KODAIRA DIMENSION AND ZEROS OF HOLOMORPHIC ONE-FORMS

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Introduction

My topic today is holomorphic one-forms on smooth complex projective varieties. Of course, you all know what happens for curves: on $\mathbb{P}^1$, there are no nontrivial one-forms; on an elliptic curve, there is a nowhere vanishing one-form; and on a curve of genus $g \geq 2$, every one-form vanishes at exactly $2g-2$ points (counted with multiplicity). For surfaces, the situation is a bit more complicated, because there more cases in the classification; but one can show that on a surface of general type, every holomorphic one-form has to vanish at some point. (You can try to prove this as an exercise, while I finish up the introduction.) Based on these examples, several people conjectured that the same result should be true on any smooth projective variety of general type. What I am going talk about is the proof of this conjecture; it is the outcome of a joint project with Mihnea Popa.

Theorem. *Let X be a smooth projective variety of general type. Then every holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ has to vanish at some point of X.*

This was conjectured in 2005 by Christopher Hacon and Sándor Kovács and independently by Tie Luo and Qi Zhang. The result was known when $\dim X \leq 3$ [LZ05], and when the canonical bundle K_X is big and nef [HK05]. We proved the general case in December of 2012.

About the problem

At first glance, the conjecture looks like a problem in birational geometry – but in fact, it is a problem about *$\mathscr{D}$-modules on abelian varieties.* So before I get to the proof, let me explain how and why $\mathscr{D}$-modules come into the picture.

Of course, there is a connection with abelian varieties because (in characteristic zero) one-forms come from abelian varieties. In fact, the Albanese variety $\mathrm{Alb}(X)$ is an abelian variety of dimension $\dim H^0(X, \Omega_X^1)$, and after choosing a base point on X, one has the Albanese mapping

$$\mathrm{alb}\colon X \to \mathrm{Alb}(X).$$

More or less by construction, it has the property that

$$\mathrm{alb}^*\colon H^0\big(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}\big) \to H^0(X, \Omega_X^1)$$

is an isomorphism. This picture immediately suggests that we could consider an arbitrary morphism from X to an abelian variety A, and only look at those one-forms that are pulled back from A. (Since any morphism to an abelian variety factors through $\mathrm{Alb}(X)$, this amounts to considering all possible quotients of the

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Albanese variety.) We can then prove a stronger result, where the assumption that X is of general type gets replaced by a positivity condition on the canonical bundle that depends on A.

Theorem. *Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. Suppose that $\omega_X^{\otimes d} \otimes f^* L^{-1}$ has a nontrivial section for some $d \geq 1$, and some ample line bundle L on A. Then for every $\omega \in H^0(A, \Omega_A^1)$, the pullback $f^*\omega$ has to vanish at some point of X.*

I hope it is clear why this implies the first theorem: Take any ample line bundle L on $\mathrm{Alb}(X)$; because the canonical bundle of X is big, $\omega_X^{\otimes d} \otimes \mathrm{alb}^* L^{-1}$ has a section for $d \gg 0$. We therefore get the desired result by applying the second theorem to the Albanese mapping of X.

Let me point out that the more general formulation can also be applied to the kind of questions that Viehweg was interested in. Here are two such results:

(1) Suppose that $f\colon X \to A$ is a smooth projective morphism to an abelian variety. Then the Kodaira dimension of X satisfies $\kappa(X) \leq \dim X - \dim A$.
(2) Suppose moreover that the fibers of f are of general type. Then f is birationally isotrivial, meaning becomes birational to a product after a generically finite base change on A.

To prove the first result, we use the Iitaka fibration of X to produce a subspace of $H^0(X, \Omega_X^1)$, of codimension at most $\dim X - \kappa(X)$, consisting of forms that all vanish somewhere on X. Because f is smooth, the image of $H^0(A, \Omega_A^1)$ has to intersect this subspace trivially, and this gives us our inequality. The second result follows from this by using Kollár's subadditivity theorem for families of varieties of general type. The inequality says in particular that there are no smooth morphisms from a variety of general type to an elliptic curve. This was first proved by Viehweg and Zuo [VZ01], and our theorem owes a lot to their work.

