Lecture 3 (February 4)

Cohomology of compact complex tori. Let $X = V/\Gamma$ be a compact complex torus of dimension n. This means that V is an n-dimensional complex vector space, and $\Gamma \subseteq V$ is a lattice of rank $2n$. Let's see how to describe the cohomology of X in terms of V and Γ. Observe that Γ generates the underlying $\mathbb{R}$-vector space

$$V_{\mathbb{R}} \cong \Gamma \otimes_{\mathbb{Z}} \mathbb{R},$$

because $\Gamma \cong \mathbb{R}^{2n}$ and $V_{\mathbb{R}} \cong \mathbb{R}^{2n}$. Over the complex numbers, we have

$$\Gamma \otimes_{\mathbb{Z}} \mathbb{C} \cong V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \bar{V},$$

where $\bar{V}$ denotes the conjugate vector space: the underlying abelian group is still $(V, +)$, but the complex numbers act via $z \cdot v = \bar{z}v$. This is true for the complexification of any complex vector space. Indeed, let $J \in \operatorname{End}_{\mathbb{R}}(V_{\mathbb{R}})$ be the endomorphism $J(v) = iv$; then $J^2 = -\operatorname{id}$. The complexification

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E_i(J) \oplus E_{-i}(J)$$

decomposes into the $\pm i$-eigenspaces of J, and the two maps

$$V \to E_i(J), \quad v \mapsto v \otimes 1 - Jv \otimes i$$
$$\bar{V} \to E_{-i}(J), \quad v \mapsto v \otimes 1 + Jv \otimes i$$

are isomorphisms of $\mathbb{C}$-vector spaces.

As complex vector spaces, $V \cong T_0 X$ is isomorphic to the holomorphic tangent space at the point $0 \in X$. Since X is a group, the holomorphic tangent bundle is trivial; this means that we have a natural isomorphism

$$T_X \cong \mathscr{O}_X \otimes_{\mathbb{C}} V.$$

Dually, we get $\Omega_X^1 \cong \mathscr{O}_X \otimes_{\mathbb{C}} V^*$, and therefore

$$\Omega_X^p \cong \mathscr{O}_X \otimes_{\mathbb{C}} \bigwedge^p V^*.$$

We can also describe the lattice Γ intrinsically:

$$\Gamma \cong \pi_1(X, 0) \cong H_1(X, \mathbb{Z}),$$

where an element $\gamma \in \Gamma$ corresponds to the homotopy class (or homology class) of the closed loop $[0, 1] \to X$, $t \mapsto t \cdot \gamma + \Gamma$. According to the universal coefficients theorem, we then have

$$H^1(X, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}\big(H_1(X, \mathbb{Z}), \mathbb{Z}\big) \cong \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) = \Gamma^*.$$

The entire integral cohomology is equally easy to describe.

Lemma 3.1. *We have* $H^k(X, \mathbb{Z}) \cong \bigwedge^k H^1(X, \mathbb{Z}) \cong \bigwedge^k \Gamma^*$.

Proof. The cup product gives us a natural map

$$\bigwedge^k H^1(X, \mathbb{Z}) \to H^k(X, \mathbb{Z}), \quad \gamma_1 \wedge \cdots \wedge \gamma_k \mapsto \gamma_1 \cup \cdots \cup \gamma_k.$$

Since $X \cong (\mathbb{S}^1)^{2n}$ as smooth manifolds, the Künneth formula implies that the map is an isomorphism. ∎

We can also describe the de Rham cohomology and the Dolbeault cohomology, by relating V and $\bar{V}$ to differential forms on X. Choose a basis $v_1, \dots, v_n \in V$, and let $z_1, \dots, z_n \in V^*$ be the dual basis; we view these linear functions as a holomorphic system of coordinates on $V \cong \mathbb{C}^n$. Their differentials are invariant under translation by Γ, and so they give us well-defined 1-forms

$$dz_1, \dots, dz_n \in A^{1,0}(X), \quad d\bar{z}_1, \dots, d\bar{z}_n \in A^{0,1}(X)$$

on $X = V/\Gamma$. Every smooth form $\alpha \in A^{p,q}(X)$ of type (p,q) can then be written as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{I,J}\, dz_I \wedge d\bar{z}_J,$$

with coefficients $\alpha_{I,J} \in A^0(X)$ that are smooth fuctions on X.

