LECTURE 9 (FEBRUARY 25)

Cohomology and base change. Our next goal is to study line bundles and their cohomology on abelian varieties. In the complex case, we saw that line bundles come in big families – because we can always tensor by line bundles in $\operatorname{Pic}^0(X)$ – and so we need to first understand how cohomology groups of line bundles behave in families. A family of line bundles parametrized by a variety T is of course just a line bundle L on the product $X \times T$, and we are interested how the cohomology of the restrictions $L_t = L|_{X \times \{t\}}$ depends on $t \in T$. The technical tool is cohomology and base change, which Mumford treats very nicely in his book.

Here is the general setting. Let $f \colon X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. For every point $y \in Y$, we have the fiber

$$X_y = X \times_Y \operatorname{Spec} k(y),$$

which is a scheme over the field field $k(y) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$. Let's denote by

$$\mathscr{F}_y = \mathscr{F}|_{X_y} = \mathscr{F} \otimes_{\mathscr{O}_Y} k(y)$$

the restriction of $\mathscr{F}$ to the closed subscheme X_y. Cohomology and base change is about the cohomology groups $H^k(X_y, \mathscr{F}_y)$, and how they relate to the higher direct image sheaves $R^k f_* \mathscr{F}$. The key assumption is *flatness.*

Definition 9.1. We say that $\mathscr{F}$ is *flat* over Y if, for every point $x \in X$, the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$ is flat over $\mathscr{O}_{Y,f(x)}$, via the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$. Since $\mathscr{F}$ is quasi-coherent, this is equivalent to saying that for every pair of affine open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$ is flat over $\mathscr{O}_Y(V)$, via the ring homomorphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$.

Example 9.2. If $A \to B$ is a ring homomorphism, and M is a B-module, then $\tilde{M}$ is a quasi-coherent sheaf on $\operatorname{Spec} B$; it is flat if and only if M is flat as an A-module.

Example 9.3. The second projection $p_2 \colon X \times T \to T$ is flat, and any locally free sheaf on $X \times T$ (such as a line bundle) is therefore flat over T.

The geometric part of cohomology and base change is the following theorem by Grothendieck. In class, I just outlined the proof, but I filled in most of the details in the notes.

Theorem 9.4. *Let $f \colon X \to Y$ be a proper morphism between noetherian schemes, with $Y = \operatorname{Spec} A$ affine. Let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then there is a bounded complex $K^\bullet$ of finitely-generated projective A-modules, of the form*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0,$$

such that for every B-algebra A, one has a functorial isomorphism

$$H^p\big(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B\big) \cong H^p\big(K^\bullet \otimes_A B\big)$$

for all $p \in \mathbb{Z}$.

Note that $K^\bullet$ is a complex of A-modules, and so its cohomology groups

$$H^p(K^\bullet) = \frac{\ker d^p \colon K^p \to K^{p+1}}{\operatorname{im} d^{p-1} \colon K^{p-1} \to K^p}$$

are again A-modules. The complex $K^\bullet$ gives us a functorial way to describe all the objects we are interested in. For example, if we take $B = A$, we get

$$H^p(K^\bullet) \cong H^p(X, \mathscr{F}),$$

which is the A-module corresponding to the sheaf $R^p f_* \mathscr{F}$. On the other hand, we can take $B = k(y)$, where $y \in Y$ is any closed point; then $X \times_Y \operatorname{Spec} k(y) = X_y$ and $\mathscr{F} \otimes_A k(y) = \mathscr{F}_y$, and so

$$H^p\big(K^\bullet \otimes_A k(y)\big) \cong H^p(X_y, \mathscr{F}_y).$$

So the theorem translates the whole problem of cohomology and base change into understanding how the cohomology groups of a bounde complex of finitely-generated projective A-modules (= locally free sheaves) change from point to point.

Here is an outline of the proof, in four steps.