The connection with D-modules

Reminder about D-modules. Now let me explain how the second theorem is connected with $\mathscr{D}$-modules. As the name indicates, a $\mathscr{D}$-module is a module over the sheaf of rings of differential operators; but for the purpose of this talk, you can think of $\mathscr{D}$-modules simply as being a generalization of vector bundles with flat connection. Their main advantage is that one can take direct and inverse images of $\mathscr{D}$-modules even in the case where the morphism is not smooth; this makes them very suitable for dealing with singularities. Note that on a smooth variety X, the structure sheaf $\mathscr{O}_X$ is a $\mathscr{D}$-module; all the other $\mathscr{D}$-modules that we use arise from $\mathscr{O}_X$ by taking direct images.

The last thing to keep in mind is that $\mathscr{D}$-modules are closely related to the cotangent bundle. With any $\mathscr{D}$-module $\mathcal{M}$ on X, one can associate a subset

$$\mathrm{Ch}(\mathcal{M}) \subseteq T^*X,$$

called its *characteristic variety*, and $\mathcal{M}$ is called *holonomic* if $\dim \mathrm{Ch}(\mathcal{M}) = \dim X$. In that case, one can say what the characteristic variety looks like: $\mathrm{Ch}(\mathcal{M})$ is a finite union of conormal varieties T_Z^*X to subvarieties $Z \subseteq X$; recall that

$$T_Z^*X = \text{closure in } T^*X \text{ of the conormal bundle to the smooth locus of } Z.$$

Vector bundles with flat connection are holonomic: their characteristic variety is the zero section of T^*X. More generally, one can have a vector bundle with flat connection on a submanifold $Z \subseteq X$, in which case the characteristic variety is $T_Z^* X$; or even a vector bundle with flat connection on the smooth locus of a subvariety $Z \subseteq X$, in which case the singularities of the connection may lead to additional components in the characteristic variety.

Some observations. Now let us return to the theorem. Since we have to show that every one-form in the image of $f^* \colon H^0(A, \Omega_A^1) \to H^0(X, \Omega_X^1)$ vanishes at some point of X, it is natural to consider the set

$$Z_f = \left\{ (x, \omega) \in X \times H^0(A, \Omega_A^1) \;\middle|\; f^*\omega \text{ vanishes at } x \right\}$$

that remembers all the zeros of all the one-forms in question. With this notation, we have to prove that the second projection $p_2 \colon Z_f \to H^0(A, \Omega_A^1)$ is surjective. Actually, it is more convenient to work instead with the set

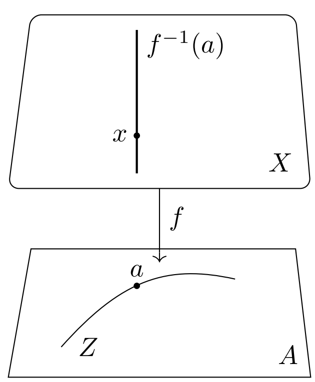
$$\begin{aligned} S_f &= (f \times \mathrm{id})(Z_f) \\ &= \left\{ (a, \omega) \in A \times H^0(A, \Omega_A^1) \;\middle|\; f^*\omega \text{ vanishes at some point } x \in f^{-1}(a) \right\}. \end{aligned}$$

The condition in brackets says that ω annihilates the image of $T_x X \to T_a A$, and so S_f is somehow related to the singularities of the morphism f.

To prove the theorem, we have to show that $p_2 \colon S_f \to H^0(A, \Omega_A^1)$ is surjective. Besides the fact that S_f is a subset of the cotangent bundle $T^* A$, there are two basic observations that suggest a connection with $\mathscr{D}$-modules on A. The first one has to do with the structure of the set S_f.

