Lecture 6 (February 13)

Translations. Our next goal is to prove a more precise version of the Kodaira embedding theorem for abelian varieties. In preparation for that, we first investigate how the group structure on a compact complex torus interacts with holomorphic line bundles.

Let $X = V/\Gamma$ be a compact complex torus. For every point $a \in X$, we have the translation automorphism

$$t_a \colon X \to X, \quad t_a(x) = a + x.$$

It is biholomorphic, with inverse t_{-a}. If we choose a vector $w \in V$ such that $q(w) = a$, where $q \colon V \to X$ is the quotient map, then t_a is induced by the linear translation $v \mapsto v + w$.

Let's consider the pullback $t_a^* L$, where L is a holomorphic line bundle on X. Write $L = L(H, \alpha)$, where (H, α) is a Appell-Humbert datum. Choose a vector $v_a \in V$ such that $q(v_a) = a$, where $q \colon V \to X$ is the quotient map. Then L is represented by the cocycle

$$\gamma \mapsto e_\gamma(v) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma)$$

and therefore $t_a^* L$ is represented by the cocycle

$$\gamma \mapsto e^{\pi H(v+w,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) = e^{\pi H(w,\gamma)} \cdot e_\gamma(v).$$

Therefore the tensor product $t_a^* L \otimes L^{-1}$ is represented by the constant cocycle

$$\gamma \mapsto e^{\pi H(w,\gamma)},$$

and is therefore an element of $\operatorname{Pic}^0(X)$. After modifying it by a coboundary

$$e^{\pi H(w,\gamma)} \cdot \frac{e^{-\pi H(v+\gamma,w)}}{e^{-\pi H(v,w)}} = e^{\pi H(w,\gamma) - \pi H(\gamma,w)} = e^{2\pi i E(w,\gamma)},$$

it becomes an Appell-Humbert datum for a unique line bundle in $\operatorname{Pic}^0(X)$, because $\gamma \mapsto e^{2\pi i E(w,\gamma)}$ is a group homomorphism from Γ to the circle group $U(1)$.

Example 6.1. If $c_1(L) = 0$, then we have $H = 0$, and therefore $t_a^* L \cong L$. So any holomorphic line bundle in $\operatorname{Pic}^0(X)$ is *translation-invariant*.

We see from these simple formulas that a holomorphic line bundle L determines a holomorphic group homomorphism

$$\varphi_L \colon X \to \operatorname{Pic}^0(X), \quad a \mapsto t_a^* L \otimes L^{-1}. \tag{6.2}$$

It is holomorphic because the cocycle $e^{\pi H(w,\gamma)}$ depends holomorphically on $w \in V$; and it is a group homomorphism because the cocycle is linear in w. Note that when $w \in \Gamma$, the cocycle $e^{2\pi i E(w,\gamma)}$ is trivial because $E(\Gamma \times \Gamma) \subseteq \mathbb{Z}$.

Lemma 6.3. *If the line bundle L is ample, the group homomorphism φ_L is surjective, and its kernel is a subgroup of X isomorphic to Γ^*/Γ. In particular, $\ker \varphi_L$ is a finite abelian group of order $(\dim H^0(X, L))^2$.*

Proof. If we again write $L = L(H, \alpha)$, then L is ample exactly when H is positive definite (and $E = \operatorname{Im} H$ is nondegenerate). This means that

$$V \to \operatorname{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C}), \quad w \mapsto H(w, -),$$

is an isomorphism of complex vector spaces. According to the discussion above, the image of φ_L therefore contains every line bundle in $\operatorname{Pic}^0(X)$ that can be represented by a cocycle of the form $e^{f(\gamma)}$, where $f \colon \bar{V} \to \mathbb{C}$ is $\mathbb{C}$-linear. But we have $\operatorname{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$ and $H^1(X, \mathscr{O}_X) \cong \operatorname{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C})$, and so this gives all line bundles in $\operatorname{Pic}^0(X)$.