Step 1. The morphism f is proper, and $\mathscr{F}$ is coherent on X, and so all the higher direct image sheaves $R^p f_* \mathscr{F}$ are coherent on Y. (This theorem is also due to Grothendieck.) Because $Y = \operatorname{Spec} A$ is affine, $R^p f_* \mathscr{F}$ is the quasi-coherent sheaf associated to the A-module $H^p(X, \mathscr{F})$, and the theorem is saying that $H^p(X, \mathscr{F})$ is a finitely-generated A-module. (If you want to see the proof, have a look at `Tag 0203` in the *Stacks Project.*)

Step 2. We can compute the cohomology of quasi-coherent sheaves using Čech cohomology. Because f is proper and Y is affine, we can cover X by finitely many affine open subsets; let $\mathcal{U} = \{U_i\}_{i \in I}$ be the open covering. Let $C^\bullet = C^\bullet(\mathcal{U}, \mathscr{F})$ be the Čech complex, with terms

$$C^p = \bigoplus_{i_0, \dots, i_p} \mathscr{F}\big(U_{i_0} \cap \dots \cap U_{i_p}\big),$$

and the usual differential. The intersections $U_{i_0} \cap \dots \cap U_{i_p}$ are affine (because X is separated), and so the flatness of $\mathscr{F}$ implies that each C^p is a flat A-module. Because affine open coverings are acyclic (for quasi-coherent sheaves), the Čech complex computes the sheaf cohomology of $\mathscr{F}$:

$$H^p(X, \mathscr{F}) \cong H^p(C^\bullet)$$

are isomorphic as A-modules. Now suppose that B is any A-algebra. Then $\mathcal{U}_B = \{U_i \times_Y \operatorname{Spec} B\}_{i \in I}$ is an affine open covering of $X \times_Y \operatorname{Spec} B$, and it is easy to deduce from the definition of the fiber product that

$$C^p\big(\mathcal{U}_B, \mathscr{F} \otimes_A B\big) \cong C^p(\mathcal{U}, \mathscr{F}) \otimes_A B$$

as B-modules. This gives us isomorphisms

$$H^p\big(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B\big) \cong H^p\big(C^\bullet \otimes_A B\big),$$

which are clearly functorial in B.

Step 3. The complex $C^\bullet$ has almost all the properties we want, except that the A-modules C^p are not finitely-generated. The following lemma allows us to replace $C^\bullet$ by a smaller complex that is finitely-generated.

Lemma 9.5. *Consider a bounded complex of A-modules $C^\bullet$, of the form*

$$0 \to C^0 \to C^1 \to \dots \to C^n \to 0,$$

whose cohomology groups $H^p(C^\bullet)$ are finitely-generated A-modules. Then there is a bounded complex of finitely-generated A-modules $K^\bullet$, of the form

$$0 \to K^0 \to K^1 \to \dots \to K^n \to 0,$$

and a morphism of complexes $\phi\colon K^\bullet \to C^\bullet$ that induces isomorphisms on cohomology. We can arrange that $K^1, \dots, K^n$ are finitely-generated free A-modules, and if $C^0, \dots, C^n$ are flat A-modules, then K^0 is also flat, hence projective.

One piece of terminology. A morphism of complexes $\phi\colon K^\bullet \to C^\bullet$ is called a *quasi-isomorphism* if it induces isomorphisms on cohomology: for every $p \in \mathbb{Z}$, the morphism $\phi\colon H^p(K^\bullet) \to H^p(C^\bullet)$ is an isomorphism.

Proof. This is a basic result in homological algebra, similar to the construction of free resolutions for A-modules. We construct the desired complex step-by-step. For simplicity, set $H^p = H^p(C^\bullet)$, which are finitely-generated A-modules, nonzero only for $p = 0, \dots, n$. Let's denote the differentials in the complex $C^\bullet$ by $\delta^p \colon C^p \to C^{p+1}$. To begin with, H^n is finitely-generated, and so we can choose a finitely-generated free A-module K^n and a surjection $K^n \to H^n$. Because $H^n = C^n / \operatorname{im} \delta^{n-1}$, we can lift this to a morphism $\phi^n \colon K^n \to C^n$. We now have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_0^n \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d, "\phi^n"] & H^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{im} \delta^{n-1} \arrow[r] & C^n \arrow[r] & H^n \arrow[r] & 0
\end{tikzcd}$$