Lemma. *We have* $\dim S_f \leq \dim A$*, and every irreducible component of dimension equal to* $\dim A$ *is a conormal variety.*

Proof. Let's consider an irreducible component $W \subseteq S_f$, and denote by $Z = p_1(W)$ its image in A; because S_f is conical, this is a closed subvariety of A. Now I claim that $W \subseteq T_Z^* A$. This is very easy. By definition, for every pair $(a, \omega) \in W$, there is a point $x \in f^{-1}(a)$ such that ω vanishes on the image of $T_x X \to T_a A$.



But at a general smooth point $a \in Z$, this image has to contain the tangent space $T_a Z$ (because Z lies in the image of f by construction), and so $(a, \omega) \in T_Z^* A$. This proves that $W \subseteq T_Z^* A$. $\square$

A typical example is when f is surjective and singular over a divisor $D \subseteq A$; in that case, $T_D^* A \subseteq S_f$. Note that we are precisely looking for a component of dimension $\dim A$, because we are trying to show that S_f maps onto $H^0(A, \Omega_A^1)$.

The second observation has to do with the direct image $\mathscr{D}$-modules $\mathcal{H}^i f_+ \mathscr{O}_X$. In fact, there is a general result, due to Kashiwara, about the characteristic variety of direct images: one always has

$$\operatorname{Ch}\bigl(\mathcal{H}^i f_+ \mathscr{O}_X\bigr) \subseteq (f \times \mathrm{id})\bigl(df^{-1} \operatorname{Ch}(\mathscr{O}_X)\bigr),$$

where the notation is as follows:

$$\begin{tikzcd} X \times H^0(A, \Omega_A^1) \arrow[r, "df"] \arrow[d, "f \times \mathrm{id}"] & T^* X \\ A \times H^0(A, \Omega_A^1) \end{tikzcd}$$

Now $\operatorname{Ch}(\mathscr{O}_X)$ is the zero section, and so $df^{-1} \operatorname{Ch}(\mathscr{O}_X) = Z_f$; this means that the characteristic varieties of the $\mathscr{D}$-modules $\mathcal{H}^i f_+ \mathscr{O}_X$ are all contained in the set S_f. This does not solve our problem, but it suggests that the set S_f has something to do with $\mathscr{D}$-modules on the abelian variety.

Strategy of the proof

The two observations about S_f suggest the following strategy for proving the theorem: Find a holonomic $\mathscr{D}$-module $\mathcal{M}$ on A, whose characteristic variety is contained in the set S_f, and such that $p_2 \colon \operatorname{Ch}(\mathcal{M}) \to H^0(A, \Omega_A^1)$ is surjective. Note that $\operatorname{Ch}(\mathcal{M})$ automatically has the correct dimension, namely $\dim A$.

For the purposes of this talk, let me pretend that this strategy works. In reality, it does not – but the idea behind the proof in [PS14] is essentially what I just said, except that we work with coherent sheaves on the cotangent bundle instead of with $\mathscr{D}$-modules. I will explain at the end what needs to be modified.

To get $\mathcal{M}$, we do a geometric construction with branched coverings (that we learned from the paper [VZ01] by Viehweg and Zuo); to show that $\operatorname{Ch}(\mathcal{M})$ maps onto $H^0(A, \Omega_A^1)$, we use results from our work about $\mathscr{D}$-modules on abelian varieties.

The geometric construction. Let me begin by explaining the geometric part. By assumption, there is an ample line bundle L on A, and an integer $d \geq 1$, such that $\omega_X^{\otimes d} \otimes f^* L^{-1}$ has a section. By making a base change by the multiplication-by-$2d$ morphism on A, we can arrange that L becomes the d-th power of an ample line bundle.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ A \arrow[r, "{[2d]}"] & A \end{tikzcd}$$

Because X' is finite étale over X, we can test the vanishing or non-vanishing of one-forms upstairs on X'; after the obvious replacements, we can therefore assume without loss of generality that the d-th power of $\omega_X \otimes f^* L^{-1}$ has a section.

