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By Steven Zucker*

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Introduction

In an unpublished manuscript on Hodge theory [7], Deligne has proved a Hodge decomposition theorem for twisted coefficient systems like those arising in algebraic geometry. Specifically, given a non-singular projective variety $\bar{S}$ over C (or more generally, a compact Kähler manifold), and a locally constant system V of complex vector spaces on $\bar{S}$ which underlies a polarizable variation of Hodge structure of weight m, Deligne constructed polarizable Hodge structures of weight $m+i$ on $H^i(\bar{S}, V)$ naturally associated to the variation of Hodge structure. In the case $V = \mathbf{C}$ (with the trivial variation of Hodge structure of weight zero), the construction reduces to classical Hodge theory; at the other extreme where $\bar{S}$ is a point,

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it recovers on $H^0(\bar{S}, V)$ the Hodge structure on V.

In practice, V is a system $R^m\bar{f}_*\mathbf{C}$ of cohomology along the fibers for some smooth family of projective varieties (more generally, compact Kähler manifolds) $\bar{f}: \bar{X} \to \bar{S}$. In this case, the Hodge structures constructed for all i and m are compatible with the Leray spectral sequence for $\bar{f}$, so we may view the construction of Deligne as giving the "right" Hodge structures for $H^*(\bar{S}, V)$.

However, a family of projective varieties over a complete base generally has some singular fibers. Assuming that the generic fiber is non-singular, there is a Zariski-open subset $S \subset \bar{S}$ over which $\bar{f}|_{\bar{f}^{-1}(S)} = f: X \to S$ is smooth; the family is said to degenerate on the complement of S in $\bar{S}$. For most applications, one would need a generalization of Deligne's construction to degenerating families. It is the purpose of this paper to provide such a generalization when $\bar{S}$ is a curve, and to give applications of the result. (The hope of allowing $\bar{S}$ to be higher dimensional will always be at the back of our minds.) We prove (7.12) (also (11.6)):

THEOREM. *Let S be a non-singular algebraic curve over* $\mathbf{C}$, *$\bar{S}$ its smooth completion, $j: S \hookrightarrow \bar{S}$, and V a locally constant system of complex vector spaces on S underlying a polarizable variation of Hodge structure of weight m. Then there is a natural polarizable Hodge structure of weight $m + i$ on $H^i(\bar{S}, j_* V)$ associated to the variation of Hodge structure.*

When $V = R^m f_*\mathbf{C}$, $j_* V$ is the sheaf of local invariant "cycles". The Hodge structure above is then most interesting when $i = 1$. A Hodge structure can always be placed extrinsically on $H^1(\bar{S}, j_*R^m f_*\mathbf{C})$ via the Leray spectral sequence for $\bar{f}$ (cf. [23, § 3]). We eventually show that the Hodge structure constructed in (7.12) agrees with this other Hodge structure. It was the lack of an intrinsic definition that created most of the technical difficulties in [23]. In fact, as a consequence of (7.12), we obtain a direct proof of the optimal form of the theorem on normal functions (9.2) (cf. [23, (3.7)]), which relates sections of the associated family of intermediate Jacobians to integral cohomology classes on $\bar{X}$ of type (p, p).

The proof of the main result depends on establishing the machinery of de Rham cohomology, in both the holomorphic and L_2 (replacing C^∞) senses, for degenerating variations of Hodge structure. We then show that cohomology classes are uniquely representable by harmonic forms (the Hodge theorem), and that these forms decompose according to bi-degree (Hodge decomposition).

In Sections 1 and 2, we review Deligne's construction for the non-degenerate case. There are two good reasons for doing this. First, his ideas are carried over to the degenerate case, and we introduce them without the complication of degenerating coefficients. Secondly, his Kähler identities are local results that we will later use. The key point in Section 1 is to use the infinitesimal period relation (horizontality) to define the Hodge filtration on the complex $\Omega^{\cdot}_{S}(V)$ resolving V. This filtration induces on cohomology the Hodge structure in question. In Section 2, the Kähler identities for the Laplacians associated to the exterior derivative on V-valued forms are deduced from the classical ones.

The construction of the Hodge theory for $H^i(\bar{S}, j_* V)$ takes place in Sections 3–7. The main theme is to use differential forms that are square-summable with respect to a suitably chosen Kähler metric on S, to compute cohomology groups on $\bar{S}$. The metric is asymptotic to the Poincaré metric of the punctured disc near the points of $\bar{S} - S$. The Hodge theorem for this setting, then, really concerns L_2 cohomology on the *non-compact* manifold S, where standard theory does not apply. We are forced to work on S, since we wish to use Deligne's Kähler identities, and the "Hodge metric" on V (from a polarization of the variation of Hodge structure) has singularities where V degenerates, as described in [17]. However, we use the presence of the compactification $\bar{S}$ to circumvent this difficulty.

Degenerating coefficients (by which we mean $j_* V$, with its degenerating variation of Hodge structure) are introduced in Section 3. In Section 4, we prove a holomorphic Poincaré Lemma, giving a resolution of $j_* V$ which restricts to the usual resolution $V \to \Omega^{\cdot}_{S}(V)$ on S. The extension is describable in two ways: either in terms of the weight filtrations of the local monodromy transformations, or, more significantly, as the germs of holomorphic V-valued forms which are square-summable near the punctures. For the latter reason, we call this complex $\Omega^{\cdot}_{S}(V)_{(2)}$. In Section 5, the L_2 condition is forced through the Hodge filtration of Section 1 to filter the above complex. Then in Section 6, the relevant L_2 Poincaré and Dolbeault Lemmas for this setting are proved, to provide complexes of weakly differentiable forms that compute the hypercohomology of $\Omega^{\cdot}(V)_{(2)}$ and its successive quotients under the Hodge filtration. In then turns out to be a relatively simple matter to see that harmonic forms represent cohomology and decompose according to type (§ 7). It should be remarked here that the choice of metric for S (up to equivalence) is dictated by the numerous properties we wish it to have. Statement (6.1) is very significant, asserting that the "linear part" of the differentiation on $\Omega^{\cdot}(V)$ is a *bounded* operator in L_2 norm.

In Section 8, we describe functorial properties of the Hodge structure we have constructed. As a blanket statement, all of the reasonable assertions that one might make are true. The simplest of these gives another way to prove that the globally invariant part of V is a constant Hodge sub-structure.

Section 9 is devoted to a proof of the theorem on normal functions (cf. [23]) for general mappings, which was for me the main motivation for Hodge theory with degenerating coefficients. Given

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ S \arrow[r, hook, "j"] & \bar{S} \end{tikzcd}$$

there is a diagram:

$$\begin{tikzcd}[column sep=large]
 & \left\{\begin{array}{l}\text{Horizontal sections of the}\\ \text{family of } p\text{-th intermediate}\\ \text{Jacobians } J^p \text{ over } S\end{array}\right. \\
\left\{\begin{array}{l}\text{Codimension } p \text{ relative cycles } Z\\ \text{on } \bar{X} \text{ with } Z\cdot X_s \text{ homologically}\\ \text{equivalent to zero on } X_s=\bar{f}^{-1}(s)\\ \text{for all } s\in\bar{S}\end{array}\right\} \arrow[ur, "\nu\ \text{Abel-Jacobi mapping}"] \arrow[r, dashed, "\bar{\nu}"] \arrow[d, "\text{Cycle class}"', "\zeta"] & \left\{\begin{array}{l}\text{Horizontal normal functions}\\ =\text{horizontal sections of the}\\ \text{family of } p\text{-th generalized}\\ \text{intermediate Jacobians } \bar{J}^p \text{ over } \bar{S}\end{array}\right. \arrow[u, hook] \arrow[d, "\kappa", "\text{cohomology class}"'] \\
\ker\{H^{2p}(\bar{X}, C) \to H^0(\bar{S}, R^{2p}\bar{f}_*C)\} \arrow[r, "\alpha", "\text{from Leray spectral sequence}"'] & H^1(\bar{S}, R^{2p-1}\bar{f}_*C) \arrow[d, "\simeq"] \\
 & H^1(\bar{S}, j_*R^{2p-1}f_*C)
\end{tikzcd}$$

On the subset of cycles for which the Abel-Jacobi mapping extends to $\bar{S}$, so that $\bar{\nu}$ is defined, the square commutes. The theorem on normal functions asserts that the image of κ is the set of integral (p, p) classes. Later (15.3), we show that α is a morphism of Hodge structures. We are then perhaps a step closer to the Hodge Conjecture that $(\operatorname{im}\zeta)\otimes\mathbf{Q}$ is the set of rational (p, p) classes in $H^{2p}(\bar{X}, \mathbf{C})$.

We continue with other applications of (7.12). In Section 10, we prove miscellaneous consequences of the Hodge theory. In Section 12, we give an interesting application of the theory: to provide an interpretation of the Hodge structures of Shimura [18]. If Γ is a Fuchsian subgroup of the first kind in $G = \mathrm{SL}(2, \mathbf{R})$, and V_m is the m^{th} symmetric power of the standard representation of G, then the parabolic cohomology $\tilde{H}^1(\Gamma, V_m)$ acquires a decomposition $Q \oplus \bar{Q}$, where Q is a space of V_m-valued automorphic forms. Letting S be the quotient of the upper half-plane by Γ, we see that S is an algebraic curve. Building a variation of Hodge structure on S from V_m, we show that Q is the $(m + 1, 0)$ component of the Hodge theory with degenerating coefficients. $\bar{Q}$ is then, of course, the $(0, m + 1)$ component, and the intermediate terms vanish.

We move on to mixed Hodge theory in Sections 13, 14. While there is something for every $\tilde{S}$ between S and $\bar{S}$, we will consider now only $\tilde{S} = S$. There are functorial mixed Hodge structures on $H^i(S, V)$ (respectively $H^i_c(S, V)$, the cohomology with compact supports) with $H^i(\bar{S}, j_* V)$ as the Hodge structure for the lowest (respectively highest) non-zero weight quotient. The most interesting case is again $i = 1$, $V = R^m f_* \mathrm{C}$.

In the last section of the paper, Section 15, we show, for $V = R^m f_* \mathrm{C}$, the compatibility of the Hodge theory of $H^*(\bar{S}, j_* V)$ with that of $H^*(\bar{X}, \mathrm{C})$. By this, we mean that certain homomorphisms of cohomology associated to the Leray spectral sequence for $\bar{f}$ are morphisms of Hodge structures; for a precise statement, see (15.16). The proof begins with the compatibility of the mixed Hodge structures for $H^*(S, V)$ and $H^*(X, \mathrm{C})$ (15.4), and it uses recent results of Steenbrink [19], [20] and Clemens [4] on mixed Hodge theory.

I am indebted to Pierre Deligne for many helpful suggestions, which both extended the scope of this paper and simplified some of the proofs. Specifically, the ideas for Sections 11, 12 and 14 fall into the first category, and the abandonment of C^∞ forms in Sections 6 and 7 the second. The idea for this paper began with a conversation with Deligne over the content of [7], in which he shared with me his belief that a Hodge theory with degenerating coefficients should exist, and that it might come from some L_2 cohomology on the noncompact curve. His optimism stemmed from a formal comparison with étale cohomology. Throughout this work, I have been guided by the desire to explain the following more earthly phenomenon. Since the methods are blind to the actual degeneration of V, assume $V = \mathrm{C}$, but take $S \neq \bar{S}$. The harmonic forms are generated by constants in degree 0, holomorphic differentials and their conjugates in degree 1, and the Kähler class in degree 2. One checks the L_2 condition to see that $1 \in L_2$ (S has finite area) and that any L_2 holomorphic differential on S actually has removable singularities, i.e., extends holomorphically to $\bar{S}$. Thus, we see the usual Hodge decomposition of $H^*(\bar{S}, \mathrm{C})$ arising in a different way.

I would also like to thank William Sweeney for several discussions concerning differential operators, relevant in Section 7, which also led to an earlier proof of the Hodge theorem for $j_* V$.

I am grateful to the referee for pointing out the need for corrections in the original manuscript, and for suggesting simplifications, clarifications and improvements. The results in this paper have been announced in [24].

Notations and conventions

An attempt has been made to use the same notation as appears in the literature, especially in [8], [9], [17] and [23], whenever feasible.

Complex projective varieties are viewed solely as analytic varieties, and thus all coherent sheaves are in the analytic sense.

A holomorphic (or anti-holomorphic, or C^∞) bundle will not be distinguished from its locally-free analytic (respectively anti-holomorphic, respectively C^∞) sheaf of germs of sections. If $\mathcal{V}$ is a holomorphic (or anti-) bundle, $\mathcal{E}(\mathcal{V})$ will denote the underlying C^∞ bundle.

$\mathcal{O}$ and Ω^p are used to denote $\mathcal{O}_S$ and Ω^p_S.

The letters C and K are used in Sections 6, 11 to represent unspecified constants which appear in the norm estimates.

The temptation to shift the weight of a Hodge structure so as to make $H^{p,q} = 0$ whenever $p < 0$ or $q < 0$ has been resisted, with the one exception of Section 10a.

In defining finite filtrations (e.g., on complexes of sheaves) indexed by **Z**, we will omit mentioning the infinitely many sub-complexes which are zero or everything.

1. Gauss-Manin and the Hodge filtration

Let $f: X \to S$ be a smooth family of complex projective varieties over the non-singular quasi-projective variety S. Let V denote a local system $R^m f_* \mathbf{C}$ of cohomology along the fibers, and $\mathcal{V} = \mathcal{O}_S \otimes_{\mathbf{C}} V$ the corresponding cohomology bundle over S. $\mathcal{V}$ carries a natural flat holomorphic connection $\nabla = \partial \otimes 1$ (*Gauss-Manin*), from which we obtain a complex $\Omega^\cdot(V) = \Omega^\cdot_S \otimes_{\mathbf{C}} V = \Omega^\cdot_S \otimes_{\mathcal{O}_S} \mathcal{V}$. On $\mathcal{V}$, there is the decreasing filtration $\{\mathcal{F}^p\}$ of Hodge filtration sub-bundles. The filtration is not preserved under Gauss-Manin, but instead, there is the so-called *infinitesimal period relation*

$$\nabla \mathcal{F}^p \subset \Omega^1 \otimes \mathcal{F}^{p-1} \tag{1.1}$$

(where we have begun to suppress the S: $\Omega^1 = \Omega^1_S$). Letting $\mathscr{G}\!r^p$ denote the successive quotient bundles $\mathcal{F}^p/\mathcal{F}^{p+1}$, we note that the induced mapping

$$\nabla : \mathcal{G}_z{}^p \longrightarrow \Omega^1 \otimes \mathcal{G}_z{}^{p-1}$$

is $\mathcal{O}$-linear. Gauss-Manin is easily extended to an operator D on $\mathcal{E}^{\cdot}(V) = \mathcal{E}_S^{\cdot} \otimes_{\mathbf{C}} V$, the C^∞ differential forms with values in V, by taking $D = d \otimes 1$, so that for a (local) C^∞ r-form ϕ and holomorphic section v of $\mathcal{V}$,

$$D(\phi \otimes v) = d\phi \otimes v + (-1)^r \phi \wedge \nabla v .$$

It follows, then, from (1.1) that

$$D\mathcal{E}^{r,s}(\mathcal{F}^p) \subset \mathcal{E}^{r+1,s}(\mathcal{F}^{p-1}) \oplus \mathcal{E}^{r,s+1}(\mathcal{F}^p) \tag{1.2}$$

where $\mathcal{E}^{r,s}$ is the sheaf of C^∞ forms on S of type (r, s).

If we pass to the underlying C^∞ bundles, which we will denote by $\mathcal{E}(\mathcal{V})$, etc., $\mathcal{E}(\mathcal{V})$ splits according to the Hodge decomposition as

$$\mathcal{E}(\mathcal{V}) \simeq \bigoplus_k \mathcal{H}^{k,m-k} \tag{1.3}$$

with

$$\mathcal{E}(\mathcal{F}^p) \simeq \bigoplus_{k \geq p} \mathcal{H}^{k,m-k} ,$$

and hence

$$\mathcal{E}(\mathcal{G}_z{}^p) \simeq \mathcal{H}^{p,m-p} .$$

The C^∞ bundles $\mathcal{H}^{p,q}$ satisfy $\mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}}$ and admit the useful description

$$\mathcal{H}^{p,q} = \mathcal{E}(\mathcal{F}^p) \cap \mathcal{E}(\overline{\mathcal{F}^q}) , \tag{1.4}$$

where $\bar{\mathcal{F}}^q$ is the conjugate bundle to $\mathcal{F}^q$, and $\mathcal{E}(\bar{\mathcal{F}}^q)$ is the underlying C^∞ sub-bundle of $\mathcal{E}(\mathcal{V})$.

Returning to the discussion of Gauss-Manin, if we view everything from the complex conjugate viewpoint, we obtain the analogues of (1.1) and (1.2), namely

$$\nabla \overline{\mathcal{F}^q} \subset \bar{\Omega}^1 \otimes \overline{\mathcal{F}^{q-1}} , \tag{1.5}$$

$$D\mathcal{E}^{r,s}(\overline{\mathcal{F}^q}) \subset \mathcal{E}^{r,s+1}(\overline{\mathcal{F}^{q-1}}) \oplus \mathcal{E}^{r+1,s}(\overline{\mathcal{F}^q}) , \tag{1.6}$$

and combining this with (1.2), we obtain

$$D\mathcal{E}^{r,s}(\mathcal{F}^p \cap \overline{\mathcal{F}^q}) \subset \mathcal{E}^{r+1,s}(\mathcal{F}^{p-1} \cap \overline{\mathcal{F}^q}) \oplus \mathcal{E}^{r,s+1}(\mathcal{F}^p \cap \overline{\mathcal{F}^{q-1}}) , \tag{1.7}$$

or

$$D\mathcal{E}^{r,s}(\mathcal{H}^{p,q}) \subset \mathcal{E}^{r+1,s}(\mathcal{H}^{p,q} \oplus \mathcal{H}^{p-1,q+1}) \oplus \mathcal{E}^{r,s+1}(\mathcal{H}^{p,q} \oplus \mathcal{H}^{p+1,q-1}) . \tag{1.8}$$

In other words, under the Hodge decomposition, the Gauss-Manin operator splits into four components:

$$\begin{aligned} &\text{(i)} & \partial' &: \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) \longrightarrow \mathcal{E}^{r+1,s}(\mathcal{H}^{p,q}) \\ &\text{(ii)} & \bar{\partial}' &: \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) \longrightarrow \mathcal{E}^{r,s+1}(\mathcal{H}^{p,q}) \\ &\text{(iii)} & \nabla' &: \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) \longrightarrow \mathcal{E}^{r+1,s}(\mathcal{H}^{p-1,q+1}) \\ &\text{(iv)} & \bar{\nabla}' &: \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) \longrightarrow \mathcal{E}^{r,s+1}(\mathcal{H}^{p+1,q-1}) . \end{aligned} \tag{1.9}$$

The following is an immediate consequence of the definitions:

(1.10) LEMMA. *The following diagrams commute*:

$$\begin{array}{ccc} \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) & \xrightarrow{\bar\partial'} & \mathcal{E}^{r,s+1}(\mathcal{H}^{p,q}) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathcal{E}^{r,s}(\mathcal{G}_z{}^p) & \xrightarrow{\bar\partial} & \mathcal{E}^{r,s+1}(\mathcal{G}_z{}^p) \ , \end{array}$$

$$\begin{array}{ccc} \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) & \xrightarrow{\nabla'} & \mathcal{E}^{r+1,s}(\mathcal{H}^{p-1,q+1}) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathcal{E}^{r,s}(\mathcal{G}_z{}^p) & \xrightarrow{1\wedge\nabla} & \mathcal{E}^{r+1,s}(\mathcal{G}_z{}^{p-1}) \ . \end{array}$$

(1.11) COROLLARY. ∇' *and* $\bar\nabla'$ *are linear over* C^∞ *functions* (i.e., ∇' *and* $\bar\nabla'$ *are* 0^{th} *order operators*).

The following construction is due to Deligne. A *Hodge filtration* is placed on $\Omega^\cdot(V)$ by setting

(1.12) $$F^p\Omega^r(V) = \Omega^r(\mathcal{F}^{p-r}) \ .$$

Because of (1.1), this defines a filtration $\{F^p\Omega^\cdot(V)\}$ of $\Omega^\cdot(V)$ by sub-complexes. For C^∞ differential forms, one defines

(1.13) $$\mathcal{E}^\cdot(V)^{P,Q} = \bigoplus_{\substack{p+r=P \\ q+s=Q}} \mathcal{E}^{r,s}(\mathcal{H}^{p,q}) \ ,$$

(1.14) $$F^p\mathcal{E}^k(V) = \bigoplus_{P\geq p} \mathcal{E}^k(V)^{P,Q} \simeq \bigoplus_{r+s=k} \mathcal{E}^{r,s}(\mathcal{F}^{p-r}) \ ,$$

i.e., $F^p\mathcal{E}^\cdot(V)$ consists of all forms having at least a total of p holomorphic parts coming from both S and $\mathcal{V}$. The $\{F^p\mathcal{E}^\cdot(V)\}$ filter the complex $\mathcal{E}^\cdot(V)$, and the successive quotients, denoted $\mathrm{Gr}^p_F\mathcal{E}^\cdot(V)$, are complexes under the differential $D'' = \bar\partial' + \nabla'$.

(1.15) LEMMA. $\mathrm{Gr}^p_F\Omega^\cdot(V) \to \mathrm{Gr}^p_F\mathcal{E}^\cdot(V)$ *is the Dolbeault resolution.*

Proof. From (1.14), $\mathrm{Gr}^p_F\mathcal{E}^\cdot(V)$ is the double complex of sheaves

$$K^{r,s} = \mathcal{E}^{r,s}(\mathcal{H}^{p-r,m-p+r})$$

with differential D''. By Lemma (1.10), this complex is isomorphic to

$$\tilde K^{r,s} = \mathcal{E}^{r,s}(\mathcal{G}_z{}^{p-r})$$

which is clearly giving the Dolbeault resolution (for each r) of $\Omega^r(\mathcal{G}_z{}^{p-r})$, the sheaf in degree r in $\mathrm{Gr}^p_F\Omega^\cdot(V)$.

(1.16) COROLLARY. (i) $\mathbf{H}^*(S, \mathrm{Gr}^p_F\Omega^\cdot(V)) \simeq H^*(\Gamma(S, \mathrm{Gr}^p_F\mathcal{E}^\cdot(V)))$
(ii) $\mathbf{H}^*(S, F^p\Omega^\cdot(V)) \simeq H^*(\Gamma(S, F^p\mathcal{E}^\cdot(V)))$.

Since $\Omega^\cdot_S$ resolves $\mathbf{C}_S$, we also have

(1.17) $$\mathbf{H}^*(S, \Omega^\cdot(V)) \simeq H^*(S, V) \ .$$

2. The Kähler identities

Before proceeding any further, it is useful to introduce the natural "abstract" setting of this paper. Observe first that the entire construction of Section 1 depends only on the existence of a local system V with an underlying real structure (i.e., there exists a real local system $V_{\mathbf{R}}$ such that $V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$), and a bounded decreasing filtration $\{\mathcal{F}^p\}$ on $\mathcal{V} = \mathcal{O}_S \otimes_{\mathbf{C}} V$ satisfying (1.1), (1.3) and (1.4). This collection of data is known as a *variation of Hodge structure of weight m*. In addition, we require that V be *polarizable*: there should exist a non-degenerate flat bilinear pairing, denoted $(,)$, defined over $\mathbf{R}$,[1] with the property that for each $s \in S$,

$$\langle v, w \rangle_s = i^{p-q}(v, \bar{w})_s$$

is a positive-definite Hermitian inner product on $H_s^{p,q}$, the subspace of V_s determined by $\mathcal{H}^{p,q}$, and that the Hodge decomposition is orthogonal. In the case $V = R^m f_* \mathbf{C}$ for $f\colon X \to S$ (the *geometric* case), the Kähler (hyperplane) class $\eta \in \Gamma(S, R^2 f_* \mathbf{Z})$ splits V into its primitive decomposition (see [22]), which is compatible with the Hodge filtration, and the primitive part of V is polarized by the pairing induced from cup product,

$$(v, w)_s = (-1)^{m(m-1)/2} \langle [X_s], v_s \smile \eta_s^{n-m} w_s \rangle \ .$$

Summing over the primitive decomposition, one gets a polarization on each $R^m f_* \mathbf{C}$. Note that in the geometric case, $\mathcal{F}^{m+1} = 0$ and $\mathcal{F}^0 = \mathcal{V}$, i.e., $\mathcal{H}^{p,q} = 0$ if p or q is negative, though we need not presume this in general.