Let's compute the kernel. We have seen that $\varphi_L(a)$ is represented by Appell-Humbert datum $(0, \gamma \mapsto e^{2\pi i E(w,\gamma)})$, and so it is trivial exactly when $E(w,\gamma) \in \mathbb{Z}$ for every $\gamma \in \Gamma$. Now E is nondegenerate, and so the map

$$V_{\mathbb{R}} \to \operatorname{Hom}_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R}), \quad w \mapsto E(w, -),$$

is an isomorphism of $\mathbb{R}$-vector spaces. Under this isomorphism, the subgroup $\Gamma^* = \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ corresponds exactly to those $w \in V_{\mathbb{R}}$ such that $E(w,\gamma) \in \mathbb{Z}$ for every $\gamma \in \Gamma$; the reason is that $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \Gamma$. Therefore

$$\ker \varphi_L = \left\{ w \in V \;\middle|\; E(w,\gamma) \in \mathbb{Z} \text{ for every } \gamma \in \Gamma \right\} / \Gamma \cong \Gamma^*/\Gamma.$$

As we saw during the proof of Theorem 4.6, this is a group of order

$$\det E = \big(\dim H^0(X,L)\big)^2,$$

and so the proof is complete. □

Example 6.4. When L is a principal polarization ($\det E = 1$), the group Γ^*/Γ is trivial; in that case, our homomorphism

$$\varphi_L \colon X \to \operatorname{Pic}^0(X)$$

is an isomorphism of abelian varieties. Later on, when we treat abelian varieties using algebraic methods, we are going to use this kind of result in order to *define* the Picard variety $\operatorname{Pic}^0(X)$.

The fact that φ_L is a group homomorphism means that

$$t_{a+b}^* L \otimes L^{-1} \cong t_a^* L \otimes L^{-1} \otimes t_b^* L \otimes L^{-1}.$$

If we clean this up a bit, it becomes

$$t_{a+b}^* L \otimes L \cong t_a^* L \otimes t_b^* L$$

for any two points $a, b \in X$. This result is known as the "theorem of the square".

The Lefschetz theorem. We are now going to prove a sharp version of the Kodaira embedding theorem.

Theorem 6.5 (Lefschetz)**.** *Let $L = L(H,\alpha)$ be a holomorphic line bundle such that the hermitian form H is positive definite.*

(a) *The line bundle L^2 is base-point free, and its global sections give a holomorphic mapping*

$$\varphi_2 \colon X \to \mathbb{P}\big(H^0(X, L^2)\big).$$

(b) *The line bundle L^3 is very ample, and its global sections give an embedding*

$$\varphi_3 \colon X \to \mathbb{P}\big(H^0(X, L^3)\big).$$

The numbers 2 and 3 are exactly as in the case of elliptic curves: any elliptic curve has a 2:1 map to $\mathbb{P}^1$, and can be embedded into $\mathbb{P}^2$ as a cubic curve. In general, by Corollary 5.3, we have

$$\dim H^0(X, L^k) = \frac{1}{n!} c_1(L^k)^n = k^n \dim H^0(X, L),$$

and so the projective spaces in question are fairly big once n gets larger.

Let's start by proving (a). According to Theorem 4.6, we have

$$\dim H^0(X, L) = \sqrt{\det E} \geq 1$$

because H is positive definite. Let $s_0 \in H^0(X,L)$ be any nontrivial section. The idea is to use translations in order to generate additional sections of L^2. Recall from above that

$$t_a^* L \otimes t_{-a}^* L \cong L^2$$

for any $a \in X$. This shows that $t_a^* s_0 \otimes t_{-a}^* s_0$ is a global section of L^2. The proof of (a) is now very easy. To show that L^2 is base-point free, we need to find, at any given point $x \in X$, a global section of L^2 that does not vanish at x. For that, we only have to choose $a \in X$ so that the two points $x \pm a$ do not lie on the zero locus of s_0; then ${t_a}^* s_0 \otimes t_{-a}^* s_0$ does the job.

It remains to prove (b). The argument that I gave in class was incomplete – as Spencer pointed out, I did not really prove that φ_3 is injective. So I am going to deviate from what I said in class, and use the notes to present Mumford's argument. Before doing that, let's briefly review a bit of general theory. Suppose that X is a compact complex manifold, and L a holomorphic line bundle that is base-point free. If we set $d = \dim H^0(X, L) - 1$, and choose a basis $s_0, \dots, s_d \in H^0(X, L)$, then we get a holomorphic mapping

$$\varphi \colon X \to \mathbb{P}^d, \quad x \mapsto \bigl(s_0(x), s_1(x), \dots, s_d(x)\bigr).$$

It is proper because X is compact. To show that φ is an embedding, we have to prove two things:

(1) φ is injective. By compactness, this ensures that φ is a homeomorphism between X and $\varphi(X)$.
(2) φ is an immersion. Concretely, this means that for every $x \in X$, the map on tangent spaces $d\varphi_x \colon T_x X \to T_{\varphi(x)} \mathbb{P}^d$ is injective. This ensures that $\varphi(X)$ is a complex manifold and φ is biholomorphic.