Lemma 3.2. *The (p,q)-forms of the shape*

$$\sum_{|I|=p,|J|=q} c_{I,J}\, dz_I \wedge d\bar{z}_J$$

with $c_{I,J} \in \mathbb{C}$ give a basis for the Dolbeault cohomology group $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

Proof. This is a consequence of the Hodge theorem that we proved last semester. Choose a hermitian inner product h on V. It determines a hermitian metric on V and on $X = V/\Gamma$, whose associated $(1,1)$-form is

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h(v_j, v_k)\, dz_j \wedge d\bar{z}_k.$$

This is obviously closed, and so the metric is Kähler. By the Hodge theorem, every de Rham (and Dolbeault) cohomology class contains a unique harmonic representative. But the harmonic forms for this metric are exactly the forms with constant coefficients, as in the statement of the lemma. (Mumford's book contains a more elementary proof, using Fourier series.) $\square$

We can also say this without choosing coordinates. In degree 1, the isomorphism $V^* \cong H^{1,0}(X)$ sends a linear functional $f\colon V \to \mathbb{C}$ to the holomorphic 1-form df; the isomorphism $\bar{V}^* \cong H^{0,1}(X)$ sends a conjugate-linear functional $f\colon V \to \mathbb{C}$ to the anti-holomorphic 1-form df. In higher degrees, we have

$$H^{p,q}(X) \cong \bigwedge^p V^* \otimes \bigwedge^q \bar{V}^*,$$

by taking wedge products.

The above description of integral cohomology (in terms of Γ) and de Rham cohomology (in terms of V and $\bar{V}$) are compatible in the following way: the diagram

$$\begin{tikzcd} H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^1(X,\mathbb{C}) \arrow[r] \arrow[d, "\cong"] & H^1(X,\mathscr{O}_X) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C}) \arrow[r] & \mathrm{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C}) \end{tikzcd}$$

is commutative. The second arrow in the bottom row is the projection

$$\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V \oplus \bar{V}, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C}).$$

The commutativity of the diagram requires a little bit of checking that I will skip.

Holomorphic line bundles. Our next goal is to describe all holomorphic line bundles on $X = V/\Gamma$, in a way that is suitable for determining their spaces of sections and deciding which line bundles are ample. In particular, this will tell us which compact complex tori can be embedded into projective space.

One way to describe holomorphic line bundles is via the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_X \xrightarrow{e^{2\pi i(-)}} \mathscr{O}_X^\times \longrightarrow 0.$$

The long exact sequence in cohomology reads

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^\times) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X).$$

Let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of holomorphic line bundles; this is a group under tensor product. We have $\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$, and so we get a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} \ker\big(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)\big) \longrightarrow 0,$$

where $\mathrm{Pic}^0(X) \cong H^1(X,\mathscr{O}_X)/H^1(X,\mathbb{Z})$ is the set of isomorphism classes of topologically trivial holomorphic line bundles. The element $c_1(L) \in H^2(X,\mathbb{Z})$ is the first Chern class of the holomorphic line bundle $L \in \mathrm{Pic}(X)$.