With the exception of (2.13), the methods and results in the remainder of this section are essentially borrowed from Deligne's manuscript [7]. Recall that we have a direct sum decomposition

$$\mathcal{E}^k(V) = \bigoplus_p \mathrm{Gr}^p \mathcal{E}^k(V) \ .$$

We aim to show that this decomposition passes to cohomology, as it does in classical Hodge theory ($V = \mathbf{C}$). $\mathcal{E}^{\bullet}(V)$ has the bigradation (1.13), under which D splits as the direct sum $D' + D''$, where $D' = \partial' + \bar{\nabla}'$ is an operator of type (1, 0) and $D'' = \bar{\partial}' + \nabla'$ is of type (0, 1).

We now set up norms for $\Gamma(S, \mathcal{E}^r(V))$. On $\mathcal{V}$, there are two pairings: the pairing $((v, w)) = (v, \bar{w})$ which is horizontal, i.e.,

$$d((v, w)) = ((Dv, w)) + ((v, Dw)) \ ,$$

[1] In practice, V is defined over $\mathbf{Z}$ and $(,)$ over $\mathbf{Q}$, so rationality is often included in the definition of a variation of Hodge structure. For analytic questions, as Schmid remarks in [17], the extra structure is only excess baggage, so we drop it for the purposes of this paper. However, if V has additional structure, it will be passed along in our constructions. Also, we have separated the notion of the polarization from the definition of a variation of Hodge structure, in contrast to [17, p. 220].

and the positive-definite Hermitian form

$$\langle v, w\rangle = \big((C_V v, w)\big) = \big(v, C_V w)\big) \tag{2.1}$$

where C_V is the *Weil operator* of V, given as the direct sum of the scalar operators i^{p-q} on $H^{p,q}$. Taking the tensor product of either of these with the usual norms on $\mathcal{E}_S^\cdot$ induced by the Kähler metric on S, we obtain (pointwise) pairings on $\mathcal{E}^\cdot(V)$. By integrating these against the volume form, one obtains pairings on the global sections (we impose a square-summability hypothesis if S is non-compact); for $\alpha, \beta \in \Gamma\big(S, \mathcal{E}^k(V)\big)$, these will also be denoted $\big((\alpha, \beta)\big)$ and $\langle \alpha, \beta\rangle$, the latter being the norm form, and the two are related by (2.1). By consideration of type alone, we have

(2.2) PROPOSITION. *The (P, Q)-decomposition of $\Gamma\big(S, \mathcal{E}^k(V)\big)$ is orthogonal under $\langle\, ,\rangle$. In fact, the spaces $\Gamma\big(S, \mathcal{E}^{r,s}(\mathcal{H}^{p,q})\big)$ are all mutually orthogonal.*

Let $L_V\colon \Gamma\big(S, \mathcal{E}^k(V)\big) \to \Gamma\big(S, \mathcal{E}^{k+2}(V)\big)$ denote the operation of exterior multiplication with the Kähler form of S. For the operators D, D', D'', and L_V respectively, let $\mathfrak{d}$, $\mathfrak{d}'$, $\mathfrak{d}''$, and Λ_V denote their (formal) adjoints with respect to the inner product $\big($so for $\alpha \in \Gamma\big(S, \mathcal{E}^k(V)\big)$ and $\beta \in \Gamma\big(S, \mathcal{E}^{k-1}(V)\big)$, with one compactly supported, $\langle \alpha, D\beta\rangle = \langle \mathfrak{d}\alpha, \beta\rangle$, etc.$\big)$, and let $\mathfrak{d}_0$, $\mathfrak{d}_0'$, $\mathfrak{d}_0''$, Λ_0 denote the corresponding adjoints under the indefinite pairing $\big((\, ,)\big)$. For the purpose of simplifying the notation, we identify an operator on $\mathcal{E}_S^\cdot$ with the induced operator on $\mathcal{E}^\cdot(V) = \mathcal{E}_S^\cdot \otimes_{\mathbf{C}} V$.

(2.3) LEMMA. (i) $\mathfrak{d}_0 = -*D*$, *where $*$ is the Hodge star operator on $\mathcal{E}_S^\cdot$.*

(ii) *Λ_0 is the usual operator Λ on $\mathcal{E}_S^\cdot$, adjoint to L.*

Proof. (i) says that $\mathfrak{d}_0$ is, under a local horizontal trivialization of V, equal to the adjoint of d on $\mathcal{E}_S^\cdot$. This is obvious, as is statement (ii).

(2.4) PROPOSITION. *Let P be an operator on $\mathcal{E}^\cdot(V)$, and let π (resp. π_0) be the adjoint of P relative to the positive-definite (resp. horizontal) pairing. Then $\pi = C_V^{-1}\pi_0 C_V$.*

Proof.

$$\begin{aligned}\langle P\alpha, \beta\rangle &= \big((P\alpha, C_V\beta)\big) = \big((\alpha, \pi_0 C_V\beta)\big)\\ &= \big((\alpha, C_V C_V^{-1}\pi_0 C_V\beta)\big)\\ &= \langle \alpha, C_V^{-1}\pi_0 C_V\beta\rangle\ .\end{aligned}$$

In particular, if the commutator $[\pi_0, C_V]$ vanishes, $\pi = \pi_0$. Thus,

(2.5) COROLLARY. $\Lambda_V = \Lambda$.

One defines the Laplacian operators of D, D', and D'' in the usual way:

$$\square_D = D\mathfrak{d} + \mathfrak{d}D$$
$$\square_{D'} = D'\mathfrak{d}' + \mathfrak{d}'D'$$
$$\square_{D''} = D''\mathfrak{d}'' + \mathfrak{d}''D'' .$$

(2.6) PROPOSITION. $\square_D$, $\square_{D'}$ *and* $\square_{D''}$ *are formally self-adjoint elliptic differential operators.*

Proof. The formal self-adjointness is obvious from the definitions. For the ellipticity, one argues as in [22, p. 120], observing that by (1.11), D'' and $\bar{\partial}$ have the same symbol.

The key to the Hodge decomposition is the relation among the various Laplacians:

(2.7) THEOREM. $\square_{D'} = \square_{D''}$ *and* $\square_D = \square_{D'} + \square_{D''} = 2\square_{D''}$.

Proof. The proof of these equations is formally identical to the proof of the corresponding formulas in classical Hodge theory. It hinges on the following computation:

(2.8) PROPOSITION. $[\Lambda, D] = -\mathfrak{d}^C$, *where* $\mathfrak{d}^C = C^{-1}\mathfrak{d}C$, *and* $C = C_0C_V$ *is the total Weil operator, in which* C_0 *is the Weil operator of* $\mathcal{E}^{\bullet}_S$.

Proof. By (2.3) and (2.5) we infer from the classical case the identity

$$[\Lambda, D] = -\mathfrak{d}_0^{C_0} = -C_0^{-1}\mathfrak{d}_0C_0 = C_0\mathfrak{d}_0C_0^{-1} .$$

Applying (2.4), we get $\mathfrak{d}_0 = C_V\mathfrak{d}C_V^{-1}$. Therefore

$$[\Lambda, D] = C_0C_V\mathfrak{d}C_V^{-1}C_0^{-1} = C\mathfrak{d}C^{-1} = -C^{-1}\mathfrak{d}C .$$

The remainder of the proof of (2.7) follows from (2.8) as in the classical case (see [22, p. 186]).

At this point, *when S is compact* one can invoke the standard machinery of elliptic operators on compact manifolds and conclude

(2.9) THEOREM (Deligne [7]). *Let S be a non-singular projective variety (compact Kähler manifold), V a locally constant system of complex vector spaces which underlies a polarizable variation of Hodge structure of weight m over S. Then* $H^i(S, V)$ *has a natural Hodge structure of weight* $i + m$ *associated to the variation of Hodge structure.*

While it would be redundant to write out a proof of this theorem, it will be useful to recall the main points of the proof, for this will serve as an outline for the next five sections of the paper.

(2.10) *Outline.* (i) From (1.17), $H^i(S, V)$ is isomorphic to the hypercohomology group $\mathbf{H}^i(S, \Omega^{\bullet}(V))$, and the latter group is computed as the cohomology of the complex of C^∞ differential forms $\Gamma(S, \mathcal{E}^\cdot(V))$.

(ii) As a graded sheaf, $\mathcal{E}^\cdot(V)$ splits as a direct sum of sub-sheaves $\mathrm{Gr}^p_F\mathcal{E}^\cdot(V)$. The latter is a complex under the differential D'', and by (1.16, i), the cohomology groups of $\Gamma(S, \mathrm{Gr}^p_F\mathcal{E}^\cdot(V))$ are also hypercohomology groups, namely $\mathbf{H}^*(S, \mathrm{Gr}^p_F\Omega^\cdot(V))$.

(iii) The cohomology groups in (i) and (ii) are isomorphic to spaces of *harmonic* forms, i.e., solutions of $\square_D\phi = 0$ or $\square_{D''}\phi = 0$ respectively. In each cohomology class, there is exactly one harmonic representative.

(iv) From (2.7), it follows that a form is harmonic if and only if all of its (P, Q)-components are harmonic (in any of the equivalent senses). If $\mathfrak{h}^k(V)$ (resp. $\mathfrak{h}^{P,Q}(V)$) denotes the space of harmonic forms of degree k (resp. of type (P, Q)) with values in V, then

$$\mathfrak{h}^k(V) = \bigoplus_{P+Q=m+k}\mathfrak{h}^{P,Q}(V)\ .$$

This decomposition passes to $H^k(S, V)$ via (ii) and (iii) above.

(2.11) COROLLARY. (i) *The spectral sequence*

$$E_1^{p,q} = \mathbf{H}^{p+q}(S, \mathrm{Gr}^p_F\Omega^\cdot(V)) \Longrightarrow \mathbf{H}^{p+q}(S, \Omega^\cdot(V)) \simeq H^{p+q}(S, V)$$

degenerates at E_1.

(ii) *The filtration on* $\mathbf{H}^i(S, \Omega^\cdot(V))$ *defined by* $\{F^p\Omega^\cdot(V)\}$ *is the Hodge filtration*

$$F^pH^i(S, V) \simeq \bigoplus_{\substack{P+Q=i+m\\ P\geq p}}\mathfrak{h}^{P,Q}(V)\ .$$

(iii) *There are canonically isomorphic short exact sequences*

$$0 \longrightarrow \mathbf{H}^q(S, F^p\Omega^\cdot(V)) \longrightarrow H^q(S, V) \longrightarrow \mathbf{H}^q(S, \Omega^\cdot(V)/F^p\Omega^\cdot(V)) \longrightarrow 0$$

and

$$0 \longrightarrow F^pH^q(S, V) \longrightarrow H^q(S, V) \longrightarrow H^q(S, V)/F^pH^q(S, V) \longrightarrow 0\ .$$

In particular, the Hodge structure on $H^i(S, V)$ *is independent of the Kähler metric on* S *and the polarization of* V*, though the two combine to determine a polarization of* $H^i(S, V)$.

There are several consequences of the Hodge theory that follow formally as in the classical case (Poincaré duality, primitive decomposition, etc.), but these corollaries will be left to the reader.

To illustrate the power of (2.9), let V underlie a variation of Hodge structure of odd weight $m = 2p - 1$, which is defined over $\mathbf{Z}$. One may associate families of complex tori, whose sheaves of germs of cross-sections $\mathcal{J}^p$ are defined by the short exact sequences

$$0 \longrightarrow V_{\mathbf{Z}} \longrightarrow \mathcal{V}/\mathcal{F}^p \longrightarrow \mathcal{J}^p \longrightarrow 0\ . \tag{2.12}$$

When we take $V = R^m f_* \mathbf{C}$ for some smooth family of Kähler manifolds, the tori above are the p^{th} *intermediate Jacobians* of the fibers. Elements of $H^0(S, \mathcal{J}^p)$ are called *normal functions*; a normal function is said to be *horizontal* if it is annihilated by the mapping induced from ∇:

$$\begin{tikzcd} \mathcal{V}/\mathcal{F}^p \arrow[r, "\nabla"] \arrow[d] & \Omega^1 \otimes \mathcal{V}/\mathcal{F}^{p-1} \\ \mathcal{J}^p \arrow[ur, dashed] & \end{tikzcd} .$$

If $\nu \in H^0(S, \mathcal{J}^p)$, then its image under the connecting homomorphism, $\delta(\nu) \in H^1(S, V_{\mathbf{Z}})$, is called the cohomology class of the normal function. (See [23] for remarks concerning the significance of these concepts in the geometric case.) This class maps into the integral lattice in $H^1(S, V)$. As the principal corollary of Theorem (2.9), we have the following:

(2.13) THEOREM. *Let V underlie a variation of Hodge structure of odd weight $m = 2p - 1$ over the complete non-singular curve S, which is defined over $\mathbf{Z}$. Then the set of all cohomology classes of horizontal normal functions is equal to the set of integral classes of type (p, p).*

Proof. Incorporating horizontality into (2.12), we have a short exact sequence of sheaves (cf. [23, p. 198]),

$$0 \longrightarrow V_{\mathbf{Z}} \longrightarrow (\mathcal{V}/\mathcal{F}^p)_h \longrightarrow \mathcal{J}^p_h \longrightarrow 0$$

from which we obtain the exact sequence

$$H^0(S, \mathcal{J}^p_h) \longrightarrow H^1_{\mathbf{Z}} \xrightarrow{\theta} H^1(S, (\mathcal{V}/\mathcal{F}^p)_h) ,$$

where we have written $H^1_{\mathbf{Z}}$ for $H^1(S, V_{\mathbf{Z}})$, so we are looking for the kernel of θ. On the other hand, the set of integral elements of type (p, p) is precisely

$$\ker \{H^1_{\mathbf{Z}} \longrightarrow H^1_{\mathbf{C}}/F^p H^1_{\mathbf{C}}\} .$$

From the Hodge theory (2.11, iii),

$$H^1_{\mathbf{C}}/F^p H^1_{\mathbf{C}} = \mathbf{H}^1(S, \Omega^\bullet(V)/F^p\Omega^\bullet(V)) ,$$

and $\Omega^\bullet(V)/F^p\Omega^\bullet(V)$ is the complex

$$\mathcal{V}/\mathcal{F}^p \xrightarrow{\nabla} \Omega^1 \otimes (\mathcal{V}/\mathcal{F}^{p-1}) . \tag{2.14}$$

Since ∇ is surjective (in fact, $\nabla: \mathcal{V} \to \Omega^1 \otimes \mathcal{V}$ is surjective), (2.14) gives a resolution of $\ker \nabla = (\mathcal{V}/\mathcal{F}^p)_h$. Thus

$$H^1_{\mathbf{C}}/F^p H^1_{\mathbf{C}} \simeq H^1(S, (\mathcal{V}/\mathcal{F}^p)_h) ,$$

and the theorem follows.

(2.15) *Remark.* Taking $V = R^{2p-1}f_*\mathbf{C}$ in the above, we obtain a sharp version of the theorem on normal functions [23, (3.7)] for smooth fibrations.

Since the Leray spectral sequence for f degenerates at E_2 [6], there is another Hodge structure induced on $H^i(S, R^m f_*\mathbf{C})$ from $H^{i+m}(X)$. (This is true for S of arbitrary dimension.) In the case $\dim S = 1$, $i = 1$, $m = 2p - 1$, it comes from the isomorphism

$$H^1(S, R^{2p-1}f_*\mathbf{C}) \simeq \ker\{H^{2p}(X) \longrightarrow H^{2p}(X_s)\}/\mathrm{im}\{H^{2p-2}(X_s) \xrightarrow{\text{Gysin}} H^{2p}(X)\}$$

for any $s \in S$.

(2.16) PROPOSITION. *For all values of i and m, the two Hodge structures on $H^i(S, R^m f_*\mathbf{C})$ coincide.*

Proof (Deligne). In addition to its Hodge filtration $\{F^p\}$, the complex $\Omega_X^\bullet$ has a filtration $\{L^p\}$ determined by the mapping $f\colon X \to S$. It is given by

$$L^p\Omega_X^\bullet = \text{image of } f^*\Omega_S^p \otimes \Omega_X^\bullet ;$$

so

$$\mathrm{Gr}_L^p\Omega_X^\bullet = f^*\Omega_S^p \otimes \Omega_{X/S}^\bullet[-p] .$$

With $Rf_*\Omega_X^\bullet$ denoting the associated object in the derived category of the category of sheaves on S, the spectral sequence determined by L for its cohomology sheaves (i.e., the hyperderived sheaves of $\Omega_X^\bullet$) is

$${}_L\mathbf{E}_1^{p,q} = \mathcal{H}^{p+q}(\mathrm{Gr}_L^p Rf_*\Omega_X^\bullet) \simeq \Omega_S^p \otimes \mathbf{R}^q f_*\Omega_{X/S}^\bullet \simeq \Omega_S^p(V^q) \qquad (V^q = R^q f_*\mathbf{C}) ,$$

with differential d_1 given by ∇ [15]. Shifting the spectral sequence via the *filtration décalée* [8, p. 15], we obtain a mapping

$${}_{\mathrm{Dec}(L)}\mathbf{E}_0^{p,q} \longrightarrow {}_L\mathbf{E}_1^{2p+q,-p} = \Omega_S^{2p+q} \otimes V^{-p} ,$$

i.e.,

$$\mathrm{Gr}_{\mathrm{Dec}(L)}^p Rf_*\Omega_X^\bullet \longrightarrow \Omega_S^\bullet(V^{-p})[p] , \tag{2.17}$$

which is a quasi-isomorphism. The associated spectral sequence of hypercohomology

$$\begin{aligned} E_1^{p,q} &= \mathbf{H}^{p+q}\big(S, \Omega_S^\bullet(V^{-p})[p]\big) \\ &= H^{2p+q}(S, R^{-p}f_*\mathbf{C}) \end{aligned}$$

is the Leray spectral sequence for f (after a shift of indices).

Since

$$\mathrm{Gr}_L^p Rf_*\Omega_X^\bullet = \Omega_S^p \otimes Rf_*\Omega_{X/S}^\bullet[-p]$$

and the spectral sequence

$$E_1^{p,q} = R^q f_*\Omega_{X/S}^p \Longrightarrow \mathbf{R}^{p+q}f_*\Omega_{X/S}^\bullet$$

degenerates at E_1 (by classical Hodge theory), it follows by [8, (1.3.15)] that the morphism (2.17) is an isomorphism in the filtered derived category (with respect to F). In particular, the filtration induced by F on the former is carried to the filtration induced by F on $\Omega_S^\cdot(V^q)$, which is easily seen to be the Hodge filtration in (1.12). Because of this, it follows by (2.11) that the cohomology spectral sequence for F on $\mathrm{Gr}^p_{\mathrm{Dec}(L)} Rf_* \Omega_X^\cdot$ degenerates at E_1: i.e., the differential in the complex $R\Gamma\, \mathrm{Gr}^p_{\mathrm{Dec}(L)} Rf_* \Omega_X^\cdot$ is strictly compatible with F. As $d_r = 0$ for all differentials in the Leray spectral sequence [6], we obtain (2.16) from Deligne's lemma on two filtrations [8, (1.3.17)] (viewing both Hodge structures as being defined on the E_∞ term).

Thus, Theorem (2.12) has topological and geometric significance.

3. Degenerating coefficients: the preliminaries

It was remarked in the introduction that smooth families of projective varieties over a complete base variety S, which would give rise to variations of Hodge structure satisfying the hypothesis of Theorem (2.9), are rather rare. We turn now to the problem on a quasi-projective base S, and let $\bar{S}$ be a projective compactification of S so that $\Sigma = \bar{S} - S$ is a union of smooth divisors with normal crossings.

The underlying theme of this paper is to use complexes of differential forms with values in V, both L_2 and holomorphic, which are regular on S and which have specified growth near Σ, to compute cohomology groups of sheaves *on* $\bar{S}$. This will generalize items (i) and (ii) in (2.10).

The first step is to construct a good Kähler metric on S with singularities along Σ, which, though not removable, are negligible in the sense that differential analysis on S is like that of $\bar{S}$. Since Σ has normal crossings on $\bar{S}$, for any point $t \in \Sigma$ there is a coordinate neighborhood U of t that is isomorphic to Δ^n, the unit polydisc, in which $S \cap U = \{z = (z_1, \cdots, z_n)\colon z_1 \neq 0, \cdots, z_l \neq 0 \text{ for some } 1 \leqq l \leqq n\} \simeq (\Delta^*)^l \times \Delta^{n-l}$, where Δ^* is the punctured disc. Such coordinates will be called *special coordinates*. Recall that the Poincaré metric of Δ^* is given by $(dx^2 + dy^2)/|z|^2 \log^2 |z|^2$, and has associated 2-form $(i/2)\, dz \wedge d\bar{z}/|z|^2 \log^2 |z|^2$. (For notational convenience, we identify a metric with its 2-form.) We regard the product $(\Delta^*)^l \times \Delta^{n-l}$ as having the metric

$$(3.1) \qquad \frac{i}{2}\left[\sum\nolimits_{k=1}^{l} \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2} + \sum\nolimits_{k=l+1}^{n} dz_k \wedge d\bar{z}_k\right],$$

which possesses the singularity of the Poincaré metric near the punctures (and away from the outer boundaries).

(3.2) PROPOSITION. *Let $\bar{S}$ be a projective variety (compact Kähler manifold), Σ a union of smooth divisors on $\bar{S}$ with only normal crossings,*

$S = \bar{S} - \Sigma$. *Then there exists a Kähler metric on* S *which in special coordinates is equivalent to the metric* (3.1), *in the sense that the two norms are mutually uniformly bounded.*

Proof. The construction is essentially the same as that appearing in [5]. Let $\Sigma = \bigcup_{i=1}^N \Sigma_i$, where each Σ_i is non-singular. Let $[\Sigma_i]$ be the line bundle on $\bar{S}$ associated to Σ_i, σ_i a holomorphic section of $[\Sigma_i]$ which vanishes to first order on Σ_i, and $\|\ \|_i$ the norm from a C^∞ Hermitian metric on $[\Sigma_i]$ normalized so that $\|\sigma_i\|_i < 1$ on $\bar{S}$. Let ω be the Kähler form of $\bar{S}$. The desired metric is

$$\eta = \Big(k\omega - \frac{1}{2}\sum_{i=1}^N \partial\bar{\partial} \log \log^2 \|\sigma_i\|_i^2\Big)$$

for k sufficiently large. In special coordinates U in which Σ_i is defined by $z = 0$,

$$\|\sigma_i\|_i^2 = |z|^2 e^u$$

for some function u that is C^∞ on U. Then

$$(3.3) \qquad -\frac{1}{2}\partial\bar{\partial}\log\log^2\|\sigma_i\|_i^2 = \frac{1}{(\log|z|^2 + u)^2}\Big(\frac{dz}{z} + \partial u\Big)\wedge\Big(\frac{d\bar{z}}{\bar{z}} + \bar{\partial}u\Big) - \frac{1}{\log|z|^2 + u}\partial\bar{\partial}u\ .$$

It is now clear that η is positive near Σ, with singularities like (3.1); the term $k\omega$ is added to make η positive on all of S. It is also evident that the two metrics are equivalent if k is taken sufficiently large.

The metric η has many excellent properties, as we shall see, but we can at least state some of the more obvious ones.

(3.4) PROPOSITION. S, *endowed with the metric* η, *is a complete manifold of finite volume.*

Proof. Because of the compactness of $\bar{S}$ and the asymptotic properties of η, it suffices to verify the statment of the proposition for the subset $\{z: 0 < |z| \leqq A\}$ of Δ^*, for any $A < 1$, with the Poincaré metric. Using polar coordinates, we see that this follows from the computations

$$\int_0^A \frac{dr}{r\log r} = \log\log r]_0^A = \infty\ ,$$

$$\int_0^{2\pi}\int_0^A \frac{r\,dr\,d\theta}{r^2\log^2 r} = 2\pi\log^{-1} r]_0^A < \infty\ .$$

We turn now to the degeneration of $(V, \{\mathcal{F}^p\})$ on Σ. This problem was studied in great detail by Wilfried Schmid. His best results were obtained in one variable (the SL_2-orbit theorem and its consequences [17, §§ 5, 6]); generalizations to several variables are anticipated, but are yet unknown.[2] We must therefore restrict ourselves to the case where S is a curve, and so Σ consists of a finite number of points. The points of Σ will be called the *singular points* of V, even though the variation of Hodge structure may extend across some of them. In fact, the discussions in this paper ignore the removability of singularities, and therefore include the case of constant coefficients $V = \mathbf{C}$.

We localize our attention to one of the singular points, and we establish, once and for all, a complex coordinate t centered at the point; thus $S = \Delta^*$ for the purposes of local arguments. Choosing a base point $s \in S$, there is a monodromy transformation γ obtained by following elements of V_s horizontally around the origin.[3] By the Monodromy Theorem [17, (6.1)], γ is quasi-unipotent if V is defined over $\mathbf{Z}$: $(\gamma^\nu - 1)^{M+1} = 0$ for some positive integers ν and M. (In the geometric case, we may take $M = m$.) We will also assume for the time being that γ is actually unipotent ($\nu = 1$). We are then in the set-up of [23, §2]. $\mathcal{V}$ has a *canonical extension* $\bar{\mathcal{V}}$ on $\bar{S}$ ($= \Delta$ locally), generated at the origin by elements of $\Gamma(\Delta^*, \mathcal{V})$ which are of the form

$$\tilde{v} = \exp\left(\frac{1}{2\pi i} N \log t\right) v \ , \tag{3.5}$$

where v is a (multi-valued) horizontal section of V, and $N = \log \gamma = -\sum_{k=1}^{m} (1 - \gamma)^k/k$.