Proof that φ_3 is injective. Let's now prove (1) for the line bundle L^3. Recall that global sections of $L = L(H, \alpha)$ are theta functions for (H, α); these are holomorphic functions $\theta \colon V \to \mathbb{C}$ that satisfy the functional equation

$$\theta(v + \gamma) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) \cdot \theta(v). \tag{6.6}$$

For any two vectors $u, w \in V$, the product

$$\theta(v - u)\theta(v - w)\theta(v + u + w)$$

is a theta function for $(3H, \alpha^3)$, and therefore a global section of L^3. Suppose that there are two points $x_1, x_2 \in X$ with $\varphi_3(x_1) = \varphi_3(x_2)$. If we lift $x_1, x_2 \in X$ to vectors $v_1, v_2 \in V$, then it follows that there is a constant $C \neq 0$ such that

$$\phi(v_1) = C\phi(v_2)$$

for every theta function ϕ for the Appell-Humbert datum $(3H, \alpha^3)$. In particular, for every pair of vectors $v, w \in V$, we will have

$$\theta(v_1 - v)\theta(v_1 - w)\theta(v_1 + v + w) = C\theta(v_2 - v)\theta(v_2 - w)\theta(v_2 + v + w) \tag{6.7}$$

for all theta function for (H, α). We are going to deduce from this condition that $v_2 - v_1 \in \Gamma$, and hence that $x_1 = x_2$.

Consider (6.7) as a function of $v \in V$. To eliminate the constant C, we take logarithmic derivatives. Let $\omega = (d\theta)/\theta$, which is a meromorphic 1-form on V. After differentiating (6.7), we obtain

$$\omega(v_1 + v + w) - \omega(v_1 - v) = \omega(v_2 + v + w) - \omega(v_2 - v),$$

and so the meromorphic 1-form $\omega(v_2 + v) - \omega(v_1 + v)$ is invariant under translation by arbitrary elements of V, hence constant. We can therefore write it as $df(v)$, where $f \colon V \to \mathbb{C}$ is $\mathbb{C}$-linear. Since $\omega(v_2 + v) - \omega(v_1 + v)$ is the logarithmic derivative of $\theta(v_2 + v)/\theta(v_1 + v)$, it follows that there is a constant $A \in \mathbb{C}$ such that

$$\theta(v + v_2) = Ae^{f(v)}\theta(v + v_1)$$

for every $v \in V$. Set $w = v_2 - v_1$, and replace v by $v - v_1$ to put this into the form

$$\theta(v + w) = Be^{f(v)}\theta(v),$$

where $B \in \mathbb{C}$ is some other constant.

If we now substitute into the functional equation in (6.6) and cancel terms that appear on both sides, we get $e^{\pi H(w,\gamma)} = e^{f(\gamma)}$ for every $\gamma \in \Gamma$. This means that

$$\pi H(w,\gamma) - f(\gamma) \in 2\pi i \cdot \mathbb{Z}.$$

Recalling that $E = \operatorname{Im} H$, we have

$$\pi H(w,\gamma) - f(\gamma) = \pi H(\gamma, w) - f(\gamma) + 2\pi i E(w,\gamma) \in 2\pi i \cdot \mathbb{Z},$$

and so $\pi H(\gamma, w) - f(\gamma) \in i \cdot \mathbb{R}$. Because it is also $\mathbb{C}$-linear in the first argument, it follows that

$$\pi H(v,w) = f(v) \qquad \text{for every } v \in V. \tag{6.8}$$

We conclude that $E(w,\gamma) \in \mathbb{Z}$ for every $\gamma \in \mathbb{Z}$, and so our vector $w = v_2 - v_1$ belongs to the larger lattice

$$\hat{\Gamma} = \left\{ v \in V \;\middle|\; E(v,\gamma) \in \mathbb{Z} \text{ for every } \gamma \in \mathbb{Z} \right\}.$$

Recall that $\hat{\Gamma} \cong \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$, and that $\hat{\Gamma}/\Gamma$ is a finite group of order $\det E$. This already shows that some integer multiple of w lies in Γ.