Now any nontrivial section of $(\omega_X \otimes f^* L^{-1})^{\otimes d}$ defines a branched covering X_d of X; it will typically be very singular. After resolving singularities, we arrive at the following picture:

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[rr, bend left, "\varphi"] \arrow[drr, bend right, "h"] & X_d \arrow[r, "\pi"] & X \arrow[d, "f"] \\ & & A \end{tikzcd}$$

In the original construction by Viehweg and Zuo, both the section and the resolution of singularities need to be chosen very carefully; our advantage is that we are working with $\mathscr{D}$-modules, and therefore do not have to worry about singularities.

In the above notation, let us consider the $\mathscr{D}$-module $\mathcal{H}^0 h_+ \mathscr{O}_Y$. By Kashiwara's result, its characteristic variety is contained in the set S_h, and so it is much too large for our purpose, because h is obviously a lot more singular than the original morphism f. We will get around this by finding a smaller $\mathscr{D}$-module $\mathcal{M} \subseteq \mathcal{H}^0 h_+ \mathscr{O}_Y$, with the property that $\mathrm{Ch}(\mathcal{M}) \subseteq S_f$.

Here is how this works. In fact, $\mathcal{H}^0 h_+ \mathscr{O}_Y$ is a so-called Hodge module; and according to a result from Saito's theory, we have an inclusion of $\mathscr{O}$-modules

$$h_* \omega_Y \subseteq \mathcal{H}^0 h_+ \mathscr{O}_Y.$$

Over a point $a \in A$ where the fiber $F = f^{-1}(a)$ is smooth, this reduces to the familiar result from Hodge theory that

$$H^0(F, \omega_F) \subseteq H^{\dim F}(F, \mathbb{C}),$$

The basic property of the branched covering X_d is that $\pi^*(\omega_X \otimes f^* L^{-1})$ has a section; after pulling back to Y, this gives us a nontrivial morphism

$$\varphi^* \omega_X^{-1} \otimes h^* L \to \mathscr{O}_Y.$$

We can tensor it with $\varphi^* \omega_X \to \omega_Y$ to obtain $h^* L \to \omega_Y$, and hence $L \to h_* \omega_Y$. Together with Saito's result, this gives us a nontrivial morphism of $\mathscr{O}$-modules from L into $\mathcal{H}^0 h_+ \mathscr{O}_Y$. Now let $\mathcal{M}$ be the $\mathscr{D}$-submodule of $\mathcal{H}^0 h_+ \mathscr{O}_Y$ generated by the image of L; more formally, we define

$$\mathcal{M} = \mathrm{im}\big(\mathscr{D}_A \otimes_{\mathscr{O}_A} L \to \mathcal{H}^0 h_+ \mathscr{O}_Y\big).$$

Although it is not actually true, let me pretend that $\mathrm{Ch}(\mathcal{M})$ is contained in S_f. Here is why. To compute the characteristic variety of $\mathcal{M}$, one chooses a good filtration $F_\bullet \mathcal{M}$ by coherent $\mathscr{O}$-modules, and looks at the support of the coherent sheaf on T^*A determined by the graded module $\mathrm{gr}_\bullet^F \mathcal{M}$. The point is that $\mathcal{H}^0 h_+ \mathscr{O}_Y$ has a natural filtration (induced by the obvious filtration on $\mathscr{O}_Y$), and that Saito's theory gives one an explicit complex to compute it. It turns out that $L \to h_* \omega_Y$ is part of a morphism between two such complexes, one coming from X, the other from Y, and so it is plausible that $\mathrm{Ch}(\mathcal{M})$ might be contained in the subset $S_f \subseteq S_h$. If we use coherent sheaves on T^*A instead of $\mathscr{D}$-modules, this argument does really work: the image of $L \to h_* \omega_Y$ generates a graded submodule of $\mathrm{gr}_\bullet^F \mathcal{H}^0 h_+ \mathscr{O}_Y$, and the support of the associated coherent sheaf on T^*A is contained in S_f. In [PS14], we work with this coherent sheaf instead of with $\mathcal{M}$.