On V_s, there is the (increasing) *monodromy weight filtration* $\{W_k\}$ defined over $\mathbf{R}$ (over $\mathbf{Q}$ if V is defined over $\mathbf{Z}$), which is γ-invariant and which therefore determines a filtration of V by locally constant sheaves, also denoted $\{W_k\}$. For the basic properties of the weight filtration, see [17, § 6]; significantly, there are isomorphisms

$$N^k \colon \mathrm{Gr}^W_{m+k}(V) \longrightarrow \mathrm{Gr}^W_{m-k}(V) \ .$$

According to Theorem 6.6′ of [17], the weight filtration is characterized by the asymptotic behavior of the norm:

$$v \in W_k \text{ if and only if } ||v||^2 = \langle v, v \rangle = O(\log^{k-m} |t|)$$
$$\text{for all } t \text{ in any angular sector of } \Delta^*.$$

Furthermore, the above estimate is sharp for $v \in W_k - W_{k-1}$. We introduce the notation "$\sim$" to mean "has the same order of growth as"; thus

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[2] Schmid has informed me that the theory goes through if Σ is smooth.

[3] We take the path to be *clockwise* around 0, so as to avoid the conflict of notation with [17], which occurs in [23, § 2], over the meaning of $N = \log \gamma$.

(3.6) PROPOSITION. *If* $v \in W_k$ *projects non-trivially in* $\mathrm{Gr}_k^W V$ *then* $\|v\|^2 \sim |\log r|^{k-m}$.

The weight filtration induces a filtration $\{\bar{\mathcal{W}}_k\}$ of $\bar{\mathcal{V}}$ by canonical extension, and we have, by making minor changes in the discussion in [17]:

(3.7) PROPOSITION. $v \in W_k - W_{k-1}$ $(\tilde{v} \in \bar{\mathcal{W}}_k - \bar{\mathcal{W}}_{k-1})$ *if and only if* $\|\tilde{v}\|^2 \sim |\log r|^{k-m}$.

4. The holomorphic Poincaré Lemma

We are ready to attack (2.10, i). Let $j: S \to \bar{S}$ be the inclusion mapping,

(4.1) PROPOSITION. *The following sequence is exact*

$$0 \longrightarrow j_* V \longrightarrow [\bar{\mathcal{W}}_m + t\bar{\mathcal{V}}] \xrightarrow{\nabla} \frac{dt}{t} \otimes [\bar{\mathcal{W}}_{m-2} + t\bar{\mathcal{V}}] \longrightarrow 0 .$$

Remark. The above is a well-defined extension to $\bar{S}$ of the exact sequence of sheaves on S,

$$0 \longrightarrow V \longrightarrow \mathcal{O}(V) \longrightarrow \Omega^1(V) \longrightarrow 0 ;$$

though written in local notation, it makes global sense (by making the indicated extension at each singular point).

Proof. We first note that we have an exact sequence

$$t\bar{\mathcal{W}}_k \xrightarrow{\nabla} dt \otimes \bar{\mathcal{W}}_k \longrightarrow 0 \tag{4.2}$$

for all k. This is precisely the content of [23, (3.12)]. It is also true that

$$\bar{\mathcal{W}}_{l+2} \xrightarrow{\nabla} \frac{dt}{t} \otimes (\bar{\mathcal{W}}_l + t\bar{\mathcal{W}}_{l-2}) \longrightarrow 0 \tag{4.3}$$

is exact whenever $l < m$: for a holomorphic function f, and $v \in W_{l+2}$, $f(dt/t \otimes Nv) = 2\pi i\,[\nabla(f\tilde{v}) - df \otimes \tilde{v}]$, so that exactness of (4.3) follows from (4.2) for $k = l + 2$, and the fact that N: $W_{l+2} \to W_l$ is surjective whenever $l < m$. Combining (4.2) and (4.3) with the inclusion $\ker N = V^\Gamma \subset W_m$, we obtain (4.1).

For reasons that will be apparent in a moment, we introduce the notation

$$\mathcal{O}(V)_{(2)} = \bar{\mathcal{W}}_m + t\bar{\mathcal{V}} ,$$
$$\Omega^1(V)_{(2)} = \frac{dt}{t} \otimes [\bar{\mathcal{W}}_{m-2} + t\bar{\mathcal{V}}] ,$$

and define $\Omega^\bullet(V)_{(2)}$ to be the complex

$$\mathcal{O}(V)_{(2)} \longrightarrow \Omega^1(V)_{(2)} .$$

(4.4) PROPOSITION. *Sections of $\Omega^{\cdot}(V)_{(2)}$ near $t = 0$ are precisely those sections of $j_*\Omega^{\cdot}(V)$ which are square-summable near $t = 0$.*

In the proof of the above proposition, we use an idea that recurs in Sections 5 and 6. Let $\mathcal{V}$ be a trivial bundle (either analytic or C^∞) and $v = \{v_1, \cdots, v_r\}$ a framing of $\mathcal{V}$. We also assume that $\mathcal{V}$ is given a Hermitian metric. Then v is said to be *L_2-adapted* if $\sum f_i v_i \in L_2$ implies $f_i v_i \in L_2$ for all i. The condition is invariant under change of scale, so we may normalize the frame without changing the issue of whether v is L_2-adapted.

(4.5) LEMMA. *Let v be a frame for $\mathcal{V}$ with $\sup\|v_i\| < \infty$ for all i. A sufficient condition that v be L_2-adapted is that the matrix of the inner product $\Xi = [\xi_{ij}]$, where $\xi_{ij} = \langle v_i, v_j\rangle$, has a bounded inverse.*

Proof. If $g = \sum f_i v_i \in L_2$, then the functions $\phi_i = \langle g, v_i\rangle \in L_2$. We have the identity $\Phi = \Xi F$, where Φ (resp. F) are the column vectors of ϕ_i's (resp. f_i's). Then $F = \Xi^{-1}\Phi$, from which the lemma follows.

We should recall the source of the norm estimates (3.6) and (3.7). Given $v \in V_s$, it determines a multi-valued horizontal section of $\mathcal{V}$, which becomes single-valued when it is lifted to the upper half-plane $H = \{z = x + iy \in \mathrm{C}: y > 0\}$, covering Δ^* by $\tau: H \to \Delta^*$, with $\tau(z) = e^{2\pi i z}$. Writing $v(z)$ as the sum of its (p, q)-components

$$v(z) = \sum\nolimits_{p+q=m} v^{p,q}(z)\ ,$$
$$\|v(z)\|^2 = \sum\nolimits_{p+q=m} \|v^{p,q}(z)\|^2\ .$$

The norms as a function of z are best understood by transporting $v(z)$ by an isometry L_z to a reference Hodge structure, and then measuring there. For a suitable basis $\{v_l\}$ of V flagged according to the weight filtration (see the proof of (5.2)), Schmid has obtained the expression [17, p. 253]

$$L_z v_l(z) = h(z)[y^{(k-m)/2}\exp(-xN)v_l + (\text{lower order terms})]\ ,$$

where the term in brackets is a Laurent series in $y^{1/2}$, $h(z)$ is strongly asymptotic to the identity, and k is some integer. From this, we obtain

$$L_z\tilde{v}_l(z) = h(z)\Big[\,y^{(k-m)/2}\Big(\sum\nolimits_{j=0}^{m}\frac{(i)^j}{j!}N^j v_l\Big) + \cdots\Big]\ , \tag{4.9}$$

since $\tilde{v}_l(z) = \exp(zN)v_l$. As $\{y^{(m-k)/2}L_z\tilde{v}_l(z)\}$ is asymptotically a basis of V as $y \to \infty$, $\{\tilde{v}_l(z)\}$ is L_2-adapted by the criterion (4.5). In fact, given any basis $\{v_l\}$ of V flagged according to the weight filtration, the leading (V-valued) coefficients in the expansion (4.6) are linearly independent, so $\{V_l\}$ is L_2-adapted. Similarly, the L_2-adaptedness of any basis for $\bar{\mathcal{V}}$ (near the origin) is determined only by its image in $\bar{\mathcal{V}}(0) = \bar{\mathcal{V}}/t\bar{\mathcal{V}}$.

Proof of (4.4). We first work with $\mathcal{O}(V)$. It will suffice to consider $\sigma = t^n\tilde{v}$ for $v \in W_k - W_{k-1}$. Then

$$\|\sigma\|^2 \sim r^{2n}|\log r|^{k-m} ,$$

so for $A < 1$, we must check the convergence or divergence of the integral

$$\int_0^{2\pi}\int_0^A r^{2n}|\log r|^{k-m}\frac{r\,dr\,d\theta}{r^2\log^2 r} .$$

It is finite precisely when $k \leqq m$ and $n \geqq 0$, or $k > m$ and $n > 0$, i.e., when σ is a section of $\mathcal{O}(V)_{(2)}$. That σ cannot have an essential singularity in $\bar{\mathcal{V}}$ at $t = 0$ follows from an argument estimating the Laurent series coefficients of σ (very similar to the calculations to come in § 6), which also gives an alternate proof of (4.4). The conclusion for $\Omega^1(V)$ follows from what we have already deduced, because

$$\left\|\frac{dt}{t}\right\|^2 \sim \log^2 r . \tag{4.7}$$

As a corollary of (4.4), we obtain

(4.8) THEOREM. *If V is the local system of a polarized variation of Hodge structure over an algebraic curve S, with unipotent local monodromy,*

$$H^i(\bar{S}, j_* V) \simeq \mathbf{H}^i(\bar{S}, \Omega^\cdot(V)_{(2)}) .$$

We will take care of the C^∞ end of (2.10, i) in Section 6.

5. The Hodge filtration on $\Omega^\cdot(V)_{(2)}$

We turn our attention now to item (ii) of (2.10). Having defined the Hodge filtration on $\Omega^\cdot(V)$ in (1.12), we define $F^p\Omega^\cdot(V)_{(2)}$ by forcing the L_2 growth condition through (1.12), to be the complex of sheaves on $\bar{S}$,

$$\mathcal{F}^p_{(2)} \xrightarrow{\nabla} (\Omega^1 \otimes \mathcal{F}^{p-1})_{(2)} , \tag{5.1}$$

where the sub-(2) means, as before, that we take the locally-L_2 germs to define the stalks of the sheaves at the singular points. The bundle $\mathcal{G}r^p = \mathcal{F}^p/\mathcal{F}^{p-1}$ on S inherits a quotient norm from $\mathcal{F}^p$ which, in fact, makes $\mathcal{E}(\mathcal{G}r^p)$ isometric to $\mathcal{H}^{p,m-p}$. We define $\mathcal{G}r^p_{(2)}$ in the obvious way. In order to compute readily with the Hodge filtration on $\Omega^\cdot(V)_{(2)}$, it is useful to know the following:

(5.2) PROPOSITION. *The sheaves $\mathcal{F}^p_{(2)}$, $\mathcal{G}r^p_{(2)}$, $(\Omega^1 \otimes \mathcal{F}^p)_{(2)}$ and $(\Omega^1 \otimes \mathcal{G}r^p)_{(2)}$ are locally free, and the sequences*

$$0 \longrightarrow \mathcal{F}^{p+1}_{(2)} \longrightarrow \mathcal{F}^p_{(2)} \longrightarrow \mathcal{G}r^p_{(2)} \longrightarrow 0 ,$$
$$0 \longrightarrow (\Omega^1 \otimes \mathcal{F}^{p+1})_{(2)} \longrightarrow (\Omega^1 \otimes \mathcal{F}^p)_{(2)} \longrightarrow (\Omega^1 \otimes \mathcal{G}r^p)_{(2)} \longrightarrow 0$$

are exact.

Remark. Before beginning the proof, I should comment that the exactness of the above sequences is *not* a priori obvious; (5.2) says that the Hodge filtration is "L_2-adapted".

Proof. The proposition at hand depends on the existence of good local bases for $\bar{\mathcal{V}}$ at the singular points, which respect both the Hodge and weight filtrations, at least to 0^{th} order. As usual, we localize at a singular point. It is necessary to do a careful reading of [17, pp. 255–263], and to make a translation of Schmid's results from V to $\bar{\mathcal{V}}$ (cf. [23, § 2]). The main point is that for $v \in V_s$, $v \in F_\infty^p$ (in the notation of [17]) if and only if there exists a holomorphic section w of $\bar{\mathcal{V}}$ such that $\tilde{v} + tw$ is a section of $\mathcal{F}^p$.

By [17, (6.20)], F_∞ is the left-translate of a suitably chosen reference Hodge structure F_0 by a linear mapping g_∞, which preserves the weight filtration and acts as the identity on $\mathrm{Gr}_k^W V$. By the SL_2-orbit theorem, F_0 has a horizontal $\mathfrak{sl}(2, \mathbf{C})$-action, which splits it (over $\mathbf{R}$) into a direct sum of orthogonal irreducible sub-structures. The basic irreducible structures are called $S(n)$, the n^{th} symmetric power of $S(1)$; the latter is obtained by taking $\mathbf{C}^2 = \mathbf{C}e_1 \oplus \mathbf{C}e_2$ with the usual $\mathfrak{sl}(2, \mathbf{C})$-action as the underlying vector space, so that $Ne_2 = e_1$, $Ne_1 = 0$, and with

$$\begin{aligned} v_+ &= e_1 + ie_2 \qquad \text{of type } (0, 1) , \\ v_- &= e_1 - ie_2 \qquad \text{of type } (1, 0) . \end{aligned}$$

For the polarization, $(v_+, v_-) = 2i$, and therefore $\{e_1, e_2\}$ is orthonormal in the associated norm (cf. (2.1)); from this it follows that $\{e_1^k e_2^{n-k}\}$ is an orthogonal basis of $S(n)$ (the one used for (4.6)). All irreducibles are realized by tensoring $S(n)$ with a trivial factor, be it $E(p, q)$ with basis $e^{p,q}$ and its conjugate $e^{q,p}$, or $H(l) = \mathbf{C}h^{l,l}$, so we first examine $S(n)$.

As a basis of $S(n)$, we choose

$$\alpha_k = N^{n-k}(v_-)^n \qquad (0 \leqq k \leqq n) .$$

This basis has been selected so that

$$\alpha_k \in F_0^k \cap W_{2k}$$

and α_k projects non-trivially in both $\mathrm{Gr}_{F_0}^k S(n)$ and $\mathrm{Gr}_{2k}^W S(n)$. Using this we build a basis for V_0, inducing bases for each $\mathrm{Gr}_{F_0}^k V_0$, by taking

$$\alpha_{k-l} \otimes h^{l,l} \ (\text{if } k \geqq l) \qquad \text{from} \quad S(m - 2l) \otimes H(l) ,$$

and

$$\begin{cases} \alpha_{k-p} \otimes e^{p,q} \ (\text{if } k \geqq p) \\ \alpha_{k-q} \otimes e^{q,p} \ (\text{if } k \geqq q) \end{cases} \qquad \text{from} \quad S(m - p - q) \otimes E(p, q) .$$

For simplicity, we assume that $V_0 = S(m)$. Then there exist sections β_p of $\bar{\mathcal{V}}$ so that

$$\sigma_p = \widetilde{g_\infty(\alpha_p)} + t\beta_p$$

represents a generator for $\mathcal{G}_{\bar{z}}{}^p$. Furthermore,

$$||\sigma_p||^2 \sim |\log r|^{2p-m} ,$$

since one can verify from (4.6) that

$$L_z(\tilde{\alpha}_p) = h(z)[y^{(2p-m)/2}\alpha_p + \cdots] , \tag{5.3}$$

and $||t\beta_p||^2 = O(r^2 \log^m r)$. By (5.3), the σ_k's give an L_2-adapted basis for the $\mathcal{F}^p$'sc, for $L_z g_\infty(\alpha_p)$ and $L_z(\alpha_p)$ are asymptotically the same, and we may apply (4.5). Then $\mathcal{F}^p_{(2)}$ is freely generated by the sections $\{t^{\varepsilon_k}\sigma_k\}_{k=p}^m$, with $\varepsilon_k = 0$ if $2k \leqq m$ and $\varepsilon_k = 1$ otherwise. Furthermore, it also follows from (5.3) that the $(p, m-p)$ component of $L_z(\sigma_p)$ is asymptotic to $y^{(2p-m)/2}\alpha_p$; i.e., σ_p carries much of its norm in this Hodge component. Therefore, a generator of $\mathcal{G}_{\bar{z}}{}^p_{(2)}$ is the projection of $t^{\varepsilon_p}\sigma_p$.

It is easy to see, again using (5.3), that if there are several irreducible factors in the decomposition of V_0, all bases constructed from $\{\sigma_k\}$ are L_2-adapted. The statement of (5.2) follows.

(5.4) *Remark*. The statement of (5.2) generalizes to the inclusion $0 \to \mathcal{F}^{p'}_{(2)} \to \mathcal{F}^p_{(2)}$ for all $p' > p$.

6. The Poincaré and Dolbeault Lemmas

The purpose of this section is to show that the cohomology groups $H^i(\bar{S}, j_* V)$, which are isomorphic by (4.8) to $\mathbf{H}^i(\bar{S}, \Omega^\cdot(V)_{(2)})$, and $\mathbf{H}^i(\bar{S}, \mathrm{Gr}^p_F\Omega^\cdot(V)_{(2)})$ are computable from complexes of L_2 differential forms. We will *not* insist on using C^∞ forms, a simplification suggested by Deligne. This is the formal link between hypercohomology and harmonic forms, as described in (2.10).

Let $\mathcal{L}^p(V)_{(2)}$ be the sheaf on $\bar{S}$ of germs of locally L_2 V-valued p-forms ϕ for which $D\phi$ exists weakly as a locally-L_2 form. We thus obtain a complex $\mathcal{L}^\cdot(V)_{(2)}$ of fine sheaves on $\bar{S}$. Similarly, we have a complex $[\mathrm{Gr}^p_F\mathcal{L}^\cdot(V)]_{(2)}$ which is formed of the sheaves $\mathcal{L}^{0,q}(\mathcal{G}_{\bar{z}}{}^p)_{(2)}$ and $\mathcal{L}^{1,q}(\mathcal{G}_{\bar{z}}{}^{p-1})_{(2)}$, consisting of forms ϕ for which ϕ and $\bar{\partial}\phi$ are L_2.[4] This suffices because

(6.1) Lemma. ∇' *(and hence also* $\bar{\nabla}'$*) is a bounded operator.*

Proof. As we see from the discussion in Section 5, ∇' acts, up to some asymptotically negligible terms, as $(1/2\pi i)\, dt/t \otimes N$. N lowers weights by

[4] Observe that $[\mathrm{Gr}^p_F\mathcal{L}^\cdot(V)]_{(2)}$ is not the same as $\mathrm{Gr}^p_F[\mathcal{L}^\cdot(V)_{(2)}]$, for in the latter complex, one is insisting on control over D'.

two, but $\|dt/t\|^2 \sim \log^2 r$, so $\|\nabla'\| \sim 1$ fiberwise, hence $\|\nabla'\|_{(2)} < \infty$.

(6.2) THEOREM (Poincaré Lemma). *$\mathcal{L}^{\cdot}(V)_{(2)}$ is a resolution of $j_* V$.*

(6.3) THEOREM (Dolbeault Lemma).

$$\mathbf{H}^i(\bar{S}, \mathrm{Gr}_F^p \Omega^{\cdot}(V)_{(2)}) \simeq H^i(\Gamma(\bar{S}, [\mathrm{Gr}_F^p \mathcal{L}^{\cdot}(V)]_{(2)})) .$$

On S, exactness follows from the solution of the Neumann problem, so the trouble comes from the singular points, and we localize to Δ^* as usual. We first prove simple approximations to (6.2) and (6.3); we work with the latter first.

(6.4) PROPOSITION. *Let $\mathcal{V}$ be a holomorphic line bundle on Δ^* with generating section σ, and with a Hermitian metric satisfying*

$$\|\sigma\|^2 \sim |\log r|^k , \qquad k \neq 1 .$$

Then for every germ of an L_2 (0, 1)-form $\phi = f d\bar{t} \otimes \sigma$ at the origin, there exists an L_2 section $u \otimes \sigma$ with $\bar{\partial} u = f d\bar{t}$.

Proof. Using polar coordinates, we write u and f as r-dependent Fourier series

$$u = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\theta} , \qquad f = \sum f_n(r) e^{in\theta} .$$

As $\partial/\partial\bar{t} = (1/2)e^{i\theta}[\partial/\partial r + (i/r)\partial/\partial\theta]$, the equation $\bar{\partial} u = f$ becomes

$$\frac{1}{2}\left[u_n' - \frac{n}{r} u_n\right] = f_{n+1} \quad \text{for all} \quad n \in \mathbf{Z} ,$$

for C^∞ germs u and f, or

$$\frac{1}{2}\frac{d}{dr}[r^{-n} u_n(r)] = r^{-n} f_{n+1}(r) .$$

We are given that for some $A < 1$,

$$\begin{aligned} \|\phi\|_{(2)}^2 &= 2\int_0^{2\pi}\int_0^A |f|^2 |\log r|^k (r\,dr\,d\theta) \\ &= 4\pi \sum_n \int_0^A |f_n|^2 |\log r|^k (r\,dr) < \infty , \end{aligned}$$

and we want to have

$$\|u\|_{(2)}^2 = 2\pi \sum_n \int |u_n|^2 |\log r|^{k-2} (r^{-1} dr) < \infty .$$

We try to arrange that $\|u\|_{(2)}^2 \leqq C \|f\|_{(2)}^2$ for some constant C.[5] Take

$$u_n = \begin{cases} 2r^n \int_0^r \rho^{-n} f_{n+1}(\rho) d\rho & \text{if} \quad n < 0 , \quad \text{or} \quad n = 0 \quad \text{and} \quad k > 1 \\ -2r^n \int_r^A \rho^{-n} f_{n+1}(\rho) d\rho & \text{if} \quad n > 0 , \quad \text{or} \quad n = 0 \quad \text{and} \quad k < 1 . \end{cases}$$

[5] We remark that we are replacing the actual norms by their asymptotic forms. This is permissible, for the two are equivalent.

The estimates will be carried out for only the first formula, the other case being similar. If $n < 0$,

$$\begin{aligned}
&\int_0^A |u_n|^2 r^{-1} |\log r|^{k-2} dr \\
&\quad \leqq \int_0^A 4r^{2n}\Big(\int_0^r \rho^{-2n} |f_{n+1}(\rho)|^2 d\rho\Big)\Big(\int_0^r d\rho\Big) r^{-1} |\log r|^{k-2} dr \\
&\quad = \int_0^A 4r^{2n} |\log r|^{k-2} \Big(\int_0^r \rho^{-2n} |f_{n+1}(\rho)|^2 d\rho\Big) dr \\
&\quad = \int_0^A \rho^{-2n} |f_{n+1}(\rho)|^2 \Big(\int_\rho^A 4r^{2n} |\log r|^{k-2} dr\Big) d\rho \\
&\quad \leqq \frac{C}{2n+1} \int_0^A |f_{n+1}(\rho)|^2 \rho |\log \rho|^{k-2} d\rho ,
\end{aligned}$$

with the last inequality holding because

$$\int_\rho^A r^{2n} |\log r|^{k-2} dr \sim \frac{1}{2n+1} \rho^{2n+1} |\log \rho|^{k-2}$$

uniformly in $n < 0$ provided A is taken to be sufficiently small. This is much better than we need. Similarly, if $n = 0$, $k > 1$,

$$\begin{aligned}
u_0 &= 2\int_0^r f_1(\rho) d\rho , \\
|u_0|^2 &\leqq 4\Big(\int_0^r |f_1(\rho)|^2 \rho |\log \rho|^{1+\varepsilon} d\rho\Big)\Big(\int_0^r \rho^{-1} |\log \rho|^{-1-\varepsilon} d\rho\Big) \\
&= \frac{4}{\varepsilon}\Big(\int_0^r |f_1(\rho)|^2 \rho |\log \rho|^{1+\varepsilon} d\rho\Big) |\log r|^{-\varepsilon}
\end{aligned}$$

where $0 < \varepsilon < k - 1$. So

$$\begin{aligned}
&\int_0^A |u_0(r)|^2 r^{-1} |\log r|^{k-2} dr \\
&\quad \leqq \frac{4}{\varepsilon} \int_0^A |f_1(\rho)|^2 \rho |\log \rho|^{1+\varepsilon} \Big(\int_\rho^A r^{-1} |\log r|^{k-2-\varepsilon} dr\Big) d\rho \\
&\quad \leqq C\int_0^A |f_1(\rho)|^2 \rho |\log \rho|^{1+\varepsilon} |\log \rho|^{k-1-\varepsilon} d\rho \\
&\quad = C\int_0^A |f_1(\rho)|^2 |\log \rho|^k (\rho d\rho) .
\end{aligned}$$

If f is C^∞ on $(0, A]$ and has compact support therein, the section u thus constructed is a C^∞ solution to $\bar\partial(u \otimes \sigma) = f d\bar t \otimes \sigma$. This can be seen by noticing that u is holomorphic near the origin, and has rapidly decreasing Fourier coefficients u_n as a function of n (because f does). Since such forms f are dense in L_2 and by the above inequality, Cauchy sequences are taken to Cauchy sequences in the solution process, by passing to the limit the same formula gives a weak solution for L_2 forms $f d\bar t \otimes \sigma$.