where we define $K_0^n = (\phi^n)^{-1}(\operatorname{im} \delta^{n-1})$ as the preimage of $\operatorname{im} \delta^{n-1}$. Note that K_0^n is again a finitely-generated A-module (because K^n and H^n are); we can therefore choose a finitely-generated A-module K^{n-1} and a surjection $K^{n-1} \to K_0^n$. Because $\operatorname{im} \delta^{n-1} = C^{n-1} / \ker \delta^{n-1}$, we can again lift the morphism from K^{n-1} to $\operatorname{im} \delta^{n-1}$ to a morphism $\phi^{n-1} \colon K^{n-1} \to C^{n-1}$, giving us another commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0^{n-1} \arrow[r] \arrow[d] & K^{n-1} \arrow[r] \arrow[d, "\phi^{n-1}"] & K_0^n \arrow[r] \arrow[d, "\phi^n"] & 0 \\
0 \arrow[r] & \ker \delta^{n-1} \arrow[r] & C^{n-1} \arrow[r, "\delta^{n-1}"] & \operatorname{im} \delta^{n-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows; of course, $K_0^{n-1} = (\phi^{n-1})^{-1}(\ker \delta^{n-1})$. Define $d^{n-1} \colon K^{n-1} \to K^n$ as the composition $K^{n-1} \to K_0^n \to K^n$; then $\ker d^{n-1} = K_0^{n-1}$ and $K^n / \operatorname{im} d^{n-1} \cong H^n$, and so ϕ^n induces an isomorphism between the n-th cohomology of our (partial) complex $K^\bullet$ and the n-th cohomology of $C^\bullet$.

The composition $K_0^{n-1} \to \ker \delta^{n-1} \to H^{n-1}$ may not be surjective, but because H^{n-1} is finitely-generated, we can add a finitely-generated free A-module to both K_0^{n-1} and K^{n-1} (and let d^{n-1} act on it as zero); this makes sure that $K_0^{n-1} \to H^{n-1}$ is surjective. After this change, the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1^{n-1} \arrow[r] \arrow[d] & \ker d^{n-1} \arrow[r] \arrow[d, "\phi^{n-1}"] & H^{n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{im} \delta^{n-2} \arrow[r] & \ker \delta^{n-1} \arrow[r] & H^{n-1} \arrow[r] & 0
\end{tikzcd}$$

is exact, where $K_1^{n-1} = (\phi^{n-1})^{-1}(\operatorname{im} \delta^{n-2})$. Since K_1^{n-1} is finitely-generated, we can map a finitely-generated free A-module K^{n-2} onto it, and so on. In other words, we keep repeating the whole procedure n times: for each $p = 1, \dots, n$, we get a morphism $\phi^p \colon K^p \to C^p$ from a finitely-generated free A-module, and a differential $d^p \colon K^p \to K^{p+1}$, such that the induced morphism from the p-th cohomology of $K^\bullet$ to the p-th cohomology of $C^\bullet$ is an isomorphism. In the final step of the construction, for $p = 0$, we need to define

$$K^0 = \left\{ (x, y) \in C^0 \times K_0^1 \;\middle|\; \delta^0(x) = \phi^1(y) \right\}$$

in order for the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker \delta^0 \arrow[r] \arrow[d, equal] & K^0 \arrow[r] \arrow[d, "\phi^0"] & K_0^1 \arrow[r] \arrow[d, "\phi^1"] & 0 \\
0 \arrow[r] & \ker \delta^0 \arrow[r] & C^0 \arrow[r, "\delta^0"] & \operatorname{im} \delta^0 \arrow[r] & 0
\end{tikzcd}$$

to be exact. Because $\ker \delta^0 = H^0$ is finitely-generated, the A-module K^0 will be finitely-generated, but not necessarily free.

It remains to show that if $C^0, \dots, C^n$ are flat A-modules, then K^0 is also flat; flat and finitely-generated implies projective, so K^0 will then be a projective A-module, as claimed. For that, we consider the mapping cone complex $L^\bullet$ for the morphism $\phi\colon K^\bullet \to C^\bullet$. This is the complex with terms

$$L^p = K^{p+1} \oplus C^p$$

and with differential

$$d\colon L^p \to L^{p+1}, \quad d(x, y) = \bigl(-dx, \delta(y) - \phi(x)\bigr).$$