Our starting point is the fact that on $V \cong \mathbb{C}^n$, all holomorphic line bundles are trivial. Let $q \colon V \to X$ be the quotient map. Given $L \in \mathrm{Pic}(X)$, the pullback

$$q^* L \cong V \times \mathbb{C}$$

is trivial. The group Γ acts on q^*L in a way that is compatible with the translation action on V. We can write this action in the form

$$\gamma \cdot (v,z) = \big(v+\gamma, e_\gamma(v)\cdot z\big),$$

where $e_\gamma \in \Gamma(V, \mathscr{O}_V^\times)$ is a nowhere vanishing holomorphic function on V. If we set $H^\times = \Gamma(V,\mathscr{O}_V^\times)$, we can write this more concisely as $e_\gamma \in H^\times$. The group Γ acts on $H^\times$ by translation, according to the rule

$$(\gamma \cdot e)(v) = e(v+\gamma).$$

Obviously, for $\gamma, \delta \in \Gamma$, we have

$$(\gamma+\delta)\cdot(v,z) = \gamma\cdot\delta\cdot(v,z),$$

and this translates into the cocycle condition

$$e_{\gamma+\delta}(v) = e_\gamma(v+\delta)\cdot e_\delta(v). \tag{3.3}$$

If we change the trivialization of q^*L by multiplying pointwise by a nowhere vanishing holomorphic function $g \in H^\times$, then our cocycle changes to

$$e'_\gamma(v) = e_\gamma(v)\cdot g(v+\gamma)/g(v). \tag{3.4}$$

If you know the definition of group cohomology, you may recognize that these two conditions are describing the first group cohomology $H^1(\Gamma, H^\times)$.

Group cohomology. Let's put our discussion of line bundles on hold for a moment and briefly review group cohomology. Let G be a group, and let M be a G-module; this means that M is an abelian group with a left action by G, or in other words, a left module over the group algebra $\mathbb{Z}G$. The subspace of G-invariants

$$M^G = \big\{\, m \in M \;\big|\; gm = m \text{ for all } g \in G \,\big\} = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$$

is a left-exact functor on G-modules, and group cohomology is the derived functors:

$$H^i(G,M) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M)$$

In practice, one uses a specific resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module to compute group cohomology. We therefore define $H^i(G,M)$ as the i-th cohomology of the following complex. For each $p \in \mathbb{N}$, set

$$C^p = C^p(G,M) = \{\text{functions } f \colon G^p \to M\},$$

and define the differential $d \colon C^p \to C^{p+1}$ by the formula

$$\begin{aligned}(df)(g_0,\dots,g_p) = g_0\cdot f(g_1,\dots,g_p) &+ \sum_{i=0}^{p-1}(-1)^{i+1} f(g_0,\dots,g_i g_{i+1},\dots,g_p) \\ &+ (-1)^{p+1} f(g_0,\dots,g_{p-1}).\end{aligned}$$

One checks that $d \circ d = 0$, and so this is indeed a complex.

Example 3.5. We have $C^0 = M$, and therefore

$$H^0(G, M) = \{ m \in M \mid gm = m \text{ for all } g \in G \} = M^G$$

is the space of G-invariants, as it should be. Let's also compute $H^1(G, M)$. Now a 1-chain is just a function $f\colon G \to M$, and because

$$(df)(g, h) = gf(h) - f(gh) + f(g),$$

the cocycle condition $df = 0$ translates into the identity

$$f(gh) = gf(h) + f(g), \qquad \text{for all } g, h \in G.$$

It follows that

$$H^1(G, M) = \frac{\{ f\colon G \to M \mid f(gh) = gf(h) + f(g) \}}{\{ g \mapsto gm - m \mid m \in M \}}.$$

In the discussion above, $H^\times = \Gamma(V, \mathscr{O}_V^\times)$ is a Γ-module, but with the group structure written multiplicatively. Taking this into account, the conditions in (3.3) and (3.4) are therefore exactly describing $H^1(\Gamma, H^\times)$.