The case $k = 1$ will pose a technical problem—for it occurs in W_{m+1}—that, fortunately resolves itself. In fact, $n = 0$ gives examples of Hardy's inequality with weights (see [16]): there is a necessary and sufficient condition for getting $\|u\|^2_{(2)} \leqq C\|f\|^2_{(2)}$, and this condition is not satisfied when $k = 1$. We define

(6.5)

$$\mathcal{M}_1 = \frac{\left\{\text{measurable functions } f\colon \int_0^A |f(r)|^2 |\log r| (r dr) < \infty \text{ for some } A < 1\right\}}{\left\{f\colon f = u' \text{ weakly with } \int_0^A |u|^2 |\log r|^{-1} (r^{-1} dr) < \infty \text{ for some } A < 1\right\}}\,.$$

$\mathcal{M}_1$ gives the L_2 $\bar\partial$-cohomology in dimension one for the case $k = 1$.

We can now prove (6.3). Recall that $\mathrm{Gr}^p_F\Omega^\cdot(V)_{(2)}$ is the complex

$$\mathscr{G}_{\flat}{}^p_{(2)} \longrightarrow (\Omega^1 \otimes \mathscr{G}_{\flat}{}^{p-1})_{(2)}\ .$$

The statement would follow by elementary homological algebra, as in (1.16, i), if

$$0 \longrightarrow \mathscr{G}_{\flat}{}^p_{(2)} \longrightarrow [\mathcal{L}^{0,0}(\mathscr{G}_{\flat}{}^p)]_{(2)} \longrightarrow [\mathcal{L}^{0,1}(\mathscr{G}_{\flat}{}^p)]_{(2)} \longrightarrow 0$$

were exact. As an L_2-adapted basis decomposes the $\bar\partial$ problem into a direct sum of $\bar\partial$ problems in a line bundle, (6.4) covers most of these problems. However, it is possible that, at a singular point, $\mathscr{G}_{\flat}{}^p_{(2)}$ has among its generators a section σ, part of an L_2-adapted basis, with $\|\sigma\|^2 \sim |\log r|$. If so, then $\nabla(\sigma)$ will be part of an L_2-adapted basis for $(\Omega^1 \otimes \mathscr{G}_{\flat}{}^{p-1})_{(2)}$, with $\|\nabla(\sigma)\|^2 \sim |\log r|$. Then ∇ establishes an isomorphism between the $\bar\partial$-cohomology groups at the singular point. This enables one to conclude that $\Gamma(\bar S, \mathrm{Gr}^p_F\mathcal{L}^\cdot(V)_{(2)})$ computes the hypercohomology of $\mathrm{Gr}^p_F\Omega^\cdot(V)_{(2)}$.

In the direction of (6.2), we first prove

(6.6) PROPOSITION. *Let V be a constant one dimensional local system over Δ^*, with generator e, and assume that $\mathcal{E}(V)$ has a Hermitian metric with $\|e\|^2 \sim |\log r|^k$. Then the cohomology sheaves for $\mathcal{L}^\cdot(V)_{(2)}$ have stalks at the origin,*

$$\mathcal{H}^0(\mathcal{L}^\cdot(V)_{(2)}) = \begin{cases} V & \text{if } k \leqq 0 \\ 0 & \text{if } k > 0\,, \end{cases}$$

$$\mathcal{H}^1(\mathcal{L}^\cdot(V)_{(2)}) = \begin{cases} \dfrac{dt}{t} \otimes V & \text{if } k < -1 \\ 0 & \text{if } k \geqq -1,\ k \neq 1 \\ \mathcal{M}_1 dr \otimes e & \text{if } k = 1\,, \end{cases}$$

$$\mathcal{H}^2(\mathcal{L}^\cdot(V)_{(2)}) = \begin{cases} 0 & \text{if } k \neq -1 \\ \mathcal{M}_1 dr \wedge \dfrac{dt}{t} \otimes e & \text{if } k = -1\,, \end{cases}$$

with $\mathfrak{M}_1$ *as in* (6.5).

Proof. The statement about $\mathcal{H}^0$ is clear. To compute $\mathcal{H}^1$, we use Fourier series again. A germ of an L^2 1-form at the origin,

$$\omega = fdr + gd\theta ,$$

with $d\omega = 0$ is approximated by compactly supported C^∞ germs $\tilde{\omega} = \tilde{f}dr + \tilde{g}d\theta$ on $(0, A]$ for some $A < 1$, with

$$d\tilde{\omega} = Edr \wedge d\theta \longrightarrow 0 \quad \text{in} \quad L_2 .$$

In terms of Fourier coefficients, with $E = \sum \varepsilon_n e^{in\theta}$,

$$\tilde{g}'_n = in\tilde{f}_n + \varepsilon_n .$$

As $E \to 0$, so do all ε_n. Letting $\varepsilon_0 \to 0$, we have $g'_0 = 0$, so g_0 is constant ($g_0 d\theta$ will be giving the $\mathcal{H}^1$ if $k < -1$). Assume that $g_0 = 0$. Taking

$$\tilde{u}_n = -\frac{i}{n}\tilde{g}_n \qquad (n \neq 0) ,$$

we obtain for $\tilde{u} = \sum_{n \neq 0} \tilde{u}_n e^{in\theta}$,

$$\begin{aligned}(d\tilde{u})_n &= \tilde{u}'_n dr + in\,\tilde{u}_n d\theta \\ &= -\frac{i}{n}\tilde{g}'_n dr + \tilde{g}_n d\theta \\ &= \tilde{f}_n dr + \tilde{g}_n d\theta - \frac{i}{n} e_n dr ;\end{aligned}$$

i.e.,

$$d\tilde{u} - \tilde{\omega} - \tilde{f}_0 dr = -i\sum_{n \neq 0} \frac{1}{n}\varepsilon_n e^{in\theta} dr .$$

By the same estimate that we will soon use in computing $\mathcal{H}^2$,

$$\| d\tilde{u} - \tilde{\omega} - \tilde{f}_0 dr \|_{(2)} \longrightarrow 0 \qquad \text{as} \quad \| d\tilde{\omega} \|_{(2)} \longrightarrow 0 .$$

Calculating the norms for $n \neq 0$, we have

$$\begin{aligned}\| \tilde{u}_n \|^2_{(2)} &= n^{-2}\int |\tilde{g}_n|^2 |\log r|^{k-2} (r^{-1} dr) \\ &\leqq Cn^{-2}\int |\tilde{g}_n|^2 r^{-2} |\log r|^k (rdr) .\end{aligned}$$

These inequalities also show that Cauchy sequences of $\tilde{\omega}$'s are taken into Cauchy sequences of $\tilde{u}$'s, so by passing to the limit, we obtain a solution to $du = \omega - f_0(r)dr$ with

$$\| u \|^2_{(2)} \leqq K \| gd\theta \|^2_{(2)} < \infty .$$

For $f_0 dr = du_0$, we must solve $u'_0 = f_0$ as in (6.4), leaving $\mathfrak{M}_1 dr \otimes e$ as the cohomology when $k = 1$.

To compute $\mathcal{H}^2$, let $\omega = f dr \wedge d\theta$. We solve $d\eta = \omega$ with $\eta = g dr$ whenever $f_0 = 0$, for it suffices to take $g_n = -(i/n)f_n$. Then

$$\begin{aligned} ||\eta||^2_{(2)} &= 2\pi \sum \int |g_n|^2 |\log r|^k (r\,dr) \\ &= 2\pi \sum n^{-2} \int |f_n|^2 |\log r|^k (r\,dr) \\ &\leqq C \sum n^{-2} \int |f_n|^2 |\log r|^{k+2} (r\,dr) \\ &\leqq K ||\omega||^2_{(2)} . \end{aligned}$$

Given $\omega_0 = f_0(r) dr \wedge d\theta$, we must try $\eta_0 = h_0(r) d\theta$. Then $h_0' = f_0$, with

$$||\eta_0||^2_{(2)} = 2\pi \int |h_0|^2 |\log r|^k (r^{-1} dr) ,$$

$$||\omega_0||^2_{(2)} = 2\pi \int |f_0|^2 |\log r|^{k+2} (r\,dr) .$$

As before, we can solve the equation in L_2 provided $k + 2 \neq 1$, i.e., $k \neq -1$; if $k = -1$, $\mathfrak{M}_1 dr \wedge d\theta \otimes e$ appears as the cohomology. (6.6) follows from these results because $dt/t = i d\theta + d(\log r)$, with $\log r \in L_2$ if $k < -1$, and $dr \wedge dt/t = i dr \wedge d\theta$.

Proof of (6.2). Use the spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p+q}\big(\mathrm{Gr}^W_{-p} \mathcal{L}^\cdot(V)_{(2)}\big) \Longrightarrow \mathcal{H}^{p+q}\big(\mathcal{L}^\cdot(V)_{(2)}\big) .$$

The differentials in the spectral sequence arise by lifting $v \in V$ to $\tilde{v}$ (as defined in (3.5), noting (3.7)), and applying D. Then $d_1 = 0$, because N lowers weights by two, and

$$d_2 \colon \mathcal{H}^0\big(\mathcal{L}^\cdot[\mathrm{Gr}^W_{l+2}(V)]_{(2)}\big) \longrightarrow \mathcal{H}^1\big(\mathcal{L}^\cdot[\mathrm{Gr}^W_l(V)]_{(2)}\big)$$

is induced by $(1/2\pi i) dt/t \otimes N$. Recall that Gr^W_l corresponds to $k = l - m$ in (6.6). If $k < -1$, $N\colon W_{k+m+2} \to W_{k+m}$ is surjective; hence by (6.6) d_2 is surjective. Likewise, d_2 sets up an isomorphism between the unpleasant cohomology groups

$$\mathcal{H}^1\big(\mathcal{L}^\cdot[\mathrm{Gr}^W_{m+1}(V)]_{(2)}\big) \quad \text{and} \quad \mathcal{H}^2\big(\mathcal{L}^\cdot[\mathrm{Gr}^W_{m-1}(V)]_{(2)}\big) .$$

Thus, the spectral sequence degenerates at E_3, for all non-zero terms are $E_3^{p,-p}$. $j_* V$ must remain as $\mathcal{H}^0\big(\mathcal{L}^\cdot(V)_{(2)}\big)$.

(6.7) *Remark.* In the non-degenerate case ($S = \bar{S}$), the usual Dolbeault Lemma and holomorphic Poincaré Lemma, i.e., the standard versions of (6.3) and (4.1), imply the C^∞ Poincaré Lemma. Here (having dropped the C^∞ condition) we have proved (6.2) and (6.3) separately so that we could avoid worrying about D' when studying D''. Thus, we are using two different complexes to compute D and D'' cohomology; however, they will have the same harmonic forms.

Having established (4.1), (6.2), and (6.3) for unipotent monodromy, we see that it is a relatively simple matter to generalize these results to variations of Hodge structure in which the local monodromy is only quasi-unipotent. When V carries a geometric variation of Hodge structure, the quasi-unipotence follows from the monodromy theorem.

Localizing as usual to $S = \Delta^*$, we let γ be the local monodromy transformation, with $(\gamma^\nu - 1)^{M+1} = 0$. Then V_s, and hence V, splits into generalized eigenspaces

$$V = \bigoplus_{a=0}^{\nu-1} V_{a/\nu}$$

with

$$V_{a/\nu} = \{v \in V_s : (\gamma - \zeta^a)^{M+1} v = 0\} ,$$

where $\zeta = \exp(2\pi i/\nu)$. Then $\mathcal{V} = V \otimes_{\mathbf{C}} \mathcal{O}$ has a *quasi-canonical extension* $\bar{\mathcal{V}}$. Writing $\gamma|_{V_{a/\nu}} = \zeta^a \gamma_a$, with γ_a unipotent, and $N_a = \log \gamma_a$, one defines [10] $\bar{\mathcal{V}}$ to be generated at the origin by

$$\tilde{v}(t) = \exp\left(\log t\left[\frac{a}{\nu} + \frac{1}{2\pi i} N_a\right]\right) v \ , \tag{6.8}$$

where v is the multi-valued flat section of $\mathcal{V}$ associated with $v \in V_{a/\nu}$ and a ranges from 0 to $\nu - 1$.

Information about $\bar{\mathcal{V}}$ can be obtained by pulling V back via the ν-sheeted covering

$$\psi : \Delta_u^* \longrightarrow \Delta^* \qquad (\psi(u) = u^\nu) \ .$$

In $\psi^*(V)$, we have γ^ν as (unipotent) monodromy transformation, hence

$$N = \bigoplus_{0 \leq a \leq \nu-1} \nu N_a \ .$$

We may assume without loss of generality that $V = V_a$, and inasmuch as the previous results cover the unipotent part V_0, we also assume that $a \neq 0$.

(6.9) Proposition. *If V has quasi-unipotent monodromy, with no unipotent part, then*

$$\mathcal{O}(V)_{(2)} = \bar{\mathcal{V}} \quad \textit{and} \quad \Omega^1(V)_{(2)} = \frac{dt}{t} \otimes \bar{\mathcal{V}} \ .$$

Furthermore, $\Omega^{\bullet}(V)_{(2)}$ resolves $j_* V$. (N.B. — *$j_* V$ has the zero stalk at $t = 0$.*)

Proof. If $v \in V_a$,

$$\psi^*(\tilde{v}) = u^a \exp\left(\frac{\nu}{2\pi i} N_a \log u\right) v \ , \tag{6.10}$$

and, of course,

$$\psi^*(t^n) = u^{n\nu}\,, \qquad \psi^*\left(\frac{dt}{t}\right) = \nu\frac{du}{u}\,. \tag{6.11}$$

As ψ respects the Poincaré metrics, the first statement follows immediately from (4.4). Since ∇ patently maps $\mathcal{O}(V)_{(2)}$ into $\Omega^1(V)_{(2)}$, in order to prove the second assertion, we must show that this mapping is surjective. But this follows by (4.1) and the explicit description of the anti-differentiation given in [23]:

$$\begin{aligned}\psi^*\left(\frac{dt}{t} \otimes t^n \tilde{v}(t)\right) &= \nu\frac{du}{u} \otimes u^{n\nu+a}\tilde{v}(u)\\ &= \nabla\big(u^{n\nu+a}\,\tilde{w}(u)\big)\end{aligned}$$

for some $w \in V_a$, and we may descend to Δ^*.

(6.12) COROLLARY. *If V has quasi-unipotent monodromy, $\Omega^\cdot(V)_{(2)}$ is a resolution of j_*V.*

The idea of pullback and descent also works effectively in the case of L_2 differential forms. Let G be the group of covering transformations for ψ, a cyclic group of order ν. The following is obvious:

(6.13) LEMMA. *There is a one-to-one correspondence between G-invariant L_2 $\psi^*(V)$-valued i-forms on Δ_u^* and L_2 V-valued i-forms on Δ^*.*

We may place the Hodge filtration of (1.12) on $\Omega^\cdot(V)_{(2)}$ to obtain complexes of locally free $\mathcal{O}_{\bar{S}}$-modules, with the analogue of (5.2) holding; that is, we can find an L_2-adapted basis of $\tilde{\mathcal{V}}$ at a singular point which respects the Hodge filtration, for the $\mathfrak{sl}_2$ decomposition of V is compatible with the eigenspace decomposition.

(6.14) COROLLARY. *The L_2 Poincaré and Dolbeault Lemmas* ((6.2) *and* (6.3)) *hold when V has quasi-unipotent monodromy.*

Proof. If η is an L_2 solution to $D\eta = \omega$ or $D''\eta = \omega$ on Δ_u^*, where ω is G-invariant, then $1/\nu\sum_{g\in G} g^*\eta$ solves the corresponding problem on Δ^*. (For D'', we need to observe that G acts as automorphisms of the variation of Hodge structure, so (1.10) implies that g^* and D'' commute.)

As in (1.16) and (1.17), we have

(6.15) COROLLARY. *If V underlies a variation of Hodge structure with quasi-unipotent monodromy*:

(i) $H^i(\bar{S}, j_*V) = \mathbf{H}^i(\bar{S}, \Omega^\cdot(V)_{(2)}) \simeq H^i(\Gamma(\bar{S}, \mathcal{L}^\cdot(V)_{(2)}))\,,$

(ii) $\mathbf{H}^i(\bar{S}, \mathrm{Gr}_F^p\Omega^\cdot(V)_{(2)}) \simeq H^i(\Gamma(\bar{S}, \mathrm{Gr}_F^p\mathcal{L}^\cdot(V)_{(2)}))\,.$

7. The strong Hodge decomposition

We will show in this section that the Laplacian operators $\square_D$, $\square_{D'}$ and $\square_{D''}$ have closed range when extended to suitable L_2-spaces of differential forms. It is this fact which enables one to conclude that the spaces of harmonic forms represent cohomology.

First, we introduce some new notation. Let

$$\begin{aligned} L_2^p(S, V) &= \{\text{square-summable global } V\text{-valued } p\text{-forms on } S \\ &\qquad \text{with measurable coefficients}\}, \\ A_2^p(S, V) &= \{\phi \in L_2^p(S, V)\colon \phi \text{ is } C^\infty \text{ on } S\}, \\ A_c^p(S, V) &= \{\phi \in A_2^p(S, V)\colon \phi \text{ has compact support}\}. \end{aligned}$$

Similarly, we can define $L_2^{P,Q}(S, V)$, $A_2^{P,Q}(S, V)$ and $A_c^{P,Q}(S, V)$.

Beginning with $A_c^p(V)$, one extends the differential operator $D\colon A_c^p(S, V) \to A_c^{p+1}(S, V)$ to a densely-defined closed operator on $L_2^p(S, V)$, which we denote by $\mathbf{D}\colon L_2^p(S, V) \to L_2^{p+1}(S, V)$. The same is done for

$$D'\colon A_c^{P,Q}(S, V) \longrightarrow A_c^{P+1,Q}(S, V) \text{ and for } D''\colon A_c^{P,Q}(S, V) \longrightarrow A_c^{P,Q+1}(S, V)\ .$$

A priori, there are several possible domains for these operators.[6] The smallest, obtained by taking the closure of the graph of D, is called the *strong* (or *minimal*) *closure* of D; it is computed by taking limits of Cauchy sequences $\{\phi_i\}$ in $A_c^p(S, V)$ for which $\{D\phi_i\}$ converges in $L_2^{p+1}(S, V)$. The largest, the *weak closure*, is defined by the adjoint condition:

$$\eta = \mathbf{D}\phi \quad \text{if for all} \quad \psi \in A_c^{p+1}(M)\ , \quad \langle \phi, \mathfrak{d}\psi \rangle = \langle \eta, \psi \rangle\ ,$$

i.e., if $\psi \to \langle \phi, \mathfrak{d}\psi \rangle$ is a bounded linear functional. The domain of the weak closure may alternatively be described as the set of L_2 forms whose distribution derivative is also in L_2. We observe that this space is equal to $\Gamma(\bar{S}, \mathfrak{L}^p(V)_{(2)})$. There is no ambiguity in our case, for we have

(7.1) PROPOSITION. *For each of the operators D, D', D'', the strong and weak closures coincide.*

This is a consequence of the following two similar results:

(7.2) THEOREM. *Let $\mathcal{V}$ be a flat vector bundle on a complete Riemannian manifold M. Then the strong and weak closures of $D = d \otimes 1$ coincide.*

The proof of (7.2) is almost identical to that of

(7.3) THEOREM [1, p. 92]. *Let $\mathcal{V}$ be a holomorphic vector bundle on a complete complex manifold. Then the weak and strong closures of $\bar\partial$ coincide.*

[6] The uninitiated reader should be forewarned that the issue over the domains must be treated with care.

Proof of (7.1). Of course, (7.2) contains the statement for D. To prove the same for D'' (and hence for D' by conjugation), we recall that $D'' = \bar{\partial}' + \nabla'$, where $\bar{\partial}'$ is unitarily equivalent to the direct sum of $\bar{\partial}$ operators (1.10), and ∇' is a bounded operator (6.1). The assertion follows from (7.3).

As strong and weak closures are necessarily adjoint to each other, it follows that the strong and weak closures of the adjoints also coincide. Having established this, one can define a closure for the Laplacians, with domain (in the case of $\Box_D$)

(7.4)

$$\mathfrak{D}(\Box_{\mathbf{D}}^{p}) = \{\phi \in L_2^p(S, V)\colon \phi \in \mathfrak{D}(\mathbf{D}^p) \cap \mathfrak{D}(\mathfrak{d}^p),\ \mathbf{D}\phi \in \mathfrak{D}(\mathfrak{d}^{p+1}),\ \text{and}\ \mathfrak{d}\phi \in \mathfrak{D}(\mathbf{D}^{p-1})\}\ ,$$

where the superscripts indicate the degree of the forms in the domain. It is well-known that $\Box_{\mathbf{D}}$ is self-adjoint (in the strict operator sense) [12].

We define the spaces of harmonic forms. Some care needs to be taken, in light of the subtle issue of the domains: a priori, it is not clear that

$$\Box_{\mathbf{D}}^{p} = \textstyle\bigoplus_{P+Q=m+p} \Box_{\mathbf{D}''}^{P,Q} \tag{7.5}$$

as operators, though, of course, one does have the identity on a dense subset of the domains. However, because S is complete, $\Box_D$ and $\Box_{D''}$ are essentially self-adjoint, i.e., their strong closures are self-adjoint [3]. We then necessarily have equality of the strong closure and the operator defined in (7.4). Since strong closure commutes with orthogonal direct sums, we do have the decomposition (7.5).

Let

$$\mathfrak{h}^p(V) = \{\phi \in \mathfrak{D}(\Box_{\mathbf{D}}^{p})\colon \Box_{\mathbf{D}}\phi = 0\}\ ,$$
$$\mathfrak{h}^{P,Q}(V) = \{\phi \in \mathfrak{D}(\Box_{\mathbf{D}''}^{P,Q})\colon \Box_{\mathbf{D}''}\phi = 0\}\ .$$

Using (7.5) and the Kähler identities of Section 2, we obtain

(7.6) PROPOSITION. $\mathfrak{h}^p(V) = \bigoplus_{P+Q=m+p} \mathfrak{h}^{P,Q}(V)$.

In order to conclude that the harmonic forms represent L_2 cohomology, we need to know that the range of $\Box_{\mathbf{D}}$ (hence also of $\Box_{\mathbf{D}''}$ and $\Box_{\mathbf{D}'}$, by (7.5)) is closed. Viewing the operator as the one defined in (7.4), we would then have

$$L_2^p(S, V) = \mathfrak{h}^p(V) \oplus \Box_{\mathbf{D}} L_2^p(S, V) \tag{7.7}$$

by self-adjointness. If $\phi \in L_2^p(S, V)$ we could write

$$\phi = h + \Box\eta\ , \tag{7.8}$$

where $h \in \mathfrak{h}^p(V)$, and $\eta \in L_2^p(S, V)$. Rewriting (7.8), we have

$$\phi = h + \mathbf{D}\mathfrak{d}\eta + \mathfrak{d}\mathbf{D}\eta\ , \tag{7.9}$$

with each term in $L_2^p(S, V)$. Moreover, if $\mathbf{D}\phi = 0$, then we must have $\mathfrak{d}\mathbf{D}\eta = 0$. From (7.4), we have that $\mathfrak{d}\eta \in L_2^{p-1}(S, V)$. This gives the isomorphism

$$\text{(7.10)} \qquad \mathfrak{H}^p(V) \simeq H^p(\Gamma(\bar{S}, \mathcal{L}^{\cdot}(V)_{(2)}))$$
$$= \{\phi \in L_2^p(S, V): D\phi = 0\}/\{D\eta \in L_2^p(S, V): \eta \in L_2^{p-1}(S, V)\} .$$

It is a well-known fact about operators that $\square^p$ has closed range if and only if $\mathbf{D}^p$ and $\mathbf{D}^{p-1}$ have closed range. The statement for $\mathbf{D}$ is, however, a simple consequence of (6.15), for the range of $\mathbf{D}^p$ is the image of the bounded linear mapping of Hilbert spaces

$$\mathbf{D}: \mathfrak{D}(\mathbf{D}^p) \longrightarrow \ker \mathbf{D}^{p+1}$$

(using the graph norm for $\mathbf{D}$ on the domain), and the cokernel is finite-dimensional.