Reduction to a problem about D-modules. By construction, we also have a nontrivial morphism from L into $\mathcal{M}$. We can now complete the proof of the theorem by applying the following general result about $\mathscr{D}$-modules on abelian varieties.

Proposition. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}$-module on an abelian variety. If there is a nontrivial morphism of $\mathscr{O}$-modules $L \to \mathcal{M}$, with L ample, then*

$$p_2 \colon \operatorname{Ch}(\mathcal{M}) \to H^0(A, \Omega_A^1)$$

is surjective.

This clearly implies that S_f projects onto $H^0(A, \Omega_A^1)$, and therefore shows that every one-form in the image of $H^0(A, \Omega_A^1)$ vanishes somewhere on X.

D-modules on abelian varieties

Let $\mathcal{M}$ be a holonomic $\mathscr{D}$-module on the abelian variety A. To decide whether or not its characteristic variety maps onto $H^0(A, \Omega_A^1)$, we can use Kashiwara's index theorem. Every irreducible component of $\operatorname{Ch}(\mathcal{M})$ actually has a well-defined multiplicity, which is determined by $\mathcal{M}$; taking these multiplicities into account, the index theorem says that

$$\deg\bigl(\operatorname{Ch}(\mathcal{M}) \colon H^0(A, \Omega_A^1)\bigr) = \chi(A, \mathcal{M}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i\bigl(A, \operatorname{DR}(\mathcal{M})\bigr)$$

is equal to the Euler characteristic of $\mathcal{M}$ (computed from the cohomology of its de Rham complex). We therefore have to understand the cohomology of $\mathcal{M}$. Here our main tool is the Fourier-Mukai transform for $\mathscr{D}$-modules on abelian varieties, developed by Laumon and Rothstein during the 1990s.

The idea is that we have a large collection of basic $\mathscr{D}$-modules of the form $(\mathcal{L}, \nabla)$, where $\mathcal{L} \in \operatorname{Pic}^0(A)$ is a holomorphic line bundle with trivial first Chern class, and $\nabla \colon \mathcal{L} \to \Omega_A^1 \otimes \mathcal{L}$ is a flat connection on it. These can be used to do "Fourier analysis", by looking at the cohomology of the various twists $\mathcal{M} \otimes_{\mathscr{O}_A} (\mathcal{L}, \nabla)$. Let $A^\natural$ be the moduli space. It is a quasi-projective variety of dimension $2 \dim A$; in fact, the natural projection $\pi \colon A^\natural \to \operatorname{Pic}^0(A)$ is a bundle of affine spaces with fiber $H^0(A, \Omega_A^1)$. On the product $A \times A^\natural$, there is a universal object $(\mathcal{P}^\natural, \nabla^\natural)$; one can show that the formula

$$\mathcal{M}^\bullet \mapsto \mathbf{R}p_{2*} \operatorname{DR}_{A \times A^\natural / A^\natural}\bigl(p_1^* \mathcal{M}^\bullet \otimes (\mathcal{P}^\natural, \nabla^\natural)\bigr)$$

defines an exact functor

$$\operatorname{FM}_A \colon D^b_{coh}(\mathscr{D}_A) \to D^b_{coh}(\mathscr{O}_{A^\natural}),$$

called the *Fourier-Mukai transform.* Laumon and Rothstein proved, independently of each other, that FM_A is an equivalence of categories [Lau96, Rot96]. Point by point, the Fourier-Mukai transform is of course just computing the cohomology of the various twists by our basic $\mathscr{D}$-modules $(\mathcal{L}, \nabla)$.

Perhaps you are more familiar with Mukai's "Fourier transform" for coherent sheaves. The relation with the construction for $\mathscr{D}$-modules is that

$$\operatorname{FM}_A\bigl(\mathscr{D}_A \otimes_{\mathscr{O}_A} \mathscr{F}\bigr) \simeq \pi^* \mathbf{R}\mathcal{S}(\mathscr{F}),$$

where $\mathscr{F}$ is a coherent sheaf on A.