It is easy to see that this fits into a short exact sequence of complexes

$$0 \to C^\bullet \to L^\bullet \to K^{\bullet+1} \to 0,$$

where the usual homological algebra convention is that the differential in the complex $K^{\bullet+1}$ is $-\delta$. The long exact sequence in cohomology reads

$$\cdots \longrightarrow H^{p-1}(L^\bullet) \longrightarrow H^p(K^\bullet) \xrightarrow{\phi} H^p(C^\bullet) \longrightarrow H^{p+1}(L^\bullet) \longrightarrow \cdots$$

and because ϕ is quasi-isomorphism, we get $H^p(L^\bullet) = 0$ for all $p \in \mathbb{Z}$, and so the complex $L^\bullet$ is exact. All the terms L^p are flat A-modules, with the possible exception of $L^{-1} = K^0$. From this and exactness, it follows readily that L^{-1} is also flat; as we said earlier, this means that K^0 is actually a projective A-module. □

In fact, we can do a little bit better. Suppose we are interested in the local behavior near a point $y_0 \in Y$. We can localize at y_0, meaning replace A by the local ring $\mathscr{O}_{Y,y_0}$. Now each time we need to choose a finitely-generated free A-module in the construction above, we can choose a *minimal* one, using Nakayama's lemma. Indeed, suppose that M is a finitely-generated A-module, where $(A, \mathfrak{m})$ is a local ring. Then $M/\mathfrak{m}M$ is a finite-dimensional vector space over $A/\mathfrak{m}$, and if we choose elements $m_1, \dots, m_n \in M$ whose images in $M/\mathfrak{m}M$ form a basis, then $m_1, \dots, m_n$ generate M by Nakayama's lemma. This gives us a surjection $A^n \to M$ with n minimal. If we use this device at each step, then K^0 will also be a free A-module (because every projective module over a local ring is free), and the differentials d^p in the complex $K^\bullet$ will have the property that $\operatorname{im} d^p \subseteq \mathfrak{m}K^{p+1}$. In other words, the complex $K^\bullet$ will be a *minimal complex*, in the following sense.

Definition 9.6. A complex of free A-modules $K^\bullet$ over a local ring $(A, \mathfrak{m})$ is *minimal* if $\operatorname{im} d^p \subseteq \mathfrak{m}K^{p+1}$ for every $p \in \mathbb{Z}$, or equivalently, if the tensor product $K^\bullet \otimes_A A/\mathfrak{m}$ has trivial differentials.

The complex $K^\bullet$ will actually make sense on some affine open set $\operatorname{Spec} A'$ containing the point y_0; at the cost of replacing $\operatorname{Spec} A$ by this smaller open set, we can therefore always achieve that the complex $K^\bullet \otimes k(y_0)$ has trivial differentials at a given point $y_0 \in Y$.

Step 4. It remains to show that we have

$$H^p\bigl(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B\bigr) \cong H^p\bigl(K^\bullet \otimes_A B\bigr)$$

for every A-algebra B. Since this holds for the Čech complex $C^\bullet$ by construction, we can apply the following general lemma.

Lemma 9.7. *Suppose that $\phi\colon K^\bullet \to C^\bullet$ is a quasi-isomorphism. If $C^0, \dots, C^n$ and $K^0, \dots, K^n$ are flat A-modules, then*

$$\phi \otimes_A B\colon K^\bullet \otimes_A B \to C^\bullet \otimes_A B$$

is also a quasi-isomorphism.

Proof. Consider again the mapping cone complex $L^\bullet$. The argument we gave earlier shows that ϕ is a quasi-isomorphism if and only if $L^\bullet$ is exact. Because every $L^p = K^{p+1} \oplus C^p$ is flat, the tensor product $L^\bullet \otimes_A B$ is still exact. But this is the mapping cone complex of $\phi \otimes_A B$, and so $\phi \otimes_A B$ is a quasi-isomorphism. □

Consequences of Grothendieck's theorem. The rest of the theory is basically just linear algebra. Let's first investigate the dimensions of the fiberwise cohomology groups.

Corollary 9.8. *Let $f\colon X \to Y$ be a proper morphism, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y.*

(a) The function $y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ is upper semicontinuous on Y.
(b) The Euler characteristic function

$$y \mapsto \chi(\mathscr{F}_y) = \sum_{p \in \mathbb{Z}} (-1)^p \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is locally constant on Y.