(7.11) *Remark*. With (7.7) established, regularity theory for elliptic operators gives the statement

$$\square\phi \in A_2^p(S, V) \quad \text{implies} \quad \phi \in A_2^p(S, V) .$$

Therefore, we may conclude that the intrinsic C^∞, L^2 cohomology groups of S with values in V:

$$H_{(2)}^p(S, V) = \{\phi \in A_2^p(S, V): D\phi = 0\}/\{D\eta \in A_2^p(S, V): \eta \in A_2^{p-1}(S, V)\} ,$$
$$H_{(2)}^{P,Q}(S, V) = \{\phi \in A_2^{P,Q}(S, V): D''\phi = 0\}/\{D''\eta \in A_2^{P,Q}(S, V): \eta \in A^{P,Q-1}(S, V)\} ,$$

are also isomorphic to the cohomology groups (6.15) and are represented by harmonic forms.

Incorporating the analogous results for D' to give complex conjugation on harmonic forms, we have finished the proof of the following:

(7.12) THEOREM. *Let V be a locally constant sheaf of vector spaces underlying a polarizable variation of Hodge structure of weight m on the algebraic curve S (with completion $j: S \to \bar{S}$), having quasi-unipotent local monodromy transformations. Then there is a Hodge structure of weight $m + i$ on $H^i(\bar{S}, j_* V)$, with classes of type (P, Q) represented by $\mathfrak{H}^{P,Q}(V)$.*

Letting $F^pH^i(\bar{S}, j_* V)$ denote the subspace of $H^i(\bar{S}, j_* V)$ generated by the L_2 harmonic forms,

$$\bigoplus_{\substack{P+Q=m+i \\ P \geq p}} \mathfrak{H}^{P,Q}(V) ,$$

we have,

(7.13) COROLLARY. (i) *The spectral sequences*

$$E_1^{p,q} = \mathbf{H}^{p+q}(\bar{S}, \mathrm{Gr}_F^p \Omega^{\cdot}(V)_{(2)}) \Longrightarrow \mathbf{H}^{p+q}(\bar{S}, \Omega^{\cdot}(V)_{(2)})$$
$$\simeq H^{p+q}(\bar{S}, j_* V)$$

degenerate at E_1.

(ii) $\mathbf{H}^i(\bar{S}, F^p\Omega^{\cdot}(V)_{(2)}) \to \mathbf{H}^i(\bar{S}, \Omega^{\cdot}(V)_{(2)})$ *is injective for all* p.

(iii) $\mathbf{H}^i(\bar{S}, F^p\Omega^{\cdot}(V)_{(2)}) = F^pH^i(\bar{S}, j_*V)$ *under the above inclusion.*

Of the above statements, only (iii) is not completely standard, because of (6.7). However, (iii) is proved by a simple "chase", which we omit. In particular, the Hodge structure is independent of the choice of Kähler metric on S and polarization of V, though these choices determine the polarization of the Hodge structure.

(7.14) *Remarks.* (i) In the language of [9], the pair $j_*V_{\mathbf{R}}$ and its resolution $\Omega^{\cdot}(V)_{(2)}$ with Hodge filtration F form a cohomological Hodge complex of weight m.

(ii) The isometries

$$L\colon \mathfrak{h}^{P,Q}(V) \xrightarrow{\sim} \mathfrak{h}^{P+1,Q+1}(V) \qquad (P + Q = m)$$

induce an isomorphism between $H^0(\bar{S}, j_*V)$ and $H^2(\bar{S}, j_*V)$ which is compatible with the Hodge decomposition.

As the hypotheses of (7.12) are automatically satisfied for geometric variations of Hodge structure, we have as the main application of the theorem:

(7.15) THEOREM. *For the geometric situation*

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ S & \xhookrightarrow{j} & \bar{S} \end{array}$$

where f *and* $\bar{f}$ *are projective and* f *is smooth over* S, *there are natural polarizable Hodge structures of weight* $m + i$ *on the cohomology groups* $H^i(\bar{S}, j_*R^mf_*\mathbf{C})$.

8. Functoriality of the Hodge structure

We have, via (7.12), associated to every variation of Hodge structure of weight m over a smooth curve S, with underlying locally constant sheaf V having quasi-unipotent monodromy, Hodge structures of weight $m + i$ on $H^i(\bar{S}, j_*V)$. These structures, being defined geometrically, figure to have functorial properties, and we will show that this is the case.

To begin with, we know that the Hodge structure is intrinsically defined, given $S \subset \bar{S}$, and is independent of the metric (within the class of metrics having the right asymptotic form at the singular points) used to define the Laplacian.

(8.1) PROPOSITION. *The Hodge structure is independent of the choice of* S.

Proof. It suffices to check that if we restrict from S to a Zariski-open subset S', we do not change the complex $\Omega^{\cdot}(V)_{(2)}$. But this is clear, since $\Omega^{\cdot}(V)_{(2)} = \Omega^{\cdot}_{\bar{S}}(V)$ at any point of $\bar{S}$ where V extends as a locally constant sheaf.

We now show that the Hodge structure is compatible with morphisms:

(8.2) PROPOSITION. *The Hodge structure defined in* (7.12) *is functorial*; i.e.,

(i) *For any diagram of variations of Hodge structure of weight* m *over a smooth curve* S,

$$\begin{tikzcd} V' \arrow[rr, "\phi"] \arrow[dr] & & V \arrow[dl] \\ & S & \end{tikzcd},$$

$\phi_*: H^i(\bar{S}, j_*V') \to H^i(\bar{S}, j_*V)$ *is a morphism of Hodge structures.*

(ii) *For any diagram*

$$\begin{tikzcd} & \psi^*(V) = V' \arrow[r, "\phi"] \arrow[d] & V \arrow[d] & \\ \bar{T} & T \arrow[l, hook', "j'"'] \arrow[r, "\psi"] & S \arrow[r, hook, "j'"] & \bar{S} \end{tikzcd} \qquad (\psi \text{ dominant}),$$

$\psi^*: H^i(\bar{S}, j_*V) \to H^i(\bar{T}, j'_*\psi^*(V))$ *is a morphism of Hodge structures.*

Proof. In either case Hodge filtrations are respected, so it suffices to check that in the morphism (i) $\phi_*: \Omega^{\cdot}(V') \to \Omega^{\cdot}(V)$, resp. (ii) $\psi^*: \psi^{-1}\Omega^{\cdot}_S(V) \to \Omega^{\cdot}_T(V')$, the L_2 conditions are preserved. We retain the notation from the end of Section 6. In case (i), we define ν relative to V', then $\phi(V'_a) \subset V_a$, and for unipotent monodromy $\phi(W'_k) \subset W_k$ (because $N \circ \phi = \phi \circ N'$), from which the desired conclusion is obvious using the definition of $\Omega^{\cdot}(V)_{(2)}$. In case (ii), the result follows as easily from (6.10) and the fact that if the local monodromy in V is unipotent, $N' = \mu N$, where μ is thelocal degree of ψ at the singular point of V, so $W'_k = \psi^*(W_k)$.

Remark. There is never any trouble from the non-unipotent part of V, since, by (6.10), the corresponding sections of $\bar{\mathcal{V}}$ "vanish" at the origin.

Applying this result to the geometric situation,

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "\phi"] & S \end{tikzcd},$$

a diagram of smooth projective morphisms over smooth curves, we obtain

(8.3) COROLLARY. *The induced mapping*

$$\psi^*: H^i(\bar{S}, j_* R^m f_* \mathbf{C}) \longrightarrow H^i(\bar{T}, j'_* R^m g_* \mathbf{C})$$

is a morphism of Hodge structures.

There is one more obvious functorial property:

(8.4) LEMMA. *If $s \in S$, the restriction mapping*

$$H^0(\bar{S}, j_* V) \longrightarrow V_s$$

is a morphism of Hodge structures.

Proof. The above mapping is induced by the morphism of filtered complexes

$$\Omega^\cdot(V)_{(2)} \longrightarrow \mathcal{O}(V) \otimes \mathcal{O}_{S,s} \simeq V_s\,.$$

From this, we obtain a purely Hodge-theoretic proof of the result of Deligne [8, (4.1.3.1)] and Schmid [17, (7.22)], and all of its consequences:

(8.5) COROLLARY. *If $\sigma \in \Gamma(S, V)$, then its Hodge components $\sigma^{p,q}$ (which a priori lie in $\Gamma(S, \mathcal{E}(\mathcal{V}))$) are elements of $\Gamma(S, V)$. Thus, if σ is of type (p, q) at one point, it is of type (p, q) everywhere.*

Finally, the following is true:

(8.6) PROPOSITION. *If V' and V'' underlie polarizable variations of Hodge structure over S, then, with $V = V' \otimes_{\mathbf{C}} V''$, the cup-product mappings*

$$H^i(\bar{S}, j_* V') \otimes_{\mathbf{C}} H^k(\bar{S}, j_* V'') \longrightarrow H^{i+k}(\bar{S}, j_* V)$$

are morphisms of Hodge structures (using the natural tensor product structure on the left hand member).

Proof. The above cup-products will be induced by the extension to $\bar{S}$ of the morphism of complexes $\Omega^\cdot_S(V') \otimes_{\mathcal{O}_S} \Omega^\cdot_S(V'') \to \Omega^\cdot_S(V)$. This mapping preserves the L_2 conditions, for in the unipotent case $W'_k \otimes W''_l \subset W_{k+l}$ and the L_2 conditions are determined by position relative to the middle weight $m = m' + m''$; and the Hodge filtration is respected since

$$\mathcal{F}'^p \otimes_{\mathcal{O}_S} \mathcal{F}''^q \subset \mathcal{F}^{p+q}\,.$$

Hence the mapping induces on cohomology morphisms of Hodge structures.

(8.7) COROLLARY. *In the geometric situation, the cup-product mappings*

$$H^i(\bar{S}, j_* R^p f_* \mathbf{C}) \otimes_{\mathbf{C}} H^k(\bar{S}, j_* R^q f_* \mathbf{C}) \longrightarrow H^{i+k}(\bar{S}, j_* R^{p+q} f_* \mathbf{C})$$

are morphisms of Hodge structures.

9. The theorem on normal functions

In Section 2, we showed that the theorem on normal functions for smooth fibrations $\bar{f}\colon \bar{X} \to \bar{S}$ (2.12) follows as an immediate consequence of the definition of the Hodge filtration. In this section, we prove the analogous result for general fibrations. It was also shown in Section 2 how the Hodge structure on $H^1(\bar{S}, R^{2p-1}\bar{f}_*\mathbf{C})$ is related to that of $H^{2p}(\bar{X}, \mathbf{C})$; however, for the discussion of this matter in the degenerating case, we defer until Section 15.

Let V underlie a variation of Hodge structure of weight $2p-1$ defined over $\mathbf{Z}$. Recall that there is an exact sequence

$$0 \longrightarrow j_* V_{\mathbf{Z}} \longrightarrow (\bar{\mathcal{V}}/\bar{\mathcal{F}}^p)_h \longrightarrow \bar{\mathcal{J}}^p_h \longrightarrow 0 .$$

We consider the filtered complex $K^\bullet$ defined by

$$\begin{aligned} K^0 &= \bar{\mathcal{V}} , \\ K^1 &= \mathcal{S} = \operatorname{im}\{\nabla\colon \bar{\mathcal{V}} \longrightarrow \Omega^1_{\bar{S}}(\log \Sigma) \otimes \bar{\mathcal{V}}\} , \end{aligned}$$

with $\{F^p K^\bullet\}$ defined by restricting the Hodge filtration (1.12) of Deligne from $j_*\Omega^\bullet(V)$ to $K^\bullet$. We show that $K^\bullet$ is a replacement for $\Omega^\bullet(V)_{(2)}$ in the cohomological Hodge complex:

(9.1) PROPOSITION. *The inclusion of complexes*

$$\Omega^\bullet(V)_{(2)} \longrightarrow K^\bullet$$

is a filtered quasi-isomorphism.

Proof. Let $Q^\bullet$ denote the quotient complex $K^\bullet/\Omega^\bullet(V)_{(2)}$, with induced filtration. Since $K^\bullet$ and $\Omega^\bullet(V)_{(2)}$ differ only at the singular points in the summand of V corresponding to eigenvalue 1, we may assume that V has unipotent monodromy. We will show that

$$\mathrm{Gr}^n_F Q^0 \longrightarrow \mathrm{Gr}^n_F Q^1$$

is an isomorphism of finite-dimensional vector spaces, hence $\mathrm{Gr}^n_F Q^\bullet$ is acyclic. As usual, the non-trivial issue is at the singular points. We use the $\mathfrak{sl}(2)$ decomposition of V, and the basis for $\bar{\mathcal{V}}$ used in the proof of (5.2). For irreducible component $S(m-p-q) \otimes E(p, q)$ (the case $S(m-2l) \otimes H(l)$ is similar), we obtain as basis elements of type $(n, m-n)$,

$$\begin{aligned} &\alpha_{n-p} \otimes e^{p,q} &&\text{if} \quad p \leqq n \leqq m - q , \\ &\alpha_{n-q} \otimes e^{q,p} &&\text{if} \quad q \leqq n \leqq m - p . \end{aligned}$$

If β_{n-p} denotes the section of $\bar{\mathcal{F}}^n \subset \bar{\mathcal{V}}$ corresponding to the former, we have β_{n-p} as one of the generators for $\mathrm{Gr}^n_F K^0$; for $\mathrm{Gr}^n_F \mathcal{O}(V)_{(2)}$, we have

$$\begin{cases} \beta_{n-p} & \text{if} \quad n - p \leqq \dfrac{1}{2}(m-p-q) \\ t\beta_{n-p} & \text{otherwise} . \end{cases}$$

Similarly, for $\mathrm{Gr}_F^n K^1$, we use $\gamma_{n-p} = \nabla\beta_{n-p}$ unless $n = m - q$, in which case $t\gamma_{n-p}$; in $\mathrm{Gr}_F^n\Omega^1(V)_{(2)}$:

$$\begin{cases} \gamma_{n-p} & \text{if} \quad n - p \leqq \dfrac{1}{2}(m - p - q) \\ t\gamma_{n-p} & \text{otherwise} . \end{cases}$$

Thus, we have, in a basis for $\mathrm{Gr}_F^n Q^0$, the element $\beta_{n-p}(0)$, and for $\mathrm{Gr}_n^F Q^1$, $\gamma_{n-p}(0)$ if $p \geqq 0$ and $n - p \geqq (1/2)(m - p - q)$; there is no contribution from $e^{p,q}$ otherwise. A similar discussion can be carried out for $e^{q,p}$. Evidently, ∇ induces an isomorphism from $\mathrm{Gr}_F^n Q^0$ to $\mathrm{Gr}_F^n Q^1$ for all n.

Proposition (9.1) asserts that there is no discrepancy arising from the use of $K^{\cdot}$ versus $\Omega^{\cdot}(V)_{(2)}$. This fact will be of use in later sections of this paper; we use it now to prove

(9.2) THEOREM ON NORMAL FUNCTIONS (optimal version). *The image of $H^0(\bar{S}, \bar{\mathcal{J}}_h^p)$ (the horizontal normal functions) in $H_{\mathbf{Z}}^1 = H^1(\bar{S}, j_* V_{\mathbf{Z}})$ is equal to the set of elements which are of type (p, p) in $H^1(\bar{S}, j_* V)$ ($V = R^{2p-1}f_*\mathbf{C}$).*

Proof. As in (2.12), we are seeking

$$\ker\{H_{\mathbf{Z}}^1 \longrightarrow H^1(\bar{S}, (\bar{\mathcal{V}}/\bar{\mathcal{F}}^p)_h)\} ,$$

and the set of integral elements of type (p, p) is precisely

$$\ker\{H_{\mathbf{Z}}^1 \longrightarrow H_{\mathbf{C}}^1/F^p H_{\mathbf{C}}^1\} .$$

But by (9.1),

$$\begin{aligned} H_{\mathbf{C}}^1/F^p H_{\mathbf{C}}^1 &= \mathbf{H}^1(\bar{S}, \Omega^{\cdot}(V)_{(2)}/F^p\Omega^{\cdot}(V)_{(2)}) \\ &= \mathbf{H}^1(\bar{S}, K^{\cdot}/F^p K^{\cdot}) \\ &= \mathbf{H}^1(\bar{\mathcal{V}}/\bar{\mathcal{F}}^p \xrightarrow{\nabla} \mathcal{S}/F^p\mathcal{S}) \\ &= H^1(\bar{S}, (\bar{\mathcal{V}}/\bar{\mathcal{F}}^p)_h) . \end{aligned}$$

(9.3) *Remark.* As in Section 2, we have the result (9.2) for arbitrary variations of Hodge structure of odd weight defined over $\mathbf{Z}$.

10. Miscellaneous results

(a) *Sections of Hodge bundles.* For notational simplicity, we assume that $\mathcal{F}^0 = \mathcal{V}$; this can always be arranged at the cost of artificially shifting the weight of the variation of Hodge structure (tensoring with a constant one-dimensional Hodge structure of sufficiently high weight). In [13], Griffiths calculates that the last Hodge bundle ($\mathcal{G}_{\geq}{}^0$) of a variation of Hodge structure is a negative (semi-definite) vector bundle. Using the Hodge theory with degenerating coefficients, we prove an analogous result for the canonical extensions. Let $\bar{\mathcal{G}}_{\geq}{}^p = \bar{\mathcal{F}}^p/\bar{\mathcal{F}}^{p+1}$.

(10.1) PROPOSITION. *Under the hypotheses of* (7.12), *every global section of* $\bar{\mathscr{G}}_{\gtrsim}^{0}$ *is induced by a flat section of* $\bar{\mathfrak{F}}^0 = \bar{\mathfrak{V}}$.

Proof. In the surjection

$$H^0(\bar{S}, j_* V) \longrightarrow (F^0/F^1)H^0(\bar{S}, j_* V) ,$$

the right hand side is, by (10.1), equal to

$$\mathbf{H}^0(\bar{S}, \mathrm{Gr}_F^0 K^\cdot) = H^0(\bar{S}, \bar{\mathscr{G}}_{\gtrsim}^{0}) .$$

(10.2) COROLLARY (*theorem of the fixed part for Abelian varieties, cf.* (9.2) *for* $p = 1$). *Let* $f: X \to S$ *be a family of Abelian varieties, and* $\bar{f}: \bar{X} \to \bar{S}$ *its Neron model. Then there is a short exact sequence*

$$0 \longrightarrow X_{\mathrm{fixed}} \longrightarrow \Gamma_0(\bar{S}, \bar{X}) \longrightarrow \mathbf{H}_{\mathbf{Z}}^{1,1}(j_* V) \longrightarrow 0 ,$$

where $V = R^1 f_* \mathbf{C}$, X_{fixed} *is the Abelian variety*

$$(F^0/F^1)H^0(\bar{S}, j_* V)/H^0(\bar{S}, j_* V_{\mathbf{Z}}) ,$$

with $\bar{S} \times X_{\mathrm{fixed}}$ *embedded in* $\bar{X}$ *over* $\bar{S}$, *and* $\Gamma_0(\bar{S}, \bar{X})$ *denotes the group of sections of* $\bar{X}$ *passing through the identity components of the singular fibers.*

Proof. $\Gamma_0(\bar{S}, \bar{X})$ is the set of sections of the sheaf $\bar{\mathfrak{A}}$ defined by

$$0 \longrightarrow j_* V_{\mathbf{Z}} \longrightarrow \bar{\mathscr{G}}_{\gtrsim}^{0} \longrightarrow \bar{\mathfrak{A}} \longrightarrow 0 .$$

Now take cohomology.

By the same argument as was used in the proof of (10.1), but applied to $\mathscr{G}_{\gtrsim F}^{p}$, we obtain the following more general statement (cf. [17, (7.19)]):

(10.3) PROPOSITION. *If* $\eta \in H^0(\bar{S}, \bar{\mathscr{G}}_{\gtrsim}^{p})$ *and* $\nabla\eta \big(\in H^0(\bar{S}, \Omega_{\bar{S}}^1(\log \Sigma) \otimes \bar{\mathscr{G}}_{\gtrsim}^{p-1})\big)$ *is zero, then there exists a flat section* σ *of* $\bar{\mathfrak{F}}^p$, *which may be taken to be everywhere of type* $(p, m - p)$, *so that* η *is induced by* σ.

(10.4) COROLLARY. *If* $\phi \in H^0(\bar{S}, \bar{\mathfrak{F}}^p)$ *and* $\nabla\phi$ *takes its values in* $\bar{\mathfrak{F}}^p$, *then there exists a flat section* σ *of* $\bar{\mathfrak{F}}^p$ *so that* $\phi - \sigma \in H^0(\bar{S}, \bar{\mathfrak{F}}^{p+1})$.

(b) *Duality.* The inner product $\langle\, , \rangle$ (defined in § 2) is non-degenerate on cohomology, as $\langle Cv, \bar{v} \rangle > 0$ for any non-zero harmonic form v. Using instead the natural cup-products (along with the polarization of V), we conclude:

(10.5) PROPOSITION. *The natural pairings*

$$H^0(\bar{S}, j_* V) \times H^2(\bar{S}, j_* V) \longrightarrow H^2(\bar{S}, \mathbf{C}) \simeq \mathbf{C} ,$$
$$H^1(\bar{S}, j_* V) \times H^1(\bar{S}, j_* V) \longrightarrow H^2(\bar{S}, \mathbf{C}) \simeq \mathbf{C}$$

are dual pairings.

11. General real variations of Hodge structure

For a general variation of Hodge structure over $\mathbf{R}$, one does not have quasi-unipotence for the local monodromy transformations, only that their eigenvalues are all of norm one. In this section, we prove the Poincaré and Dolbeault Lemmas for the general case. Of course, this gives an alternate proof of (6.14). Consequently, we immediately obtain the existence of a functorial Hodge structure on $H^i(\bar{S}, j_* V)$.

At a singular point, we decompose V into its generalized eigenspaces

$$V = \bigoplus_{0 \leq \alpha < 1} V_\alpha$$

so that $(T - e^{2\pi i \alpha} I)$ is nilpotent on V_α.

The quasi-canonical extension $\bar{\mathcal{V}}$ of $\mathcal{V}$ is defined as in (6.9), with generators

$$\text{(11.1)} \qquad \begin{cases} \tilde{v}(t) = \exp\left(\left[\alpha + \dfrac{1}{2\pi i} N_\alpha\right] \log t\right) v & (v \in V_\alpha) \\ \qquad = t^\alpha \exp\left(\dfrac{1}{2\pi i} N_\alpha \log t\right) v \; . \end{cases}$$

The theory of [17] generalizes to arbitrary variations of Hodge structure (unpublished correspondence between Deligne and Schmid), and we have

(11.2) LEMMA. $|\tilde{v}|^2 \sim r^{2\alpha} \log^{k-m} r$ *if* $v \in W_k(V_\alpha)$.

The statement here is the natural extension of the norm estimates that come from [17]. Assuming (11.2) and the existence of L_2-adapted bases achieving these norms, we proceed as in Sections 6 and 7. We work with a fixed summand V_α with $\alpha \neq 0$. If we grade V_α according to N_α, and write forms in $\mathrm{Gr}^k_W(V_\alpha)$ as $\omega\tilde{v}$, D becomes the operator

$$L_\alpha(\omega) = d\omega + \alpha \frac{dt}{t} \wedge \omega \; .$$

Generalizing (6.6), we have

(11.3) PROPOSITION. *Let* $\omega\tilde{v}$ *be an* L_2 *i-form* ($i = 1$ *or* 2) *with* $L_\alpha(\omega) = 0$. *Then there exists an* L_2 $(i-1)$*-form* $\eta\tilde{v}$ *with* $L_\alpha(\eta) = \omega$.

Proof. We will carry out the argument only for the case $i = 1$, the other case being similar. From our experience in Section 6, we work explicitly in polar coordinates, and we may permit ourselves to compute formally. Writing $\omega = f\, dr + g\, d\theta$, and using the identity

$$\frac{dt}{t} = \frac{dr}{r} + i d\theta \; ,$$

we see that the condition $L_\alpha(\omega) = 0$ becomes

$$\frac{\partial g}{\partial r} - \frac{\partial f}{\partial \theta} + \alpha\left(\frac{g}{r} - if\right) = 0 ,$$

or, expressing this in terms of Fourier coefficients, with

$$f = \sum f_n(r)e^{in\theta} , \qquad g = \sum g_n(r)e^{in\theta} ,$$

(11.4) $$g_n' + \frac{\alpha g_n}{r} = i(n + \alpha)f_n .$$

If $u = \sum u_n(r)e^{in\theta}$ is to satisfy $L_\alpha u = \omega$, then

(11.5) $$\text{(i)}\ \ u_n' + \frac{\alpha}{r}u_n = f_n$$

$$\text{(ii)}\ \ i(n + \alpha)u_n = g_n .$$

Because $\alpha \notin \mathbf{Z}$, we may use (11.5, ii) to obtain

$$u_n = -\frac{i}{n + \alpha}g_n .$$

Then (11.4) guarantees that (11.5, i) is simultaneously satisfied. Estimating L_2 norms, we have

$$\begin{aligned} ||u_n\tilde{v}||_{(2)}^2 &= (n + \alpha)^{-2}\int |g_n|^2 r^{2\alpha}|\log r|^{k-2}(r^{-1}dr) \\ &\leqq C(n + \alpha)^{-2}\int |g_n|^2 r^{-2}\left(r^{2\alpha}|\log r|^k\right)(rdr) \\ &= C(n + \alpha)^{-2}||g_n d\theta||_{(2)}^2 . \end{aligned}$$

So $||u\tilde{v}||_{(2)}^2 \leqq K||\omega\tilde{v}||^2$.