We are going to finish the proof of (1) by showing that $w \in \Gamma$. Observe that θ is actually a theta function for the larger lattice $\Gamma' = \Gamma + \mathbb{Z}w$. The reason is that, because of (6.8), we have

$$\theta(v+w) = Be^{\pi H(v,w)}\theta(v) = Be^{-\frac{\pi}{2}H(w,w)} \cdot e^{\pi H(v,w) + \frac{\pi}{2}H(w,w)}\theta(v).$$

Because an integer multiple of w lies in Γ, the constant $Be^{-\frac{\pi}{2}H(w,w)}$ must be of absolute value 1, and so we can extend $\alpha \colon \Gamma \to U(1)$ uniquely to $\alpha' \colon \Gamma' \to U(1)$ by requiring that $\alpha'(w) = Be^{-\frac{\pi}{2}H(w,w)}$ and $\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)e^{i\pi E(\gamma,\delta)}$ for all $\gamma, \delta \in \Gamma'$. With this choice, every theta function θ for the pair (H, α) and the lattice Γ is then also a theta function for the pair (H, α') and the bigger lattice Γ'.

The dimension of the space of theta functions for (H, α) and Γ is, according to Theorem 4.6, equal to the square root of the order of the group Γ^*/Γ. If $\Gamma' \neq \Gamma$, then this is strictly larger than the order of the group Γ'^*/Γ', and so for dimension reasons, it is not possible for every theta function for Γ to also be a theta function for Γ'. The conclusion is that $\Gamma' = \Gamma$, and hence that $w \in \Gamma$. This proves that φ_3 is injective.

Proof that φ_3 is an immersion. Next, we prove (2) for φ_3. Suppose there is a point $x_0 \in X$ and a tangent vector $\xi \in T_{x_0}X$ that is mapped to zero under the differential of φ_3. Choose a basis $v_1, \dots, v_n \in V$ and let $z_1, \dots, z_n \in V^*$ be the dual basis; as usual, we view $z_1, \dots, z_n$ as coordinates on V, and hence as local coordinates on X. Write $\xi = \sum_{j=1}^n c_j \partial/\partial_j$. Choose a lifting of $x_0 \in X$ to a vector $v_0 \in V$. After computing the derivatives in an affine coordinate chart on projective space, we find that there is a constant $c_0 \in \mathbb{C}$ such that

$$\sum_{j=1}^n c_j \frac{\partial \phi}{\partial z_j}(v_0) = c_0 \phi(v_0)$$

for every theta function ϕ for the pair $(3H, \alpha^3)$. As before, we apply this to functions of the form $\phi(v) = \theta(v-u)\theta(v-w)\theta(v+u+w)$ with $u, w \in V$, where θ is any theta function for the pair (H, α). For given θ, consider the meromorphic function

$$f = \theta^{-1} \sum_{j=1}^n c_j \frac{\partial \theta}{\partial z_j}.$$

After substituting into the relation above, we get

$$f(v_0 - u) + f(v_0 - w) + f(v_0 + u + w) = c_0$$

for all $u, w \in V$. By the usual argument with first derivatives, it follows that $f(v) = \ell(v) + f(0)$ for a linear functional $\ell\colon V \to \mathbb{C}$.

Define $c = \sum_{j=1}^{n} c_j v_j \in V$. We compute that

$$\frac{d}{dt}\theta(v+tc) = \sum_{j=1}^{n} c_j \frac{\partial \theta}{\partial z_j}(v+tc) = \bigl(t\ell(c) + f(v)\bigr) \cdot \theta(v+tc).$$

After integration, this leads to the identity

$$\theta(v+tc) = e^{\frac{1}{2}t^2\ell(c)+tf(v)}\theta(v)$$

for every $v \in V$ and every $t \in \mathbb{C}$. If we now plug this into the functional equation in (6.6) and cancel terms that appear on both sides, we find that

$$e^{\pi H(tc,\gamma)} = e^{\frac{1}{2}t^2\ell(c)+tf(v)}.$$

By varying $v \in V$, we conclude that $f = 0$, and hence that $\ell = 0$. By varying $t \in \mathbb{C}$, it follows that $H(c,\gamma) = 0$ for every $\gamma \in \Gamma$. Because H is nondegenerate, this finally gives $c = 0$. We conclude that $\xi = 0$, and hence that φ_3 is indeed an immersion.