Proof. The problem is local on Y, and so we may assume that $Y = \operatorname{Spec} A$ is affine, and that we have a bounded complex $K^\bullet$ of finitely-generated free A-modules as in the theorem. (This works because projective A-modules are locally free.) So each differential $d^p\colon K^p \to K^{p+1}$ is now just a matrix with entries in the ring A. For every point $y \in Y$, we have

$$\begin{aligned}
\dim H^p(X_y, \mathscr{F}_y) &= \dim H^p\big(K^\bullet \otimes_A k(y)\big) \\
&= \dim \ker d^p \otimes_A k(y) - \dim \operatorname{im} d^{p-1} \otimes_A k(y) \\
&= \dim K^p \otimes_A k(y) - \dim \operatorname{im} d^p \otimes_A k(y) - \dim \operatorname{im} d^{p-1} \otimes_A k(y).
\end{aligned}$$

Taking the alternating sum over $p \in \mathbb{Z}$, we get

$$\sum_{p \in \mathbb{Z}} (-1)^p \dim H^p(X_y, \mathscr{F}_y) = \sum_{p \in \mathbb{Z}} (-1)^p \dim K^p \otimes_A k(y)$$

which is independent of y because each K^p is a free A-module. This gives (b).

For (a), we need to prove that each set

$$\big\{\, y \in Y \;\big|\; \dim H^p(X_y, \mathscr{F}_y) \geq \ell \,\big\}$$

is the set of closed points of a closed subscheme of Y. By the computation above, it suffices to show that the same is true for the sets

$$\big\{\, y \in Y \;\big|\; \dim \operatorname{im} d^p \otimes_A k(y) \leq \ell \,\big\}.$$

But this set is defined by all the minors of size $(\ell + 1)$ of the matrix representing the differential d^p, and so it is a closed subscheme. □

The next corollary is the actual base change theorem. It says that if the dimensions of the cohomology groups $H^p(X_y, \mathscr{F}_y)$ are constant, then they fit together into a locally free sheaf, namely $R^p f_* \mathscr{F}$.

Corollary 9.9. *Let $f\colon X \to Y$ be a proper morphism, with Y reduced. Let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then the following two conditions are equivalent:*

(a) The function $y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ is constant.
(b) The coherent sheaf $R^p f_* \mathscr{F}$ is locally free, and the base change morphism

$$R^p f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$

is an isomorphism for every $y \in Y$.

If this happens, then the base change morphism

$$R^{p-1}f_*\mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^{p-1}(X_y, \mathscr{F}_y)$$

in the next lower degree is also an isomorphism.

Proof. The problem is again local on Y, and so we may assume that $Y = \operatorname{Spec} A$ is affine, and that we have a complex $K^\bullet$ as in the theorem. To simplify the notation, let us set $K^\bullet(y) = K^\bullet \otimes_A k(y)$ and $d^p(y) = d^p \otimes_A k(y)$. This time, though, we also choose a point $y_0 \in Y$, and arrange that the complex $K^\bullet$ is minimal at y_0, in the sense that $K^\bullet(y_0)$ has trivial differentials. Obviously, this means that

$$H^p(X_{y_0}, \mathscr{F}_{y_0}) \cong H^p\big(K^\bullet(y_0)\big) \cong K^p(y_0).$$

Now suppose that $\dim H^p(X_y, \mathscr{F}_y)$ is constant, and therefore equal to $\dim K^p(y_0) = \dim K^p(y)$. Because this is the cohomology of

$$K^{p-1}(y) \xrightarrow{d^{p-1}(y)} K^p(y) \xrightarrow{d^p(y)} K^{p+1}(y),$$

we must have $d^{p-1}(y) = d^p(y) = 0$ for every $y \in Y$, which means that the entries of the matrices for d^{p-1} and d^p vanish at every point $y \in Y$. Because Y is reduced, this means that $d^{p-1} = d^p = 0$. But then

$$H^p(X, \mathscr{F}) = H^p(K^\bullet) = K^p$$

is a free A-module. The associated coherent sheaf is $R^p f_*\mathscr{F}$, which is therefore locally free. It is clear from this that the base change morphism is an isomorphism.