Likewise, extending (6.4), we have

(11.5) PROPOSITION. *Let $\mathcal{V}$ be a holomorphic line bundle on Δ^* with generating section σ, and with a Hermitian metric satisfying*

$$||\sigma||^2 \sim r^{2\alpha}|\log r|^k \qquad (0 < \alpha < 1) .$$

Then for every L_2 (0, 1)-form $\phi = fd\bar{t} \otimes \sigma$ there exists an L_2 section $u \otimes \sigma$ with $\bar{\partial}u = fd\bar{t}$.

Proof. As in (6.4), we define u to have Fourier coefficients

$$u_n = \begin{cases} 2r^n\int_0^r \rho^{-n}f_{n+1}(\rho)d\rho & \text{if} \quad n < 0 \\ -2r^n\int_A^r \rho^{-n}f_{n+1}(\rho)d\rho & \text{if} \quad n \geqq 0 \end{cases}$$

for some $A < 1$, so that $\bar{\partial}u = fd\bar{t}$. Checking L_2 norms in the first instance (the other being similar), we see that

$$\begin{aligned}
&\int_0^A |u_n|^2 r^{-1+2\alpha} |\log r|^{k-2} dr \\
&\leqq \int_0^A 4r^{2n} \left(\int_0^r \rho^{-2n} |f_{n+1}(\rho)|^2 d\rho\right)\left(\int_0^r d\rho\right) r^{-1+2\alpha} |\log r|^{k-2} dr \\
&= \int_0^A 4r^{2(n+\alpha)} |\log r|^{k-2} \left(\int_0^r \rho^{-2n} |f_{n+1}(\rho)|^2 d\rho\right) dr \\
&= \int_0^A \rho^{-2n} |f_{n+1}(\rho)|^2 \left(\int_\rho^A 4r^{2(n+\alpha)} |\log r|^{k-2} dr\right) d\rho \\
&\leqq \frac{C}{2(n+\alpha)+1} \int_0^A |f_{n+1}(\rho)|^2 \rho^{1+2\alpha} |\log \rho|^{k-2} d\rho \\
&\leqq K \int_0^A |f_{n+1}(\rho)|^2 \rho^{1+2\alpha} |\log \rho|^k d\rho\ .
\end{aligned}$$

So $\|u \otimes \sigma\|^2_{(2)} \leqq K \|f d\bar{t} \otimes \sigma\|^2_{(2)}$.

The rest of the theory proceeds as in Section 7, so we may conclude

(11.6) THEOREM. *If V is the sheaf of local constants underlying a polarizable variation of Hodge structure of weight m (defined over* $\mathbf{R}$*) on the smooth algebraic curve S, then there are associated functorial Hodge structures of weight $m+i$ on $H^i(\bar{S}, j_* V) \simeq \mathbf{H}^i(\bar{S}, \Omega^\cdot(V)_{(2)})$, whose Hodge filtration is induced by the Hodge filtration* (1.12) *on the complex $\Omega^\cdot(V)_{(2)}$.*

(11.7) *Remark.* As in (9.1), we may use the complex $K^\cdot$ instead to give the Hodge filtration. (This affects only the unipotent summand.)

12. The Hodge structures of Shimura

In [18], Shimura shows the existence of what is, in effect, a Hodge structure (defined over $\mathbf{R}$) on some group cohomology of Fuchsian subgroups of $SL(2, \mathbf{R})$ in terms of vector-valued automorphic forms (see also [21, (4.2.6)] and [11, (2.10)]). The Hodge structure appears in a somewhat ad hoc manner.[7] In this section, we show that these Hodge structures are examples of the Hodge theory associated to a variation of Hodge structure over a curve.

We follow for the most part [21]. Let $G = SL(2, \mathbf{R})$ with standardized notation

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for the elements of G, and let H denote the upper half-plane. Using $i \in H$ as the base-point, we identify H as the quotient G/K where

$$K = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbf{R} \right\}$$

[7] If the subgroup is cocompact, the Hodge theory for vector-valued forms on Hermitian symmetric spaces (see [25]) provides another interpretation.

is the stabilizer of i. If Γ is a Fuchsian group of the first kind (Γ is a discrete subgroup of G such that G/Γ has finite invariant measure), by putting $S = H/\Gamma$ we obtain an algebraic curve whose smooth completion $\bar{S}$ is obtained by adjoining a finite number of *cusps* to S.

Let (ρ_m, V_m) denote the m^{th} standard irreducible representation of G; namely, $V_1 = \mathbf{C}^2$ with G operating in the usual way, and (ρ_m, V_m) is the m^{th} symmetric power of (ρ_1, V_1). It will suffice to consider V_1 for most purposes. Since K is a circle, ρ_1 necessarily diagonalizes under K, and the eigenvectors are computed to be $e_1 = u_1 + iu_2$ and $e_{-1} = \bar{e}_1 = u_1 - iu_2$, where $\{u_1, u_2\}$ is the standard basis of $\mathbf{C}^2$. It follows that V_m has a basis of K-eigenvectors $\{\varepsilon_m, \varepsilon_{m-2}, \cdots, \varepsilon_{-m}\}$ where

$$\varepsilon_{m-2l} = (e_1)^{m-l}(e_{-1})^l \,, \tag{12.1}$$

the vector ε_p corresponding to eigenvalue $e^{ip\theta}$ under the matrix $k(\theta)$.

Let $S_k(\Gamma, m)$ denote the space of V_m-valued cusp forms for Γ. Explicitly, these are holomorphic functions $f\colon H \to V_m$ which satisfy

$$f(\gamma z) = (cz + d)^k \rho_m(\gamma) f(z) \qquad \text{for all} \quad \gamma \in \Gamma \,, \tag{12.2}$$

and, in addition, a *cusp condition* [21, (A_3)] which will not be repeated now. If

$$-I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma \,,$$

we must insist that $m + k$ be even in order that the construction to be done shortly make sense. (Also, if $m + k$ is odd, $S_k(\Gamma, m) = 0$.) In fact, for simplicity, we assume $-I_2 \notin \Gamma$. We expand a cusp form as

$$f(z) = \textstyle\sum_p f_p(z)(ci + d)^p \rho_m(g)\varepsilon_p \qquad \text{if} \quad z = gi \,.$$

One easily verifies that this expression does indeed define $f_p(z)$, independent of the choice of g.

(12.3) LEMMA. *If $f \in S_k(\Gamma, m)$ and $f_r = 0$ for all $r > p$, then $f_p \in S_{k-p}(\Gamma, 0)$.*

Proof. See [21], specifically (1.1.9) and (2.3).

In particular, there is an injection

$$Q\colon S_{k+m}(\Gamma, 0) \longrightarrow S_k(\Gamma, m)$$

given by $Q(F) = F(z)(ci + d)^{-m} \rho_m(g)\varepsilon_{-m}$.

For any value of m one constructs a locally constant system, also denoted V_m, on S by taking $H \times V_m/\Gamma$, where Γ acts by the product action. Let $\mathcal{V}_m = V_m \otimes_{\mathbf{C}} \mathcal{O}_S$. The space $S_0(\Gamma, m)$ may be identified with a subspace of the global sections of $\mathcal{V}_m$ over S (equivalently, the Γ-equivariant holomorphic V_m-valued functions on H). Similarly, $S_2(\Gamma, m)$ is contained in the space of

V_m-valued holomorphic one-forms on S. From the short exact sequence of Γ-modules,

$$0 \longrightarrow V_m \longrightarrow H^0(H, \mathcal{O}_H \otimes_{\mathbf{C}} V_m) \longrightarrow H^0(H, \Omega^1_H \otimes_{\mathbf{C}} V_m) \longrightarrow 0 ,$$

one obtains a mapping

$$\begin{tikzcd} H^0(H, \Omega^1_H \otimes_{\mathbf{C}} V_m)^\Gamma \arrow[r] & H^1(\Gamma, V_m) \arrow[d, phantom, "\wr"] \\ S_2(\Gamma, m) \arrow[u, hook] \arrow[r, "\delta"] & H^1(S, V_m) . \end{tikzcd}$$

It can be checked that δ is injective, so we identify $S_2(\Gamma, m)$ with its image under δ.

Let Γ_0 be a parabolic subgroup of Γ. This means that Γ_0 is the stabilizer in Γ of a cusp of $\bar{S}$, and necessarily Γ_0 is cyclic with $H/\Gamma_0 \simeq \Delta^*$. One defines the *parabolic* (or *Eichler*) *cohomology* group

$$\tilde{H}^1(\Gamma, V_m) = \textstyle\bigcap_{\Gamma_0 \subset \Gamma} \ker\{H^1(\Gamma, V_m) \longrightarrow H^1(\Gamma_0, V_m)\} .$$

Then the *Hodge structure of Shimura* is given by the following:

(12.4) THEOREM [18]. $Q[S_{2+m}(\Gamma, 0)] \oplus \overline{Q[S_{2+m}(\Gamma, 0)]} = \tilde{H}^1(\Gamma, V_m)$.

We begin to translate into the context of this paper:

(12.5) PROPOSITION. *$\tilde{H}^1(\Gamma, V_m)$ is naturally isomorphic to $H^1(\bar{S}, j_* V)$.*

Proof. From the Leray spectral sequence for j, we obtain the exact sequence

$$0 \longrightarrow H^1(\bar{S}, j_* V_m) \longrightarrow H^1(S, V_m) \xrightarrow{\ \pi\ } H^0(\bar{S}, R^1 j_* V_m) .$$

Since $R^1 j_* V_m$ is supported on the set of cusps $\Sigma \subset \bar{S}$, we may write

$$H^0(\bar{S}, R^1 j_* V_m) = \textstyle\bigoplus_{s \in \Sigma} H^1\big(\Delta^*(s), V_m\big) ,$$

where $\Delta^*(s)$ is a small punctured disc around the cusp s. The mapping π then decomposes into a direct sum of restriction mappings

$$H^1(S, V_m) \longrightarrow H^1\big(\Delta^*(s), V_m\big) . \tag{12.6}$$

But $\Delta^*(s)$ is a deformation retract of the larger punctured disc H/Γ_0 (Γ_0 being the parabolic subgroup for s). Thus, the mapping (12.6) is none other than

$$H^1(\Gamma, V_m) \longrightarrow H^1(\Gamma_0, V_m) .$$

Now, (12.5) follows immediately.

In order to apply Hodge theory with degenerating coefficients, we need to put a polarized variation of Hodge structure on $\mathcal{V}_m$. The structure on $\mathcal{V}_m$ will be of weight m, and it will be induced by symmetric product from $\mathcal{V}_1$.

At the point $z = gi \in H$, define

$$H_z^{p,m-p}(V_m) = \mathbf{C}\rho_m(g)\varepsilon_{m-2p} .$$

Since K acts diagonally with respect to $\{\varepsilon_p\}$, this is a well-defined subspace of V_m. It is clear that

$$H_z^{m-p,p} = \overline{H_z^{p,m-p}}$$

and

$$V_m = \bigoplus_{p=0}^m H_z^{p,m-p} .$$

By construction, whatever Γ is, the variation of Hodge structure descends to $S = H/\Gamma$, giving a variation of Hodge structure on $\mathcal{V}_m$. To polarize this structure, we observe that ρ_m is isomorphic as a representation to its contragredient, so there is a G-equivariant isomorphism

$$\phi_m \colon V_m \longrightarrow V_m^*$$

(determined uniquely up to a multiplicative constant). Since ϕ_m is necessarily (proportional to) the m^{th} symmetric power of ϕ_1, we need only polarize V_1, and this may be checked at the one point i, where e_1 is of type (0, 1) and e_{-1} is of type (1, 0). Explicitly, if $\{e_1^*, e_{-1}^*\}$ is the dual basis for V_1^*,

$$\phi_1(e_1) = -2ie_{-1}^* ,$$
$$\phi_1(e_{-1}) = 2ie_1^* ,$$

so we obtain a skew-symmetric $\mathbf{C}$-linear pairing with

(12.7) $$(e_1, e_{-1}) = \langle e_1, \phi_1(e_{-1})\rangle = 2i .$$

It is easy to see that this pairing is, in fact, defined over $\mathbf{R}$. Then V_1 is polarized by the Hermitian form $\langle\, ,\rangle$, with

(12.8) $$\langle e_1, e_1\rangle = (Ce_1, \bar{e}_1) = (i^{-1}e_1, e_{-1}) = 2 ,$$
$$\langle e_{-1}, e_{-1}\rangle = 2 .$$

(12.9) Lemma. *$\omega(z) = zu_1 + u_2$ is a nowhere-vanishing holomorphic section of the first Hodge bundle $\mathcal{F}_1^1$ on H.*

Proof. We know that at $z = gi$, F^1 is spanned by ge_{-1}. If $z = x+iy$, we may choose

$$g = y^{-1/2}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} .$$

Then

$$ge_{-1} = y^{-1/2}\begin{pmatrix} y - ix \\ -i \end{pmatrix} = -iy^{-1/2}\begin{pmatrix} z \\ 1 \end{pmatrix} .$$

Thus, $zu_1 + u_2$ is a section of $\mathcal{F}_1^1$, and it is patently holomorphic and nowhere-vanishing on H.

Now let $s \in \bar{S}$ be a cusp, and Γ_0 the associated parabolic subgroup. As was remarked earlier, X/Γ_0 is a punctured disc Δ^*, and the natural mapping $X/\Gamma_0 \to X/\Gamma$ is an embedding on a sufficiently small deleted neighborhood of s. We may assume without loss of generality that s is the image of the parabolic point ∞. A generator γ of Γ_0 must then be of the form

$$\lambda\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},$$

where $\lambda = \pm 1$ and $h \in \mathbf{R}$. By the definition of V_m, γ is the local monodromy transformation around the cusp. The associated nilpotent logarithm is

$$N = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix},$$

and the covering $\tau: H \to \Delta^*$ is given by

$$t = \tau(z) = e^{2\pi i z/h} .$$

(12.10) LEMMA. *If $\lambda = 1$, $\omega(z) = zu_1 + u_2$ and u_1 are (local) generating sections of the canonical extension bundle $\bar{\mathcal{V}}_1$; if $\lambda = -1$,*

$$e^{\pi i z/h}\omega(z) = t^{1/2}(zu_1 + u_2) \qquad \text{and} \quad t^{1/2}u_1$$

are generators.

Proof. This is obvious, once one writes

$$\begin{aligned}\omega(z) &= (z/h)(hu_1) + u_2 \\ &= \exp\Big(\frac{1}{2\pi i}N\log t\Big)u_2 .\end{aligned}$$

(12.11) COROLLARY. *If $\lambda = 1$, or if $\lambda = -1$ and m is even, $\sigma_p = [\omega(z)]^p u_1^{m-p}$ generates $\bar{\mathcal{F}}_m^p$ modulo $\bar{\mathcal{F}}_m^{p+1}$; if $\lambda = -1$ and m is odd, $\sigma_p = t^{1/2}[\omega(z)]^p u_1^{m-p}$ is a generator.*

This takes care of the Hodge filtration. Though it is not essential for the discussion, we briefly take note of the weight filtration. For V_1 (the rest being induced by symmetric product), we have trivially:

(12.12) LEMMA. *On V_1, u_1 spans $W_0(= W_1)$, and u_2 spans $\mathrm{Gr}_2^W(V_1)$.*

We can compute directly the norms of the two elements u_1 and u_2. Breaking each into its Hodge components, we have

$$\begin{aligned}u_1 &= (2iy)^{-1}[\omega(z) - \overline{\omega(z)}] , \\ u_2 &= (2iy)^{-1}[z\bar{\omega}(z) - \bar{z}\omega(z)] ,\end{aligned}$$

so by (12.8), we calculate, as $\omega(z) = iy^{1/2}ge_1$, that

$$\|u_1\|_z^2 = (4y^2)^{-1}[2\|\omega\|^2] = (2y^2)^{-1}[y\langle e_1, e_1\rangle] = y^{-1} ,$$
$$\|u_2\|_z^2 = (4y^2)^{-1}[2|z|^2\|\omega\|^2] = y^{-1}|z|^2 \sim y^1 .$$

(12.13) *Remark.* It deserves to be mentioned that we are looking here at perhaps the "purest" example of Schmid's asymptotic description of the degeneration of a variation of Hodge structure. In fact, the asymptotic and the actual coincide here. The structure on V_1 is the model $S(1)$, with $v_+ = e_1$ and $v_- = e_{-1}$, and there is a correspondence of polarizations (compare (12.7) to [17, (6.23)]).

We are ready to analyze the Hodge structure on $H^1(\bar{S}, j_* V_m)$.

(12.14) LEMMA. *The complex* $\mathrm{Gr}_F^p K^\cdot$ *is acyclic except for* $p = 0, m + 1$.

Proof. This is very simple: for $0 < p \leqq m$, $\mathrm{Gr}_F^p K^\cdot$ is the complex

$$\mathscr{G}_z{}^p \xrightarrow{\nabla} \Omega_{\bar{S}}^1(\log \Sigma) \otimes \overline{\mathscr{G}_z{}^{p-1}} .$$

Even at a cusp (which we take to be given by $z = \infty$), $\bar{\mathscr{G}}_z{}^p$ is generated by σ_p, and $\nabla\sigma_p = pdz \otimes \sigma_{p-1}$ generates $\Omega_{\bar{S}}^1 (\log \Sigma) \otimes \overline{\mathscr{G}_z{}^{p-1}}$, according to (12.11). Since ∇ is $\mathcal{O}_{\bar{S}}$-linear, it is an isomorphism, i.e., $\mathrm{Gr}_F^p K^\cdot$ is acyclic.

As $\mathbf{H}^1(\bar{S}, \mathrm{Gr}_F^p K^\cdot)$ gives the subspace $H^{p,m+1-p}$ in the Hodge decomposition of $H^1(\bar{S}, j_* V_m)$, most of the components are zero, and we have

$$\begin{aligned} H^1(\bar{S}, j_* V) &= H^{m+1,0} \oplus H^{0,m+1} \\ &= H^{m+1,0} \oplus \overline{H^{m+1,0}} . \end{aligned}$$

We complete the interpretation of (12.4) by showing

(12.15) LEMMA. $H^{m+1,0} = Q[S_{2+m}(\Gamma, 0)]$.

Proof.
$$\begin{aligned} H^{m+1,0} &= F^{m+1}H^1(\bar{S}, j_* V_m) \\ &= \mathbf{H}^1(\bar{S}, F^{m+1}\Omega^\cdot(V_m)_{(2)}) \\ &= H^0(\bar{S}, (\Omega^1 \otimes \mathcal{F}_m^m)_{(2)}) . \end{aligned}$$

Thus, it remains to verify that the $\mathcal{F}^m$-valued one-forms in $Q[S_{2+m}(\Gamma, 0)]$ are those with the requisite growth at the cusps. Write $f \in Q[S_{2+m}(\Gamma, 0)]$ as

$$f(z) = F(z)(ci + d)^{-m}\rho_m(g)\varepsilon_{-m} = F(z)[\omega(z)]^m ,$$

which represents

$$\eta = [F(z)dz] \otimes [\omega(z)]^m \in H^0(S, \Omega_S^1 \otimes \mathcal{F}_m^m) .$$

At a cusp (taking $z = \infty$ again), we must distinguish two cases:

Case 1: $\lambda = 1$, or $\lambda = -1$ and m is even. Then $F(z)$ is a cusp form if and only if it has a power series expansion in $t = e^{2\pi iz/h}$, with constant term zero. Thus, η is of the form $tg(t)(dt/t) \otimes \omega^m$, where $g(t)$ is analytic at the origin. As $|\omega^m|^2 \sim \log^m|t|$, this gives $(\Omega^1 \otimes \mathcal{F}^m_m)_{(2)}$ at the cusp.

Case 2: $\lambda = -1$ and m is odd. Then F is a cusp form if and only if F has a power series expansion in (odd powers of) $t^{1/2}$. The leading term of η is then proportional to $dt/t \otimes t^{1/2}\omega^m$, which generates $(\Omega^1 \otimes \mathcal{F}^m_m)_{(2)}$ at the cusp.

In either case, this completes the proof of (12.15).

13. Mixed Hodge theory

We return to the general set-up where V underlies a polarizable variation of Hodge structure of weight m over the smooth curve S, $j: S \hookrightarrow \bar{S}$, $\Sigma = \bar{S} - S$. Let $\tilde{S}$ be any open subvariety of $\bar{S}$; restricting S if necessary (noting (8.1)), we may assume that $S \subset \tilde{S} \subset \bar{S}$. In this section, we show that there is a functorial mixed Hodge structure on $H^i(\tilde{S}, j_* V|_{\tilde{S}})$. Since we are working over $\mathbf{R}$, a mixed Hodge structure on a complex vector space E, with real structure $E_{\mathbf{R}}$ (so $E = E_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$), consists of two filtrations—an increasing weight filtration $\{W_k\}$ defined over $\mathbf{R}$, and a decreasing Hodge filtration $\{F^p\}$—such that $\{F^p\}$ induces on $\mathrm{Gr}^W_k(E)$ a Hodge structure of weight k. Again, if V is defined over $\mathbf{Z}$, the weight filtration will be defined over $\mathbf{Q}$.

Let $\tilde{j}: \tilde{S} \to \bar{S}$, $j_0: S \to \tilde{S}$, $\tilde{\Sigma} = \bar{S} - \tilde{S}$, $\Sigma_0 = \Sigma - \tilde{\Sigma}$. We build a two-term complex $K(\tilde{S})^\cdot$ by setting $K(\tilde{S})^0 = \bar{\mathcal{V}}$, and letting $K(\tilde{S})^1$ be the extension to $\bar{S}$ of $\Omega^1_S(V)$ given by $\Omega^1_{\bar{S}}(\log \Sigma) \otimes \bar{\mathcal{V}}$ at points of $\tilde{\Sigma}$, and by $K^1 = \mathcal{S}$ (see § 9) at the points of Σ_0. Then we have

(13.1) LEMMA. $\mathbf{H}^i(\bar{S}, K(\tilde{S})^\cdot) \simeq H^i(\tilde{S}, j_{0*}V)$.

Proof. The assertion follows by combination of the fact that $K^\cdot$ resolves $j_* V$ with [10, p. 105] (that $\tilde{j}_*\tilde{j}^*K(\tilde{S})^\cdot$ is quasi-isomorphic to $K(\tilde{S})^\cdot$), since $j_{0*}V = j_*V|_{\tilde{S}}$.

We can use the Hodge filtration (1.12), as before, to define $\{F^pK(\tilde{S})^\cdot\}$. To define a *weight filtration*, we start by letting

$$W_kK(\tilde{S})^\cdot = 0 \qquad \text{if } k < m\,,$$
$$W_mK(\tilde{S})^\cdot = K^\cdot\,;$$

to define the higher weights, we must first introduce some notation. As $K(\tilde{S})^\cdot$ and $K^\cdot(=K(\bar{S})^\cdot)$ agree for non-unipotent monodromy, the non-unipotent local summands of V will not enter into the discussion, so we may assume without loss of generality that the local monodromy transformations are unipotent. At each $s \in \tilde{\Sigma}$, V decomposes into irreducible $\mathfrak{sl}_2$-components (cf. § 5), each of which contains *real* basis elements of highest weight filtration —viz., e_2^m in $S(m)$, etc.—which span the cokernel of

$$\nabla \colon \bar{\mathcal{V}} \longrightarrow \frac{dt}{t} \otimes \bar{\mathcal{V}}$$

(via the *residue* of ∇) or, equivalently, the cokernel of N. Let P denote their span. For $0 \leqq r \leqq M+1$ define (at $s \in \widetilde{\Sigma}$)

$$Z_{m+r} = NV \oplus \big(P \cap W_{m+r-1}(V)\big) \qquad (13.2)$$
$$= W_{m+r-1} + NV$$

and let $\bar{\mathcal{Z}}_{m+r}$ be the corresponding sub-bundle of $\bar{\mathcal{V}}$. In this notation, $K(\widetilde{S})^1$ is equal to

$$\frac{dt}{t} \otimes [\bar{\mathcal{Z}}_m + t\bar{\mathcal{V}}] \, .$$

Define $W_{m+r}K(\widetilde{S})^\cdot$ to be the complex

$$\bar{\mathcal{V}} \longrightarrow \frac{dt}{t} \otimes [\bar{\mathcal{Z}}_{m+r} + t\bar{\mathcal{V}}] \, .$$

(On $\widetilde{S}$, we use $K^\cdot$.) As in Section 4, this gives well-defined complexes of sheaves globally on $\bar{S}$, such that taking F^p or Gr_F^p gives complexes of locally-free $\mathcal{O}_S$-modules (cf. (5.2)).