There are two other useful facts. The first is that a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of G-modules gives rise to long exact sequence in group cohomology (as usual for the functors Ext^i). The second is that group cohomology can be used to compute sheaf cohomology. Suppose that $\mathscr{F}$ is a sheaf on $X = V/\Gamma$. Assuming that the pullback sheaf $q^*\mathscr{F}$ has no higher cohomology, one has

$$H^i(X, \mathscr{F}) \cong H^i\big(\Gamma, H^0(V, q^*\mathscr{F})\big),$$

where $H^0(V, q^*\mathscr{F})$ is a Γ-module. This says, for example, that

$$H^1(X, \mathscr{O}_X^\times) \cong H^1(\Gamma, H^\times),$$

as suggested by the discussion above.

Holomorphic line bundles, continued. We return to our study of holomorphic line bundles. From $L \in \mathrm{Pic}(X)$, we get a cohomology class in $H^1(\Gamma, H^\times)$, represented by the cocycle e_γ from (3.3). Conversely, a cocycle determines a holomorphic line bundle by letting Γ act on $V \times \mathbb{C}$ according to the rule

$$\gamma \cdot (v, z) = \big(v + \gamma, e_\gamma(v) \cdot z\big).$$

The quotient $(V \times \mathbb{C})/\Gamma \to V/\Gamma$ is then a holomorphic line bundle on $X = V/\Gamma$. So all we need is nice description of these cocycles.

Let's start by describing the possible first Chern classes

$$c_1(L) \in \ker\big(H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X)\big).$$

We know that

$$H^2(X, \mathbb{Z}) \cong \bigwedge^2 H^1(X, \mathbb{Z}) \cong \bigwedge^2 \Gamma^*,$$

and so each cohomology class is represented uniquely by an alternating form

$$E\colon \Gamma \times \Gamma \to \mathbb{Z}.$$

When is such a class in the kernel of $H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X)$? We can extend E uniquely to an alternating bilinear form on

$$\Gamma_{\mathbb{Z}} \times \mathbb{C} \cong V \oplus \bar{V},$$

and since $H^2(X, \mathscr{O}_X) \cong \bigwedge \bar{V}^*$, this extension needs to be trivial on $\bar{V} \times \bar{V}$. This translates into the condition that

$$E\big(v \otimes 1 + Jv \otimes i, w \otimes 1 + Jw \otimes i\big) = 0$$

for $v, w \in V$. Expanding and looking at the real part, we deduce that

$$E(Jv, Jw) = E(v, w) \qquad \text{for all } v, w \in V. \tag{3.6}$$

It is easy to see that this condition is equivalent to the existence of a hermitian bilinear form

$$H \colon V \times V \to \mathbb{C}$$

such that $E = \operatorname{Im} H$. Indeed, H must be given by the formula

$$H(v, w) = E(Jv, w) + iE(v, w),$$

and the condition in (3.6) ensures that H is hermitian symmetric. We can summarize this in the following lemma.

Lemma 3.7. *An alternating bilinear form $E \colon \Gamma \times \Gamma \to \mathbb{Z}$ represents the first Chern class of a holomorphic line bundle on X iff there is a hermitian form $H \colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$.*

Equivalently, we can start from the hermitian form $H \colon V \times V \to \mathbb{C}$, and then the condition is that $E = \operatorname{Im} H$ needs to take integer values on the subset $\Gamma \times \Gamma$.

Note. If this seems too abstract, here is a more concrete way of thinking about the lemma. Let's start from a hermitian form $H \colon V \times V \to \mathbb{C}$. Choose a basis $v_1, \dots, v_n \in V$, and let $z_1, \dots, z_n \in V^*$ be the dual basis. Setting $h_{j,k} = H(v_j, v_k)$, we get a closed $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k \in A^{1,1}(X),$$

and the fact that H is hermitian ensures that $\omega \in A^2(X, \mathbb{R})$. In order for ω to be the first Chern class of a holomorphic line bundle, the cohomology class $[\omega] \in H^2(X, \mathbb{R})$ needs to be in the image of $H^2(X, \mathbb{Z})$, which means that the integral of ω over every homology class in $H_2(X, \mathbb{Z})$ should be an integer. A basis for $H_2(X, \mathbb{Z})$ is given by the images of the maps