Now let's see where the (somewhat unexpected) additional assertion comes from. We have $d^{p-1} = 0$, and so $R^{p-1}f_*\mathscr{F}$ is the coherent sheaf associated to

$$H^{p-1}(X, \mathscr{F}) = H^{p-1}(K^\bullet) = K^{p-1}/\operatorname{im} d^{p-2}.$$

In other words, we have an exact sequence

$$K^{p-2} \xrightarrow{d^{p-2}} K^{p-1} \longrightarrow H^{p-1}(X, \mathscr{F}) \longrightarrow 0$$

Because tensor product is right exact, we can tensor with $k(y)$ and

$$K^{p-2}(y) \xrightarrow{d^{p-2}(y)} K^{p-1}(y) \longrightarrow H^{p-1}(X, \mathscr{F}) \otimes_A k(y) \longrightarrow 0$$

is still exact. This gives the desired isomorphism between $H^{p-1}(X, \mathscr{F}) \otimes_A k(y)$ and $H^{p-1}\big(K^\bullet(y)\big)$. □

This has many nice consequences. For example, suppose that $H^{p+1}(X_y, \mathscr{F}_y) = 0$ for every $y \in Y$; this will happen for example if p is the maximum of the fiber dimensions $\dim X_y$. Then the base change morphism

$$R^p f_*\mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$

is an isomorphism for every $y \in Y$. For that reason, base change always holds for the cohomology groups in the largest possible degree. Similarly, suppose that we have the vanishing $H^p(X_y, \mathscr{F}_y) = 0$ for every $y \in Y$ and every $p \geq p_0$. By repeatedly applying Corollary 9.9, we conclude that $R^p f_*\mathscr{F} = 0$ for all $p \geq p_0$. In this way, we can turn a fiberwise vanishing statement into the vanishing of the higher direct image sheaves, which will be useful if we are, for example, doing computations with the Leray spectral sequence.

The seesaw theorem. We now apply the base change theorem to the case of line bundles. Let X be a complete variety, T an arbitrary variety, and suppose we have a line bundle L on the product $X \times T$. For every $t \in T$, denote by L_t the restriction of L to $X \times \{t\}$. In the notation from above, we are working with the second projection $p_2 \colon X \times T \to T$, which is clearly proper and flat.

Theorem 9.10. *Under these assumptions, the set*

$$T_1 = \left\{\, t \in T \;\middle|\; L_t \textit{ is trivial on } X \times \{t\} \,\right\}$$

is closed in T, and there is a line bundle M on T_1 such that

$$L|_{X \times T_1} \cong p_2^* M.$$

Proof. We observe that a line bundle L on a proper variety X is trivial if and only if $H^0(X, L) \neq 0$ and $H^0(X, L^{-1}) \neq 0$. The reason is that a nonzero section $s \in H^0(X, L)$ gives a nonzero morphism $s \colon \mathscr{O}_X \to L$, and a nonzero section $t \in H^0(X, L^{-1})$ gives a nonzero morphism $t \colon L \to \mathscr{O}_X$. Their composition $t \circ s$ is a nonzero morphism from $\mathscr{O}_X$ to itself, hence a nonzero constant by properness. After multiplying by the inverse of this constant, we can assume that $t \circ s = 1$. But then $s \colon \mathscr{O}_X \to L$ is an isomorphism with inverse $t \colon L \to \mathscr{O}_X$.

This observation proves that

$$T_1 = \left\{\, t \in T \;\middle|\; \dim H^0(X \times \{t\}, L_t) \geq 1 \text{ and } \dim H^0(X \times \{t\}, L_t^{-1}) \geq 1 \,\right\}.$$

By Corollary 9.8, this is a closed subset of T. To prove the other half, we can replace T by T_1 and assume without loss of generality that L_t is trivial for every $t \in T$. Then $\dim H^0(X \times \{t\}, L_t) = 1$ is constant, and so the direct image sheaf

$$M = (p_2)_* L$$

is a locally free sheaf of rank 1, hence a line bundle. By construction, the induced morphism $p_2^* M \to L$ is an isomorphism on every fiber $X \times \{t\}$ (because L_t is trivial), and therefore an isomorphism on $X \times T$. ∎