Let N_s denote the nilpotent monodromy logarithm associated to $s \in \widetilde{\Sigma}$. Then the following is immediate:

(13.3) LEMMA. *For* $0 < r \leqq M+1$,

$$\mathrm{Res} \colon \mathrm{Gr}^W_{m+r} K(\widetilde{S})^\cdot \xrightarrow{\sim} \bigoplus_{s \in \widetilde{\Sigma}} \mathbf{P}_{m+r-1}(s)[-1]$$

where $\mathbf{P}_k(s)$ *is the primitive part of* $\mathrm{Gr}^W_k(V)$ *at* s [17, (6.4)]:

$$\mathbf{P}_{m+i} = \ker\{N^{i+1} \colon \mathrm{Gr}^W_{m+i}(V) \longrightarrow \mathrm{Gr}^W_{m-i-2}(V)\} \, .$$

(13.4) *Remarks*: (i) The isomorphism in (13.3) is non-canonical, in the sense that the identification $Z_{m+r}/NV \simeq P \cap W_{m+r-1}(V)$ involves a splitting; intrinsically,

$$Z_{m+r}/NV = W_{m+r-1}/(W_{m+r-1} \cap NV) \, .$$

(ii) In the case where V is non-degenerate at s, $N = 0$ and $V = W_m$, so $Z_m = 0$, and $\big(W_m K(\widetilde{S})^1\big)_s = \big(\Omega^1_{\bar{S}}(V)\big)_s$. Similarly, $\big(W_{m+1}K(\widetilde{S})^1\big)_s = \big(\Omega^1_{\bar{S}}(\log \widetilde{\Sigma})(V)\big)_s$.

In the spectral sequence

$$E_1^{p,q} = \mathbf{H}^{p+q}\big(\bar{S}, \mathrm{Gr}^W_{-p}K(\widetilde{S})^\cdot\big) \Longrightarrow \mathbf{H}^*\big(\bar{S}, K(\widetilde{S})^\cdot\big) \simeq H^*(\widetilde{S}, j_{0*}V) \, ,$$

we see that the E_1 terms are zero except for

$$-p = m \, , \qquad q = m, m+1, m+2$$
$$-(M+m-1) \leqq p < -m \, , \qquad q = -p+1 \, ,$$

and therefore, the only possible non-zero differentials are

$$(13.5) \qquad d_r\colon \mathbf{H}^1(\bar{S}, \mathrm{Gr}^W_{m+r}K(\tilde{S})^\bullet) \longrightarrow E_r^{-m,m+2} \qquad (1 \leqq r \leqq M+1)\,.$$

In particular, there is

$$(13.6) \qquad d_1\colon \bigoplus\nolimits_{s\in\Sigma} \mathbf{P}_m(s) \longrightarrow H^2(\bar{S}, j_* V)\,.$$

Define the *coarse weight filtration* $\{\tilde{W}_k\}$ on $K(\tilde{S})^\bullet$ by setting

$$(13.7) \qquad \tilde{W}_k = \begin{cases} W_k & \text{if} \quad k \leqq m \quad \text{or} \quad k \geqq M+m+1 \\ W_m & \text{if} \quad m < k < M+m+1\,. \end{cases}$$

(13.8) PROPOSITION. *The spectral sequence associated to $\{\tilde{W}_k\}$ on $K(\tilde{S})^\bullet$ is the Leray spectral sequence for $\tilde{j}$ and the sheaf $j_{0*}V$, shifted so that the E_{M+1} term of the former equals the E_2 term of the latter. Moreover, the following diagram is commutative*:

$$\begin{tikzcd}
\bigoplus_{s\in\tilde{\Sigma}} \mathbf{P}_m(s) \arrow[d, hook] & \mathbf{H}^1(\bar{S}, \mathrm{Gr}^W_{m+1}K(\tilde{S})^\bullet) \arrow[l, "\text{Residue}"', "\sim"] \arrow[d, hook] \arrow[dr, "{}_W d_1"] & \\
\bigoplus_{s\in\tilde{\Sigma}} [\bigoplus_{r=0}^M \mathbf{P}_{m+r}(s)] & \mathbf{H}^1(\bar{S}, \mathrm{Gr}^{\tilde{W}}_{m+M+1}K(\tilde{S})^\bullet) \arrow[l, "\sim"'] \arrow[r, "{}_{\tilde{W}} d_{M+1}"'] & \mathbf{H}^2(\bar{S}, j_* V)\,;
\end{tikzcd}$$

and the mapping ${}_{\tilde{W}}d_{+M1}$ is equal to the composite

$$\begin{aligned}
&\bigoplus\nolimits_{s\in\tilde{\Sigma}} [\bigoplus\nolimits_{r=0}^M \mathbf{P}_{m+r}(s)] \\
&\quad \simeq \bigoplus\nolimits_{s\in\tilde{\Sigma}} V/N_s V \simeq \bigoplus\nolimits_{s\in\tilde{\Sigma}} (\ker N_s)^* \longrightarrow (\bigcap\nolimits_{s\in\tilde{\Sigma}} \ker N_s)^* \longrightarrow (\bigcap\nolimits_{s\in\Sigma} \ker N_s)^* \\
&\quad \xrightarrow{\sim} (H^0(\bar{S}, j_* V))^* \simeq H^2(\bar{S}, j_* V)\,,
\end{aligned}$$

with the unlabeled arrows indicating surjective mappings provided $\tilde{\Sigma} \neq \varnothing$ (i.e., $\tilde{S} \neq \bar{S}$).

Proof. The first assertion follows as in [8, (3.18)], using [10, p. 85 (3.15)]. (We remark that $j_* V = \tilde{j}_*(j_{0*}V)$.) The second assertion is obvious, so it remains to check the last statement. Let $\Delta^*(s)$ be a small disc centered at $s \in \tilde{\Sigma}$. Given $v \in V/N_s V$, let $\tilde{v}$ be a generating section for $\mathcal{V}$ which projects onto v at the origin. As the differential in the spectral sequence is equal to the connecting homomorphism δ in the cohomology sequence of

$$0 \longrightarrow W_m \longrightarrow K(\tilde{S})^\bullet \longrightarrow \mathrm{Gr}^{\tilde{W}}_{m+M} \longrightarrow 0\,,$$

we use Dolbeault cohomology to compute that

$${}_{\tilde{W}}d_{M+1}\Big(\frac{1}{2\pi i}\frac{dt}{t} \otimes v\Big) = \delta\Big(\frac{1}{2\pi i}\frac{dt}{t} \otimes v\Big) = \frac{1}{2\pi i}\bar{\partial}\Big(\rho\,\frac{dt}{t} \otimes \tilde{v}\Big)\,,$$

where ρ is a C^∞ function supported in $\Delta^*(s)$ that is identically 1 in a neighborhood of the origin. If $w \in H^0(\bar{S}, j_* V)$, then

$$\begin{aligned}\delta\Big(\frac{1}{2\pi i}\frac{dt}{t}\otimes\bar v\Big)\smile w &= \frac{1}{2\pi i}\int_S \bar\partial\Big(\rho\,\frac{dt}{t}\Big)(\tilde v, w)\\ &= \frac{1}{2\pi i}\int_S \bar\partial\Big(\rho\,\frac{dt}{t}\Big)(v, w)\\ &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|t|=\varepsilon}\rho\,\frac{dt}{t}(v, w)\\ &= (v, w)\ .\end{aligned}$$

Summing over $\tilde\Sigma$, we see that the proof is complete.

As a consequence of the preceding, we have the following results of a topological nature, which may also be deduced directly.

(13.9) COROLLARY. *If* $\tilde\Sigma \neq \varnothing$, $H^2(\tilde S, j_{0*}V) = 0$.

(13.10) COROLLARY. *If V is a locally constant sheaf on Δ^* and V_0 is the unipotent summand, with associated N, then*

$$H^1(\Delta^*, V) \simeq \frac{1}{2\pi i}\frac{dt}{t}\otimes(V_0/NV_0)\ .$$

(The statement of (13.10) remains true even if the monodromy has arbitrary non-zero eigenvalues, as can be seen directly using Čech cohomology.)

With the aid of Proposition (13.8), we see that the spectral sequence associated to the weight filtration $\{W_k\}$ on $K(\tilde S)^\cdot$ is a refinement of the Leray spectral sequence for $\tilde j_*$ and $j_{0*}V$. Each of the non-zero $E_1^{p,q}$ terms possesses a Hodge structure induced by $\{F^pK(\tilde S)^\cdot\}$. For $p = -m$, it is Hodge theory with degenerating coefficients. For $E_1^{-(m+r),m+r+1} \simeq \bigoplus_{s\in\tilde\Sigma}\mathbf{P}_{m+r-1}(s)$ $(0 < r \leqq M+1)$, there is a Hodge structure of weight $m+r+1$ which can be described as follows. On a Δ^*, the mapping $v\to\tilde v$ sets up an isomorphism $V\simeq\bar{\mathcal{V}}(0)$, imparting a real structure to the latter vector space. The filtration $\{F^p_\infty\}$ of [17, (6.15)] on V is carried into $\{\bar{\mathcal{F}}^p(0)\}$ under the above isomorphism. With the monodromy weights $\{W_k(V)\}$, there is then a mixed Hodge structure on V such that $\mathbf{P}_{m+r-1}$ is a Hodge sub-structure of $\mathrm{Gr}^W_{m+r-1}(V)$ [17, (6.16)]. The isomorphisms in (13.3) decrease Hodge filtration levels by one, so we obtain a Hodge structure of weight $m+r+1$ on $E_1^{-(m+r),m+r+1}$, with the isomorphisms (13.3) now appearing as morphisms of type $(-1, -1)$.

We have a complex (defined over $\mathbf{C}$) that seems ripe for mixed Hodge theory. However, we need to know that the weight filtration is defined over $\mathbf{R}$. To do this, we will complete $(K(\tilde S)^\cdot, W, F)$ to the data of a cohomological mixed Hodge complex [9] by showing that we may refine the canonical filtration $\{\tau_{\leq k}\}$ on Rj_*V to obtain a weight filtration consistent with the above. Since the singular points are isolated, there is no harm in assuming $S = \tilde{S}$. Furthermore, it will suffice to make the construction locally, so we assume that $S = \Delta^*$, $\bar{S} = \Delta$. Associated to $\{Z_k\}$ in (13.2) are locally constant sub-sheaves of V, which we also denote $\{Z_k\}$. Define

$$W_m(Rj_* V) = \tau_{\leqq 0} Rj_* V \simeq j_* V \,;$$

for $0 < r \leqq M + 1$, define $W_{m+r}(Rj_* V)$ to be the complex whose non-zero terms are

$$\begin{aligned} (Rj_* V)^0 \qquad & \text{in degree } \; 0 \,, \\ d(Rj_* V)^0 + (\ker d) \cap (Rj_* Z_{m+r})^1 \qquad & \text{in degree } \; 1 \,; \end{aligned}$$

for $i > 1$, $W_{m+M+i}(Rj_* V) = \tau_{\leqq i}(Rj_* V)$.

Then

$$\begin{aligned} \mathrm{Gr}^W_m(Rj_* V) &\simeq j_* V \,, \\ \mathrm{Gr}^W_{m+r}(Rj_* V) &\simeq R^1 j_*(Z_{m+r}/Z_{m+r-1})[-1] \,, \qquad 0 < r \leqq M + 1 \\ &\simeq \mathbf{P}_{m+r-1}[-1] \,, \end{aligned}$$

and for $i > 1$, $\mathrm{Gr}^W_{M+m+i}(Rj_* V) \simeq R^i j_* V = 0$. The weight filtration is evidently induced by the analogous filtration on $Rj_* V_{\mathbf{R}}$. Moreover, the mappings

$$K(S)^\cdot \hookrightarrow j_* \Omega^\cdot(V) \xleftarrow{\;\sim\;} Rj_* V$$

are easily seen to be W-filtered quasi-isomorphisms when we filter $j_*\Omega^\cdot(V)$ by setting $W_m\big(j_*\Omega^\cdot(V)\big) = j_* V$ and for $0 < r \leqq M + 1$,

$$\begin{aligned} W_{m+r}\big(j_*\Omega^\cdot(V)\big) &= j_*\mathcal{O}(V) \longrightarrow [\nabla\big(j_*\mathcal{O}(V)\big) + j_*\Omega^1(Z_{m+r})] \\ &= j_*\mathcal{O}(V) \longrightarrow \left[\nabla\big(j_*\mathcal{O}(V)\big) + \frac{dt}{t} \otimes W_{m+r-1}\right] . \end{aligned}$$

As in (8.2), one easily verifies that the bi-filtered complex $K(\tilde{S})^\cdot$ is functorial in its construction, and therefore we conclude

(13.11) THEOREM. *With notation as above, there is a functorial mixed Hodge structure on $H^i(\tilde{S}, j_{0*} V)$ induced by W and F.*

Expressed for the geometric situation, this theorem reads

(13.12) COROLLARY. *If we have the diagram*

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \\ \big\downarrow f & & \big\downarrow \tilde{f} \\ S & \xhookrightarrow{\;j_0\;} & \tilde{S} \end{array}$$

with f smooth projective, then there is a natural mixed Hodge structure on $H^i(\tilde{S}, j_{0*} R^m f_* \mathbf{C})$, for all i and m.

Most significantly, taking $S = \tilde{S}$, we state for emphasis

(13.13) COROLLARY. *If $f\colon X \to S$ is a smooth projective morphism, there is a natural mixed Hodge structure on $H^i(S, R^m f_* \mathbf{C})$ for all i and m.*

Because of (13.8) and the fact that d_1 is the only non-zero differential in the spectral sequence for W, we necessarily have that d_1 is surjective. Taking $\tilde{\Sigma}$ to consist of a single point, we obtain the following, which is also a consequence of (8.4):

(13.14) PROPOSITION. *If $v \in \Gamma(S, V)$ and v annihilates every element of $\mathbf{P}_m(s)$ for some $s \in \Sigma$, then $v = 0$.*

(13.15) COROLLARY. *If $v \in \Gamma(S, V)$ and $v \in W_{m-1}(V)$ for some $s \in \Sigma$, then $v = 0$.*

14. Cohomology with compact supports

There is a natural mixed Hodge structure on cohomology with compact supports, whose existence we derive in this section (14.3).

We first do the simplest case $S \overset{j}{\hookrightarrow} \bar{S}$. Using standard notation, we write $j_! V$ for the extension by zero of the sheaf V from S to $\bar{S}$. Then

$$H^i_c(S, V) = H^i(\bar{S}, j_! V)\ .$$

At a point $s \in \Sigma$, write $V = \bigoplus_{0 \leq \alpha < 1} V_\alpha$. The following is immediate:

(14.1) PROPOSITION. *There is a diagram of exact sequences*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & j_! V \arrow[r] \arrow[d] & t\bar{\mathcal{V}}_0 \oplus [\bigoplus_{\alpha \neq 0} \bar{\mathcal{V}}_\alpha] \arrow[r, "\nabla"] \arrow[d] & \frac{1}{2\pi i}\frac{dt}{t} \otimes \left(t\bar{\mathcal{V}}_0 \oplus [\bigoplus_{\alpha \neq 0} \bar{\mathcal{V}}_\alpha]\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_* V \arrow[r] \arrow[d] & \bar{\mathcal{V}} \arrow[r] \arrow[d, "\rho"] & \mathcal{S} \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \bigoplus_{s \in \Sigma} (j_* V)_s \arrow[r] \arrow[d] & \bigoplus_{s \in \Sigma} (V_0)_s \arrow[r, "N = \operatorname{Res} \nabla"] \arrow[d] & \bigoplus_{s \in \Sigma} (\operatorname{im} N_s) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(*where again we are using local notation in the first row to define global objects*).

(14.2) COROLLARY. *The cohomology groups of $j_! V$ on $\bar{S}$ are given as the hypercohomology of any of the following complexes*:

(i) $t\bar{\mathcal{V}}_0 \oplus [\bigoplus_{\alpha \neq 0} \bar{\mathcal{V}}_\alpha] \xrightarrow{\nabla} \dfrac{dt}{t} \otimes \left(t\bar{\mathcal{V}}_0 \oplus [\bigoplus_{\alpha \neq 0} \bar{\mathcal{V}}_\alpha]\right)$,

(ii) $j_* V \longrightarrow \bigoplus_{s \in \Sigma} (j_* V)_s$,

(iii) $\bar{\mathcal{V}} \xrightarrow{\rho \oplus \nabla} [\bigoplus_{s \in \Sigma} (V_0)_s] \oplus \mathcal{S} \xrightarrow{[\oplus N_s] \oplus -\rho} \bigoplus_{s \in \Sigma} (\operatorname{im} N_s)$.

Define for $k < m$,

$W_k(j_! V) = [(\ker N_s) \cap W_{k+1}(V)_s][-1]$ (using (14.2, ii) to resolve $j_! V$; as before, we may tacitly assume that all local monodromy is unipotent), and let W_m be the whole thing. Then for $k < m$,

$$\begin{aligned} \mathrm{Gr}_k^W(j_! V) &= \bigoplus_{s \in \Sigma} \mathrm{Gr}_{k+1}^W(V \cap \ker N_s)[-1] ; \\ \mathrm{Gr}_m^W(j_! V) &= j_* V . \end{aligned}$$

In the complex (14.2, iii) (call it $K^{\cdot}$), we define for $k < m$,

$$\begin{aligned} W'_k &= \bigoplus_{s \in \Sigma} \left(W_{k+1}(V_0)_s \longrightarrow W_{k-1}(V_0)_s \right)[-1] \\ W'_m &= K^{\cdot} . \end{aligned}$$

Then $\mathrm{Gr}_k^{W'} = \bigoplus_{s \in \Sigma} (\mathrm{Gr}_{k+1}^W(V_0)_s \xrightarrow{N} \mathrm{Gr}_{k-1}^W(V_0)_s)[-1]$, which is quasi-isomorphic to $\mathrm{Gr}_k^W(j_! V)$; and $\mathrm{Gr}_m^{W'}$ is (the single complex associated to)

$$\begin{tikzcd} \bar{\mathcal{V}} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \bigoplus_{s \in \Sigma} [V_0 / W_m(V_0)]_s \arrow[r] & \bigoplus_{s \in \Sigma} [\mathrm{im}\, N_s / W_{m-2}(V_0)_s] , \end{tikzcd}$$

which is a resolution of $\Omega^{\cdot}(V)_{(2)}$, hence is quasi-isomorphic to $\mathrm{Gr}_m^W(j_! V)$. As in Section 13, it follows from [17, (6.16)] that F induces on $H^1(\bar{S}, \mathrm{Gr}_k^W(j_! V))$ a Hodge structure of weight $k + 1$, whereas for $H^i(\bar{S}, \mathrm{Gr}_m^W(j_! V)) = H^i(\bar{S}, j_* V)$, it induces Hodge theory with degenerating coefficients, of weight $m + i$. Thus, the data $(j_! V_{\mathbf{R}}, W)$, $(K^{\cdot}, W', F)$ determine a cohomological mixed Hodge complex over $\mathbf{R}$, and we have

(14.3) THEOREM. *The filtrations $W[i]$ and F induce a functorial mixed Hodge structure on $H_c^i(S, V)$. Furthermore, the exact cohomology sequence*

$$0 \longrightarrow H^0(\bar{S}, j_* V) \longrightarrow \bigoplus_{s \in \Sigma} (j_* V)_s \longrightarrow H_c^1(S, V) \longrightarrow H^1(\bar{S}, j_* V) \longrightarrow 0$$

is an exact sequence of mixed Hodge structures.

We can combine the content of Theorems (13.11) and (14.3) to get the following (where we are retaining the notation from the beginning of § 13):

(14.4) THEOREM. *There is a natural mixed Hodge structure on $H_c^i(\tilde{S}, j_{0*} V) \simeq H^i(\bar{S}, \tilde{j}_! j_{0*} V)$, and an exact sequence of mixed Hodge structures*

$$0 \longrightarrow H^0(\bar{S}, j_* V) \longrightarrow \bigoplus_{s \in \tilde{\Sigma}} (j_* V)_s \longrightarrow H_c^1(\tilde{S}, j_{0*} V) \longrightarrow H^1(\bar{S}, j_* V) \longrightarrow 0 .$$

(14.5) *Remark.* The above exact sequence is dual (up to a shift of weights) to

$$0 \longrightarrow H^1(\bar{S}, j_* V) \longrightarrow H^1(\tilde{S}, j_{0*} V) \longrightarrow \bigoplus_{s \in \tilde{\Sigma}} V_0 / N_s V_0 \longrightarrow H^2(\bar{S}, j_* V) .$$

15. Compatibility of the Hodge structures with the Leray spectral sequence

We wish to demonstrate that in the geometric situation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ S \arrow[r, hook] & \bar{S} \end{tikzcd}$$

the Hodge structures on $H^i(\bar{S}, j_*R^m f_*\mathbf{C})$ are induced by those of $H^{m+i}(\bar{X})$.

As the case $S = \bar{S}$ is covered by (2.14), we assume that $S \neq \bar{S}$. We further assume initially that the singular fibers $\bar{f}^{-1}(\Sigma)$ are unions of smooth divisors with normal crossings, denoted Y, which we can always arrange via a resolution of singularities from a general situation. The case $i = 0$ is well-known, and the case $i = 2$ comes rather easily by duality (15.11), so we will concentrate on the more interesting group $H^1(\bar{S}, j_*R^m f_*\mathbf{C})$. Ascertaining this result will impart full significance to the theorem on normal functions (9.2).

The proof is based on the mixed Hodge theories for $H^1(S, R^m f_*\mathbf{C})$ given in Section 13, and for $H^{m+1}(X)$ [8]. The relations among the various cohomology groups involved are given by the diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\alpha"] & H^{m+1}(\bar{X}) \arrow[r] \arrow[ddd] & H^0(\bar{S}, R^{m+1}\bar{f}_*\mathbf{C}) \arrow[ddd] & \\ & H^1(\bar{S}, R^m\bar{f}_*\mathbf{C}) \arrow[d, "\gamma", "\simeq"'] & & & \\ & H^1(\bar{S}, j_*R^m f_*\mathbf{C}) \arrow[d, hook] & & & \\ 0 \arrow[r] & H^1(S, R^m f_*\mathbf{C}) \arrow[r, "\beta"] & H^{m+1}(X) \arrow[r] & H^0(S, R^{m+1}f_*\mathbf{C}) \arrow[r] & 0 . \end{tikzcd} \tag{15.1}$$

That γ is an isomorphism follows from the local invariant cycle theorem (see [4, (3.7)]), which asserts that $R^m\bar{f}_*\mathbf{C} \to j_*R^m f_*\mathbf{C}$ is surjective (with kernel supported on Σ).

(15.2) *Remark.* There is no need to assume that the singular fibers are reduced, as in [4], for the proof needs only the existence of mixed Hodge structures on the hypercohomology of certain complexes, all of which are defined in general.

With the notation from (15.1), we are asserting

(15.3) THEOREM. *The mapping $\gamma \circ \alpha$ is a morphism of Hodge structures.*

We will continue and prove the following stronger result:

(15.4) THEOREM. *The mapping β is a morphism of mixed Hodge structures.*

While (15.4) contains (15.3), we will first prove (15.3) and then use another argument to establish the rest.

The proof begins with a discussion of the Hodge filtrations F, proceeding along the lines of (2.14).