$$c_{\gamma,\delta} \colon [0,1]^2 \to X, \quad (s,t) \mapsto s\gamma + t\delta + \Gamma$$

for $\gamma, \delta \in \Gamma$. Writing $\gamma = \sum_j \gamma_j v_j$ and $\delta = \sum_j \delta_j v_j$, the integral of ω over the image of the map $c_{\gamma,\delta}$ is then

$$\int_{[0,1]^2} c_{\gamma,\delta}^* \omega = \int_0^1 \int_0^1 \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} (\gamma_j ds + \delta_j dt) \wedge \bigl(\bar{\gamma}_k ds + \bar{\delta}_k dt\bigr) = \operatorname{Im} H(\gamma, \delta).$$

So the condition is precisely that $E = \operatorname{Im} H$ should take integer values on $\Gamma \times \Gamma$.

Now let's compute the first Chern class from the cocycle e_γ in (3.3). Setting $H = \Gamma(V, \mathscr{O}_V)$, we have a short exact sequence of Γ-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \xrightarrow{e^{2\pi i(-)}} H^\times \longrightarrow 0,$$

and therefore a long exact sequence in cohomology. The connecting homomorphism $\delta \colon H^1(\Gamma, H^\times) \to H^2(\Gamma, \mathbb{Z})$ fits into a commutative diagram

$$\begin{tikzcd}
H^1(\Gamma, H^\times) \arrow[r, "\delta"] \arrow[d, "\cong"] & H^2(\Gamma, \mathbb{Z}) \arrow[d, "\cong"] \arrow[dr, bend left, "\cong"] & \\
H^1(X, \mathscr{O}_X^\times) \arrow[r, "c_1"] & H^2(X, \mathbb{Z}) \arrow[r, "\cong"] & \bigwedge^2 \Gamma^*.
\end{tikzcd}$$

Our cocycle $e = \{e_\gamma\}$ is an element in $C^1(\Gamma, H^\times)$. To compute its image under the connecting homomorphism, we need to lift it to $f = \{f_\gamma\} \in C^1(\Gamma, H)$, and then apply the differential $d\colon C^1(\Gamma, H) \to C^2(\Gamma, H)$. So we write

$$e_\gamma(v) = e^{2\pi i f_\gamma(v)}$$

with $f_\gamma \in H$, and then

$$F(\gamma, \delta) = (df)(\gamma, \delta) = f_\delta(v + \gamma) - f_{\gamma+\delta}(v) + f_\gamma(v) \in \mathbb{Z}.$$

Under the isomorphism $H^2(\Gamma, \mathbb{Z}) \cong \bigwedge^2 \Gamma^*$, this 2-cocycle then goes to the alternating form $E\colon \Gamma \times \Gamma \to \mathbb{Z}$ given by the formula

$$E(\gamma, \delta) = F(\gamma, \delta) - F(\delta, \gamma) = \big(f_\delta(v + \gamma) - f_\delta(v)\big) - \big(f_\gamma(v + \delta) - f_\gamma(v)\big). \tag{3.8}$$

Since this is the first Chern class of a line bundle, we have $E = \operatorname{Im} H$ for a hermitian form $H\colon V \times V \to \mathbb{C}$.

The Appel-Humbert theorem. We can now solve our problem in the following way. Fix a hermitian form $H\colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. Let's describe all line bundles $L \in \operatorname{Pic}(X)$ such that $c_1(L) \in H^2(X, \mathbb{Z})$ is represented by E. There are two cases.

