we needed to show. □