(15.5) PROPOSITION. *The Hodge filtration on $H^1(S, R^m f_* \mathrm{C})$ is induced by that of $H^{m+1}(X)$.*

Proof. The cohomology of X is computed by means of the log complex

$$M^{\cdot} = \Omega^{\cdot}_{\bar{X}}(\log Y) .$$

Filtering this complex by

$$L^p M^{\cdot} = \text{image of } f^* \Omega^p_{\bar{S}}(\log \Sigma) \otimes \Omega^{\cdot}_{\bar{X}}(\log Y) ,$$

so that

$$\mathrm{Gr}^p_L M^{\cdot} = f^* \Omega^p_{\bar{S}}(\log \Sigma) \otimes \Omega^{\cdot}_{\bar{X}/\bar{S}}(\log Y) ,$$

we get a spectral sequence abutting to the cohomology sheaves on $\bar{S}$ of $R^{\cdot} = Rf_* M^{\cdot}$, with

$$\begin{aligned} {}_L E_1^{p,q} &= \Omega^p_{\bar{S}}(\log \Sigma) \otimes \mathbf{R}^q \bar{f}_* \Omega^{\cdot}_{\bar{X}/\bar{S}}(\log Y) \\ &\simeq \Omega^p_{\bar{S}}(\log \Sigma) \otimes \bar{\mathcal{V}}^q \qquad \text{by [19, (2.20)]} . \end{aligned}$$

Therefore, there is a quasi-isomorphism

$$\mathrm{Gr}^p_{\mathrm{Dec}\,(L)} Rf_* M^{\cdot} \longrightarrow \Omega^{\cdot}_{\bar{S}}(\log \Sigma) \otimes \bar{\mathcal{V}}^{-p}[p] \tag{15.6}$$

(cf. (2.15)). The hypercohomology of the right-hand side gives $H^*(S, R^{-p} f_* \mathrm{C})$ [10, II (6.10)]. Hence, the spectral sequence associated to the filtration $D = \mathrm{Dec}(L)$ is, as before, the Leray spectral sequence for f. Furthermore, we have the relative Hodge spectral sequence

$${}_F E_1^{p,q} = R^q \bar{f}_* \Omega^p_{\bar{X}/\bar{S}}(\log Y) \Longrightarrow \mathbf{R}^{p+q} \bar{f}_* \Omega^{\cdot}_{\bar{X}/\bar{S}}(\log Y)$$

degenerating at ${}_F E_1$, because all sheaves above are locally free on $\bar{S}$ [20, (2.11)], and all differentials vanish on S by classical Hodge theory. Consequently, (15.6) is actually a *filtered* quasi-isomorphism with respect to the filtrations induced by F; on the latter complex therein, it is the filtration given by (1.12).

The remainder of the argument can be simplified from (2.14). We have both

$${}_D E_1^{p,q} = H^{2p+q}(S, R^{-p} f_* \mathrm{C}) \Longrightarrow H^{p+q}(X)$$

and

$$ {}_{F(D)}E_1^{p,q} = \mathbf{H}^{p+q}\big(\bar{S},\, \mathrm{Gr}_F^p\Omega_{\bar{S}}^{\cdot}(\log\Sigma)\otimes\bar{\mathcal{V}}^m\big) \Longrightarrow H^{p+q}(S,\, V^m)\ , $$

with $V^m = R^m f_* \mathbf{C}$, degenerating at E_1, the first by [6] (note the shift), and the second by the mixed Hodge theory [9, (8.19)]. Necessarily the $D(F)$ spectral sequence degenerates at E_1 (as does the F spectral sequence, which is precisely [8, (3.2.13, ii)]), and therefore all spectral sequences associated to D and F degenerate at E_1. From this, using the natural identifications that follows from the degeneration, we conclude that

$$ \begin{aligned} F^p\mathbf{H}^k(\mathrm{Gr}_D^q R^{\cdot}) &= \mathbf{H}^k\big((D^q\cap F^p)R^{\cdot}/(D^{q+1}\cap F^p)R^{\cdot}\big) \\ &= \mathbf{H}^k\big((D^q\cap F^p)R^{\cdot}\big)/\mathbf{H}^k\big((D^{q+1}\cap F^p)R^{\cdot}\big) \\ &= F^p\big[\mathrm{Gr}_D^q H^k(X)\big]\ , \end{aligned} $$

which is the desired result.

We turn now to the weight filtrations W. On $M^{\cdot}$, $W_0M^{\cdot} = \Omega_{\bar{X}}^{\cdot}$, so

$$ W_kH^k(X) = \text{image of}\quad H^k(\bar{X})\ ; $$

and the assertion contained in (15.4) that β respects W_{m+1} follows from the fact that α in (15.1) is surjective ($\bar{S}$ is a curve). It is now a simple diagram chase to conclude that (15.3) holds.

We will work now modulo the lowest non-zero weight. There is a natural injection

$$ \begin{array}{c} B\colon \mathbf{H}^1\big(\bar{S},\, K(S)^{\cdot}/W_mK(S)^{\cdot}\big) \longrightarrow \mathbf{H}^{m+1}(\bar{X},\, M^{\cdot}/W_0M^{\cdot}) \\ \big\wr\big| \\ \bigoplus_{s\in\Sigma}(V_0/NV_0)_s \end{array} \tag{15.7} $$

which we will now describe. Since both quotient complexes are supported on or over Σ, we may restrict to a small disc Δ around a point of Σ. As is implicit in [19, (4.3)], there is a natural isomorphism

$$ V_0 \simeq \mathbf{H}^m\big(\bar{f}^{-1}(\Delta),\, \Omega_{\bar{X}/\bar{S}}^{\cdot}(\log Y)\otimes\mathcal{O}_{Y^{\mathrm{red}}}\big)\ , $$

and it fits into an exact sequence

$$ V_0 \xrightarrow{\ N\ } V_0 \xrightarrow{\ \wedge(1/2\pi i)dt/t\ } \mathbf{H}^{m+1}\big(\bar{f}^{-1}(\Delta),\, M^{\cdot}/W_0M^{\cdot}\big)\ . \tag{15.8} $$

Each vector space in (15.8) carries a mixed Hodge structure. As is implied in [4], this is an exact sequence of mixed Hodge structures. Taking the cokernel by N gives (15.7).

We will be essentially finished with the proof of (15.4) once we know that the mapping B fits into a diagram of exact sequences of mixed Hodge structures:

$$(15.9)\qquad \begin{tikzcd} 0 \arrow[r] & W_{m+1}\mathbf{H}^1(\bar{S}, K(S)^{\boldsymbol{\cdot}}) \arrow[r] \arrow[d] & \mathbf{H}^1(\bar{S}, K(S)^{\boldsymbol{\cdot}}) \arrow[r, "\sigma"] \arrow[d, "\beta"] & \mathbf{H}^1(\bar{S}, K(S)^{\boldsymbol{\cdot}}/W_m K(S)^{\boldsymbol{\cdot}}) \arrow[d, "B"] \\ 0 \arrow[r] & W_{m+1}\mathbf{H}^{m+1}(\bar{X}, M^{\boldsymbol{\cdot}}) \arrow[r] & \mathbf{H}^{m+1}(\bar{X}, M^{\boldsymbol{\cdot}}) \arrow[r, "\rho"] & \mathbf{H}^{m+1}(\bar{X}, M^{\boldsymbol{\cdot}}/W_0 M^{\boldsymbol{\cdot}}) \end{tikzcd} ;$$

if $\xi \in W_k\mathbf{H}^1(\bar{S}, K(S^{\boldsymbol{\cdot}}))$, and $k > m+1$, then

$$\rho[\beta(\xi)] = B[\sigma(\xi)] \in W_k\mathbf{H}^{m+1}(M^{\boldsymbol{\cdot}}/W_0M^{\boldsymbol{\cdot}}) ,$$

so by strictness $\beta(\xi)$ lies in

$$\big(W_k\mathbf{H}^{m+1}(M^{\boldsymbol{\cdot}}) + \ker\rho\big) = W_k\mathbf{H}^{m+1}(M^{\boldsymbol{\cdot}}) .$$

So we must prove

(15.10) PROPOSITION. *The right-hand square in* (15.9) *commutes.*

Proof. It suffices to work on Δ, as (15.7) is defined locally on Σ. We consider the filtration L on $M^{\boldsymbol{\cdot}}$. There is an exact sequence

$$\mathbf{H}^m\big(\bar{f}^{-1}(\Delta), \Omega^{\boldsymbol{\cdot}}_{\bar{X}/\bar{S}}(\log Y)\big) \longrightarrow \mathbf{H}^{m+1}\Big(\bar{f}^{-1}(\Delta), \frac{dt}{t} \otimes \Omega^{\boldsymbol{\cdot}}_{\bar{X}/\bar{S}}(\log Y)[-1]\Big) \longrightarrow \mathbf{H}^{m+1}\big(\bar{f}^{-1}(\Delta), M^{\boldsymbol{\cdot}}\big) .$$

Since Δ is a disc, we may rewrite this as

$$H^0\big(\Delta, \mathbf{R}^m\bar{f}_*\Omega^{\boldsymbol{\cdot}}_{\bar{X}/\bar{S}}(\log Y)\big) \longrightarrow H^0\Big(\Delta, \frac{dt}{t} \otimes \mathbf{R}^m\bar{f}_*\Omega^{\boldsymbol{\cdot}}_{\bar{X}/\bar{S}}(\log Y)\Big) \longrightarrow \mathbf{H}^{m+1}\big(\bar{f}^{-1}(\Delta), M^{\boldsymbol{\cdot}}\big) .$$

Taking the cokernel of the first mapping, we obtain

$$0 \longrightarrow \mathbf{H}^1\big(\Delta, \Omega^{\boldsymbol{\cdot}}_{\bar{S}}(\log\Sigma) \otimes \bar{\mathcal{V}}^m\big) \longrightarrow \mathbf{H}^{m+1}\big(\bar{f}^{-1}(\Delta), M^{\boldsymbol{\cdot}}\big) .$$

Thus over small open subsets of $\bar{S}$, we have an isomorphism of filtration levels $L^1 = D^{-m}$ on $\mathbf{H}^{m+1}\big(\Omega^{\boldsymbol{\cdot}}_{\bar{X}}(\log Y)\big)$. We then follow the commutative diagram

$$\begin{tikzcd}[column sep=small] H^1(\Delta, \Omega^{\boldsymbol{\cdot}}_{\bar{S}}(\log\Sigma) \otimes \bar{\mathcal{V}}^m) \arrow[rrr] \arrow[dd, "\beta|_\Delta"] & & & V_0/NV_0 \arrow[dd, "B|_\Delta"] \\ & H^m(\bar{f}^{-1}(\Delta), \frac{dt}{t} \otimes \Omega^{\boldsymbol{\cdot}}_{\bar{X}/\bar{S}}(\log Y)) \arrow[r] \arrow[ul] \arrow[dl] & H^m(\bar{f}^{-1}(\Delta), \Omega^{\boldsymbol{\cdot}}_{\bar{X}/\bar{S}}(\log Y) \otimes \mathcal{O}_{Y^{\mathrm{red}}}) \arrow[ur] \arrow[dr, "\wedge \frac{1}{2\pi i}\frac{dt}{t}"] & \\ H^{m+1}(\bar{f}^{-1}(\Delta), \Omega^{\boldsymbol{\cdot}}_{\bar{X}}(\log Y)) \arrow[rrr] & & & H^{m+1}(\bar{f}^{-1}(\Delta), \Omega^{\boldsymbol{\cdot}}_{\bar{X}}(\log Y)/\Omega^{\boldsymbol{\cdot}}_{\bar{X}}) \end{tikzcd}$$

to reach the desired conclusion (15.10).

For the cohomology group with $i = 2$, there is a mapping

$$\iota\colon H^2(\bar{S}, j_*R^mf_*\mathbf{C}) \longrightarrow H^{m+2}(\bar{X})$$

induced by the Leray spectral sequence for $\bar{f}$, which fits into a diagram of pairings (for $m < n = \dim X_t$, the other cases being similar),

$$
\begin{tikzcd}[column sep=small]
H^{m+2}(\bar{X}) \arrow[rrr] & & & H^{m+2}(\bar{X})/\ker\{H^{m+2}(\bar{X}) \to H^{m+2}(X_t)\} \arrow[dl, "\pi"'] \\
H^m(\bar{X}) \arrow[u, "\cup\omega"] \arrow[rrr] \arrow[ddr] & & & H^m(\bar{X})/\ker\{H^m(\bar{X}) \to H^m(X_t)\} \arrow[u, "\cup\omega"'] \\
& & H^0(\bar{S}, j_*R^{m+2}f_*C) & \\
& H^0(\bar{S}, R^m\bar{f}_*C) \arrow[r, "\text{onto}"] & H^0(\bar{S}, j_*R^m f_*C) \arrow[u, hook, "\cup\omega"'] & \\
\times & \times & \times & \times \\
& H^2(\bar{S}, R^m\bar{f}_*C) \arrow[r, "\sim"] \arrow[dl] & H^2(\bar{S}, j_*R^m f_*C) \arrow[dd, hook, "\cup\omega^{n-m-1}"] \arrow[dr, "\iota"] & \\
H^{m+2}(\bar{X}) \arrow[dd, "\cup\omega^{n-m-1}"] & & & \ker\{H^{m+2}(\bar{X}) \to H^{m+2}(\bar{X}-X_t)\} \arrow[lll, hook'] \arrow[dd, hook, "\cup\omega^{n-m-1}"] \\
& & H^2(\bar{S}, j_*R^{2n-m-2}f_*C) \arrow[dr] & \\
H^{2n-m}(\bar{X}) \simeq H_{m+2}(\bar{X}) & & & \operatorname{im}\{H_{m+2}(X_t) \to H_{m+2}(\bar{X})\} \simeq \ker\{H^{2n-m}(\bar{X}) \to H^{2n-m}(\bar{X}-X_t)\} \arrow[lll, hook']
\end{tikzcd}
$$

where $\omega \in H^2(\bar{X}) \to H^0(\bar{S}, R^2\bar{f}_*\mathrm{C})$ is the hyperplane class, and cohomology is always taken with complex coefficients. We use the facts that cup-products are compatible with the Leray spectral sequence [2, p. 143], and that the duality (10.5) uses the polarization of $R^m f_*\mathrm{C}$, which involves cup-product with ω^{n-m}. With the aid of the diagram, we conclude that the mapping ι is included via morphisms of type $(n-m-1, n-m-1)$ in the dual of the known morphism of Hodge structures π. Thus

(15.11) THEOREM. *ι: $H^2(\bar{S}, j_*R^m f_*\mathrm{C}) \to H^{m+2}(\bar{X})$ is a morphism of Hodge structures.*

We can now drop the hypothesis that $\bar{X}$ have only normal crossings for singular fibers.

(15.12) PROPOSITION. *In the geometric situation*

$$
\begin{array}{ccc}
X & \hookrightarrow & \bar{X} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\
S & \xhookrightarrow{j} & \bar{S}
\end{array}
$$

with $\bar{X}$ and $\bar{S}$ non-singular projective varieties, $\bar{S}$ a curve, and f proper and smooth,

$$R^m\bar{f}_*\mathrm{C} \longrightarrow j_*R^m f_*\mathrm{C}$$

is surjective for all m (i.e., *the local invariant cycle theorem holds*).

Proof. Let Δ be a small disc around a critical value (singular point for the cohomology system on $\bar{S}$), $\tilde{f}$: $\tilde{X} \to \bar{S}$ a resolution of singularities for the fibers of $\bar{f}$, $\bar{X}_\Delta = \bar{f}^{-1}(\Delta)$, $\tilde{X}_\Delta = \tilde{f}^{-1}(\Delta)$. The assertion (15.12) is that $H^m(\bar{X}_\Delta) \to H^m(X_t)$ has as its image the cohomology classes invariant under the local monodromy transformation. We have a diagram (∂ denotes boundary):

$$\begin{tikzcd}
H^m(\bar{X}_\Delta) \arrow[rr] \arrow[d, "\wr"'] & & H^m(\tilde{X}_\Delta) \arrow[d, "\wr"] \arrow[dll] \\
H_{2n+2-m}(\bar{X}_\Delta, \partial\bar{X}_\Delta) \arrow[d] & & H_{2n+2-m}(\tilde{X}_\Delta, \partial\tilde{X}_\Delta) \arrow[ll] \arrow[d] \\
H_{2n+2-m}(\bar{X}_\Delta, \partial\bar{X}_\Delta \cup \bar{X}_0) & & H_{2n+2-m}(\tilde{X}_\Delta, \partial\tilde{X}_\Delta \cup \tilde{X}_0) \arrow[ll] \\
H^m(\bar{X}_\Delta - \bar{X}_0) \arrow[rr, "\sim"] \arrow[u, "\wr"] \arrow[dr] & & H^m(\tilde{X}_\Delta - \tilde{X}_0) \arrow[u, "\wr"'] \arrow[dl] \\
& H^m(X_t) &
\end{tikzcd}$$

The vertical isomorphisms are given by Lefschetz duality, and the squares containing these isomorphisms need not commute. By the standard local invariant cycle theorem for $\tilde{X}$, for any invariant $\eta \in H^m(X_t)$ there exists $\tilde{\eta} \in H^m(\tilde{X}_\Delta)$ with $\tilde{\eta}$ restricting to η. Mapping $\tilde{\eta}$ into $H_{2n+2-m}(\bar{X}_\Delta, \partial\bar{X}_\Delta)$ and returning to $H^m(\bar{X}_\Delta)$ by duality, we obtain a pre-image of η as desired.

Using (15.12), we obtain Hodge structures on $H^i(\bar{S}, R^m\bar{f}_*\mathbf{C}) \simeq H^i(\bar{S}, j_*R^m f_*\mathbf{C})$ for $i = 1, 2$. For $i = 0$, we begin by noting that $H^0(\bar{S}, R^m\bar{f}_*\mathbf{C})$ is a vector space extension of spaces carrying mixed Hodge structures,

$$0 \longrightarrow A \longrightarrow H^0(\bar{S}, R^m\bar{f}_*\mathbf{C}) \longrightarrow H^0(\bar{S}, j_*R^m f_*\mathbf{C}) \longrightarrow 0 ,$$

where

$$A = \ker\{\textstyle\bigoplus_{s\in\Sigma} H^m(X_s) \longrightarrow \bigoplus_{s\in\Sigma} H^m(\tilde{X}_s) \longrightarrow \mathbf{H}^m(\tilde{X}_\Delta, \Omega^\bullet_{\tilde{X}_\Delta/\Delta}(\log Y) \otimes \mathcal{O}_{Y^{\mathrm{red}}})\} .$$

However, with the aid of the basic exact sequence [4, (7.61)], we can see that A is a *pure* Hodge structure. $H^m(X_s) \to H^m(\tilde{X}_s)$ is injective because of the existence of a left-inverse (with notation as in the proof of (15.12)),

$$H^m(\tilde{X}_s) \simeq H_{2n-m}(\tilde{X}_\Delta, \partial\tilde{X}_\Delta) \longrightarrow H_{2n-m}(\bar{X}_\Delta, \partial\bar{X}_\Delta) \simeq H^m(X_s) .$$

Thus we may assume without loss of generality that $\tilde{X} = X$. Then

$$A = \textstyle\bigoplus_{s\in\Sigma} \ker\{H^m(\tilde{X}_s) \to V\} = \bigoplus_{s\in\Sigma} \mathrm{im}\{H^m(\tilde{X}_\Delta, \partial\tilde{X}_\Delta) \to H^m(\tilde{X}_\Delta) \xrightarrow{\sim} H^m(\tilde{X}_s)\}$$

is of pure weight m, for the mixed Hodge structure on $H^m(\tilde{X}_\Delta, \partial\tilde{X}_\Delta)$ [4, § 9] has its weights $\geqq m$ and $H^m(\tilde{X}_s)$ has its weights $\leqq m$.

(15.13) PROPOSITION. $A \subset \mathrm{im}\{H^m(\bar{X}) \to H^0(\bar{S}, R^m \bar{f}_* \mathbf{C})\}$.

Proof. Use the diagram

$$\begin{array}{ccc} H^m(\bar{X}, X) & \xrightarrow{\sim} & \bigoplus_{s\in\Sigma} H^m(\bar{X}_\Delta, \partial\bar{X}_\Delta) \\ \downarrow & & \downarrow \\ H^m(\bar{X}) & \longrightarrow & \bigoplus_{s\in\Sigma} H^m(X_s) \end{array} \qquad \text{(by excision)}$$

and the definition of A.

Coupled with the well-known fact [8] that

$$H^m(\bar{X}) \longrightarrow H^0(\bar{S}, j_* R^m f_* \mathbf{C}) \simeq H^0(S, R^m f_* \mathbf{C})$$

is surjective for all m, (15.13) yields

(15.14) COROLLARY. *$H^m(\bar{X}) \to H^0(\bar{S}, R^m \bar{f}_* \mathbf{C})$ is surjective for all m.*

This places, albeit extrinsically, a Hodge structure of $H^0(\bar{S}, R^m \bar{f}_* \mathbf{C})$ for any projective morphism onto a smooth algebraic curve with general fiber non-singular. Interpreting (15.14), we obtain an interesting consequence.

(15.15) COROLLARY. *Let $\bar{f}: \bar{X} \to \bar{S}$ be a morphism of smooth projective varieties, with* $\dim \bar{S} = 1$. *Then the Leray spectral sequence for $\bar{f}$* ($\mathbf{Q}$*-coefficients) degenerates at E_2.*[8]

Proof.
$$\begin{aligned} &\mathrm{Im}\{H^m(\bar{X}) \longrightarrow H^0(\bar{S}, R^m \bar{f}_* \mathbf{C})\} \\ &\qquad = \ker\{d_2 : H^0(\bar{S}, R^m \bar{f}_* \mathbf{C}) \longrightarrow H^2(\bar{S}, R^{m-1} \bar{f}_* \mathbf{C})\}, \end{aligned}$$

so we have $d_2 = 0$, and all higher d_r are trivially zero by reason of degree.

As the final compatibility theorem we have

(15.16) THEOREM. *If $\bar{f}: \bar{X} \to \bar{S}$ is as in* (15.12), *the natural mappings of cohomology,*

(i) $H^m(\bar{X}) \longrightarrow H^0(S, R^m f_* \mathbf{C})$,

(ii) $\ker\{H^{m+1}(\bar{X}) \longrightarrow H^0(\bar{S}, R^{m+1} \bar{f}_* \mathbf{C})\}$
$= \bigcap_{s\in\Sigma} \ker\{H^{m+1}(\bar{X}) \longrightarrow H^{m+1}(X_s)\} \longrightarrow H^1(\bar{S}, R^m \bar{f}_* \mathbf{C})\}$,

(iii) $H^2(\bar{S}, R^m \bar{f}_* \mathbf{C}) \longrightarrow \ker\{H^{m+2}(\bar{X}) \longrightarrow H^{m+2}(X)\}$,

are all morphisms of Hodge structures.

Proof. All of these mappings factor through or compose with the *injection* of Hodge structures

$$H^{m+i}(\bar{X}) \longrightarrow H^{m+i}(\tilde{X}) ,$$

where $\tilde{X}$ is a resolution of the singularities of the fibers of $\bar{f}$. The theorem follows from the statement for $\tilde{X}$.

[8] It is an easy step to generalize this result to allow $\bar{X}$ and $\bar{S}$ to be singular, so long as the general fiber is smooth.

(15.17) *Remark.* There is another mapping of the form (15.16, (iii)) obtained by extending the diagram (15.9) one step to the right. One would expect that the two agree.

We conclude by considering the bearing of Theorem (15.16) on the Hodge Conjecture. We have a diagram

$$\begin{tikzcd}
 & \left\{\begin{array}{l}\text{Horizontal sections of the family}\\ \text{of } p\text{-th intermediate Jacobians}\\ J^{p} \text{ over } S\end{array}\right\} \\
\left\{\begin{array}{l}\text{Codimension } p \text{ relative cycles } Z\\ \text{on } \bar{X} \text{ with } Z\cdot X_s \text{ homologically}\\ \text{equivalent to zero on } X_s \text{ for all}\\ s\in\bar{S}\end{array}\right\} \arrow[ur, "\substack{\text{Abel-Jacobi}\\ \text{mapping}\\ \nu}"] \arrow[r, dashed, "\bar{\nu}"] \arrow[d, "\text{cycle class}"', "\zeta"] & \left\{\begin{array}{l}\text{Horizontal normal functions (hori-}\\ \text{zontal sections of the family of}\\ p\text{-th generalized intermediate}\\ \text{Jacobians } \bar{J}^{p} \text{ over } \bar{S}\end{array}\right\} \arrow[d, "\substack{\text{cohomology}\\ \text{class}}"', "\kappa"] \\
\ker\{H^{p,p}(\bar{X}, Z) \longrightarrow H^{0}(\bar{S}, R^{2p}\bar{f}_{*}C)\} \arrow[r, "\alpha"] & H^{1}(\bar{S}, R^{2p-1}\bar{f}_{*}Z)^{p,p}
\end{tikzcd}$$

On the subset of cycles for which the Abel-Jacobi mapping ν (see [13] for definitions) extends to $\bar{S}$, so that $\bar{\nu}$ is defined, the square commutes [23]. This includes all cycles which do not intersect the singular loci of the bad fibers (cf. [23, (4.58)]). We now know that κ is surjective (9.2). Thus, if one can invert the Abel-Jacobi mapping for a given $\bar{X}$ (a rare phenomenon), one can obtain examples where the Hodge Conjecture is true; otherwise, the conjecture remains consistent with the present state of knowledge.

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Added in proof:

1. (Re Section 12) For curves $S = H/\Gamma$, the G-invariant Poincaré metric $y^{-2}dx \wedge dy$ on H induces on S a metric with the desired singularity at Σ.

2. The assertion after (12.3) that the system V_m is locally constant presumes that Γ contains no non-trivial elements of finite order. In general, one should base-change, via a normal subgroup $\Gamma' \subset \Gamma$, and take (Γ/Γ')-invariant elements of cohomology.

3. The author has found further applications of the results of this paper. For applications to elliptic surfaces, see Section 3 of D. Cox and S. Zucker, Intersection numbers of sections of elliptic surfaces, to appear in Inv. Math.