The first (and easier) case is when $H = 0$. Here we are looking for line bundles $L \in \operatorname{Pic}(X)$ with $c_1(L) = 0$. Recall that

$$\operatorname{Pic}^0(X) \cong H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathscr{O}_X^\times).$$

From Hodge theory, we have an isomorphism of $\mathbb{R}$-vector spaces

$$H^1(X, \mathbb{R}) \to H^1(X, \mathscr{O}_X)$$

and therefore $\operatorname{Pic}^0(X) \cong H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) \cong H^1(X, \mathbb{R}/\mathbb{Z})$. Since $x \mapsto e^{2\pi i x}$ maps $\mathbb{R}/\mathbb{Z}$ isomorphically to the circle group $U(1) = \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$, the image of $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ in $\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$ is therefore isomorphic to

$$\operatorname{Hom}_{\mathbb{Z}}\big(H_1(X, \mathbb{Z}), U(1)\big) \cong \operatorname{Hom}_{\mathbb{Z}}\big(\Gamma, U(1)\big).$$

In terms of cocycles, this means that every group homomorphism

$$\alpha\colon \Gamma \to U(1)$$

gives us a constant cocycle $e_\gamma(v) = \alpha(\gamma)$; this obviously satisfies the cocycle condition in (3.3). So we get

$$\operatorname{Hom}_{\mathbb{Z}}\big(\Gamma, U(1)\big) \cong \operatorname{Pic}^0(X), \quad \alpha \mapsto e_\gamma \equiv \alpha(\gamma).$$

The general case is when $H \neq 0$. Here the best possible choice of cocycle is

$$e_\gamma(v) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \cdot \alpha(\gamma), \tag{3.9}$$

where $\alpha\colon \Gamma \to U(1)$. This needs to satisfy the cocycle condition in (3.3), and so

$$e^{\pi H(v,\gamma+\delta) + \frac{\pi}{2} H(\gamma+\delta,\gamma+\delta)} \alpha(\gamma + \delta) = e^{\pi H(v+\delta,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) e^{\pi H(v,\delta) + \frac{\pi}{2} H(\delta,\delta)} \alpha(\delta).$$

After cancelling common factors and remembering that $E = \operatorname{Im} H$, this turns into

$$\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta) \cdot e^{i\pi E(\gamma,\delta)}. \tag{3.10}$$

So $\alpha\colon \Gamma \to U(1)$ is no longer a group homomorphism, but it is not off by very much because $e^{i\pi E(\gamma,\delta)} = \pm 1$.

We also need to make sure that the first Chern class is represented by $E = \operatorname{Im} H$. Going back to (3.8), the condition is that

$$\big(f_\delta(v + \gamma) - f_\delta(v)\big) - \big(f_\gamma(v + \delta) - f_\gamma(v)\big) = E(\gamma, \delta).$$

For e_γ as in (3.9), the lifting is

$$f_\gamma = \frac{H(v, \gamma)}{2i} + \frac{H(\gamma, \gamma)}{4i} + \frac{1}{2\pi i} \log \alpha(\gamma),$$

which is of course only determined up to $\mathbb{Z}$ (because of the logarithm). After plugging this into the formula above, we get

$$\frac{H(\gamma,\delta) - H(\delta,\gamma)}{2i} = E(\gamma,\delta),$$

which is correct because $E = \operatorname{Im} H$. (In fact, (3.9) is determined uniquely if we look for a lifting f_γ that is affine linear in v and satisfies the equation above.)

Definition 3.11. Let $H\colon V \times V \to \mathbb{C}$ be a hermitian form such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. For any $\alpha\colon \Gamma \to U(1)$ such that (3.10) holds, we define the holomorphic line bundle

$$L(H,\alpha) = (V \times \mathbb{C})/\Gamma$$

over $X = V/\Gamma$, where the Γ-action is given by

$$\gamma \cdot (v,z) = \Big(v + \gamma,\, e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) \cdot z\Big).$$

Then (H,α) is called the *Appel-Humbert datum* for the line bundle $L(H,\alpha)$.

The main result (that we have almost proved at this point) is that

$$\operatorname{Pic}(X) = \big\{\, L(H,\alpha) \;\big|\; (H,\alpha) \text{ is an Appel-Humbert datum} \,\big\}$$

describes all holomorphic line bundles on X. More on this next time.