Now suppose that we are in the situation of the proposition, and have a nontrivial morphism $L \to \mathcal{M}$ for some ample line bundle L. Because the Fourier-Mukai transform is an equivalence of categories, we get a nontrivial morphism

$$\pi^* \mathbf{R}\mathcal{S}(L) \simeq \mathrm{FM}_A\big(\mathscr{D}_A \otimes_{\mathscr{O}_A} L\big) \to \mathrm{FM}_A(\mathcal{M}); \tag{$*$}$$

note that $\mathbf{R}\mathcal{S}(L)$ is a vector bundle, concentrated in degree zero. It turns out that when $\mathcal{M}$ is a holonomic $\mathscr{D}$-module, the complex of sheaves $\mathrm{FM}_A(\mathcal{M})$ has several remarkable properties; here are two:

(1) The support of every cohomology sheaf $\mathcal{H}^i \,\mathrm{FM}_A(\mathcal{M})$ is a finite union of *linear subvarieties*, by which we mean subvarieties of the form
$$(\mathcal{L}, \nabla) \otimes \mathrm{im}\big(f^\natural \colon B^\natural \to A^\natural\big),$$
for a morphism of abelian varieties $f \colon A \to B$.

(2) Both $\mathrm{FM}_A(\mathcal{M})$ and its dual $\mathbf{R}\mathcal{H}om\big(\mathrm{FM}_A(\mathcal{M}, \mathscr{O}_{A^\natural}\big)$ satisfy the inequalities
$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^i \,\mathrm{FM}_A(\mathcal{M}) \geq 2i$$
for every $i \geq 0$; they are so-called *perverse coherent sheaves* (introduced by Kashiwara and Arinkin-Bezrukavnikov).

Popa and I proved this first for direct summands of Hodge modules of geometric origin [PS13]; later on, I discovered that it remains true for arbitrary holonomic $\mathscr{D}$-modules [Sch13]. In our case, $\mathcal{M}$ is a direct summand of the Hodge module $\mathcal{H}^0 h_+ \mathscr{O}_Y$, and so we actually have two different proofs for both properties.

In any case, the second property implies, with the help of some results about regular local rings, that the complex $\mathrm{FM}_A(\mathcal{M})$ is concentrated in nonnegative degrees; and that either $\mathcal{H}^0 \,\mathrm{FM}_A(\mathcal{M}) = 0$, or $\operatorname{Supp} \mathcal{H}^0 \,\mathrm{FM}_A(\mathcal{M}) = A^\natural$. (Deducing this from the inequalities above is a pleasant exercise.)

In our situation, we have the nontrivial morphism in ($*$) from a sheaf in degree zero to $\mathrm{FM}_A(\mathcal{M})$, and so $\mathcal{H}^0 \,\mathrm{FM}_A(\mathcal{M}) \neq 0$. But then the support of $\mathcal{H}^0 \,\mathrm{FM}_A(\mathcal{M})$ must be all of $A^\natural$. If we go to a general point $(\mathcal{L}, \nabla) \in A^\natural$, we get

$$\dim H^i\big(A, \mathcal{M} \otimes_{\mathscr{O}_A} (\mathcal{L}, \nabla)\big) = \operatorname{rk} \mathcal{H}^i \,\mathrm{FM}_A(\mathcal{M}) = \begin{cases} 0 & \text{for } i \neq 0, \\ \geq 1 & \text{for } i = 0. \end{cases}$$

Because the Euler characteristic is invariant under deformations, we conclude that

$$\chi(A, \mathcal{M}) = \chi\big(A, \mathcal{M} \otimes_{\mathscr{O}_A} (\mathcal{L}, \nabla)\big) = \operatorname{rk} \mathcal{H}^0 \,\mathrm{FM}_A(\mathcal{M}) \geq 1,$$

and together with the index theorem calculation from above, this proves that $p_2 \colon \mathrm{Ch}(\mathcal{M}) \to H^0(A, \Omega_A^1)$ is indeed surjective. The proof in [PS14] relies on some additional results about Hodge modules to carry essentially the same argument through in the case of certain coherent sheaves on T^*A.

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